Unbounded Normal Algebras and Spaces of Fractions

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Abstract. We consider arbitrary families of unbounded normal operators, commuting in a strong sense, in particular algebras consisting of unbounded normal operators, and investigate their connections with some algebras of fractions of continuous functions on compact spaces. New examples and properties of spaces of fractions are also given.

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1. Introduction

The classical Cayley transform $\kappa(t) = (t-i)(t+i)^{-1}$ is a bijective map between the real line $\mathbb{R}$ and the set $\mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ is the unit circle in the complex plane. If $p(t) = \sum_{k=0}^{n} a_k t^k$ is a polynomial of one real variable, generally with complex coefficients, then the function

$$p \circ \kappa^{-1}(z) = \sum_{k=0}^{n} i^k a_k (1+z)^k (1-z)^{-k} = \sum_{k=0}^{n} (-1)^k a_k (\Im z)^k (1 - \Re z)^{-k},$$

defined on $\mathbb{T} \setminus \{1\}$, is a sum of fractions with denominators in the family $\{(1-z)^k; k \geq 0\}$, or in the family $\{(1-\Re z)^k; k \geq 0\}$, the latter consisting of positive functions on $\mathbb{T}$. This remark allows us to identify the algebra of polynomials of one real variable with sub-algebras of some algebras of fractions. As a matter of fact, a similar identification can be easily obtained for polynomials in several real variables. We shall use this idea to describe the structure of some algebras of unbounded operators.

We start with some notation and terminology for Hilbert space linear operators. For the properties of the unbounded operators, in particular unbounded self-adjoint and normal operators, and the Cayley transform, we refer to [**?**], Chapter 13.

Let $\mathcal{H}$ be a complex Hilbert space. For a linear operator T acting in $\mathcal{H}$, we denote by $\mathcal{D}(T)$ its domain of definition. If T is closable, the closure of T will be denoted by $\bar{T}$. If T is densely defined, let T^* be its adjoint. We write $T_1 \subset T_2$ to designate that T_2 extends T_1.

We recall that a densely defined closed operator T is said to be *normal* (resp. *self-adjoint*) if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T^*T = TT^*$ (resp. $T = T^*$).

If $\mathcal{D}(T) = \mathcal{D}(T^*)$, the equality $T^*T = TT^*$ is equivalent to $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{D}(T)$ (see [11], Part II). Clearly, every self-adjoint operator is normal.

Let $\mathcal{D}$ be a dense linear subspace of $\mathcal{H}$, let $\mathcal{L}(\mathcal{D})$ be the algebra of all linear mappings from $\mathcal{D}$ into $\mathcal{D}$, and let, for further use, $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$.

Set

$$\mathcal{L}^{\#}(\mathcal{D}) = \{T \in \mathcal{L}(\mathcal{D}); \mathcal{D}(T^*) \supset \mathcal{D}, T^*\mathcal{D} \subset \mathcal{D}\}.$$

If we put $T^{\#} = T^*|\mathcal{D}$, the set $\mathcal{L}^{\#}(\mathcal{D})$ is an algebra with involution $T \to T^{\#}$.

Let $\mathcal{N}$ be a subalgebra of $\mathcal{L}^{\#}(\mathcal{D})$ such that for every $T \in \mathcal{N}$ one has $T^{\#} \in \mathcal{N}$. We also assume that the identity on $\mathcal{D}$ belongs to $\mathcal{N}$. With the terminology from [20], the algebra $\mathcal{N}$ is an *O^*-algebra*.

As in the bounded case (see [18], Definition 11.24), we say that an O^*-algebra $\mathcal{N}$ is a *normal algebra* if $\mathcal{N}$ consists of (not necessarily bounded) operators whose closures are normal operators.

One of the main aims of this work is to describe normal algebras in terms of algebras of fractions (see Theorem 4.8). We briefly recall the definition of the latter algebras in a commutative framework. In the next section, a more general framework will be presented.

Let $\mathcal{A}$ be a commutative unital complex algebra and let $\mathcal{M} \subset \mathcal{A}$ be a *set of denominators*, that is, a subset closed under multiplication, containig the unit and such that if $ma = 0$ for some $m \in \mathcal{M}$ and $a \in \mathcal{A}$, then $a = 0$. Under these conditions, we can form the algebra of fractions $\mathcal{A}/M$ consisting of the equivalence classes modulo the relation $(a_1, m_1) \sim (a_2, m_2)$ if $a_1m_2 = a_2m_1$ for $a_1, a_2 \in \mathcal{A}$ and $m_1, m_2 \in \mathcal{M}$, endowed with a natural algebraic structure (see [6] or [24] for some details; see also the next section for a more complete discussion). The equivalence class of the pair (a, m) will be denoted by a/m.

If $\mathcal{A}$ is a commutative unital normed complex algebra whose completion $\mathcal{B}$ is a semisimple Banach algebra, and $\mathcal{M} \subset A$ is a set of denominators, the Gelfand representation allows us to replace $\mathcal{A}/\mathcal{M}$ by an algebra of fractions of continuous functions. Specifically, if $\hat{a}$ denotes the Gelfand transform of any $a \in \mathcal{A}$ (computed in $\mathcal{B}$), and $\hat{\mathcal{S}}$ stands for the set $\{\hat{a};\, a \in \mathcal{S}\}$ for a subset $\mathcal{S} \subset A$, then $\hat{\mathcal{M}}$ is a set of denominators in $\hat{\mathcal{A}}$ and the natural assignement $\mathcal{A}/\mathcal{M} \ni a/m \to \hat{a}/\hat{m} \in \hat{\mathcal{A}}/\hat{\mathcal{M}}$ is a unital algebra isomorphism, as one can easily see.

The remark from above shows that the algebras of fractions of continuous functions on compact spaces are of particular interest. Their role in the study of scalar and operator moment problems, and in extension problems, has been already emphasized in [2] and [25]. In the present work, we continue the investigation concerning the algebras of fractions of continuous functions, showing their usefulness in the representation of normal algebras.

We shall briefly describe the contents of this article. In the next section, following some ideas from [6], we present a general construction of spaces of fractions, sufficiently large to allow a partially noncommutative context. Some exemples, more or less related to moment problems, are also given.

As mentioned before, algebras of fractions of continuous functions on compact spaces are of particular interest. In the third section, we describe the dual of some spaces of fractions of continuous functions (Theorem 3.1), extending a result from [25], stated as Corollary 3.2. Other examples of algebras of fractions of continuous functions, needed in the next sections, some of them considering functions depending on infinitely many variables, are also presented.

In the fourth section, we introduce and discuss the algebras of unbounded normal operators. To obtain a representation theorem of such an algebra, identifying it with a subalgebra of an algebra of fractions of continuous functions (see Theorem 4.8), we need versions of the spectral theorem for infinitely many unbounded commuting self-adjoint or normal operators (see Theorem 4.2 and Theorem 4.3). This subject has been approached by many authors (see, for instance, [4] and [19]). A recent similar result of this type, which we are aware of, is the main theorem in [16], obtained in the context of semi-groups. Inspired by a result in [18] stated for one operator, our main tool to obtain the versions of the spectral theorem mentioned above is the Cayley transform, leading to an approach seemingly different from other approaches for infinitely many operators.

Some examples of unbounded normal algebras are also given in this section.

The last section contains an extension result of the so-called subnormal families of unbounded operators to normal ones. We use the main result from [2] to get a version of another result from [2], which, unlike in the quoted work, is proved here for families having infinitely many members (see Theorem 5.2), using our Theorem 4.3. The statement of Theorem 5.2 needs another type of an algebra of fractions, which is also included in this section.

Let us mention that fractional transformations have been recently used, in various contexts, in [2], [9], [10], [14], [15], [25] etc. See also [11], [23], [26] for other related results.

2. Spaces of fractions

In this section we present a general setting for a construction of spaces of fractions, associated with real or complex vector spaces, with denominators in appropriate families of linear maps. We adapt some ideas from [6].

Let $\mathbb{K}$ be either the real field $\mathbb{R}$ or the complex one $\mathbb{C}$, let E be a vector space over $\mathbb{K}$, and let $\mathcal{L}(E)$ be the algebra of all linear maps from E into itself.

Definition 2.1. Let $\mathcal{M} \subset \mathcal{L}(E)$ have the following properties:

(1) for all $M_1, M_2 \in \mathcal{M}$ we have $M_1 M_2 \in \mathcal{M}$;
(2) the identity map I_E on E belongs to $\mathcal{M}$;
(3) $\mathcal{M}$ is commutative;
(4) every map $M \in \mathcal{M}$ is injective.

Such a family $\mathcal{M} \subset \mathcal{L}(E)$ will be called a *set of denominators*.

Definition 2.2. Let $\mathcal{M} \subset \mathcal{L}(E)$ be a set of denominators. Two elements (x_1, M_1), (x_2, M_2) from $E \times \mathcal{M}$ are said to be *equivalent*, and we write $(x_1, M_1) \sim (x_2, M_2)$, if $M_1 x_2 = M_2 x_1$.

Remark 2.3. The relation $\sim$ given by Definition 2.2 is clearly reflexive and symmetric. It is also transitive because if $(x_1, M_1) \sim (x_2, M_2)$ and $(x_2, M_2) \sim (x_3, M_3)$, we infer easily that $M_2(M_1 x_3 - M_3 x_1) = 0$, via condition (3), whence $M_1 x_3 = M_3 x_1$, by condition (4). Consequently, the relation $\sim$ is an equivalence relation. This allows us to consider the set of eqivalence classes $E \times \mathcal{M}/\sim$, which will be simply denoted by $E/\mathcal{M}$. The eqivalence class of the element (x, M) will be denoted x/M.

The set $E/\mathcal{M}$ can be organized as a vector space with the algebraic operations

$$x_1/M_1 + x_2/M_2 = (M_2 x_1 + M_1 x_2)/M_1 M_2,\ x_1, x_2 \in E,\ M_1, M_2 \in \mathcal{M},$$

and

$$\lambda(x/M) = (\lambda x)/M,\ \lambda \in \mathbb{K},\ x \in E,\ M \in \mathcal{M},$$

which are easily seen to be correctly defined.

Definition 2.4. The vector space $E/\mathcal{M}$ will be called the *space of fractions* of E *with denominators in* $\mathcal{M}$.

Note that, if $\mathcal{A}_{\mathcal{M}}$ is the (commutative) algebra generated by $\mathcal{M}$ in $\mathcal{L}(E)$, the linear space $E/\mathcal{M}$ is actually an $\mathcal{A}_{\mathcal{M}}$-module, with the action given by

$$N(x/M) = (Nx)/M,\ x \in E,\ N \in \mathcal{A}_{\mathcal{M}},\ M \in \mathcal{M}.$$

If we regard the multiplication by $M_0 \in \mathcal{M}$ as a linear map on $E/\mathcal{M}$, then M_0 has an inverse on $E/\mathcal{M}$ defined by

$$M_0^{-1}(x/M) = x/(M_0 M),\ x \in E,\ M \in \mathcal{M}.$$

We also note that the map $E \ni x \mapsto x/I_E \in E/\mathcal{M}$ is injective, which allows the identification of E with the subspace $\{x/I_E,\, x \in E\}$ of $E/\mathcal{M}$. For this reason, the fraction x/M may be denoted by $M^{-1}x$ for all $x \in E,\ M \in \mathcal{M}$.

We define the subspaces

$$E/M = \{\xi \in E/\mathcal{M};\, M\xi \in E\},\ M \in \mathcal{M}.$$

We clearly have

$$E/\mathcal{M} = \bigcup_{M \in \mathcal{M}} E/M. \tag{2.1}$$

A set of denominators $\mathcal{M} \subset \mathcal{L}(E)$ has a natural division. Namely, if $M', M'' \in \mathcal{M}$, we write $M'|M''$, and say that M' *divides* M'' if there exists $M_0 \in \mathcal{M}$ such that $M'' = M'M_0$.

A subset $\mathcal{M}_0$ of $\mathcal{M}$ is said to be *cofinal* if for every $M \in \mathcal{M}$ we can find an $M_0 \in \mathcal{M}_0$ such that $M|M_0$.

If $M'|M''$, and so $M'' = M'M_0$, the map $E/M' \ni x/M' \mapsto M_0x/M'' \in E/M''$ is the restriction of the identity to E/M', showing that E/M' is a subspace of E/M''. This also shows that the vector space $E/\mathcal{M}$ is the inductive limit of the family of vector spaces $(E/M)_{M\in\mathcal{M}}$.

Remark 2.5. If there is a norm $\| * \|$ on the vector space E, each space E/M can be endowed with the norm

$$\|\xi\|_M = \|M\xi\|,\ \xi \in E/M,\ \ M \in \mathcal{M}. \tag{2.2}$$

Assuming also that $\mathcal{M}$ consists of bounded operators on E, it is easily seen that

$$\|\xi\|_{M''} \le \|M_0\|\|\xi\|_{M'},\ \xi \in E/M',$$

whenever $M'' = M'M_0$, showing that the inclusion $E/M' \subset E/M''$ is continuous, and so $E/\mathcal{M}$ can be viewed as an inductive limit of normed spaces. If F is a topological vector space and $T : E/\mathcal{M} \mapsto F$ is a linear map, then T is *continuous* if $T|_{E/M}$ is continuous for each $M \in \mathcal{M}$ (see [17] for details).

Remark 2.6. Assume that E is an ordered vector space and let E_+ be the positive cone of E. Let also $\mathcal{M} \subset \mathcal{L}(E)$ be a set of denominators. If one has $M(E_+) \subset E_+$ for all $M \in \mathcal{M}$ (i.e., every $M \in \mathcal{M}$ is a positive operator), we say that $\mathcal{M}$ is a set of *positive* denominators. If $\mathcal{M}$ is a set of positive denominators, we may define a positive cone $(E/M)_+$ in each space E/M by setting

$$(E/M)_+ = \{\xi \in E/M;\ M\xi \in E_+\}.$$

If F is another ordered vector space with the positive cone F_+, a linear map $\phi : E/\mathcal{M} \mapsto F$ is said to be *positive* if $\phi((E/M)_+) \subset F_+$ for all $M \in \mathcal{M}$.

Remark 2.7. Assume that $E = A$ is an algebra. Let $\mathcal{M} \subset \mathcal{L}(A)$ be a set of denominators. Also assume that M is an A-module map of the A-module A for all $M \in \mathcal{M}$. In other words, $M(ab) = aM(b)$ for all $a, b \in A$ and $M \in \mathcal{M}$. Then the fraction space $A/\mathcal{M}$ becomes an algebra, with the multiplication given by the relation

$$(a'/M')(a''/M'') = (a'a'')/(M'M''),\ a', a'' \in A,\ M', M'' \in \mathcal{M}.$$

Particularly, let A be an algebra with unit 1. For each $c \in A$ we set $M_c(a) = ca$, $a \in A$, i.e., the left multiplication map by c on A. A subset $Q \subset A$ is said to be a *set of denominators* if the family $\mathcal{M}_Q = \{M_q;\ q \in Q\} \subset \mathcal{L}(A)$ is a set of denominators. In this case, we identify Q and $\mathcal{M}_Q$ and write $A/\mathcal{M}_Q$ simply A/Q. If Q is in the center of A, then A/Q is an algebra.

Example 2.8. Let $\mathcal{A}$ be a complex $*$-algebra with unit 1, and let $L : \mathcal{A} \mapsto \mathbb{C}$ be a positive form on $\mathcal{A}$. This pair can be associated, in a canonical way, with a certain pre-Hilbert space, via the classical construction due to Gelfand and Naimark. To briefly recall this construction, let $\mathcal{N} = \{a \in \mathcal{A}; L(a^*a) = 0\}$. Since L satisfies the Cauchy-Schwarz inequality, it follows that $\mathcal{N}$ is a left ideal of $\mathcal{A}$. Moreover, the quotient $\mathcal{D} = \mathcal{A}/\mathcal{N}$ is a pre-Hilbert space, whose inner product is given by $\langle a+\mathcal{N}, b+\mathcal{N}\rangle = L(b^*a)$, $a, b \in \mathcal{A}$.

Note also that $\mathcal{D}$ is an $\mathcal{A}$-module. Therefore, we may define a linear map M_a on $\mathcal{D}$ associated to any $a \in \mathcal{A}$, via the relation $M_a(\tilde{x}) = (ax)\tilde{}$, $x \in \mathcal{A}$, where $\tilde{x} = x + \mathcal{N}$.

Let $\mathcal{C}$ be the center of $\mathcal{A}$, and fix $\mathcal{C}_0 \subset \mathcal{C}$ nonempty. The map M_{r_c} is injective if $r_c = 1 + c^*c$ for each $c \in \mathcal{C}_0$. Consequently, the set $\mathcal{M}$ of all maps of the form

$$M_{r_{c_1}}^{\alpha_1} \cdots M_{r_{c_m}}^{\alpha_m},$$

where $\alpha_1, \dots \alpha_m$ are arbirary nonnegative integres and $c_1, \dots, c_m$ are arbitrary elements from $\mathcal{C}_0$, is a set of denominators in $\mathcal{L}(\mathcal{D})$. This shows that we can consider the space of fractions $\mathcal{D}/\mathcal{M}$.

The following particular case is related to the classical Hamburger moment problem in several variables. Let us denote by $\mathbb{Z}_+^n$ the set of all multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, i.e., $\alpha_j \in \mathbb{Z}_+$ for all $j = 1, ..., n$. Let $\mathcal{P}_n$ be the algebra of all polynomial functions on $\mathbb{R}^n$, with complex coefficients, endowed with its natural involution. We denote by t^α the monomial $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, where $t = (t_1, ..., t_n)$ is the current variable in $\mathbb{R}^n$, and $\alpha \in \mathbb{Z}_+^n$.

If an n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ of real numbers is given, we associate it with the functional $L_\gamma : \mathcal{P}_n \to \mathbb{C}$, where $L_\gamma(t^\alpha) = \gamma_\alpha, \alpha \in \mathbb{Z}_+^n$. Assuming L_γ to be positive, we may perform the construction described above, and obtain a pre-Hilbert space $\mathcal{D}_\gamma = \mathcal{D}$. The spaces of fractions obtained as above will be related to the family $\mathcal{C}_0 = \{t_1, \dots, t_n\}$. Setting $A_j\tilde{p} = (t_jp)\tilde{}$, $p \in \mathcal{D}_\gamma$, $j = 1, ..., n$, the denominator set as above , say $\mathcal{M}_\gamma$, will be given by the family of maps of the form

$$(1 + A_1^2)^{\alpha_1} \cdots (1 + A_n^2)^{\alpha_m},$$

where $\alpha_1, \dots \alpha_m$ are arbirary nonnegative integres. And so, we can form the space of fractions $\mathcal{D}_\gamma/\mathcal{M}_\gamma$, as a particular case of the previous construction.

Example 2.9. Let $C[0,1]$ (resp. $C[0,1)$) be the algebra of all complex-valued continuous functions on the interval [0,1] (resp. [0,1)). We consider the Volterra operator $Vf(t) = \int_0^t f(t)dt,\ t \in [0,1)$, $f \in C[0,1)$, which is an injective map. Let $C_V[0,1)$ be the subspace of $C[0,1)$ consisting of those functions f such that $V^nf \in C[0,1]$ for some integer $n \geq 0$ (depending on f). Let $\mathcal{V} = \{V^n; n \geq 0\}$, which, regarded as a family of linear maps on $C[0,1]$, is a family of denominators. Therefore, we may form the space of fractions $C[0,1]/\mathcal{V}$. Note that the space $C_V[0,1)$ may be identified with a subspace of $C[0,1]/\mathcal{V}$. Indeed, if $f \in C_V[0,1)$ and $n \geq 0$ is such that $V^nf \in C[0,1]$, we identify f with the element $V^nf/V^n \in C[0,1]/V^n$, and this assignment is linear and injective.

Example 2.10. Let Ω be a compact space and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, endowed with the natural norm $\|f\|_\infty = \sup_{\omega\in\Omega}|f(\omega)|$, $f \in C(\Omega)$. We consider a collection $\mathcal{P}$ of complex-valued functions p, each defined and continuous on an open set $\Delta_p \subset \Omega$. Let μ be a positive measure on Ω such that $\mu(\Omega \setminus \Delta_p) = 0$, and p (extended with zero on $\Omega \setminus \Delta_p$) is μ-integrable for all $p \in \mathcal{P}$. Via a slight abuse of notation, we may define the numbers $\gamma_p = \int_\Omega p d\mu$, $p \in \mathcal{P}$, which can be called the $\mathcal{P}$-*moments* of μ. A very general, and possibly hopeless moment problem at this level, might be to characterize those families of numbers $(\gamma_p)_{p\in\mathcal{P}}$ which are the $\mathcal{P}$-moments of a certain positive measure.

Let us add some natural supplementary conditions. First of all, assume that $\Omega_0 = \cap_{p\in\mathcal{P}}\Delta_p$ is a dense subset of Ω. Also assume that there exists $\mathcal{R} \subset \mathcal{P}$ a family containing the constant function 1, closed under multiplication in the sense that if $r', r'' \in \mathcal{R}$ then $r'r''$ defined on $\Delta_{r'} \cap \Delta_{r''}$ is in $\mathcal{R}$, and each $r \in \mathcal{R}$ is nonnull on its domain of definition. Finally, we assume that for every function $p \in \mathcal{P}$ there exists a function $r \in \mathcal{R}$ such that the function p/r, defined on $\Delta_p \cap \Delta_r$, has a (unique) continuous extension to Ω. In particular, all functions from the family $\mathcal{Q} = \{1/r; r \in \mathcal{R}\}$ have a continuous extension to Ω. Moreover, the set $\mathcal{Q}$, identified with a family in $C(\Omega)$, is a set of denominators. This allows us to identify each function $p \in \mathcal{P}$ with a fraction from $C(\Omega)/\mathcal{Q}$, namely with h/q, where h is the continuous extension of p/r and $q = 1/r$ for a convenient $r \in \mathcal{R}$. With these conditions, the above $\mathcal{P}$-moment problem can be approached with our methods (see [25]; see also Corollary 3.2).

Summarizing, for a given subspace $\mathcal{P}$ of the algebra of fractions $C(\Omega)/\mathcal{Q}$, and a linear functional ϕ on $\mathcal{P}$, we look for necessary and sufficient conditions on $\mathcal{P}$ and ϕ to insure the existence of a solution, that is, a positive measure μ on Ω such that each p be μ-almost everywhere defined and $\phi(p) = \int_\Omega p d\mu$, $p \in \mathcal{P}$. We may call such a problem a *singular moment problem*, when no data are specified. With this terminology, the classical moment problems of Stieltjes and Hamburger, in one or several variables, are singular moment problems.

3. Spaces of fractions of continuous functions

Let Ω be a compact space and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, endowed with the natural norm $\|f\|_\infty = \sup_{\omega\in\Omega}|f(\omega)|$, $f \in C(\Omega)$. We denote by $M(\Omega)$ the space of all complex-valued Borel measures on Ω, sometimes identified with the dual of $C(\Omega)$. For an arbitrary function $h \in C(\Omega)$, we set $Z(h) = \{\omega \in \Omega; h(\omega) = 0\}$, which is obviously a compact subset of Ω. If $\mu \in M(\Omega)$, we denote by $|\mu| \in M(\Omega)$ the variation of μ.

We discuss certain spaces of fractions, which were considered in [25]. Let $\mathcal{Q}$ be a family of nonnegative elements of $C(\Omega)$. The set $\mathcal{Q}$ is said to be a *multiplicative family* if (i) $1 \in \mathcal{Q}$, (ii) $q', q'' \in \mathcal{Q}$ implies $q'q'' \in \mathcal{Q}$, and (iii) if $qh = 0$ for some $q \in \mathcal{Q}$ and $h \in C(\Omega)$, then $h = 0$. As in Remark 2.7, a multiplicative family is a set of denominators, and so we can form the algebra of fractions $C(\Omega)/\mathcal{Q}$. To unify the terminology, a multiplicative family $\mathcal{Q} \subset C(\Omega)$ will be called, with no ambiguity, a set of (positive) denominators.

To define a natural topological structure on $C(\Omega)/\mathcal{Q}$, we use Remark 2.5. If

$$C(\Omega)/q = \{f \in C(\Omega)/\mathcal{Q}; qf \in C(\Omega)\},$$

then we have $C(\Omega)/\mathcal{Q} = \cup_{q\in\mathcal{Q}} C(\Omega)/q$. Setting $\|f\|_{\infty,q} = \|qf\|_\infty$ for each $f \in C(\Omega)/q$, the pair $(C(\Omega)/q, \| * \|_{\infty,q})$ becomes a Banach space. Hence, $C(\Omega)/\mathcal{Q}$ is an inductive limit of Banach spaces (see [17], Section V.2).

As in Remark 2.6, in each space $C(\Omega)/q$ we have a positive cone $(C(\Omega)/q)_+$ consisting of those elements $f \in C(\Omega)/q$ such that $qf \geq 0$ as a continuous function.

We use in this text sometimes the notation q^{-1} to designate the element $1/q$ for any $q \in \mathcal{Q}$.

Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty. Let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, that is, the subspace of $C(\Omega)/\mathcal{Q}$ generated by the subspaces $(C(\Omega)/q)_{q\in\mathcal{Q}_0}$, which is itself an inductive limit of Banach spaces. Let also $\psi : \mathcal{F} \to \mathbb{C}$ be linear. As in Remark 2.5, the map ψ is continuous if the restriction $\psi|C(\Omega)/q$ is continuous for all $q \in \mathcal{Q}_0$.

We note that the values of ψ do not depend on the particular representation of the elements of $\mathcal{F}$.

Let us also note that the linear functional $\psi : \mathcal{F} \to \mathbb{C}$ is positive (see Remark 2.6 or [25]) if $\psi|(C(\Omega)/q)_+ \geq 0$ for all $q \in \mathcal{Q}_0$.

The next result, which is an extension of the Riesz representation theorem, describes the dual of a space of fractions, defined as above.

Theorem 3.1. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty, let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to \mathbb{C}$ be linear. The functional ψ is continuous if and only if there exists a uniquely determined measure $\mu_\psi \in M(\Omega)$ such that $|\mu_\psi|(Z(q)) = 0$, q^{-1} is $|\mu_\psi|$-integrable for all $q \in \mathcal{Q}_0$ and $\psi(f) = \int_\Omega f d\mu_\psi$ for all $f \in \mathcal{F}$.*

The functional $\psi : \mathcal{F} \to \mathbb{C}$ is positive, if and only if it is continuous and the measure μ_ψ is positive.

Proof. Let $\mu \in M(\Omega)$ be a measure such that $|\mu|(Z(q)) = 0$ and q^{-1} (which is $|\mu|$-almost everywhere defined) is $|\mu|$-integrable for all $q \in \mathcal{Q}_0$. We set $\psi(f) = \int_\Omega f d\mu$ for all $f \in \mathcal{F}$. This definition is correct. Indeed, if $f = \sum_{j\in J} h'_j/q'_j = \sum_{k\in K} h''_k/q''_k$ are two (finite) representations of $f \in \mathcal{F}$, with $h'_j, h''_k \in C(\Omega)$ and $q'_j, q''_k \in \mathcal{Q}_0$, setting $Z = \cup_{j\in J} Z(q'_j) \cup \cup_{k\in K} Z(q''_k)$, we easily derive that $f(\omega) = \sum_{j\in J} h'_j(\omega)/q'_j(\omega) = \sum_{k\in K} h''_k(\omega)/q''_k(\omega)$ for all $\omega \in \Omega \setminus Z$. As $|\mu|(Z) = 0$, we infer that f is a function defined $|\mu|$-almost everywhere and the integral $\psi(f) = \int_\Omega f d\mu$ does not depend on the particular representation of f.

Clearly, the functional ψ is linear. Note also that

$$|\psi(f)| = \left|\int_\Omega hq^{-1} d\mu\right| \leq \int_\Omega |h| q^{-1} d|\mu| \leq \|f\|_{\infty,q} \int_\Omega q^{-1} d|\mu| \tag{3.1}$$

for all $f = h/q \in C(\Omega)/q$, showing the continuity of ψ.

Conversely, let $\psi : \mathcal{F} \to \mathbb{C}$ be linear and continuous. For every $q \in \mathcal{Q}_0$ we set $\theta_q(h) = \psi(h/q)$, $h \in C(\Omega)$. Since the map $C(\Omega) \ni h \to h/q \in C(\Omega)/q$ is continuous (in fact, it is an isometry), the map θ_q is a continuous linear functional on $C(\Omega)$. Therefore, there exists a measure $\nu_q \in M(\Omega)$ such that $\theta_q(h) = \int_\Omega h\, d\nu_q$, $h \in C(\Omega)$, for all $q \in \mathcal{Q}_0$.

Note that $\psi(hq_1/q_1) = \psi(hq_2/q_2)$ for all $q_1, q_2 \in \mathcal{Q}_0$ and $h \in C(\Omega)$, because hq_1/q_1 and hq_2/q_2 are two representations of the same element in $\mathcal{F}$. Therefore, $\theta_{q_1}(hq_1) = \psi(hq_1/q_1) = \psi(hq_2/q_2) = \theta_{q_2}(hq_2)$ for all $q_1, q_2 \in \mathcal{Q}_0$ and $h \in C(\Omega)$, implying the equality $q_1\nu_{q_1} = q_2\nu_{q_2}$. Consequently, there exists a measure μ such that $\mu = q\nu_q$ for all $q \in \mathcal{Q}_0$.

The equality $\mu = q\nu_q$ implies the equality $|\mu| = q|\nu_q|$. This shows the set $Z(q)$ must be $|\mu|$-null. Moreover, the function q^{-1} is $|\mu|$-integrable for all $q \in \mathcal{Q}_0$. Consequently, $\nu_q = q^{-1}\mu$, and the function q^{-1} is μ-integrable for all $q \in \mathcal{Q}_0$.

If $f \in \mathcal{F}$ is arbitrary, then $f = \sum_{j\in J} h_j q_j^{-1}$, with $h_j \in C(\Omega)$, $q_j \in \mathcal{Q}_0$ for all $j \in J$, J finite. We can write

$$\psi(f) = \sum_{j\in J} \theta_{q_j}(h_j) = \sum_{j\in J} \int_\Omega h_j\, d\nu_{q_j} = \int_\Omega f d\mu,$$

giving the desired integral representation for the functional ψ.

As we have $\int_\Omega h d\mu = \psi(hq/q)$ for all $h \in C(\Omega)$ and $q \in \mathcal{Q}_0$, and $\psi(hq/q)$ does not depend on q, it follows that the measure μ is uniquely determined. If we put $\mu = \mu_\psi$, we have the measure whose existence and uniqueness are asserted by the statement.

Let $\psi : \mathcal{F} \to \mathbb{C}$ be linear be linear and positive. Then, as one expects, ψ is automatically continuous. Indeed, if $h \in C(\Omega)$ is real-valued and $q \in \mathcal{Q}_0$, the inequality $-\|h\|_\infty/q \le h/q \le \|h\|_\infty/q$ implies, via the positivity of ψ, the estimate $|\psi(h/q)| \le \|h/q\|_{\infty,q}\psi(1/q)$. If $h \in C(\Omega)$ is arbitrary, the estimate above leads to $|\psi(h/q)| \le 2\|h/q\|_{\infty,q}\psi(1/q)$, showing that ψ is continuous. Finally, the equality $\psi(f) = \int f d\mu_\psi$, $f \in C(\Omega)/q$, $q \in \mathcal{Q}_0$ shows that ψ is positive if and only if μ_ψ is positive. □

The next result is essentially Theorem 3.2 from [25].

Corollary 3.2. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty, let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to \mathbb{C}$ be linear. The functional ψ is positive if and only if*

$$\sup\{|\psi(hq^{-1})|;\, h \in C(\Omega),\, \|h\|_\infty \le 1\} = \psi(q^{-1}),\ q \in \mathcal{Q}_0. \tag{3.2}$$

Proof. Set $\psi_q = \psi|C(\Omega)/q$ for all $q \in \mathcal{Q}_0$. Then, since $\|f\|_{\infty,q} = \|h\|_\infty$ whenever $f = h/q \in C(\Omega)/q$, we clearly have

$$\|\psi_q\| = \sup\{|\psi(hq^{-1})|;\, h \in C(\Omega),\, \|h\|_\infty \le 1\}.$$

If ψ is positive, and μ_ψ is the associated positive measure given by Theorem 3.1, the estimate (3.1) implies that

$$\|\psi_q\| \le \int_\Omega q^{-1} d\mu_\psi = \psi(q^{-1}) \le \|\psi_q\|,$$

because $\|1/q\|_{\infty,q} = 1$. This shows that (3.2) holds.

Conversely, we use the well-known fact that a linear functional $\theta : C(\Omega) \to \mathbb{C}$ is positive if and only if it is continuous and $\|\theta\| = \theta(1)$

Assuming that (3.2) holds, with the notation of the proof of Theorem 3.1, we have

$$\|\theta_q\| = \|\psi_q\| = \psi(1/q) = \theta_q(1),$$

showing that θ_q is positive for all q. Therefore, the measures ν_q are all positive, implying that the measure $\mu = \mu_\psi$ is positive. Consequently, ψ must be positive. □

Definition 3.3. Let $\mathcal{Q} \subset C(\Omega)$ be a set of denominators. A measure $\mu \in M(\Omega)$ is said to be $\mathcal{Q}$-*divisible* if for every $q \in \mathcal{Q}$ there is a measure $\nu_q \in M(\Omega)$ such that $\mu = q\nu_q$.

Theorem 3.1 shows that a functional on $C(\Omega)/\mathcal{Q}$ is continuous if and only if it has an integral representation via a $\mathcal{Q}$-divisible measure. In addition, Corollary 3.2 asserts that a functional is positive on $C(\Omega)/\mathcal{Q}$ if and only if it is represented by a $\mathcal{Q}$-divisible positive measure μ such that $\mu = q\nu_q$ with $\nu_q \in M(\Omega)$ positive for all $q \in \mathcal{Q}$.

As a matter of fact, the concept given by Definition 3.3 can be considerably extended, as shown by the next example.

Example 3.4. This is a continuation of the discussion started in Example 2.9, whose notation will be kept. We fix an indefinitely differentiable function ϕ, with support in [0,1]. Note the identity

$$\int_0^1 h(t)\phi(t)dt = (-1)^n \int_0^1 V^n h(t)\phi^{(n)}(t)dt, \tag{3.3}$$

valid for all $h \in C[0,1]$ and all integers $n \geq 0$. If we set $d\mu(t) = \phi(t)dt$ and $d\nu_n(t) = (-1)^n\phi^{(n)}(t)dt$, and referring to Definition 3.3, we may say, by (3.3), that the measure μ is $\mathcal{V}$-*divisible*.

It is plausible that the study of $\mathcal{M}$-divisible measures, defined in an appropriate manner for a set of denominators $\mathcal{M}$ consisting of linear and continuous operators on $C(\Omega)$, can be related to the study of continuous linear functionals on the space $C(\Omega)/\mathcal{M}$, via a possible extension of Theorem 3.1.

Example 3.5. Let $\mathcal{S}_1$ be the algebra of polynomials in $z, \bar{z}$, $z \in \mathbb{C}$. We will show that this algebra, which is used to characterize the moment sequences in the complex plane, can be identified with a subalgebra of an algebra of fractions of continuous functions. This exemple will be extended to infinitely many variables in the last section (similar, yet different examples were considered in [2]). Let $\mathcal{R}_1$ be the set of functions $\{(1+|z|^2)^{-k}; z \in \mathbb{C}, k \in \mathbb{Z}_+\}$, which can be continously extended to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Identifying $\mathcal{R}_1$ with the set of their extensions in $C(\mathbb{C}_\infty)$, the family $\mathcal{R}_1$ becomes a set of denominators in $C(\mathbb{C}_\infty)$. This will allows us to identify the algebra $\mathcal{S}_1$ with a subalgebra of the algebra of fractions $C(\mathbb{C}_\infty)/\mathcal{R}_1$.

Let $\mathcal{S}_{1,k}^{(1)}$, $k \geq 1$, be the space generated by the monomials $z^j \bar{z}^l$, $0 \leq j+l < 2k$. We put $\mathcal{S}_{1,0}^{(1)} = \mathbb{C}$. Let also $\mathcal{S}_{1,k}^{(2)}$, $k \geq 1$, be the space generated by the monomials $|z|^{2j}$, $0 < j \leq k$. Put $\mathcal{S}_{1,0}^{(2)} = \{0\}$.

Set $\mathcal{S}_{1,k} = \mathcal{S}_{1,k}^{(1)} + \mathcal{S}_{1,k}^{(2)}$, $k \geq 0$. We clearly have $\mathcal{S}_1 = \sum_{k \geq 0} \mathcal{S}_{1,k}$. Since $\mathcal{S}_{1,k}$ may be identified with a subspace of $C(\mathbb{C}_\infty)/r_k$, where $r_k(z) = (1+|z|^2)^{-k}$ for all $k \geq 0$, the space $\mathcal{S}_1$ can be viewed as a subalgebra of the algebra $C(\mathbb{C}_\infty)/\mathcal{R}_1$. Note also that $r_k^{-1} \in \mathcal{S}_{1,k}$ for all $k \geq 1$ and $\mathcal{S}_{1,k} \subset \mathcal{S}_{1,l}$ whenever $k \leq l$.

According to Theorem 3.4 from [25], a linear map $\phi : \mathcal{S}_1 \mapsto \mathbb{C}$ has a positive extension $\psi : C(\mathbb{C}_\infty)/\mathcal{R}_1 \mapsto \mathbb{C}$ with $\|\phi_k\| = \|\psi_k\|$ if and only if $\|\phi_k\| = \phi(r_k^{-1})$, where $\phi_k = \phi|\mathcal{S}_{1,k}$ and $\psi_k = \psi|C(\mathbb{C}_\infty)/r_k$, for all $k \geq 0$ (the norms of the functionals are computed in the sense discussed in Remark 2.5).

This result can be used to characterize the Hamburger moment problem in the complex plane, in the spirit of [25]. Specifically, given a sequence of complex numbers $\gamma = (\gamma_{j,l})_{j \geq 0, l \geq 0}$ with $\gamma_{0,0} = 1$, $\gamma_{k,k} \geq 0$ if $k \geq 1$ and $\gamma_{j,l} = \bar{\gamma}_{l,j}$ for all $j \geq 0, l \geq 0$, the Hamburger moment problem means to find a probability measure on $\mathbb{C}$ such that $\gamma_{j,l} = \int z^j \bar{z}^l d\mu(z)$, $j \geq 0, l \geq 0$.

Defining $L_\gamma : \mathcal{S}_1 \mapsto \mathbb{C}$ by setting $L_\gamma(z^j \bar{z}^l) = \gamma_{j,l}$ for all $j \geq 0, l \geq 0$ (extended by linearity), if L_γ has the properties of the functional ϕ above insuring the existence of a positive extension to $C(\mathbb{C}_\infty)/\mathcal{R}_1$, then the measure μ is provided by Corollary 3.2.

For a fixed integer $m \geq 1$, we can state and characterize the existence of solutions for a truncated moment problem (for an extensive study of such problems see [7] and [8]). Specifically, given a finite sequence of complex numbers $\gamma = (\gamma_{j,l})_{j,l}$ with $\gamma_{0,0} = 1$, $\gamma_{j,j} \geq 0$ if $1 \leq j \leq m$ and $\gamma_{j,l} = \bar{\gamma}_{l,j}$ for all $j \geq 0, l \geq 0$, $j \neq l$, $j + l < 2m$, find a probability measure on $\mathbb{C}$ such that $\gamma_{j,l} = \int z^j \bar{z}^l d\mu(z)$ for all indices j, l. As in the previous case, a necessary and sufficient condition is that the corresponding map $L_\gamma : \mathcal{S}_{1,m} \mapsto \mathbb{C}$ have the property $\|L_\gamma\| = L_\gamma(r_m^{-1})$, via Theorem 3.4 from [25]. Note also that the actual truncated moment problem is slightly different from the usual one (see [7]).

The similar space $\mathcal{T}_1$, introduced in [2], can be used to characterize the following moment problem: Find a probability measure μ on $\mathbb{C}$ such that the double sequence of the form $\gamma = (\gamma_{j,0}, \gamma_{k,k})_{j \geq 0,\, k \geq 1}$ (with $\gamma_{0,0} = 1$ and $\gamma_{k,k} \geq 0$ if $k \geq 1$) be a moment sequence in the sense that $\gamma_{j,0} = \int z^j d\mu(z), \gamma_{k,k} = \int |z|^{2k} d\mu(z)$.

Example 3.6. We are particularly interested in some special algebras of fractions of continuous functions, depending on infinitely many variables, necessary for our further discussion.

Let $\mathcal{I}$ be a (nonempty) family of indices. We consider the space $\mathbb{R}^{\mathcal{I}}$, where, as before, $\mathbb{R}$ is the real field. Denote by $t = (t_\iota)_{\iota \in \mathcal{I}}$ the independent variable in $\mathbb{R}^{\mathcal{I}}$. Let $\mathbb{Z}_+^{(\mathcal{I})}$ be the set of all collections $\alpha = (\alpha_\iota)_{\iota \in \mathcal{I}}$ of nonnegative integers, with finite support. Setting $t^0 = 1$ for $0 = (0)_{\iota \in \mathcal{I}}$ and $t^\alpha = \prod_{\alpha_\iota \neq 0} t_\iota^{\alpha_\iota}$ for $t = (t_\iota)_{\iota \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$, $\alpha = (\alpha_\iota)_{\iota \in \mathcal{I}} \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we may consider the algebra of complex-valued *polynomial functions* $\mathcal{P}_{\mathcal{I}}$ on $\mathbb{R}^{\mathcal{I}}$, consisting of expressions of the form $\sum_{\alpha\in\mathcal{J}} c_\alpha t^\alpha$, with c_α complex numbers for all $\alpha \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{Z}_+^{(\mathcal{I})}$ is finite. Note also that the map $\mathcal{P}_{\mathcal{I}} \ni p \to \bar{p} \in \mathcal{P}_{\mathcal{I}}$ is an involution on $\mathcal{P}_{\mathcal{I}}$, where $\bar{p}(t) = \sum_{\alpha\in\mathcal{J}} \bar{c}_\alpha t^\alpha$ if $p(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha t^\alpha$.

When dealing with finite measures, an appropriate framework is the space of all continuous functions on a compact topological space. But neither the space $\mathbb{R}^{\mathcal{I}}$ is compact nor the functions from $\mathcal{P}_{\mathcal{I}}$ are bounded. If we consider the one-point compactification $\mathbb{R}_\infty$ of $\mathbb{R}$, then we can embed $\mathbb{R}^{\mathcal{I}}$ into the compact space $(\mathbb{R}_\infty)^{\mathcal{I}}$. This operation leads us to consider the space $\mathcal{P}_{\mathcal{I}}$ as a subspace of an algebra of fractions derived from the basic algebra $C((\mathbb{R}_\infty)^{\mathcal{I}})$, via a suitable multiplicative family. Specifically, we consider the family $\mathcal{Q}_{\mathcal{I}}$ consisting of all rational functions of the form $q_\alpha(t) = \prod_{\alpha_\iota\neq 0}(1+t_\iota^2)^{-\alpha_\iota}$, $t = (t_\iota)_{\iota\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$, where $\alpha = (\alpha_\iota) \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, is arbitrary (see also [9]). Of course, we set $q_0 = 1$. The function q_α can be continuously extended to $(\mathbb{R}_\infty)^{\mathcal{I}} \setminus \mathbb{R}^{\mathcal{I}}$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$. Moreover, the set $\mathcal{Q}_{\mathcal{I}}$ becomes a set of denominators in $C((\mathbb{R}_\infty)^{\mathcal{I}})$. Set also $p_\alpha(t) = q_\alpha(t)^{-1}$, $t \in \mathbb{R}^{\mathcal{I}}$, $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$.

Let $\mathcal{P}_{\mathcal{I},\alpha}$ be the vector space generated by the monomials t^β, with $\beta_\iota \leq 2\alpha_\iota$, $\iota \in \mathcal{I}$, $\alpha, \beta \in \mathbb{Z}_+^{(\mathcal{I})}$. It is clear that p/p_α can be continuously extended to $(\mathbb{R}_\infty)^{\mathcal{I}} \setminus \mathbb{R}^{\mathcal{I}}$ for every $p \in \mathcal{P}_{\mathcal{I},\alpha}$, and so it can be regarded as an element of $C((\mathbb{R}_\infty)^{\mathcal{I}})$. Therefore, $\mathcal{P}_{\mathcal{I},\alpha}$ is a subspace of $C((\mathbb{R}_\infty)^{\mathcal{I}})/q_\alpha = p_\alpha C((\mathbb{R}_\infty)^{\mathcal{I}})$ for all $\alpha \in \mathbb{Z}_+^{\mathcal{I}}$.

Example 3.7. We give now another example of an algebra of fractions, which will be used in the next section to describe the normal algebras.

As before, let $\mathcal{I}$ be a (nonempty) family of indices. We consider the space $\mathbb{T}^{\mathcal{I}}$, where $\mathbb{T}$ is the unit circle in the complex plane. Denote by $z = (z_\iota)_{\iota\in\mathcal{I}}$ the independent variable in $\mathbb{T}^{\mathcal{I}}$. Let $\mathbb{Z}_+^{(\mathcal{I})}$ be defined as in the previous example. Setting $(\Re z)^0 = 1$ for $0 = (0)_{\iota\in\mathcal{I}}$, $(\Re z)^\alpha = \prod_{\alpha_\iota\neq 0}(\Re z_\iota)^{\alpha_\iota}$ and similar formulas for $(\Im z)^\alpha$ whenever $z = (z_\iota)_{\iota\in\mathcal{I}} \in \mathbb{T}^{\mathcal{I}}$, $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}} \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we may consider the algebra of complex-valued functions $\mathcal{R}_{\mathcal{I}}$ on $\mathbb{T}^{\mathcal{I}}$, consisting of expressions of the form $\sum_{\alpha,\beta\in\mathcal{J}} c_{\alpha,\beta}(\Re z)^\alpha(\Im z)^\beta$, with $c_{\alpha,\beta}$ complex numbers for all $\alpha, \beta \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{Z}_+^{(\mathcal{I})}$ is finite.

We may take in the algebra $\mathcal{R}_{\mathcal{I}}$ a set of denominators $\mathcal{S}_{\mathcal{I}}$ consisting of all functions of the form $s_\alpha(t) = \prod_{\alpha_\iota\neq 0}(1-\Re z_\iota)^{\alpha_\iota}$, $z = (z_\iota)_{\iota\in\mathcal{I}} \in \mathbb{T}^{\mathcal{I}}$, where $\alpha = (\alpha_\iota) \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, is arbitrary. We put $s_0 = 1$. Clearly, $\mathcal{S}_{\mathcal{I}}$ is a set of denominators also in $C((\mathbb{T})^{\mathcal{I}})$.

Note that the map

$$\mathcal{P}_{\mathcal{I}} \ni p \mapsto p \circ \tau \in \mathcal{R}_{\mathcal{I}}/\mathcal{S}_{\mathcal{I}},$$

where $\tau : (\mathbb{T}\setminus\{1\})^{\mathcal{I}} \mapsto \mathbb{R}^{\mathcal{I}}$ is given by $\tau(z)_\iota = -\Im z_\iota/(1-\Re z_\iota)$, $\iota \in \mathcal{I}$, is an injective algebra homomorphism, allowing the identification of $\mathcal{P}_{\mathcal{I}}$ with a subalgebra of $\mathcal{R}_{\mathcal{I}}/\mathcal{S}_{\mathcal{I}}$.

Remark 3.8. Let Ω be a compact Hausdorff space, let A be unital C^*-algebra, and let $C(\Omega, A)$ be the C^*-algebra of all A-valued functions, continuous on Ω. Let $\mathcal{Q}$ be set of denominators in $C(\Omega)$, which can be identified with a set of denominators in $C(\Omega, A)$. Therefore, we may consider the algebra of fractions $C(\Omega, A)/\mathcal{Q}$. Let $\mathcal{Q}_0$ be an arbitrary subset of $\mathcal{Q}$ and let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} \mathcal{F}_q$ be a subspace of $\sum_{q\in\mathcal{Q}_0} C(\Omega, A)/q$ such that $q^{-1} \in \mathcal{F}_q$ et $\mathcal{F}_q \subset C(\Omega, A)/q$ for all $q \in \mathcal{Q}_0$.

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{D}$ be a dense linear subspace of $\mathcal{H}$, and denote by $SF(\mathcal{D})$ the space of all sesquilinear forms on $\mathcal{D}$. Let $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ be linear. Suppose that $\phi(q^{-1})(x, x) > 0$ for all $x \in \mathcal{D}\setminus\{0\}$ and $q \in \mathcal{Q}_0$. Then $\phi(q^{-1})$ induces an inner product on $\mathcal{D}$, and let $\mathcal{D}_q$ be the space $\mathcal{D}$, endowed with the norm given by $\| * \|_q^2 = \phi(q^{-1})(*, *)$. Set $\phi_q = \phi|\mathcal{F}_q$. We may define the quantities

$$\|\phi_q(f)\| = \sup\{|\phi_q(f)(x, y)|; \|x\|_q \le 1, \|y\|_q \le 1\},$$

and

$$\|\phi_q\| = \sup\{\|\phi_q(f)\|; \|qf\|_\infty \le 1\}.$$

We say that the map $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ is *contractive* if $\|\phi_q\| \le 1$ for all $q \in \mathcal{Q}_0$.

4. Normal algebras

We keep the notation and terminology from the Introduction.

Let A_1, A_2 be self-adjoint in $\mathcal{H}$. Trying to avoid, at this moment, any involvement of the concept of a spectral measure for unbounded operators, we say that A_1, A_2 *commute* if the bounded operators $(A_1 + iI_\mathcal{H})^{-1}$ and $(A_2 + iI_\mathcal{H})^{-1}$ commute, where $I_\mathcal{H}$ is the identity on $\mathcal{H}$, which is one of the possible (classical) definitions of the commutativity of (unbounded) self-adjoint operators. It is known that the operator N in normal in $\mathcal{H}$ if and only if one has $N = A_1 + iA_2$, where A_1, A_2 are commuting self-adjoint operators (see [11], Part II, or [20]). For this reason, given two normal operators N', N'' having the decompositions $N' = A_1' + iA_2'$, $N'' = A_1'' + iA_2''$, with A_1', A_2', A_1'', A_2'' self-adjoint, we say that N', N'' *commute* if the self-adjoint operators A_1', A_2', A_1'', A_2'' mutually commute.

Let $\mathcal{N} \subset \mathcal{L}^\#(\mathcal{D})$ be an O^*-algebra. As mentioned in the Introduction, we say that $\mathcal{N}$ is *normal* if $\bar{N}$ is normal for each $N \in \mathcal{N}$.

A homonymic concept, defined in the framework of bouneded operators, can be found in [18]. The aim of this section is to describe the structure of normal algebras, extending Theorem 12.22 from [18]. Let us recall the essential part of that result.

Theorem A. *Let $\mathcal{N}$ be a closed normal algebra of $\mathcal{B}(\mathcal{H})$ containing the identity, and let Δ be the maximal ideal space of $\mathcal{N}$. Then there exists a unique spectral measure E on the Borel subsets of Δ such that $N = \int_\Delta \hat{N} dE$ for every $N \in \mathcal{N}$.*

This result leads us to some versions of the Spectral Theorem, called here Theorem 4.2 and Theorem 4.3, valid for an arbitrary family of not necessarily bounded, commuting self-adjoint or normal operators (see also [4], [16], [19] etc. for various approaches to this topic).

Remark 4.1. Let Ω be a topological space and let E be a spectral measure, defined on the family of all Borel subsets of Ω, with values in $\mathcal{B}(\mathcal{H})$. For every Borel function $f:\Omega\to\mathbb{C}$ we set $\mathcal{D}_f=\{x\in\mathcal{H};\int_\Omega|f|^2dE_{x,x}<\infty\}$, where $E_{x,y}(*)=\langle E(*)x,y\rangle$. Then the formula

$$\langle\Psi(f)x,y\rangle=\int_\Omega fdE_{x,y},\ x\in\mathcal{D}_f,\ y\in\mathcal{H}$$

defines a normal operator $\Psi(f)$ for each f, with $\Psi(f)^*=\Psi(\bar f)$ (see [18], especially Theorem 13.24, for this and other properties to be used in this text).

If f is E-almost everywhere defined (in particular, almost everywhere finite) on Ω, we keep the notation $\int_\Omega fdE_{x,y}$ whenever the integral is well defined.

In the following, for every complex number λ, the operator $\lambda I_\mathcal{H}$ will be simply written as λ.

The next statement is a version of Theorem 13.30 from [18] (stated and proved for one self-adjoint operator). We adapt some ideas from the proof of quoted theorem, to get a statement for infinitely many variables.

Theorem 4.2. *Let $(A_\iota)_{\iota\in\mathcal{I}}$ be a commuting family of self-adjoint operators in $\mathcal{H}$. Then there exists a unique spectral measure E on the Borel subsets of $(\mathbb{R}_\infty)^\mathcal{I}$ such that each coordinate function $(\mathbb{R}_\infty)^\mathcal{I}\ni t\to t_\iota\in\mathbb{R}_\infty$ is E-almost everywhere finite. In addition,*

$$\langle A_\iota x,y\rangle=\int_{(\mathbb{R}_\infty)^\mathcal{I}}t_\iota dE_{x,y}(t),\ \ x\in\mathcal{D}(A_\iota),\ y\in\mathcal{H},$$

where

$$\mathcal{D}(A_\iota)=\{x\in\mathcal{H};\int_{(\mathbb{R}_\infty)^\mathcal{I}}|t_\iota|^2dE_{x,x}(t)<\infty\},$$

for all $\iota\in\mathcal{I}$.

If the set $\mathcal{I}$ is at most countable, then the measure E has support in $\mathbb{R}^\mathcal{I}$.

Proof. Let U_ι denote the Cayley transform of the self-adjoint operator A_ι for each $\iota\in\mathcal{I}$. Since

$$U_\iota=(A_\iota-i)(A_\iota+i)^{-1}=1-2i(A_\iota+i)^{-1},\ \iota\in\mathcal{I},$$

the collection $(U_\iota)_{\iota\in\mathcal{I}}$ is a commuting family of unitary operators in $\mathcal{B}(\mathcal{H})$. Moreover, the point 1 does not belong to the point spectrum of $1-U_\iota$ for all ι, as follows from the general properties of the Cayley transform [18].

Let $\mathcal{B}$ be the closed unital algebra generated by the family $(U_\iota,U_\iota^*)_{\iota\in\mathcal{I}}$ in $\mathcal{B}(\mathcal{H})$, which is a commutative unital C^*-algebra. Let $\Gamma(\mathcal{B})$ be the space of characters of the algebra $\mathcal{B}$.

Standard arguments from the Gelfand theory allow us to assert that the map

$$\Gamma(\mathcal{B})\ni\gamma\mapsto(\hat U_\iota(\gamma))_{\iota\in\mathcal{I}}\in\mathbb{T}^\mathcal{I}$$

is a homeomorphism, and so the space $\Gamma(\mathcal{B})$ may be identified with a closed subspace, say Ω, of the compact space $\mathbb{T}^{\mathcal{I}}$.

By virtue of the Theorem A above, there exists a spectral measure F defined on the Borel subsets of Ω such that $U_\iota = \int_\Omega z_\iota dF(z)$, where $z = (z_\iota)_{\iota\in\mathcal{I}}$ is the variable in $\mathbb{T}^{\mathcal{I}}$ and the map $\hat{U}_\iota$ has been identified with the coordinate function $z \to z_\iota$ for all ι.

Set $C_\iota = \{z \in \Omega; z_\iota = 1\}$. We want to prove that $F(C_\iota) = 0$ for all ι. Assuming that this is not the case for some ι, we could find a nonnull vector x_ι such that $E(C_\iota)x_\iota = x_\iota$. In addition,

$$x_\iota - U_\iota x_\iota = \int_\Omega (1 - z_\iota)dF(z)x_\iota = \int_{C_\iota} (1 - z_\iota)dF(z)x_\iota = 0,$$

which contradicts the fact that the kernel of $1 - U_\iota$ is null.

In particular, this shows that the function $(1 - z_\iota)^{-1}$ is defined F-almost everywhere for all ι.

For technical reasons, we may extend the set function F to the family of all Borel subsets of $\mathbb{T}^{\mathcal{I}}$ by putting $\tilde{F}(\beta) = F(\beta \cap \Omega)$ for every Borel subset β of $\mathbb{T}^{\mathcal{I}}$. Let $\kappa_{\mathcal{I}} : (\mathbb{R}_\infty)^{\mathcal{I}} \to \mathbb{T}^{\mathcal{I}}$ be the map given by

$$(\mathbb{R}_\infty)^{\mathcal{I}} \ni t = (t_\iota)_{\iota\in\mathcal{I}} \mapsto w = (w_\iota)_{\iota\in\mathcal{I}} \in \mathbb{T}^{\mathcal{I}}$$

where $w_\iota = \kappa_\iota(t) = (t_\iota - i)(t_\iota + i)^{-1}$ if $t_\iota \neq \infty$ and $w_\iota = 1$ if $t_\iota = \infty$. The map $\kappa_{\mathcal{I}}$ is a homeomorphism and the superposition $E = \tilde{F} \circ \kappa_{\mathcal{I}}$ is a spectral measure on the Borel subsets of $(\mathbb{R}_\infty)^{\mathcal{I}}$.

If $D_\iota = \{t \in (\mathbb{R}_\infty)^{\mathcal{I}}; t_\iota = \infty\}$, then $E(D_\iota) = \tilde{F}(\kappa(D_\iota)) = F(C_\iota) = 0$, as noticed above. In other words, the coordinate function $(\mathbb{R}_\infty)^{\mathcal{I}} \ni t \to t_\iota \in \mathbb{R}_\infty$ is E-almost everywhere finite. Moreover, if $\theta_\iota(z) = i(1+z_\iota)(1-z_\iota)^{-1} = -\Im z_\iota(1-\Re z_\iota)^{-1}$, then θ_ι is $\tilde{F}$-almost everywhere defined, and one has

$$\int_{(\mathbb{R}_\infty)^{\mathcal{I}}} t_\iota dE_{x,y}(t) = \int_{\mathbb{T}^{\mathcal{I}}} \theta_\iota(z)d\tilde{F}_{x,y}(z) = \int_\Omega \theta_\iota(z)dF_{x,y}(z),$$

for all $x \in \mathcal{D}_{\theta_\iota}$ and $y \in \mathcal{H}$, via a change of variable and the Remark above. As the function θ_ι is real-valued, the operator $\tilde{A}_\iota$ given by the equality

$$\langle \tilde{A}_\iota x, y\rangle = \int_{(\mathbb{R}_\infty)^{\mathcal{I}}} t_\iota dE_{x,y}(t),\ x \in \mathcal{D}_{\theta_\iota},\ y \in \mathcal{H}$$

is self-adjoint. The arguments from the last part of Theorem 13.30 in [18] show that $\tilde{A}_\iota$ must be precisely A_ι. For the convenience of the reader, we sketch this argument. The equality $(1 - z_\iota)\theta_\iota(z) = i(1 + z_\iota)$ leads to the equality $\tilde{A}_\iota(1 - U_\iota) = i(1 + U_\iota)$. This shows that $\tilde{A}_\iota$ is a self-adjoint extension of the inverse Cayley transform A_ι of U_ι. But any self-adjoint operator is maximally symmetric ([18], Theorem 13.15), and so $A_\iota = \tilde{A}_\iota$.

The equality

$$\mathcal{D}(A_\iota) = \{x \in \mathcal{H}; \int_{(\mathbb{R}_\infty)^{\mathcal{I}}} |t_\iota|^2 dE_{x,x}(t) < \infty\},$$

for all $\iota \in \mathcal{I}$, is a consequence of [18], Theorem 13.24.

To prove the uniqueness of the measure E, we consider another spectral measure G, defined on the Borel subsets of $(\mathbb{R}_\infty)^\mathcal{I}$ such that the G-measure of set $D_\iota = \{t \in (\mathbb{R}_\infty)^\mathcal{I}; t_\iota = \infty\}$ is null for all ι. In other words, each coordinate function $(\mathbb{R}_\infty)^\mathcal{I} \ni t \to t_\iota \in \mathbb{R}_\infty$ is G-almost everywhere finite, and

$$\langle A_\iota x, y\rangle = \int_{(\mathbb{R}_\infty)^\mathcal{I}} t_\iota dG_{x,y}(t),\ x \in \mathcal{D}(A_\iota),\ y \in \mathcal{H},\ \iota \in \mathcal{I}.$$

Then the map $H = G \circ \kappa_\mathcal{I}^{-1}$ defines a spectral measure on the Borel subsets of the compact space $(\mathbb{R}_\infty)^\mathcal{I}$. Because $\kappa_\mathcal{I}^{-1} = \theta$, where $\theta(z) = (\theta_\iota(z))_{\iota\in\mathcal{I}}$, we must have

$$U_\iota = \int_{(\mathbb{R}_\infty)^\mathcal{I}} \kappa_\iota(t) dG(t) = \int_{\mathbb{T}^\mathcal{I}} z_\iota dH(z),$$

where $z = (z_\iota)_{\iota\in\mathcal{I}}$ is the variable in $\mathbb{T}^\mathcal{I}$. As the unital C^*-algebra generated by the polynomials in $z_\iota, \bar{z}_\iota$, $\iota \in \mathcal{I}$, is dense in both $C(\Omega)$ and $C(\mathbb{T}^\mathcal{I})$, via the Weierstrass-Stone density theorem, the scalar measures $F_{x,y}$ and $H_{x,y}$ are equal for all $x, y \in \mathcal{H}$. Therefore, the spectral measures F and H must be equal too. Consequently, the spectral measures measures E and G obtained from F and H respectively, must be equal, which completes the proof of the theorem.

Finally, if the set $\mathcal{I}$ is at most countable, then the measure E has support in the Borel set

$$(\mathbb{R}_\infty)^\mathcal{I} \setminus \bigcup_{\iota\in\mathcal{I}} D_\iota = \mathbb{R}^\mathcal{I}.$$

□

We may consider the one-point compactification $\mathbb{C}_\infty$ of the complex plane $\mathbb{C}$. Then, for every family of indices $\mathcal{I}$, the space $(\mathbb{C}_\infty)^\mathcal{I}$ is compact. A version of the previous result, valid for normal operators, is given by the following.

Theorem 4.3. *Let $(N_\iota)_{\iota\in\mathcal{I}}$ be a commuting family of normal operators in $\mathcal{H}$. Then there exists a unique spectral measure G on the Borel subsets of $(\mathbb{C}_\infty)^\mathcal{I}$ such that each coordinate function $(\mathbb{C}_\infty)^\mathcal{I} \ni z \to z_\iota \in \mathbb{C}_\infty$ is G-almost everywhere finite. In addition,*

$$\langle N_\iota x, y\rangle = \int_{(\mathbb{C}_\infty)^\mathcal{I}} z_\iota dG_{x,y}(z),\ x \in \mathcal{D}(N_\iota),\ y \in \mathcal{H},$$

where

$$\mathcal{D}(N_\iota) = \{x \in \mathcal{H}; \int_{(\mathbb{C}_\infty)^\mathcal{I}} |z_\iota|^2 dG_{x,x}(z) < \infty\},$$

for all $\iota \in \mathcal{I}$.

If the set $\mathcal{I}$ is at most countable, then the measure G has support in $\mathbb{C}^\mathcal{I}$.

Proof. We write $N_\iota = A'_\iota + iA''_\iota$, where $(A'_\iota, A''_\iota)_{\iota\in\mathcal{I}}$ is a commuting family of self-adjoint operators. Set $\mathcal{L} = (\mathcal{I}\times\{0\})\cup(\{0\}\times\mathcal{I})$, $B_\lambda = A'_\iota$ if $\lambda = (\iota, 0)$, and $B_\lambda = A''_\iota$ if $\lambda = (0, \iota)$. We may apply the previous theorem to the family of commuting self-adjoint operators $(B_\lambda)_{\lambda\in\mathcal{L}}$. Therefore, there exists a spectral measure E on the

Borel subsets of $(\mathbb{R}_\infty)^{\mathcal{L}}$ such that each coordinate function $\mathbb{R}^{\mathcal{L}} \ni t \to t_\lambda \in \mathbb{R}$ is E-almost everywhere defined. Moreover,

$$\langle B_\lambda x, y\rangle = \int_{(\mathbb{R}_\infty)^{\mathcal{L}}} t_\lambda dE_{x,y}(t),\ x \in \mathcal{D}(B_\lambda),\ y \in \mathcal{H},\ \lambda \in \mathcal{L}.$$

We define a map $\tau : (\mathbb{R}_\infty)^{\mathcal{L}} \mapsto (\mathbb{C}_\infty)^{\mathcal{I}}$ in the following way. If $t = (t_\lambda)_{\lambda\in\mathcal{L}}$ is an arbitrary point in $(\mathbb{R}_\infty)^{\mathcal{I}}$, we put $\tau(t) = (z_\iota)_{\iota\in\mathcal{I}}$, where $z_\iota = t_{(\iota,0)} + it_{(0,\iota)}$ for all $\iota \in \mathcal{I}$. Clearly, we put $z_\iota = \infty$ if either $t_{(\iota,0)} = \infty$ or $t_{(0,\iota)} = \infty$.

We consider the spectral measure given by $G = E \circ \tau^{-1}$, defined on the Borel subsets of $(\mathbb{C}_\infty)^{\mathcal{I}}$. Setting

$$A_\iota = \{z \in (\mathbb{C}_\infty)^{\mathcal{I}}; z_\iota = \infty\},$$

$$B_\iota = \{t \in (\mathbb{R}_\infty)^{\mathcal{L}}; t_{(\iota,0)} = \infty\},\ C_\iota = \{t \in (\mathbb{R}_\infty)^{\mathcal{L}}; t_{(0,\iota)} = \infty\},$$

we obtain $\tau^{-1}(A_\iota) \subset B_\iota \cup C_\iota$ for all $\iota \in \mathcal{I}$. Since $E(B_\iota \cup C_\iota) = 0$, it follows that $G(A_\iota) = 0$, that is, the coordinate function $z \mapsto z_\iota$ is G-almost everywhere finite. In addition,

$$\int_{(\mathbb{C}_\infty)^{\mathcal{I}}} z_\iota dG_{x,y}(z) = \int_{(\mathbb{R}_\infty)^{\mathcal{L}}} (t_{(\iota,0)} + it_{(0,\iota)}) dE_{x,y}(z)$$
$$= \langle (A'_\iota + iA''_\iota)x, y\rangle = \langle N_\iota x, y\rangle$$

for all $x \in \mathcal{D}(N_\iota) = \mathcal{D}(A'_\iota) \cap \mathcal{D}(A''_\iota), y \in \mathcal{H}$ and $\iota \in \mathcal{I}$.

We also have

$$\mathcal{D}(N_\iota) = \{x \in \mathcal{H}; \int_{(\mathbb{C}_\infty)^{\mathcal{I}}} |z_\iota|^2 dG_{x,x}(z) < \infty\},$$

for all $\iota \in \mathcal{I}$, as a consequence of [18], Theorem 13.24.

Because $G(A_\iota) = 0$ for all $\iota \in \mathcal{I}$, the last assertion follows as in the proof of Theorem 4.2. □

Example 4.4. Let Ω be a Hausdorff space, let $C(\Omega)$ be the space of all complex-valued continuous functions on Ω, and let E be a spectral measure, defined on the family of all Borel subsets of Ω, with values in $\mathcal{B}(\mathcal{H})$. Assume that the measure E is Radon, that is, the scalar positive measures $E_{x,x}$, $x \in \mathcal{H}$, are all Radon measures (see [5] and [16] for details). We shall see that the set $\mathcal{D}_\infty = \cap_{f\in C(\Omega)} \mathcal{D}_f$ is a dense subspace in $\mathcal{H}$ and

$$\mathcal{N}_\infty = \{\Psi(f)|\mathcal{D}_\infty; f \in C(\Omega)\}$$

is a normal algebra.

That $\mathcal{N}_\infty$ is an O^*-algebra follows easily from [18], Theorem 13.24. To prove that $\mathcal{D}_\infty$ is a dense subspace in $\mathcal{H}$, we adapt some ideas from [16], page 225. We fix a function $f \in C(\Omega)$ and a compact set $K \subset \Omega$. Fix also $y = E(K)x \in E(K)\mathcal{H}$. As the function $f\chi_K$ is bounded, where χ_K is the characteristic function of K, we have $\int_\Omega |f|^2 dE_{y,y} = \int_K |f|^2 dE_{y,y} = \int_K |f|^2 dE_{x,x}$, via the standard properties of the spectral measure E (see [18], Theorem 13.24). In particular, $y \in \mathcal{D}_f$, and so $\cup_{K\in\mathcal{K}(\Omega)} E(K)\mathcal{H} \subset \mathcal{D}_\infty$, where $\mathcal{K}(\Omega)$ is the family of all compact subsets of Ω. To prove the density of $\mathcal{D}_\infty$ in $\mathcal{H}$, it sufficient to prove that the set $\cup_{K\in\mathcal{K}(\Omega)}E(K)\mathcal{H}$ is dense in $\mathcal{H}$.

Fix an $x \in \mathcal{H}$. It follows from [5], Proposition 2.1.7, that

$$\sup_{K\in\mathcal{K}(\Omega)} \|E(K)x\|^2 = \sup_{K\in\mathcal{K}(\Omega)} \int_K dE_{x,x} = \int_\Omega dE_{x,x} = \|x\|^2.$$

In particular, there is a sequence $(K_k)_{k\geq 1}$ of compact subsets in Ω such that $\lim_{k\to\infty} \|E(K_k)x\| = \|x\|$. As the vectors $x - E(K_k)x = E(\Omega\setminus K_k)x$ and $E(K_k)x$ are orthogonal, we infer that

$$\|x - E(K_k)x\|^2 = \|x\|^2 - \|E(K_k)x\|^2,$$

for all $k \geq 1$. Therefore, $\lim_{k\to\infty} E(K_k)x = x$, showing the desired density.

We have only to show that the closure of the operator $\Psi(f)|\mathcal{D}_\infty$ equals $\Psi(f)$ for each $f \in C(\Omega)$. With $f \in C(\Omega)$ and $x \in \mathcal{D}_f$, we refine a previous argument. Because we have

$$\sup_{K\in\mathcal{K}(\Omega)} \int_K dE_{x,x} = \int_\Omega dE_{x,x}, \ \sup_{K\in\mathcal{K}(\Omega)} \int_K |f|^2 dE_{x,x} = \int_\Omega |f|^2 dE_{x,x},$$

(via [5], Proposition 2.1.7), and left side integrals depend increasingly on the compact set K, there is a sequence $(K_k)_{k\geq 1}$ of compact subsets in Ω such that $\lim_{k\to\infty} \|E(K_k)x\| = \|x\|$, and $\lim_{k\to\infty} \|\Psi(f)E(K_k)x\| = \|\Psi(f)x\|$. The orthogonality of the vectors $x - E(K_k)x$ and $E(K_k)x$ on one side, and that of the vectors $\Psi(f)x - \Psi(f)E(K_k)x$ and $\Psi(f)E(K_k)x$ on the other side show, as above, that $\lim_{k\to\infty} E(K_k)x = x$ and $\lim_{k\to\infty} \Psi(f)E(K_k)x = \Psi(f)x$, which implies that the closure of the operator $\Psi(f)|\mathcal{D}_\infty$ is equal $\Psi(f)$.

Consequently, $\mathcal{N}_\infty$ is a normal algebra.

Example 4.5. Let Ω be a topological space of the form $\Omega = \cup_{n\geq 1}\Omega_n$, where $(\Omega_n)_{n\geq 1}$ is an increasing sequence of Borel subsets. Let also $\mathcal{A}$ be an algebra of Borel functions on Ω, containing the constant functions and the complex conjugate of every given function from $\mathcal{A}$. Also assume that $f|_{\Omega_n}$ is bounded for all $f \in \mathcal{A}$ and $n \geq 1$.

Let E be a spectral measure (not necessarily Radon), defined on the family of all Borel subsets of Ω, with values in $\mathcal{B}(\mathcal{H})$. Setting $\mathcal{D} = \cap_{f\in\mathcal{A}}\mathcal{D}_f$, then $\mathcal{D}$ is dense in $\mathcal{H}$ and $\mathcal{N} = \{\Psi(f)|\mathcal{D}; f \in \mathcal{A}\}$ is a normal algebra. Indeed, if $f \in \mathcal{A}$, and so $f|_{\Omega_n}$ is bounded for all $n \geq 1$, we have $E(\Omega_n)x \in \mathcal{D}_f$. In addition, $\lim_{n\to\infty} E(\Omega_n)x = x$ for all $x \in \mathcal{H}$, showing that $\mathcal{D}$ is dense in $\mathcal{H}$. Moreover, if $x \in \mathcal{D}_f$, then the sequence $\Psi(f)E(\Omega_n)x$ is convergent to $\Psi(f)x$, implying $\overline{\Psi(f)|\mathcal{D}} = \Psi(f)$ for each $f \in \mathcal{A}$. It is clear that $\mathcal{N}$ is an O^*-algebra, and therefore, $\mathcal{N}_\infty$ is a normal algebra.

Lemma 4.6. *If $\mathcal{A}$ is a normal algebra and N_1, N_2 are arbitrary elements, then $\bar{N}_1, \bar{N}_2$ commute.*

Proof. We first note that if $S \in \mathcal{N}$ is symmetric, then $\bar{S}$ is selfadjoint, which is clear because $\bar{S}$ is normal. Note also that for an arbitrary $N \in \mathcal{N}$ we have $NN^\# = N^\# N$, as a consequence of the fact that $\bar{N}(\bar{N})^* = (\bar{N})^*\bar{N}$.

Now, let $S_1, S_2 \in \mathcal{N}$ be symmetric. Note that

$$2i(S_1S_2 - S_2S_1)x = (S_1 - iS_2)(S_1 + iS_2)x - (S_1 + iS_2)(S_1 - iS_2)x$$
$$= (\overline{S_1 + iS_2})^*\overline{(S_1 + iS_2)}x - \overline{(S_1 + iS_2)}(\overline{S_1 + iS_2})^*x = 0,$$

for all $x \in \mathcal{D}$, since $\overline{(S_1 + iS_2)}$ is normal and $(\overline{S_1 + iS_2})^* = (S_1 + iS_2)^* \supset (S_1 - iS_2)$. It follows from Proposition 7.1.3 in [20] that $\bar{S}_1, \bar{S}_2$ commute.

If $N_1, N_2 \in \mathcal{N}$ are arbitrary, we write $N_1 = S_{11} + iS_{12}, N_2 = S_{21} + iS_{22}$, with S_{jk} symmetric. The previous argument shows that the selfadjoint operators $\bar{S}_{11}, \bar{S}_{12}, \bar{S}_{21}, \bar{S}_{22}$ commute. Therefore, the normal operators $\bar{N}_1, \bar{N}_2$ also commute. □

Lemma 4.7. *Let $A_1, \dots, A_n$ be commuting self-adjoint operators and let $U_1, \dots, U_n$ be the Cayley transforms of $A_1, \dots, A_n$ respectively. Then we have:*

$$\prod_{j=1}^{n}(2 - U_j - U_j^*)^{k_j} = \prod_{j=1}^{n} 4^{k_j}(1 + A_j^2)^{-k_j},$$

for all integers $k_1 \geq 1, \dots, k_n \geq 1$.

In particular, the operator $\prod_{j=1}^{n}(2 - U_j - U_j^*)^{k_j}$ is positive and injective for all integers $k_1 \geq 1, \dots, k_n \geq 1$.

Proof. We have the equality $U_j = (A_j - i)(A_j + i)^{-1}$ for all indices j, by the definition of Cayley transform, whence $1 - U_j = 2i(A_j + i)^{-1}$. Because one has $(2i(A_j + i)^{-1})^* = -2i(A_j - i)^{-1}$, via the selfadjointness of A_j, we infer $1 - U_j^* = -2i(A_j - i)^{-1}$. Therefore,

$$2 - U_j - U_j^* = 2i(A_j + i)^{-1}((A_j - i) - (A_j + i))(A_j - i)^{-1} = 4(A_j^2 + 1)^{-1}.$$

Taking into account that the operators $(A_j^2 + 1)^{-1}$, $j = 1, \dots, n$, are all bounded and commute, we deduce easily the stated formula. The remaining assertions are direct consequences of that formula. □

Theorem 4.8. *Let $\mathcal{N} \subset \mathcal{L}^{\#}(\mathcal{L})$ be a normal algebra. Then there exists a family of indices $\mathcal{I}$, a compact subspace $\Omega \subset \mathbb{T}^{\mathcal{I}}$, a set of denominators $\mathcal{M} \subset C(\Omega)$, and an injective $*$-homomorphism $\mathcal{N} \ni N \mapsto \phi_N \in C(\Omega)/\mathcal{M}$.*

In addition, there exists a uniquely determined spectral measure F on the Borel subsets of Ω such that ϕ_N is F-almost everywhere defined and

$$\langle Nx, y\rangle = \int_\Omega \phi_N(z)dF_{x,y}(z),\ x \in \mathcal{D},\ y \in \mathcal{H}.$$

Proof. Let $A = (A_\iota)_{\iota\in\mathcal{I}}$ be a family of hermitian generators of the algebra $\mathcal{N}$. Such a family obviously exists because $N = (N + N^{\#})/2 + i(N - N^{\#})/2i$ and both $(N + N^{\#})/2$, $(N - N^{\#})/2i$ are hermitian. Every A_ι can be associated with the Cayley transform U_ι of $\bar{A}_\iota$. As in the proof of Theorem 4.2, since the selfadjoint operators $(\bar{A}_\iota)_{\iota\in\mathcal{I}}$ mutually commute, the corresponding Cayley transforms, as well as their adjoints, mutually commute. Let $\mathcal{B}$ be the closed unital algebra generated by the family $(U_\iota, U_\iota^*)_{\iota\in\mathcal{I}}$ in $\mathcal{B}(\mathcal{H})$, which is a commutative unital C^*-algebra.

It follows from Lemma 4.6 that each operator $2-U_\iota-U_\iota^*$ is injective. If $\tilde{\mathcal{M}}$ is the family of all possible finite products of operators of the form $2-U_\iota-U_\iota^*$, $\iota\in\mathcal{I}$, and the identity 1, then $\tilde{\mathcal{M}}$ is a set of denominators consisting of positive operators.

We define a map from $\mathcal{N}$ to $\mathcal{B}/\tilde{\mathcal{M}}$ by associating to each generator A_ι the fraction $i(U_\iota-U_\iota^*)/(2-U_\iota-U_\iota^*)$, and extending this assignment to $\mathcal{N}$ by linearity and multiplicativity. Let us do this operation properly.

Setting $A^0 = 1$ for $0 = (0)_{\iota\in\mathcal{I}}$ and $A^\alpha = \prod_{\alpha_\iota\neq 0} A_\iota^{\alpha_\iota}$ for $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}} \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha\neq 0$, we may define $p(A) = \sum_{\alpha\in\mathcal{J}} c_\alpha A^\alpha$ for every polynomial $p(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha t^\alpha$ in $\mathcal{P}_\mathcal{I}$. As a matter of fact, we have the equality $\mathcal{N} = \{p(A);\, p\in\mathcal{P}_\mathcal{I}\}$ because A is family of generators of $\mathcal{N}$. If we put $W_\iota = i(U_\iota-U_\iota^*)/(2-U_\iota-U_\iota^*)$ and $W = (W_\iota)_{\iota\in\mathcal{I}}$, then $\Phi(p(A)) = p(W)$ for each polynomial p. To show that this definition does not depend on a particular representation of $p(A)$, it suffices to show that $p(W)=0$ implies $p(A)=0$ for a fixed polynomial $p\in\mathcal{P}_\mathcal{I}$ such that $p(A) = \sum_{\alpha\in\mathcal{J}} c_\alpha A^\alpha$. Let $\beta\in\mathbb{Z}_+^{(\mathcal{I})}$ be such that $\alpha_\iota\leq\beta_\iota$ for all $\iota\in\mathcal{I}$ and $\alpha\in\mathcal{J}$. Also set $V_\gamma = \prod_\iota (2-U_\iota-U_\iota^*)^{\gamma_\iota}$ and $D_\alpha = \prod_\iota (i(U_\iota-U_\iota^*))^{\alpha_\iota}$ for all γ and α in $\mathbb{Z}_+^{(\mathcal{I})}$. Then

$$V_\beta p(W) = \sum_{\alpha\in\mathcal{J}} c_\alpha V_\beta W^\alpha = \sum_{\alpha\in\mathcal{J}} c_\alpha V_{\beta-\alpha} D_\alpha = 0.$$

It follows from Lemma 4.7 and the fact that $\mathcal{D}$ is invariant under A^α for all α, that the operator V_β^{-1} is defined on $\mathcal{D}$ and leaves it invariant, for all β. Consequently, for an arbitrary $x\in\mathcal{D}$ we have:

$$p(A)x = \sum_{\alpha\in\mathcal{J}} c_\alpha D_\alpha V_\alpha^{-1} x = \sum_{\alpha\in\mathcal{J}} c_\alpha V_{\beta-\alpha} D_\alpha V_\beta^{-1} x = 0.$$

This allows us to define correctly an injective unital $*$-homomorphism from the algebra $\mathcal{N}$ into the algebra $\mathcal{B}/\tilde{\mathcal{M}}$.

As in the proof of Theorem 4.2, the space $\Gamma(\mathcal{B})$ of characters of the algebra $\mathcal{B}$ may be identified with a closed subspace Ω of the compact space $\mathbb{T}^\mathcal{I}$. Then $\mathcal{B}$ is identified with $C(\Omega)$, and the function $\hat{U}_\iota$ with the coordinate function $\mathbb{T}^\mathcal{I}\ni z\to z_\iota\in\mathbb{C}$ for all ι. Hence, the set of denominators $\mathcal{M}$ corresponding to $\tilde{\mathcal{M}}$ will be the set of all possible finite products of functions of the form $2-z_\iota-\bar{z}_\iota$, $\iota\in\mathcal{I}$, and the constant function 1. As noticed in the Introduction, the algebra of fractions $\mathcal{B}/\tilde{\mathcal{M}}$ can be identified with the algebra of fractions $C(\Omega)/\mathcal{M}$. The image of the algebra $\mathcal{N}$ in $C(\Omega)/\mathcal{M}$ will be the unital algebra generated by the fractions $\theta_\iota(z) = -\Im z_\iota/(1-\Re z_\iota)$, $\iota\in\mathcal{I}$. Specifically, if $N\in\mathcal{N}$ is arbitrary, and if $p_N\in\mathcal{P}_\mathcal{I}$ is a polynomial such that $N = p_N(A)$, then, setting $\phi_N = p_N\circ\theta$, where $\theta = (\theta_\iota)_{\iota\in\mathcal{I}}$, the map $\mathcal{M}\ni N\mapsto\phi_N\in C(\Omega)/\mathcal{M}$ is a $*$-homomorphism.

As in the proof of Theorem 4.2, there exists a spectral measure F defined on the Borel subsets of Ω such that $U_\iota = \int_\Omega z_\iota dF(z)$ for all ι. Moreover, if $C_\iota = \{z\in\Omega;\, z_\iota = 1\}$, we have $F(C_\iota) = 0$ for all ι. This shows that the function ϕ_N, which is not defined on a finite union of the sets C_ι, is almost everywhere defined.

Fix an $N\in\mathcal{N}$, $N = p_N(A)$, with $p_N(t)) = \sum_{\alpha\in\mathcal{J}} c_\alpha t^\alpha$. Because we have

$$\langle \bar{A}_\iota x, y\rangle = \int_\Omega \theta_\iota(z) dF_{x,y}(z),\ x \in \mathcal{D}(\bar{A}_\iota),\ y \in \mathcal{H},$$

we can write the following equations:

$$\int_\Omega \phi_N(z) dF_{x,y}(z) = \int_\Omega (p_N \circ \theta)(z) dF_{x,y}(z)$$
$$= \sum_{\alpha\in\mathcal{J}} c_\alpha \int_\Omega \prod_{\alpha_\iota \neq 0} \theta_\iota^{\alpha_\iota}(z) dF_{x,y}(z) = \sum_{\alpha\in\mathcal{J}} c_\alpha \int_\Omega \prod_{\alpha_\iota\neq 0} A_\iota^{\alpha_\iota}(z) dF_{x,y}(z)$$
$$= \langle p_N(A)x, y\rangle = \langle Nx, y\rangle,$$

for all $x \in \mathcal{D}$ and $y \in \mathcal{H}$, via the usual properties of the unbounded functional calculus (see [18]).

The measure F is uniquely determined on Ω, via the uniqueness in Theorem 4.2. □

Remark 4.9. We may apply Theorem 4.3 to the family $\{\bar{N}; N \in \mathcal{N}\}$, $\mathcal{N}$ a normal algebra, which is a commuting family of normal operators in $\mathcal{H}$. According to this result, there exists a unique spectral measure E on the Borel subsets of $(\mathbb{C}_\infty)^{\mathcal{N}}$ such that each coordinate function $(\mathbb{C}_\infty)^{\mathcal{N}} \ni z \to z_N \in \mathbb{C}_\infty$ is E-almost everywhere finite, and

$$\langle \bar{N}x, y\rangle = \int_{(\mathbb{C}_\infty)^{\mathcal{N}}} z_N dE_{x,y}(z),\ x \in \mathcal{D}(\bar{N}),\ y \in \mathcal{H},\ N \in \mathcal{N}.$$

5. Normal extensions

In this section we present a version of a Theorem 3.4 in [2], concerning the existence of normal extensions. Unlike in [2], we prove it here for infinitely many operators.

Fix a Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D}$ of $\mathcal{H}$, and a compact Hausdorff space Ω. As before, we denote by $SF(\mathcal{D})$ the space of all sesquilinear forms on $\mathcal{D}$.

For the convenience of the reader, we shall reproduce some statements from [2], to be used in the sequel. We start by recalling some terminology from [2].

Let $\mathcal{Q} \subset C(\Omega)$ be a set of positive denominators. Fix a $q \in \mathcal{Q}$. A linear map $\psi : C(\Omega)/q \to SF(\mathcal{D})$ is called *unital* if $\psi(1)(x, y) = \langle x, y\rangle$, $x, y \in \mathcal{D}$. We say that ψ is *positive* if $\psi(f)$ is positive semidefinite for all $f \in (C(\Omega)/q)_+$.

More generally, let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty. Let $\mathcal{C} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{C} \to SF(\mathcal{D})$ be linear. The map ψ is said to be *unital* (resp. *positive*) if $\psi|C(\Omega)/q$ is unital (resp. positive) for all $q \in \mathcal{Q}_0$.

We start with a part from Theorem 2.2 in [2].

Theorem B. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty, let $\mathcal{C} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{C} \to SF(\mathcal{D})$ be linear and unital. The map ψ is positive if and only if*

$$\sup\{|\psi(hq^{-1})(x, x)|;\ h \in C(\Omega),\ \|h\|_\infty \leq 1\} = \psi(q^{-1})(x, x),\ q \in \mathcal{Q}_0,\ x \in \mathcal{D}.$$

Let again $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty and let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} \mathcal{F}_q$, where $q^{-1} \in \mathcal{F}_q$ and $\mathcal{F}_q$ is a vector subspace of $C(\Omega)/q$ for all $q \in \mathcal{Q}_0$. Let $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ be linear. Suppose that $\phi(q^{-1})(x,x) > 0$ for all $x \in \mathcal{D}\setminus\{0\}$ and $q \in \mathcal{Q}_0$. Then $\phi(q^{-1})$ induces an inner product on $\mathcal{D}$, and let $\mathcal{D}_q$ be the space $\mathcal{D}$, endowed with the norm given by $\| * \|_q^2 = \phi(q^{-1})(*,*)$.

Let $M_n(\mathcal{F}_q)$ (resp. $M_n(\mathcal{F})$) denote the space of $n\times n$-matrices with entries in $\mathcal{F}_q$ (resp. in $\mathcal{F}$) . Note that $M_n(\mathcal{F}) = \sum_{q\in\mathcal{Q}_0} M_n(\mathcal{F}_q)$ may be identified with a subspace of the algebra of fractions $C(\Omega, M_n)/\mathcal{Q}$, where M_n is the C^*-algebra of $n\times n$-matrices with entries in $\mathbb{C}$. Moreover, the map ϕ has a natural extension $\phi^n : M_n(\mathcal{F}) \mapsto SF(\mathcal{D}^n)$, given by

$$\phi^n(\mathbf{f})(\mathbf{x},\mathbf{y}) = \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k, y_j),$$

for all $\mathbf{f} = (f_{j,k}) \in M_n(\mathcal{F})$ and $\mathbf{x} = (x_1,\ldots,x_n), \mathbf{y} = (y_1,\ldots,y_n) \in \mathcal{D}^n$.

Let $\phi_q^n = \phi^n \mid M_n(\mathcal{F}_q)$. Endowing the Cartesian product $\mathcal{D}^n$ with the norm $\|\mathbf{x}\|_q^2 = \sum_{j=1}^n \phi(q^{-1})(x_j,x_j)$ if $\mathbf{x} = (x_1,\ldots,x_n) \in \mathcal{D}^n$, and denoting it by $\mathcal{D}_q^n$, we are in the conditions of Remark 3.8, with M_n for A and $\mathcal{D}_q^n$ for $\mathcal{D}_q$. Hence we say that the map ϕ^n is contractive if $\|\phi_q^n\| \le 1$ for all $q \in \mathcal{Q}_0$. Using the standard norm $\| * \|_n$ in the space of M_n, the space $M_n(\mathcal{F}_q)$ is endoved with the norm $\|(qf_{j,k})\|_{n,\infty} = \sup_{\omega\in\Omega} \|(q(\omega)f_{j,k}(\omega))\|_n$, for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$.

Following [3] and [13] (see also [12]), we shall say that the map $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ is *completely contractive* if the map $\phi^n : M_n(\mathcal{F}) \mapsto SF(\mathcal{D}^n)$ is contractive for all integers $n \ge 1$.

Note that a linear map $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ with the property $\phi(q^{-1})(x,x) > 0$ for all $x \in \mathcal{D}\setminus\{0\}$ and $q \in \mathcal{Q}_0$ is completely contractive if and only if for all $q \in \mathcal{Q}_0$, $n \in \mathbb{N}$, $x_1,\ldots,x_n,y_1,\ldots,y_n \in \mathcal{D}$ with

$$\sum_{j=1}^{n} \phi(q^{-1})(x_j,x_j) \le 1, \quad \sum_{j=1}^{n} \phi(q^{-1})(y_j,y_j) \le 1,$$

and for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$ with $\|(qf_{j,k})\|_{n,\infty} \le 1$, we have

$$\left| \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k,y_j) \right| \le 1. \tag{5.1}$$

Let us now recall the main result of [2], namely Theorem 2.5, in a shorter form (see also [21] and [25] for some particular cases).

Theorem C. *Let Ω be a compact space and let $\mathcal{Q} \subset C(\Omega)$ be a set of positive denominators. Let also $\mathcal{Q}_0$ be a cofinal subset of $\mathcal{Q}$, with $1 \in \mathcal{Q}_0$. Consider $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} \mathcal{F}_q$, where $\mathcal{F}_q$ is a vector subspace of $C(\Omega)/q$ such that $r^{-1} \in \mathcal{F}_r \subset \mathcal{F}_q$ for all $r \in \mathcal{Q}_0$ and $q \in \mathcal{Q}_0$, with $r|q$. Let also $\phi : \mathcal{F} \to SF(\mathcal{D})$ be linear and unital, and set $\phi_q = \phi|\mathcal{F}_q$, $\phi_{q,x}(*) = \phi_q(*)(x,x)$ for all $q \in \mathcal{Q}_0$ and $x \in \mathcal{D}$.*

The following two statements are equivalent:

(a) *The map* ϕ *extends to a unital, positive, linear map* ψ *on* $C(\Omega)/\mathcal{Q}$ *such that, for all* $x \in \mathcal{D}$ *and* $q \in \mathcal{Q}_0$*, we have:*

$$\|\psi_{q,x}\| = \|\phi_{q,x}\|, \quad \textit{where} \quad \psi_q = \psi|C(\Omega)/q,\ \psi_{q,x}(*) = \psi_q(*)(x,x).$$

(b) (i) $\phi(q^{-1})(x,x) > 0$ *for all* $x \in \mathcal{D} \setminus \{0\}$ *and* $q \in \mathcal{Q}_0$.
(ii) *The map* ϕ *is completely contractive.*

Remark. A "minimal" subspace of $C(\Omega)/\mathcal{Q}$ to apply Theorem C is obtained as follows. If $\mathcal{Q}_0$ is a cofinal subset of $\mathcal{Q}$ with $1 \in \mathcal{Q}_0$, we define $\mathcal{F}_q$ for some $q \in \mathcal{Q}_0$ to be the vector space generated by all fractions of the form r/q, where $r \in \mathcal{Q}_0$ and $r|q$. It is clear that the subspace $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} \mathcal{F}_q$ has the properties required to apply Theorem C.

We also need Corollary 2.7 from [2].

Corollary D. *Suppose that, with the hypotheses of Theorem B, condition (b) is satisfied. Then there exists a positive* $B(\mathcal{H})$*–valued measure* F *on the Borel subsets of* Ω *such that*

$$\phi(f)(x,y) = \int_\Omega f\, dF_{x,y},\ f \in \mathcal{F},\ x,y \in \mathcal{D}. \tag{5.2}$$

For every such measure F *and every* $q \in \mathcal{Q}_0$*, we have* $F(Z(q)) = 0$.

Example 5.1. We extend to infinitely many variables the Example 3.5. Let $\mathcal{I}$ be a (nonempty) family of indices. Denote by $z = (z_\iota)_{\iota\in\mathcal{I}}$ the independent variable in $\mathbb{C}^\mathcal{I}$. Let also $\bar z = (\bar z_\iota)_{\iota\in\mathcal{I}}$. As before, let $\mathbb{Z}_+^{(\mathcal{I})}$ be the set of all collections $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}}$ of nonnegative integers, with finite support. Setting $z^0 = 1$ for $0 = (0)_{\iota\in\mathcal{I}}$ and $z^\alpha = \prod_{\alpha_\iota\neq 0} z_\iota^{\alpha_\iota}$ for $z = (z_\iota)_{\iota\in\mathcal{I}} \in \mathbb{C}^\mathcal{I}$, $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}} \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we may consider the algebra of complex-valued functions $\mathcal{S}_\mathcal{I}$ on $\mathbb{C}^\mathcal{I}$, consisting of expressions of the form $\sum_{\alpha,\beta\in\mathcal{J}} c_{\alpha,\beta} z^\alpha \bar z^\beta$, with $c_{\alpha,\beta}$ complex numbers for all $\alpha,\beta \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{Z}_+^{(\mathcal{I})}$ is finite.

We can embed the space $\mathcal{S}_\mathcal{I}$ into the algebra of fractions derived from the basic algebra $C((\mathbb{C}_\infty)^\mathcal{I})$, using a suitable set of denominators. Specifically, we consider the family $\mathcal{R}_\mathcal{I}$ consisting of all rational functions of the form $r_\alpha(t) = \prod_{\alpha_\iota\neq 0}(1+|z_\iota|^2)^{-\alpha_\iota}$, $z = (z_\iota)_{\iota\in\mathcal{I}} \in \mathbb{C}^\mathcal{I}$, where $\alpha = (\alpha_\iota) \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, is arbitrary. Of course, we set $r_0 = 1$. The function r_α can be continuously extended to $(\mathbb{C}_\infty)^\mathcal{I} \setminus \mathbb{C}^\mathcal{I}$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$. In fact, actually the function $f_{\beta,\gamma}(z) = z^\beta \bar z^\gamma r_\alpha(z)$ can be continuously extended to $(\mathbb{C}_\infty)^\mathcal{I} \setminus \mathbb{C}^\mathcal{I}$ whenever $\beta_\iota + \gamma_\iota < 2\alpha_\iota$, and $\beta_\iota = \gamma_\iota = 0$ if $\alpha_\iota = 0$, for all $\iota \in \mathcal{I}$ and $\alpha,\beta,\gamma \in \mathbb{Z}_+^{(\mathcal{I})}$. Moreover, the family $\mathcal{R}_\mathcal{I}$ becomes a set of denominators in $C((\mathbb{C}_\infty)^\mathcal{I}))$. This shows that the space $\mathcal{S}_\mathcal{I}$ can be embedded into the algebra of fractions $C((\mathbb{C}_\infty)^\mathcal{I})/\mathcal{R}_\mathcal{I}$.

To be more specific, for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we denote by $\mathcal{S}_{\mathcal{I},\alpha}^{(1)}$ the linear spaces generated by the monomials $z^\beta \bar z^\gamma$, with $\beta_\iota + \gamma_\iota < 2\alpha_\iota$ whenever $\alpha_\iota > 0$, and $\beta_\iota = \gamma_\iota = 0$ if $\alpha_\iota = 0$. Put $\mathcal{S}_{\mathcal{I},0}^{(1)} = \mathbb{C}$.

We also define $\mathcal{S}^{(2)}_{\mathcal{I},\alpha}$, for $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, to be the linear space generated by the monomials $|z|^{2\beta} = \prod_{\beta_\iota \neq 0}(z_\iota \bar{z}_\iota)^{\beta_\iota}$, $0 \neq \beta$, $\beta_\iota \leq \alpha_\iota$ for all $\iota \in \mathcal{I}$ and $|z| = (|z_\iota|)_{\iota \in \mathcal{I}}$. We define $\mathcal{S}^{(2)}_{\mathcal{I},0} = \{0\}$.

Set $\mathcal{S}_{\mathcal{I},\alpha} = \mathcal{S}^{(1)}_{\mathcal{I},\alpha} + \mathcal{S}^{(2)}_{\mathcal{I},\alpha}$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$. Note that, if $f \in \mathcal{S}_{\mathcal{I},\alpha}$, the function $r_\alpha f$ extends continuously to $(\mathbb{C}_\infty)^{\mathcal{I}}$ and that $\mathcal{S}_{\mathcal{I},\alpha} \subset \mathcal{S}_{\mathcal{I},\beta}$ if $\alpha_\iota \leq \beta_\iota$ for all $\iota \in \mathcal{I}$.

It is now clear that the algebra $\mathcal{S}_{\mathcal{I}} = \sum_{\alpha \in \mathbb{Z}_+^{(\mathcal{I})}} \mathcal{S}_{\mathcal{I},\alpha}$ can be identified with a subalgebra of $C((\mathbb{C}_\infty)^{\mathcal{I}})/\mathcal{R}_{\mathcal{I}}$. This algebra has the properties of the space $\mathcal{F}$ appearing in the statement of Theorem C.

Let now $T = (T_\iota)_{\iota \in \mathcal{I}}$ be a family of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that $T_\iota(\mathcal{D}) \subset \mathcal{D}$ and $T_\iota T_\kappa x = T_\kappa T_\iota x$ for all $\iota, \kappa \in \mathcal{I}$, $x \in \mathcal{D}$.

Setting T^α as in the case of complex monomials, which is possible because of the commutativity of the family T on $\mathcal{D}$, we may define a unital linear map $\phi_T : \mathcal{S}_{\mathcal{I}} \to SF(\mathcal{D})$ by

$$\phi_T(z^\alpha \bar{z}^\beta)(x, y) = \langle T^\alpha x, T^\beta y\rangle\ , \quad x, y \in \mathcal{D},\ \alpha, \beta \in \mathbb{Z}_+^{(\mathcal{I})}, \tag{5.3}$$

which extends by linearity to the subspace $\mathcal{S}_{\mathcal{I}}$ generated by these monomials.

An easy proof shows that, for all α, β in $\mathbb{Z}_+^{(\mathcal{I})}$ with $\beta - \alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, and $x \in \mathcal{D} \setminus \{0\}$, we have

$$0 < \langle x, x\rangle \leq \phi_T(r_\alpha^{-1})(x, x) \leq \phi_T(r_\beta^{-1})(x, x)\,. \tag{5.4}$$

The polynomial r_α^{-1} will be denoted by s_α for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$.

The family $T = (T_\iota)_{\iota \in \mathcal{I}}$ is said to have a *normal extension* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a family $N = (N_\iota)_{\iota \in \mathcal{I}}$ consisting of commuting normal operators in $\mathcal{K}$ such that $\mathcal{D} \subset \mathcal{D}(N_\iota)$ and $N_\iota x = T_\iota x$ for all $x \in \mathcal{D}$ and $\iota \in \mathcal{I}$.

A family $T = (T_\iota)_{\iota \in \mathcal{I}}$ having a normal extension is also called a *subnormal family* (see, for instance, [1]).

The following result is a version of Theorem 3.4 from [2], valid for an arbitrary family of operators. We mention that the basic space of fractions from [2] is slightly modified.

Theorem 5.2. *Let $T = (T_\iota)_{\iota \in \mathcal{I}}$ be a family of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under T_ι for all $\iota \in \mathcal{I}$ and that T is a commuting family on $\mathcal{D}$. The family T admits a normal extension if and only if the map $\phi_T : \mathcal{S}_{\mathcal{I}} \mapsto SF(\mathcal{D})$ has the property that for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with*

$$\sum_{j=1}^m \phi_T(s_\alpha)(x_j, x_j) \leq 1, \quad \sum_{j=1}^m \phi_T(s_\alpha)(y_j, y_j) \leq 1,$$

and for all $p = (p_{j,k}) \in M_m(\mathcal{S}_{\mathcal{I},\alpha})$ *with* $\sup_z \|r_\alpha(z)p(z)\|_m \leq 1$*, we have*

$$\left|\sum_{j,k=1}^{m} \phi_T(p_{j,k})(x_k, y_j)\right| \leq 1. \tag{5.5}$$

Proof. We follow the lines of the proof of Theorem 3.4 in [2].

If the condition of the theorem is fulfilled, and so we have a linear and unital map $\phi_T : \mathcal{S}_{\mathcal{I}} \to SF(\mathcal{D})$ induced by (5.3), then conditions (i) (by (5.4)) and (ii) of Theorem C are satisfied for ϕ_T. Hence, by that theorem and Corollary D, there exists a positive $\mathcal{B}(\mathcal{H})$-valued measure F on the Borel sets of $\Omega = (\mathbb{C}_\infty)^{\mathcal{I}}$, such that (5.2) holds for ϕ_T. Because ϕ_T is unital, $F(\Omega)$ is the identity operator on $\mathcal{H}$. By the classical Naimark dilation theorem (see, for instance [12]), there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace and a spectral measure E on the Borel subsets of Ω with values in $\mathcal{B}(\mathcal{K})$, such that $F(*) = PE(*)|\mathcal{H}$, where P denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. As in Remark 4.1, for each $\iota \in \mathcal{I}$, let N_ι be the normal operator with domain

$$\mathcal{D}(N_\iota) = \Big\{x \in \mathcal{K}\,;\ \int_{K_\iota} |z_\iota|^2 dE_{x,x}(z)\} < \infty\Big\}$$

and

$$N_\iota x = \int_{K_\iota} z_\iota dE(z)x\,, \quad x \in \mathcal{D}(N_\iota)\,,$$

where $K_\iota = \{z \in \Omega; z_\iota \neq \infty\}$. For all $x, y \in \mathcal{D}$, $\iota \in \mathcal{I}$, we have

$$\langle PN_\iota x, y\rangle = \langle N_\iota x, y\rangle = \int_{K_\iota} z_\iota dE_{x,y}(z)$$

$$= \int_{\Omega} z_\iota dF_{x,y}(z) = \phi_T(z_\iota)(x,y) = \langle T_\iota x, y\rangle\,,$$

because $F(K_\iota) = F(\Omega)$. Indeed, $F(\Omega \setminus K_\iota) = F(\{z \in \Omega; z_\iota = \infty\}) = F(Z((1+|z_\iota|^2)^{-1}) = 0$, by Corollary D. Hence, $PN_\iota x = T_\iota x$ for all $x \in \mathcal{D}$, $\iota \in \mathcal{I}$. Note also that

$$\|T_\iota x\|^2 = \phi_T(|z_\iota|^2)(x,x) = \int_{\Omega} |z_\iota|^2 dF_{x,x}(z) =$$

$$\int_{K_\iota} |z_\iota|^2 dE_{x,x}(z) = \|N_\iota x\|^2.$$

for all $x \in \mathcal{D}$, $\iota \in \mathcal{I}$, which shows that $N = (N_\iota)_{\iota\in\mathcal{I}}$ is a normal extension of $T = (T_\iota)_{\iota\in\mathcal{I}}$, via the following:

Remark. Let $S : \mathcal{D}(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be an arbitrary linear operator. If $B : \mathcal{D}(B) \subset \mathcal{K} \mapsto \mathcal{K}$ is a normal operator such that $\mathcal{H} \subset \mathcal{K}$, $\mathcal{D}(S) \subset \mathcal{D}(B)$, $Sx = PBx$ and $\|Sx\| = \|Bx\|$ for all $x \in \mathcal{D}(S)$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$, then we have $Sx = Bx$ for all $x \in \mathcal{D}(S)$. Indeed, $\langle Sx, Sx\rangle = \langle Sx, Bx\rangle$ and $\langle Bx, Sx\rangle = \langle PBx, Sx\rangle = \langle Sx, Sx\rangle = \langle Bx, Bx\rangle$. Hence, we have $\|Sx - Bx\| = 0$ for all $x \in \mathcal{D}(S)$ (see [2], Remark 3.1).

We continue the proof of Theorem 5.2.

Conversely, if $T = (T_\iota)_{\iota\in\mathcal{I}}$ admits a normal extension $N = (N_\iota)_{\iota\in\mathcal{I}}$, the latter has a spectral measure E with support in $\Omega = (\mathbb{C}_\infty)^{\mathcal{I}}$, via Theorem 4.3. Then for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, the space $\mathcal{D}$ is contained in

$$\mathcal{D}(T^\alpha) \subset \mathcal{D}(N^\alpha) = \left\{x \in \mathcal{K}\,;\, \int_\Omega |z^\alpha|^2 dE_{x,x}(z) < \infty\right\}.$$

It follows that, for all $f \in C(\Omega)/r_\alpha$, the function f is integrable on Ω with respect to the positive scalar measure $E_{x,x}$. Using the decomposition $4E_{x,y} = E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy}$, we see that $\psi : C(\Omega)/\mathcal{R}_{\mathcal{I}} \mapsto SF(\mathcal{D})$, defined by

$$\psi(f)(x,y) = \int_\Omega f(z)dE_{x,y}(z),\ x, y \in \mathcal{D},\ f \in C(\Omega)/\mathcal{R}_{\mathcal{I}},$$

is a linear map which is obviously unital and positive. Moreover,

$$\psi(z^\alpha \bar{z}^\beta)(x,y) = \langle N^\alpha x, N^\beta y\rangle = \langle T^\alpha x, T^\beta y\rangle = \phi_T(z^\alpha \bar{z}^\beta)(x,y),$$

for all $x, y \in \mathcal{D}$ and $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, because N^α extends T^α, showing that ψ is an extension of ϕ_T.

Setting $\phi = \phi_T$, $\psi_{r_\alpha} = \psi|C(\Omega)/r_\alpha$, $\psi_{r_\alpha,x}(*) = \psi_{r_\alpha}(*)(x,x)$, $\phi_{r_\alpha} = \phi|\mathcal{S}_{r_\alpha}$, $\phi_{r_\alpha,x}(*) = \psi_{r_\alpha}(*)(x,x)$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$ and $x \in \mathcal{D}$, we have:

$$\phi(r_\alpha^{-1})(x,x) = \psi(r_\alpha^{-1})(x,x) = \|\psi_{r_\alpha,x}\| \geq \|\phi_{r_\alpha,x}\| \geq \phi(r_\alpha^{-1})(x,x),$$

via Theorem B. This shows that the map $\phi : \mathcal{S}_{\mathcal{I}} \to SF(\mathcal{D})$ satisfies condition (a) in Theorem C. We infer that the condition in the actual statement, derived from condition (b) in Theorem C, should be also satisfied. This completes the proof of Theorem 5.2. □

Remark 5.3. Let $T = (T_\iota)_{\iota\in\mathcal{I}}$ be a family of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under T_ι and that T is a commuting family on $\mathcal{D}$. If the map $\phi_T : \mathcal{S}_{\mathcal{I}} \mapsto SF(\mathcal{D})$ has the property (5.5), the family has a proper quasi-invariant subspace. In other words, there exists a proper Hilbert subspace $\mathcal{L}$ of the Hilbert space $\mathcal{H}$ such that the subspace $\{x \in \mathcal{D}(T_\iota) \cap \mathcal{L}; Tx \in \mathcal{L}\}$ is dense in in $\mathcal{L}$ for each $\iota \in \mathcal{I}$. This is a consequence of Theorem 5.3 and Theorem 11 from [1].

We use Example 3.5 for the particular case of a single operator. We take $\mathcal{I} = \{1\}$ and put $\mathcal{S}_{\mathcal{I}} = \mathcal{S}_1$, which is the set of all polynomials in z and $\bar{z}$, $z \in \mathbb{C}$. The set $\mathcal{R}_{\mathcal{I}} = \mathcal{R}_1$ consists of all functions of the form $r_k(z) = (1 + |z|^2)^{-k}$, with $z \in \mathbb{C}$ and $k \in \mathbb{Z}_+$.

Considering a single operator S, we may define a unital linear map $\phi_S : \mathcal{S}_1 \to SF(\mathcal{D})$ by

$$\phi_S(z^j \bar{z}^k)(x,y) = \langle S^j x, S^k y\rangle\ , \quad x, y \in \mathcal{D},\ j \in \mathbb{Z}_+,$$

extended by linearity to the subspace $\mathcal{S}_1$. The next result is a version of Corollary 3.5 from [2] (stated for a different basic space of fraction).

Corollary 5.4. *Let $S : \mathcal{D}(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be a densely defined linear operator such that $S\mathcal{D}(S) \subset \mathcal{D}(S)$. The operator S admits a normal extension if and only if for all $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{D}(S)$ with*

$$\sum_{j=1}^{n}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\langle S^k x_j, S^k x_j\rangle \le 1, \quad \sum_{j=1}^{n}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\langle S^k y_j, S^k y_j\rangle \le 1,$$

and for all $p = (p_{j,k}) \in M_n(\mathcal{S}_m)$, with $\sup_{z\in\mathbb{C}} \|(1+|z|^2)^{-m}p(z)\|_n \le 1$, we have

$$\left|\sum_{j,k=1}^{n}\langle \phi_S(p_{j,k})x_k, y_j\rangle\right| \le 1.$$

Corollary 5.3 is a direct consequence of Theorem 5.2.

The case of one operator, covered by our Corollary 5.3, is also studied in [22], via a completely different approach.

References

[1] E. Albrecht and F.-H. Vasilescu, *Invariant subspaces for some families of unbounded subnormal operators*, Glasgow Math. J. 45 (2003),53-67.

[2] E. Albrecht and F.-H. Vasilescu, *Unbounded extensions and operator moment problems*, Preprint, 2004.

[3] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta. Math. 123 (1969), 141-224.

[4] Ju. M. Berezanskii, *Self-adjoint operators in spaces of functions of an infinite number of variables* (Russian), Naukova Dumka, Kiev, 1978.

[5] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic analysis on semigroups*, Springer-Verlag, New York/Berlin/Heidelberg/Tokyo, 1984.

[6] N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1961.

[7] R. E. Curto and L. A. Fialkow, *Solution of the truncated complex moment problem for flat data*, Memoirs of the A. M. S. Number 568, 1996.

[8] R. E. Curto and L. A. Fialkow, *Flat extensions of positive moment matrices: Recursively generated relations*, Memoirs of the A. M. S. Number 648, 1998.

[9] O. Demanze and A. Mouze, *Dénominateurs universels pour la décomposition de polynômes positifs sur un espace de Hilbert*, J. Operator Theory, 56 (2006), 273-290.

[10] M. A. Dritschel, *On factorization on trigonometric polynomials*, Integr. equ. oper. theory 49 (2004), 11-42.

[11] N. Dunford and J.T. Schwartz, *Linear Operators*, Part I, Interscience Publishers, New York, London, 1958; Part II, Interscience Publishers John Wiley and sons, New York, London, 1963; Part III, Wiley-Interscience, New York, 1971.

[12] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge, 2002.

[13] R. T. Powers, *Selfadjoint algebras of unbounded operators.* II, Trans. Amer. Math. Soc., 187 (1974), 261-293.

[14] M. Putinar and F.-H. Vasilescu, *Solving moment problems by dimensional extension*, Annals of Math. 149 (1999), 1087-1107

[15] M. Putinar and F.-H. Vasilescu, *A uniqueness criterion in the multivariate moment problem*, Math. Scand., 92 (2003), 295-300.

[16] P. Ressel and W. J. Ricker, *An integral representation for semi-groups of unbounded normal operators*, J. Operator Theory, 47 (2002), 219-243.

[17] A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge University Press, Cambridge, 1973.

[18] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York, 1973.

[19] Yu. S. Samoilenko, *The spectral theory of sets of selfadjoint operators* (Russian), Naukova Dumka, Kiev, 1984.

[20] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Operator Theory: Advances and Applications Vol. 37, Birkhäuser Verlag, Basel/Boston/Berlin, 1990.

[21] J. Stochel, *Solving the truncated moment problem solves the full moment problem*, Glasgow Math. J. 43 (2001), 335-341.

[22] J. Stochel and F. H. Szafraniec, *The complex moment problem and subnormality: A polar decomposition approach*, J. Functional Analysis 159 (1998), 432-491.

[23] J. Stochel and F. H. Szafraniec, *Domination of unbounded operators and commutativity*, J. Math. Soc. Japan 55 (2003), no. 2, 405-437.

[24] B. L. van der Waerden, *Algebra*, Vol. 2, Frederick Ungar Publ. Co., New York, 1970.

[25] F.-H. Vasilescu, *Spaces of fractions and positive functionals*, Math. Scand. 96 (2005), 257-279.

[26] M. Woytylak, *Algebras dense in L^2 spaces: an operator approach*, Glasgow Math. J. 47 (2005), 155-165.

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