Square Positive Functionals in an Abstract Setting

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Abstract

Using techniques from the theory of finite-dimensional commutative Banach algebras, we discuss integral representations of square positive functionals in an absttract setting, extending and completing some results concerning the positive Riesz functionals in finite-dimensional spaces of polynomials.

0 Introduction

Let Ω be a Hausdorff (topological) space, and let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on Ω. We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, and following [14, 15], we say that $\mathcal{S}$, having these properties, is a *function space* (on Ω). The symbol $\mathcal{RS}$ will designate the "real part" of $\mathcal{S}$, that is $\{f \in \mathcal{S}; f = \bar{f}\}$.

Now, let $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

Important examples of function spaces are associated with the space $\mathcal{P}$ of all polynomials in $n \geq 1$ real variables, with complex coefficients. For every integer $m \geq 0$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. Note that $\mathcal{P}_m^{(2)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

We occasionally use the notation $\mathcal{P}_m^n$ instead of $\mathcal{P}_m$ and $\mathcal{P}^n$ instead of $\mathcal{P}$ when the number n should be specified.

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:

(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$;
(3) $\Lambda(1) = 1$.

Adapting some terminology from [10] to our context (see also [14], [15]), a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.

If $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality:*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ f, g \in \mathcal{S}. \tag{1}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle \hat{f}, \hat{g} \rangle = \Lambda(f\bar{g}), \tag{2}$$

where $\hat{f} = f + \mathcal{I}_\Lambda$ is the equivalence class of $f \in \mathcal{S}$ modulo $\mathcal{I}_\Lambda$.

In fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\ \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$.

If $\mathcal{S}$ is finite dimensional, then $\mathcal{H}_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

We also put $\mathcal{RH}_\Lambda := \{f + \mathcal{I}_\Lambda; f \in \mathcal{RS}\}$, which is a real Hilbert space. When $\hat{f} \in \mathcal{RH}_\Lambda$, we may and shall always assume that $f \in \mathcal{RS}$.

When the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is given, we shall use the notation $\mathcal{I}_\Lambda$, $\mathcal{H}_\Lambda$, $\hat{f}$, with the meaning from above, if not otherwise specified.

The *(abstract) moment problem* for a given uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{S}$ is a fixed function space on a Hausdorff space Ω, means to find conditions insuring the existence of a probability measure μ with support in Ω, such that $\Lambda(f) = \int f d\mu$, $f \in \mathcal{S}^{(2)}$. When such a measure μ exists, it is said to be a *representing measure* for Λ. Such a framework for stating and solving moment problems has been already used by other authors (see for instance [9]).

Note that the map $\mathcal{S}^{(2)} \ni f \mapsto \int f d\mu \in \mathbb{C}$, where μ is a probability measure with support in Ω, is a uspf, as one can easily see.

In some special cases, a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ may have an *atomic representing measure in* Ω, which in this text means that there exists a finite subset $\Omega_\Lambda = \{\omega_1, \ldots, \omega_d\} \subset \Omega$ and positive numbers $\lambda_1, \ldots, \lambda_d$, with $\lambda_1 + \cdots + \lambda_d = 1$, such that $\Lambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$ for all $f \in \mathcal{S}^{(2)}$.

When $\mathcal{S}$ is finite dimensional and the uspf Λ on $\mathcal{S}^{(2)}$ has an arbitrary representing measure, then one expects that this measure may be replaced by an atomic one. Such a property, going back to Tchakaloff (see Corollary 2 in [12]; see also [3, 4, 11, 15] etc.), will be discussed in Section 5 (see especially our Theorem 4).

In a previous work (see [15]), the author presented a new approach to truncated moment problems in several variables, via a concept of idempotent with respect to a Riesz type functional, naturally associated to the problem.

In this work we exhibit some generalizations of the results from [15], in a more abstract context, and present some new assertions and examples.

Next chapter contains comments, definitions and results (without proofs), necessary for further development. Some new definitions are also given (see Subsection 1.4).

In the second chapter we recapture some results from [15], in the present context. Generally, we follow the lines of the corresponding results from [15], but we give complete and sometimes improved arguments, often different from those in [15]. As a consequence of Theorem 2, some uspf's with no representing measure may have certain integral representations (see Example 3).

The third chapter deals with two abstract approaches to the existence of representing measure for square positive functionals. They are rather explicit characterizations but their consequences for concrete solutions of the moment problems are still to be proved.

The fourth chapter extends the concept of an orthogonal basis consisting of relative idempotents that is "multiplicative" with respect to a uspf and a tuple generating the given function space. The main result (Theorem 3) generalizes Theorem 2 from [15].

A discussion related to Tchakaloff's Theorem is presented in the fifth section. In particular, we extend (see our Theorem 2) the well-known result asserting that if a uspf has a representing measure on a finite dimensional function space of polynomials then it necessarily has an atomic representing measure.

The final chapter is an approach to the moment problem on finite dimensional function spaces when the number of atoms of a representing measure is not necessarily equal to the cardinal of an orthogonal basis consisting of idempotents. The basic result of this section (Theorem 5) presents an extension of Theorem 4 from [15].

1 Preliminaries

1.1 Tchakaloff's Theorem

Cubature formulas are of particular interest both in the abstract analysis as well as in numerical analysis for the computation or approximation of some integrals.

In 1957, V. Tchakaloff essentially proved the following cubature formula (see [12], Théorème II):

Theorem A *Let $F \subset \mathbb{R}^2$ be a bounded closed set. Fixing an integer $n \geq 1$ and setting $N = (n+1)(n+2)/2$, there are points $(u_k, v_k) \in \mathbb{R}^2$ and constants $a_k \geq 0$, $k = 1, \ldots, N$, such that*

$$\int\int_F p(u,v)dudv = \sum_{k=1}^{N} a_k p(u_k, v_k)$$

for all polynomials p in two variables, of total degre $\leq n$.

This result was improved by several authors: R. Curto and L. Fialkow [4], M. Putinar [11], C. Bayer and J. Teichmann [1] etc.

In 2006, C. Bayer and J. Teichmann proved the following version of Tchakaloff's result (see [1], Corollary 1):

Theorem B *Let μ be a positive Borel measure on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be measurable, with $\mu(\mathbb{R}^n \setminus A) = 0$. If*

$$\int_{\mathbb{R}^n} (t_1^2 + \cdots + t_n^2)^{1/2} d\mu(t) < +\infty,$$

there exist an integer $k \leq n$, k points $\xi_1, \ldots, \xi_k \in A$, and weights $\lambda_1 > 0, \ldots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^n} p(t) d\mu(t) = \sum_{j=1}^{k} \lambda_j p(\xi_j),$$

for every polynomial p of total degree equal to 1.

In the proof of our Theorem 4, we shall use a consequence of Theorem B, namely Corollary 2 from [1].

Using different methods, we have obtained (see [15], Corollary 4) another version of Tchakaloff's theorem:

Theorem C *Let μ be a positive Borel measure on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} (t_1^2 + \cdots + t_n^2) d\mu(t) < +\infty,$$

Then there exist a subset $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and positive numbers $\lambda_1, \ldots, \lambda_d$, where $d \leq n+1$, such that

$$\int_{\mathbb{R}^n} p(t) d\mu(t) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}).$$

for all polynomials p of total degree at most 2.

Although the hypothesis of Theorem C is stronger than the that of Theorem B, its consequences seem to be useful in concrete situations, because the weights $\lambda_1, \ldots, \lambda_d$, and the nodes $\xi^{(1)}, \ldots, \xi^{(d)}$ as well, can be given by explicit formulas.

1.2 Idempotents with respect to a uspf

In this subsection we recall the concept of idempotent element with respect to a given uspf, as defined in [15].

Example 1 Let $\Omega = \{\omega_1, \ldots, \omega_d\}$ be an arbitrary (finite) set and let $C(\Omega)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ω, endowed with the sup-norm. Assume that $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple in $C(\Omega)$ generating this algebra. In other words, every element of $C(\Omega)$ is a polynomial in $\theta_1, \ldots, \theta_n$. Moreover, with the standard notation for multi-indices, the set $\{\theta^\alpha; \alpha \in \mathbb{Z}_+^n\}$ must contain a subset $\{\theta^\alpha; \alpha \in \mathbb{Z}_+^n, |\alpha| \leq m\}$, for some integer $m \geq 1$, which spans the vector space $C(\Omega)$.

Fixing an integer $m \geq 1$ as above, the linear map $\mathcal{P}_m \ni p \mapsto p \circ \theta \in C(\Omega)$ is surjective.

Consider the measure $\nu = \sum_{j=1}^d \lambda_j \delta_j$, with δ_j the Dirac measure at ω_j, $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^d \lambda_j = 1$.

Setting $\xi^{(j)} := \theta(\omega_j)$, $\mu(\{\xi^{(j)}\}) = \nu(\{\omega_j\}) = \lambda_j$, $j = 1, \ldots, d$, and $\Xi := \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$, we put

$$\Lambda(p) = \int_\Xi p d\mu = \int_\Omega p \circ \theta d\nu,\ p \in \mathcal{P}_{2m},$$

which is a uspf, for which μ is a representing measure.

Let now $f \in C(\Omega)$ be an idempotent. In other words, f is the caracteristic function of a subset of Ω. Our previous discussion implies the existence of a polynomial $p \in \mathcal{P}_m$, which may be supposed to have real coefficients, such that $p \circ \theta = f$. Consequently, $\Lambda(p^2) = \int_\Omega p^2 \circ \theta d\nu = \int_\Omega p \circ \theta d\nu = \Lambda(p)$. This shows that some of the solutions the equation $\Lambda(p^2) = \Lambda(p)$, which can be expressed only in terms of Λ, play an important role when trying to reconstruct the representing measure μ.

This remark leads us to the concept of idempotent with respect to a uspf, recalled in the following (see [15], Definition 1).

Let $\mathcal{S}$ be a finite dimensional function space on a Hausdorff space Ω. Fixing a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, let $\mathcal{I}_\Lambda$, $\mathcal{H}_\Lambda = \mathcal{S}/\mathcal{I}_\Lambda$ be defined as in Introduction. We also denote by $\langle *, * \rangle$, $\| * \|$ the inner product, as in (2), and the corresponding norm induced on $\mathcal{H}_\Lambda$, respectively.

We recall that $\mathcal{RH}_\Lambda$ designate the subspace $\{\hat{f} \in \mathcal{H}_\Lambda; f \in \mathcal{RS}\}$, which is a real Hilbert space, and that if $\hat{f} \in \mathcal{RH}_\Lambda$, we always suppose the representative $f \in \mathcal{RS}$.

Definition 1 ([15]) An element $\hat{p} \in \mathcal{RH}_\Lambda$ is said to be Λ-*idempotent* (or simply *idempotent* if Λ is fixed) if it is a solution of the equation

$$\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle. \tag{3}$$

Remark 1 Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is idempotent if and only if $\Lambda(p^2) = \Lambda(p)$, via (2). Set

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{4}$$

which is the family of nonnull idempotent elements from $\mathcal{RH}_\Lambda$. This family is nonempty because $\hat{1} \in \mathcal{ID}(\Lambda)$.

Note that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

The existence of orthogonal bases consisting of idempotents with respect to a given uspf follows from the following result, which is part of Lemma 4 from [15].

Lemma 1 *Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf, and let $\mathcal{T}_\Lambda = \{\hat{v} \in \mathcal{RH}_\Lambda; \|\hat{v}\| = 1\}$.*

If the set $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{T}_\Lambda$ is an orthonormal basis of $\mathcal{H}_\Lambda$ with $\langle \hat{v}_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$, the set $\{\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. Moreover,

$$\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1 + \cdots + \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d = \hat{1}.$$

From Lemma 1, it follows readily the next result (which is Theorem 1 from [15]).

Theorem 1 *For every uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, the Hilbert space $\mathcal{H}_\Lambda$ has infinitely many orthogonal bases consisting of idempotent elements.*

Corollary 1 *Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Then there are functions $b_1, \dots, b_d \in \mathcal{RS}$ such that $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ for all $j, k = 1, \dots, d$, $j \neq k$, and every $f \in \mathcal{S}$ can be uniquely represented as*

$$f = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(f b_j) b_j + f_0,$$

with $f_0 \in \mathcal{I}_\Lambda$ and $d = \dim \mathcal{H}_\Lambda$.

This assertion is Corollary 1 from [15].

1.3 Multiplicative Structures

As in [15], the bases consisting of idempotents can be associated with multiplicative structures. We recall in the following this construction.

Remark 2 Let $\mathcal{S}$ be a finite dimensional function space on a Hausdorff space, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Let $\mathcal{B} \subset \mathcal{ID}(\Lambda)$ be a collection of nonnull mutually orthogonal elements whose sum is $\hat{1}$ (in particular an orthogonal basis), and let $\mathcal{H}_\mathcal{B}$ be the complex vector space spanned by $\mathcal{B}$ in $\mathcal{H}_\Lambda$. Using the basis $\mathcal{B}$ of the space $\mathcal{H}_\mathcal{B}$, we may define a multiplication, an involution, and a norm on $\mathcal{H}_\mathcal{B}$, making it a unital, commutative, finite dimensional C^*-algebra. Specifically, if $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ with $\hat{1} = \sum_{j=1}^d \hat{b}_j$, and if $\hat{f} = \sum_{j=1}^d \alpha_j \hat{b}_j$, $\hat{g} = \sum_{j=1}^d \beta_j \hat{b}_j$, are elements from $\mathcal{H}_\mathcal{B}$, their product is given by $\hat{f} \cdot \hat{g} = \sum_{j=1}^d \alpha_j \beta_j \hat{b}_j$. The involution is defined by $\hat{f}^* = \sum_{j=1}^d \overline{\alpha_j} \hat{b}_j$, and the norm is given by $\|\hat{f}\|_\infty = \max_{1 \le j \le d} |\alpha_j|$, for $\hat{f} = \sum_{j=1}^d \alpha_j \hat{b}_j$.

Having in mind this construction, we may speak about the C^*-*algebra (structure of)* $\mathcal{H}_\mathcal{B}$ *induced by* $\mathcal{B}$.

When $\mathcal{B}$ is actually a basis, we clearly have $\mathcal{H}_\mathcal{B} = \mathcal{H}_\Lambda$.

If for some $f, g \in \mathcal{S}$ we also have $fg \in \mathcal{S}$, the element $\hat{f} \cdot \hat{g}$, computed for instance in the algebra $\mathcal{H}_\Lambda$, is, in general, different from $\widehat{fg}$. In particular, if $\theta = (\theta_1, \dots, \theta_n)$ is an n-tuple of elements of $\mathcal{S}$, then $\hat{\theta}^\alpha = \widehat{\theta_1}^{\alpha_1} \cdots \widehat{\theta_n}^{\alpha_n}$ is, in general, different from $\widehat{\theta^\alpha}$, when $\theta^\alpha \in \mathcal{S}$ for some multi-index $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}_+^n$, where, as usually, we put $\theta^\alpha := \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}$.

It is easily seen that the space of characters of the C^*-algebra $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$, say $\Delta = \{\delta_1, \dots, \delta_d\}$, coincides with the dual basis of $\mathcal{B}$. As $\mathcal{H}_\mathcal{B}$ is also a Hilbert space as a subspace of $\mathcal{H}_\Lambda$, we note that $\delta_j(\hat{f}) = \Lambda(b_j)^{-1} \langle \hat{f}, \hat{b}_j \rangle$, $\hat{f} \in \mathcal{H}_\mathcal{B}$, $j = 1, \dots, d$. This also shows that C^*-algebra $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$ is semisimple.

1.4 Finitely Generated Function Spaces

Let $\mathcal{S}$ be a function space on a Hausdorff space Ω. We assume that there exist an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RS}$, and an integer $m \geq 1$, such that such that the family $\Theta_m := \{\theta^\alpha; |\alpha| \leq m\}$ spans the space $\mathcal{S}$.

When such a pair (θ, m) exists, we shortly say that the function space $\mathcal{S}$ is *m-generated by θ*. Clearly, in this case $\mathcal{S}$ is of finite dimension, and the family Θ_{2m} spans the space $\mathcal{S}^{(2)}$. In fact, $\mathcal{S} = \{p \circ \theta; p \in \mathcal{P}_m\}$, where $\mathcal{P}_m$, as mentioned before, is the space of polynomials of total degree less or equal to m.

In particular, $\mathcal{S}$ is 1-generated by θ if and only if $\mathcal{S}$ is the span of $\{\theta, 1\}$. Clearly, if $\mathcal{S}$ is a finite dimensional function space, it is at least 1-generated by a basis $\theta = (\theta_1, \ldots, \theta_n)$. Nevertheless, the case of an $\mathcal{S}$ m-generated by a tuple θ with $m > 1$ is also of some interest. Also note that if $\mathcal{S}$ is a function space that is m-generated by an tuple θ of elements of $\mathcal{S}$, and if $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, the Hilbert space $\mathcal{H}_\Lambda$ must be of finite dimension less or equal to the cardinal of the set Θ_m.

Example 2 As before, let $\mathcal{P} = \mathcal{P}^n$ be the algebra of all polynomials in n real variables, say $t_1, \ldots, t_n$, with complex coefficients, and fix a k-tuple $\theta = (\theta_1, \ldots, \theta_k)$ of elements of $\mathcal{P}$. Also let $\mathcal{P}_{\theta,m}$ be the span of the set $\Theta_m := \{\theta^\alpha; |\alpha| \leq m\}$ in $\mathcal{P}$, that is, $\mathcal{P}_{\theta,m}$ is a function space m-generated by θ. It consists of certain polynomials of degree less or equal to $\max\{\alpha_1 d_1 + \cdots + \alpha_k d_k; |\alpha| \leq m\}$, with $d_j := \deg(\theta_j); j = 1, \ldots, k$. In particular, if $\theta_j = t_j$, $j = 1, \ldots, n$, then $\mathcal{P}_m = \mathcal{P}_{\theta,m}$, is a function space m-generated by $t := (t_1, \ldots, t_n)$.

1.5 Continuous Point Evaluations

A discussion concerning the point evaluations in the context of function spaces on $\mathbb{R}^n$ can be found in [15], Section 4. Some assertions of interest in the context of Hausdorff spaces can be obtained from those in [15], with minor modifications.

Let $\mathcal{S}$ be a finite dimensional function space on the Hausdorff space Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. For every point $\omega \in \Omega$, we denote by δ_ω the point evaluation at ω, that is, $\delta_\omega(f) = f(\omega)$, for every function $f \in \mathcal{S}$.

Definition 2 The point evaluation δ_ω is said to be Λ-*continuous* if there exists a constant $c_\omega > 0$ such that

$$|\delta_\omega(f)| \leq c_\omega \Lambda(|f|^2)^{1/2}, \ f \in \mathcal{S}.$$

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\omega \in \Omega$ such that δ_ω is Λ-continuous. For every function f let us denote by $\mathcal{Z}(f)$ the set of its zeros.

Lemma 2 *We have the equality*

$$\mathcal{Z}_\Lambda = \cap_{f \in \mathcal{I}_\Lambda} \mathcal{Z}(f).$$

The proof of Lemma 2 is similar to that of Lemma 6 from [15], and will be omitted.

Note that $\mathcal{I}_\Lambda = \{0\}$ implies $\mathcal{Z}_\Lambda = \Omega$.

Remark 3 (1) The previous lemma shows that the set $\mathcal{Z}_\Lambda$ extends the concept of algebraic variety of the moment sequence associated to Λ (see for instance (1.6) from [3]).

(2) It also follows from Lemma 2 that for each $\omega \in \mathcal{Z}_\Lambda$, the functional δ_ω induces a functional $\hat{\delta}_\omega$ on $\mathcal{H}_\Lambda$ given by $\hat{\delta}_\omega(\hat{f}) = f(\omega)$, $f \in \mathcal{S}$.

(3) When the map $\Omega \ni \omega \mapsto \delta_\omega$ is injective, we may regard the elements of $\mathcal{H}_\Lambda$ as functions on $\mathcal{Z}_\Lambda$, putting $\hat{f}(\omega) = f(\omega), \hat{f} \in \mathcal{H}_\Lambda$, $\omega \in \mathcal{Z}_\Lambda$.

The next result extends an assertion from [2] (see also [15], Lemma 7).

Lemma 3 *Suppose that the uspf* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ *has an atomic representing measure* μ *in* Ω. *Then* $\mathrm{supp}(\mu) \subset \mathcal{Z}_\Lambda$.

The proof of Lemma 3 is similar to that of Lemma 7 from [15] and will be omitted.

2 Integral Representations

In this Section we recapture, in the context of function spaces on a Hausdorff space Ω, some integral representation, stated in [15] for polynomial function spaces on the real n-dimensional Euclidean space ($n \geq 1$). As in Example 1, for every finite set Ξ, we denote by $C(\Xi)$ the finite dimensional C^*-algebra of all complex-valued functions defined on Ξ.

Remark 4 Let $\mathcal{S}$ be a finite dimensional function space on Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Let also $\mathcal{C} = \{\hat{c}_1, \ldots, \hat{c}_v\}$ be an orthogonal family in $\mathcal{H}_\Lambda$ consisting of nonnull idempotents, and let $\mathcal{H}_\mathcal{C}$ be the span of the set $\mathcal{C}$ in $\mathcal{H}_\Lambda$. We denote by $\mathcal{S}_\mathcal{C}$ the set of those $f \in \mathcal{S}$ with $\hat{f} \in \mathcal{H}_\mathcal{C}$, and by $\mathcal{G}$ the linear span of the (linearly independent) family $\{c_1, \ldots, c_v\}$ in $\mathcal{S}$. Then we have $\mathcal{S}_\mathcal{C} = \mathcal{G} + \mathcal{I}_\Lambda$, which is obvious, and $\mathcal{G} \cap \mathcal{I}_\Lambda = \{0\}$. Indeed, if $g = \sum_{j=1}^{v} \beta_j c_j \in \mathcal{I}_\Lambda$, with β_j scalars, then

$$\Lambda(|g|^2) = \sum_{j,k=1}^{v} \beta_j \overline{\beta_k} \Lambda(c_j c_k) = \sum_{j=1}^{v} |\beta_j|^2 \Lambda(c_j) = 0,$$

whence $g = 0$, because $\Lambda(c_j) > 0$ for all j. See also Corollary 1.

Proposition 1 *Let* $\mathcal{S}$ *be a finite dimensional function space on* Ω, *and let* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ *be a uspf. Assume that the space* $\mathcal{H}_\Lambda$ *is endowed with the* C^**-algebra structure induced by an orthogonal basis consisting of idempotent elements. Let also* $\mathcal{H}_\theta$ *be the sub-*C^**-algebra generated by the set* $\{\hat{\theta}_0 = 1, \hat{\theta}_1, \ldots, \hat{\theta}_n\}$ *in* $\mathcal{H}_\Lambda$, *where* $\theta := (\theta_1, \ldots, \theta_n)$ *is a fixed* n*-tuple of* $\mathcal{RS}$.

Then there exist a finite subset Ξ *of* $\mathbb{R}^n$, *whose cardinal is* $\leq \dim \mathcal{H}_\Lambda$, *and a linear map* $\mathcal{S}_\theta \ni u \mapsto u^\# \in C(\Xi)$, *whose kernel is* $\mathcal{I}_\Lambda$, *such that*

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi), \ u \in \mathcal{S}_\theta,$$

where $\mathcal{S}_\theta = \{u \in \mathcal{S}; \hat{u} \in \mathcal{H}_\theta\}$, *and* μ *is a probability measure on* Ξ.

Proof. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, inducing the C^*-algebra structure of $\mathcal{H}_\Lambda$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the set of all characters of the C^*-algebra $\mathcal{H}_\Lambda$. First of all, we shall deal with the structure of the sub-C^*-algebra $\mathcal{H}_\theta$, which consists of arbitrary polynomials in $\hat{\theta}_1, \ldots, \hat{\theta}_n$. We can write $\hat{\theta}_k = \tau_{k1}\hat{b}_1 + \cdots + \tau_{kd}\hat{b}_d$, where $\tau_{kj} = \delta_j(\hat{\theta}_k)$, $k = 1, \ldots, n$, $j = 1, \ldots, d$. Put $\tau^{(j)} = (\tau_{1j}, \ldots, \tau_{nj}) \in \mathbb{R}^n$, $j = 1, \ldots, d$. With the notation from Remark 2, let us show, by recurrence, that

$$\hat{\theta}^\alpha = \sum_{j=1}^{d} (\tau^{(j)})^\alpha \hat{b}_j, \tag{5}$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$. As (5) clearly holds if $|\alpha| = 1$, assuming that (5) holds if $|\alpha| < m$ for an integer $m > 1$, we have, for some fixed $k \in \{1, \ldots, n\}$,

$$\hat{\theta}_k \cdot \hat{\theta}^\alpha = \sum_{l=1}^{n} \sum_{j=1}^{d} \tau_{kl} (\tau^{(j)})^\alpha \hat{b}_l \cdot \hat{b}_j = \sum_{j=1}^{d} \tau_{kj} (\tau^{(j)})^\alpha \hat{b}_j = \hat{\theta}^{\alpha_{(k)}},$$

where $\alpha_{(k)} = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1} \ldots, \alpha_n)$, showing that (5) holds whenever $|\alpha| \le m$. Consequently, $p(\hat{\theta}) = \sum_{j=1}^{d} p(\tau^{(j)})\hat{b}_j$ for every $p \in \mathcal{P}$.

Let $\Xi = \{\xi^{(1)} \ldots, \xi^{(v)}\}$ be the distinct points from the set $\{\tau^{(1)} \ldots, \tau^{(d)}\}$, with $v \le d$. Let also $I_j = \{k; \tau^{(k)} = \xi^{(j)}\}$, $j = 1, \ldots, v$.

If $p \in \mathcal{P}$ is arbitrary, then, as above,

$$p(\hat{\theta}) = \sum_{j=1}^{v} p(\xi^{(j)})\hat{c}_j, \tag{6}$$

where $\hat{c}_j = \sum_{k \in I_j} \hat{b}_k$, $j = 1, \ldots, v$, which is a family of mutually orthogonal idempotents, whose sum is $\hat{1}$.

Consider now the space $\mathcal{S}'_\theta$ given by

$$\mathcal{S}'_\theta = \{\sum_{j=1}^{v} p(\xi^{(j)})c_j + r;\, p \in \mathcal{P}|\Xi,\, r \in \mathcal{I}_\Lambda\} = \mathcal{G}_\theta + \mathcal{I}_\Lambda,$$

with $\mathcal{G}_\theta = \{\sum_{j=1}^{v} p(\xi^{(j)})c_j;\, p \in \mathcal{P}|\Xi\}$, where $\mathcal{P}|\Xi$ is the space of all restrictions of arbitrary polynomials to the set Ξ.

Let us remark that the sum $\mathcal{G}_\theta + \mathcal{I}_\Lambda$ is direct. Indeed, because $\mathcal{S}'_\theta$ is a subspace of the space spanned by $\{c_1, \ldots, c_v\}$, it follows that the intersection $\mathcal{G}_\theta \cap \mathcal{I}_\Lambda = \{0\}$, via Remark 4. In particular, if $u = \sum_{j=1}^{v} p(\xi^{(j)})c_j + r \in \mathcal{S}'_\theta$, the function $p|\Xi$ is uniquely determined.

Further, we have a linear map $\mathcal{S}'_\theta \ni u \mapsto u^\# \in C(\Xi)$, defined in the following way. Taking an element $u = \sum_{j=1}^{v} p(\xi^{(j)})c_j + r \in \mathcal{S}'_\theta$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$, we put $u^\#(\xi) = p(\xi)$, $\xi \in \Xi$. As the function $p|\Xi$ is uniquely determined by u, the definition of $u^\#$ is correct, the assignment $u \mapsto u^\#$ is linear, and its kernel is precisely $\mathcal{I}_\Lambda$. In addition, $\mathcal{S}'_\theta = \{u \in \mathcal{P}_m; \hat{u} \in \mathcal{H}_\theta\} = \mathcal{S}_\theta$. Indeed, if $\hat{u} \in \mathcal{H}_\theta$ there exists an integer $m \geq 0$ such that $\hat{u} = \sum_{|\alpha|\leq m} a_\alpha \hat{\theta}^\alpha = \sum_{j=1}^v p_u(\xi^{(j)})\hat{c}_j$, via (6) aplied to t^α for each α, where $p_u(t) = \sum_{|\alpha|\leq m} a_\alpha t^\alpha$.

Consequently, if $u = \sum_{j=1}^v p(\xi^{(j)})c_j + r$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$, we have

$$\Lambda(u) = \sum_{j=1}^{v} p(\xi^{(j)})\Lambda(c_j) = \int_\Xi u^\#(\xi)d\mu(\xi),$$

where μ is the measure with weights $\Lambda(c_j)$ at $\xi^{(j)}$, $j = 1, \ldots, v$, which concludes the proof. ■

Theorem 2 *Let $\mathcal{S}$ be a finite dimensional function space on Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements. Also assume that the elements $\{\hat{1}, \hat{\theta}_1, \ldots, \hat{\theta}_n\}$ generate the C^*-algebra $\mathcal{H}_\Lambda$ where $\theta = (\theta_1, \ldots, \theta_n)$ is a given n-tuple of $\mathcal{S}$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal equals $\dim \mathcal{H}_\Lambda$, and a surjective linear map $\mathcal{S} \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, with the property*

$$\Lambda(u) = \int_\Xi u^\#(\xi)d\mu(\xi),\ u \in \mathcal{S},$$

where μ is a probability measure on Ξ.

Moreover, the map $\mathcal{S} \ni u \mapsto u^\# \in C(\Xi)$ induces a $*$-isomorphism between C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$.

Proof. We follow the lines and use the notation of the proof of Proposition 1. We must have $\mathcal{H}_\theta = \mathcal{H}_\Lambda$, and $\mathcal{S}_\theta = \mathcal{S}$. Moreover, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is the orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements given by the hypothesis, and $\Delta = \{\delta_1, \ldots, \delta_d\}$ is the set of the characters of the C^*-algebra $\mathcal{H}_\Lambda$, the points $\tau^{(j)} \in \mathbb{R}$, $j = 1, \ldots, d$, are distinct because the family of generators $\{\hat{\theta}_1, \ldots, \hat{\theta}_n\}$ separates the points of Δ, so

$$\delta_j(\hat{\theta}) = \tau^{(j)} = \xi^{(j)} = (\xi_1^{(j)}, \ldots, \xi_n^{(j)}) \in \mathbb{R}^n,\ j = 1, \ldots, d,$$

and also $c_j = b_j$, $\xi_k^{(j)} = \delta_j(\hat{\theta}_k)$, $j = 1, \ldots, d$, $k = 1, \ldots, n$.

Note that the space $\mathcal{S}$ can be written as

$$\mathcal{S} = \{\sum_{j=1}^{d} p(\xi^{(j)})b_j + r;\ p \in \mathcal{P},\ r \in \mathcal{I}_\Lambda\} = \mathcal{G} + \mathcal{I}_\Lambda,$$

with $\mathcal{G} = \{\sum_{j=1}^d p(\xi^{(j)})b_j;\ p \in \mathcal{P}\}$, and where the sum of spaces is direct. Consequently, if $u \in \mathcal{S}$, we must have $u = \sum_{j=1}^d p(\xi^{(j)})b_j + r$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$. Moreover, the function $p|\Xi$ is uniquely determined by u, and setting

$u^{\#} = p|\Xi$, we have a linear map $\mathcal{S} \ni u \mapsto u^{\#} \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$. In addition, as in Proposition 1, we also have the formula

$$\Lambda(u) = \int_\Xi u^{\#}(\xi) d\mu(\xi),\ u \in \mathcal{S},$$

where μ is the measure with weights $\Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \dots, d$.

Note that the map $\mathcal{S} \ni u \mapsto u^{\#} \in C(\Xi)$ is also surjective because, taking an arbitrary element of $C(\Xi)$ written under the form $p|\Xi$ for some $p \in \mathcal{P}$, the function $u = \sum_{j=1}^d p(\xi^{(j)}) b_j \in \mathcal{G}$ has the property $u^{\#} = p|\Xi$.

Since the map $\mathcal{S} \ni u \mapsto u^{\#} \in C(\Xi)$ is surjective, and its kernel is precisely $\mathcal{I}_\Lambda$, the induced map $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^{\#} \in C(\Xi)$ is correctly defined and bijective, where $\hat{u}^{\#}(\xi) = u^{\#}(\xi)$, $\xi \in \Xi$. This map is actually a $*$-isomprphism.

To prove this assertion, let us first choose the functions $p_k \in \mathcal{P}$ and $r_k \in \mathcal{I}_\Lambda$ with the property $b_k = \sum_{j=1}^d p_k(\xi^{(j)}) b_j + r_k$, $k = 1, \dots, d$. The uniqueness of this representation shows that $r_k = 0$, $p_k(\xi^{(j)}) = 1$ if $k = j$, and $= 0$ otherwise, for all $k, j = 1, \dots, d$. In addition, $\hat{b}_k^{\#} = p_k|\Xi$, $k = 1, \dots, d$.

Because $(\hat{b}_j \cdot \hat{b}_k)^{\#}(\xi) = 0 = p_j(\xi) p_k(\xi)$ if $j \neq k$, and $(\hat{b}_j \cdot \hat{b}_j)^{\#}(\xi) = \hat{b}_j^{\#}(\xi) = p_j(\xi) = p_j(\xi)^2$, for all $\xi \in \Xi$ and $j, k = 1, \dots, d$, it follows that the map $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^{\#} \in C(\Xi)$ is multiplicative. Taking into account the definitions given in Remark 2, the equalities

$$\hat{1}^{\#}(\xi) = \sum_{j=1}^d (\hat{b}_j)^{\#}(\xi) = \sum_{j=1}^d p_j(\xi) = 1,$$

as well as $(\hat{u}^*)^{\#} = \overline{\hat{u}^{\#}}$, show that the map $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^{\#} \in C(\Xi)$ is a unital $*$-morphism. In addition, if $\hat{u} = \sum_{j=1}^d p(\xi^{(j)}) \hat{b}_j \in \mathcal{H}_\Lambda$ is arbitrary,

$$\|\hat{u}\|_\infty = \max_{1 \le j \le d} |p(\xi^{(j)})| = \|\hat{u}^{\#}\|_\infty,$$

proving that $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^{\#} \in C(\Xi)$ is a $*$-isomorphism. ■

Remark 5 We keep the conditions and notation from Theorem 2 and its proof. First of all, because $\mathcal{H}_\Lambda$ is finite dimensional, there exists a nonnegative integer m such that the set $\hat{\Theta}_m := \{\hat{\theta}^\alpha; |\alpha| \le m\}$ generates the C^*-algebra $\mathcal{H}_\Lambda$. Consequently,

$$\mathcal{S} = \{\sum_{j=1}^d p(\xi^{(j)}) b_j + r;\ p \in \mathcal{P}_m,\ r \in \mathcal{I}_\Lambda\} = \mathcal{G} + \mathcal{I}_\Lambda,$$

with $\mathcal{G} = \{\sum_{j=1}^d p(\xi^{(j)}) b_j;\ p \in \mathcal{P}_m\}$, where the sum of spaces is direct.

In addition, the algebra $\mathcal{P}_m$ contains a family $\{p_1, \dots, p_d\}$, which interpolates the set Ξ, where $p_j = \hat{b}_j^{\#}$ for all j.

Example 3 As in [5], Example 2.1 (see also [15], Examples 1 and 3), we consider the uspf $\Lambda : \mathcal{P}_4^1 \mapsto \mathbb{C}$, given by $\Lambda(t^k) = 1$, $k = 0, 1, 2, 3$, and $\Lambda(t^4) = 2$, extended to $\mathcal{P}_4^1$ by linearity. It is known that this uspf has no representing measure. Nevertheless, according to Theorem 2, the restriction $\Lambda|\mathcal{P}_2^1$ has a certain integral representations, with respect to any fixed orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents, to be explicitly presented in the following. We use some information from [15], Examples 1 and 3. We have $\mathcal{I}_\Lambda = \{p(t) = u - ut; u \in \mathbb{C}\}$, and $\mathcal{H}_\Lambda = \{\hat{p}\,;\, p(t) = u + ut + vt^2,\ u, v \in \mathbb{C}\}$. In particular, $\hat{1} = \hat{t}$, and so $\mathcal{H}_\Lambda = \{u\hat{1} + v\widehat{t^2},\ u, v \in \mathbb{C}\}$.

We fix the elements $b_1 = t^2/2$ and $b_2 = 1/2 + t/2 - t^2/2$, satisfying $\Lambda(b_1^2) = \Lambda(b_1) = 1/2, \Lambda(b_2^2) = \Lambda(b_2) = 1/2$, and $\Lambda(b_1 b_2) = 0$.

As we have $\dim\mathcal{H}_\Lambda = 2$, it follows that $\{\hat{b}_1, \hat{b}_2\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents, with $\hat{b}_1 = \widehat{t^2}/2$, $\hat{b}_2 = \hat{1} - \widehat{t^2}/2$.

Put $\theta_1 = t$, $\theta_2 = t^2$. Clearly, the set $\{\hat{\theta}_1, \hat{\theta}_2\}$ generates the C^*-algebra $\mathcal{H}_\Lambda$.

Using the dual basis $\Delta = \{\delta_1, \delta_2\}$, we infer that

$$\xi^{(1)} = (\delta_1(\hat{\theta}_1), \delta_1(\hat{\theta}_2)) = \Lambda(b_1)^{-1}(\Lambda(\theta_1 b_1), \Lambda(\theta_2 b_1)) = (1, 2),$$

$$\xi^{(2)} = (\delta_2(\hat{\theta}_1), \delta_2(\hat{\theta}_2)) = \Lambda(b_2)^{-1}(\Lambda(\theta_1 b_2), \Lambda(\theta_2 b_2)) = (1, 0).$$

In other words, $\Xi = \{(1, 2), (1, 0)\} \subset \mathbb{R}^2$. If $p = u + wt + vt^2$ is an arbitrary element of $\mathcal{P}_2^1$, then

$$\hat{p} = u\hat{1} + v\widehat{t^2} = (u + 2v)\hat{b}_1 + u\hat{b}_2 \in \mathcal{H}_\Lambda,\ u, v \in \mathbb{C},$$

and we have

$$\Lambda(p) = p^\#(\xi^{(1)})\Lambda(b_1) + p^\#(\xi^{(2)})\Lambda(b_2) = u + v,$$

where $p^\#(x) = ux_1 + vx_2$, $x = (x_1, x_2) \in \mathbb{R}^2$, is computed by formula (6). Therefore, $\Lambda(p) = \int_\Xi p^\#(\xi)d\nu(\xi)$, $p \in \mathcal{P}_2^1$, where ν is the atomic measure with weights $\Lambda(b_1), \Lambda(b_2)$ at $\xi^{(1)}, \xi^{(2)}$, respectively. In addition, the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p^\#|\Xi \in C(\Xi)$ is a $*$-isomprphism.

Finally, a similar procedure may be applied to any pair of idempotents $\{\hat{b}_1, \hat{b}_2\}$, which is an orthogonal basis of $\mathcal{H}_\Lambda$.

3 Representing Measures: Abstract Approaches

The previous section offers some integral representation formulas for certain uspf's. The existence of a representing measure for a general uspf is still an open question, even in $\mathbb{R}^n$. For a given uspf, the problem may have no solution, a unique solution or infinitely many solutions. In this section we present some abstract criteria concerning the existence of a representing measure in the context of Hausdorff spaces (see [2, 3, 14, 15] etc.).

3.1 Contractive USPF's

For a given uspf Λ, the associated set $\mathcal{Z}_\Lambda$ is defined in Subsection 1.5.

Let $\mathcal{S}$ be a finite dimensional function space on a Hausdorff space Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. We say that Λ is *contractive* if there exists a finite set $F \subset \mathcal{Z}_\Lambda$ such that $|\Lambda(f)| \le \|f\|_F$, $f \in \mathcal{S}^{(2)}$, where $\|f\|_F = \max_{\omega\in F} |f(\omega)|$.

Proposition 2 *The uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has an atomic representing measure if and only if it is contractive.*

Proof. The existence of an atomic representing measure of Λ implies an equation of the form $\Lambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$ for each $f \in \mathcal{S}^{(2)}$, where $\lambda_1 > 0, \ldots, \lambda_d > 0$, $\sum_{j=1}^d \lambda_j = 1$, and $\omega_1, \ldots, \omega_d$ are (distinct) points in Ω. Setting $F := \{\omega_1, \ldots, \omega_d\}$, we deduce easily that $|\Lambda(f)| \le \|f\|_F$, $f \in \mathcal{S}^{(2)}$.

Conversely, assume that Λ is contractive, so $|\Lambda(f)| \le \|f\|_F$, $f \in \mathcal{S}^{(2)}$, for some $F \subset \mathcal{Z}_\Lambda$. Setting $\mathcal{J}_F := \{f \in \mathcal{S}^{(2)}; f|F = 0\}$, we may define the map $\mathcal{S}^{(2)}/\mathcal{J}_F \ni \tilde f \mapsto f|F \in C(F)\}$, where $\tilde f = f + \mathcal{J}_F$, $f \in \mathcal{S}^{(2)}$. Clearly, this map is correctly defined and linear. Moreover, putting $\|\tilde f\|_F = \|f\|_F$ whenever $\tilde f = f + \mathcal{J}_F$, $f \in \mathcal{S}^{(2)}$, we obtain a norm on the space $\mathcal{S}^{(2)}/\mathcal{J}_F$. In addition, the map $\tilde f \mapsto f|F$ is an isomatry.

On the subspace $\mathcal{R}_\Lambda := \{f|F \in C(F); \tilde f \in \mathcal{S}^{(2)}/\mathcal{J}_F\}$, which is isometrically isomorphic to $\mathcal{S}^{(2)}/\mathcal{J}_F$, we may define the map $\tilde\Lambda(f|F) = \Lambda(f)$, which is correctly defined because Λ is contractive. This also shows that $\|\tilde\Lambda\| \le 1$. In fact, as we have $\tilde\Lambda(1) = 1$, we must have $\|\tilde\Lambda\| = 1$.

We denote also by $\tilde\Lambda$ a norm preserving extension of $\tilde\Lambda$ to $C(F)$, which therefore should be a positive functional. Denoting by δ_j the point evaluation at ω_j, $j = 1, \ldots, d$, we deduce the existence of scalars $\lambda_1 > 0, \ldots, \lambda_d > 0$ such that $\tilde\Lambda = \sum_{j=1}^d \lambda_j \delta_j$, and thus

$$\Lambda(f) = \tilde\Lambda(f|F) = \sum_{j=1}^d \lambda_j f(\omega_j),\ \ f \in \mathcal{S}^{(2)},$$

In addition, $\sum_{j=1}^d \lambda_j = \tilde\Lambda(1) = 1$, showing that Λ has an atomic representing measure on Ω. ■

Remark 6 Assume that the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is contractive, so we have $|\Lambda(f)| \le \|f\|_F$, $f \in \mathcal{S}^{(2)}$ for a certain finite subset F in Ω. In particular, Λ has representing measure, that is $\Lambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$, $f \in \mathcal{S}^{(2)}$, via Proposition 2. Let us define the quantities

$$\sigma(f) := \sup_{g\in\mathcal{R}\mathcal{S}^{(2)}} [-\Lambda(g) - \|f+g\|_F],$$

$$\tau(f) = \inf_{g\in\mathcal{R}\mathcal{S}^{(2)}} [\|f+g\|_F - \Lambda(g)],$$

where $f \in \mathcal{R}C(F)$ is arbitrary. It follows from the standard proof of the Hahn-Banach Theorem (see for instance [7]) that the equality $\sigma(f) = \tau(f)$ for all

$f \in \mathcal{R}C(F)$ implies the uniqueness of the extension, that is, the uniqueness of the representing measure.

3.2 An Interpolation Approach

The existence of a representing measure can be also characterized in terms of an interpolation property. A similar connection already appears in Remark 5. An even stronger connection is given the following proposition, whose proof uses some arguments from Subsection 1.5 (see especially Remark 3(2)).

Proposition 3 *Let $\mathcal{S}$ be a finite dimensional function space on Ω. A uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω with $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exist an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$, and a set $\Omega_\Lambda = \{\omega_1, \dots, \omega_d\} \subset \mathcal{Z}_\Lambda$ such that $b_j(\omega_j) = 1$ and $b_k(\omega_j) = 0$ for all $j, k = 1, \dots, d$, $j \neq k$.*

Proof. First assume that the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω, say μ, given by

$$\Lambda(f) = \sum_{j=1}^{d} \lambda_j f(\omega_j),\ \ f \in \mathcal{S}^{(2)},$$

with $\lambda_j > 0$ for all $j = 1, \dots, d$, and $\sum_{j=1}^{d} \lambda_j = 1$, where $d = \dim \mathcal{H}_\Lambda$, and the points $\omega_1, \dots, \omega_d$ are distinct. Set $\Omega_\Lambda := \{\omega_1, \dots, \omega_d\}$, which is a subset of $\mathcal{Z}_\Lambda$, via Lemma 3.

Note that $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; f|\Omega_\Lambda = 0\}$. This shows that there exists a map $\rho : \mathcal{H}_\Lambda \mapsto C(\Omega_\Lambda)$ given by $\hat{f} \mapsto f|\Omega_\Lambda$, which is correctly defined, linear and injective. This map is also surjective because we clearly have $\dim(\mathcal{H}_\Lambda) = \dim(C(\Omega_\Lambda))$.

Let $\chi_k \in C(\Omega_\Lambda)$ be the characteristic function of the set $\{\omega_k\}$ and let $\hat{b}_k \in \mathcal{H}_\Lambda$ be the element with $\rho(\hat{b}_k) = \chi_k$, $k = 1, \dots, d$. As $\Lambda(b_j b_k) = \int_{\Omega_\Lambda} \chi_j \chi_k d\mu$ for all $j, k = 1, \dots, d$, we deduce that the set $\{\hat{b}_1, \dots, \hat{b}_d\}$ is a family of orthogonal idempotents in $\mathcal{H}_\Lambda$, which is actually a basis. Moreover, $b_j(\omega_j) = 1$ and $b_k(\omega_j) = 0$ for all $j, k = 1, \dots, d$, $j \neq k$, proving the necessity of the condition in the statement.

Conversely, if there exist an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$, and a set $\Omega_\Lambda = \{\omega_1, \dots, \omega_d\} \subset \mathcal{Z}_\Lambda$ such that $b_j(\omega_j) = 1$ and $b_k(\omega_j) = 0$ for all $j, k = 1, \dots, d$, $j \neq k$, then Λ has a representing measure whose support is Ω_Λ. Indeed, as we have for every $f \in \mathcal{S}$

$$f = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(f b_j) b_j + r,$$

with $r \in \mathcal{I}_\Lambda$, we obtain

$$f(\omega_k) = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(f b_j) b_j(\omega_k) = \Lambda(b_k)^{-1} \Lambda(f b_k),$$

because $r(\omega_k) = 0$, for all $k = 1, \dots, d$. Therefore

$$\Lambda(f) = \sum_{j=1}^{d} \lambda_j f(\omega^{(j)}),\ f \in \mathcal{S},$$

with $\lambda_j = \Lambda(b_j)$ for all $j = 1, \dots, d$.

Now, if $h = \sum_{l \in L} f_l g_l \in \mathcal{S}^{(2)}$ is arbitrary, with $f_l, g_l \in \mathcal{S}$ for all $l \in L$, L finite, then $f_l = \sum_j f_l(\omega_j) b_j + r_l$, $g_l = \sum_k g_l(\omega_k) b_k + s_l$, with $r_l, s_l \in \mathcal{I}_\Lambda$ for all $l \in L$, we must have

$$\Lambda(h) = \sum_{l \in L} \Lambda(f_l g_l) = \sum_{l \in L} \sum_{j,k=1}^{d} f_l(\omega_j) g_l(\omega_k) \Lambda(b_j b_k) =$$

$$\sum_{j=1}^{d} \lambda_j \sum_{l \in L} f_l(\omega_j) g_l(\omega_j) = \sum_{j=1}^{d} \lambda_j h(\omega_j),$$

showing the existence of a representing measure of Λ in Ω having d atoms. ■

Remark 7 We keep the notation from Proposition 3 and its proof. Assuming that there the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω with support in $\Omega_\Lambda := \{\omega_1, \dots, \omega_d\} \subset \mathcal{Z}_\Lambda$, we have constructed an orthogonal basis $\{\hat{b}_1, \dots, \hat{b}_d\}$ consisting of idempotents in $\mathcal{H}_\Lambda$. If $\mathcal{H}_\Lambda$ is given the C^*-algebra structure induced by $\{\hat{b}_1, \dots, \hat{b}_d\}$, then $\mathcal{H}_\Lambda$ and $C(\Omega_\Lambda)$ are isomorphic as C^*-algebras, via the map $\mathcal{H}_\Lambda \ni \hat{f} \mapsto f|F \in C(F)$, which is easily seen. Fixing a point $\omega_l \in \Omega_\Lambda$, we may correctly define a linear map $\kappa_l : \mathcal{H}_\Lambda \mapsto \mathbb{C}$ by the equation $\kappa_l(\hat{f}) = f(\omega_l)$, $f \in \mathcal{S}$. Taking $\hat{f} = \sum_j c_j \hat{b}_j$, $\hat{g} = \sum_k d_k \hat{b}_k$ in $\mathcal{H}_\Lambda$, as we have

$$\kappa_l(\hat{f} \cdot \hat{g}) = \sum_j c_j d_j \kappa_l(\hat{b}_j) = c_l d_l = f(\omega_l) g(\omega_l) = \kappa_l(\hat{f}) \kappa_l(\hat{g}),\ f, g \in \mathcal{S},$$

the map κ_l is also multiplicative on the algebra $\mathcal{H}_\Lambda$, so it is a character.

Let $\Delta = \{\delta_1, \dots, \delta_d\}$ be the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$. Then for each index j there exists a unique point $\omega_j \in \Omega_\Lambda$ such that $\delta_j(\hat{f}) = f(\omega_j)$, $f \in \mathcal{S}$, $j = 1, \dots, d$. In other words, we can identify the sets Δ and Ω_Λ.

If $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{RS}$ provides a family of generators $\{\hat{\theta}_1, \dots, \hat{\theta}_n\}$ of the C^*-algebra $\mathcal{H}_\Lambda$, setting $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\}$, $\xi^{(j)} = \delta_j(\hat{\theta}) \in \mathbb{R}^n$, we also have that the algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$ are isomorphic as C^*-algebras (as in the proof of Theorem 2), with $u^{\#}(\xi^{(j)}) = u(\omega_j)$, $j = 1, \dots, d$.

4 Relative Multiplicativity

As in [15], we may characterize the existence of a representing measure in terms of idempotents. We start with a basic concept, which generalizes the corresponding one from [15], Definition 3.

Definition 3 Let $\mathcal{S}$ be a function space m-generaterd by the n-tuple θ, let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* (*with respect to* θ) if

$$\Lambda(\theta^\alpha b_j)\Lambda(\theta^\beta b_j) = \Lambda(b_j)\Lambda(\theta^{\alpha+\beta} b_j) \tag{7}$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \dots, d$.

Lemma 4 *Let $\mathcal{S}$ be a function space m-generaterd by the n-tuple θ, let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. The basis $\mathcal{B}$ is Λ-multiplicative if and only if $\delta(\widehat{\theta^\alpha}) = \delta(\hat{\theta}^\alpha)$ whenever $|\alpha| \leq m$ and δ is any character of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$.*

The proof of this lemma, using the formula of a character given in Remark 2 (and similar to that of Lemma 5 from [15]), will be omitted.

The next result is a generalization of Theorem 2 from [15]. Because of some differences, we exhibit complete arguments.

Theorem 3 *Let $\mathcal{S}$ be a function space on Ω, supposed to be m-generaterd by the n-tuple $\theta = (\theta_1, \dots, \theta_n)$. A uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω with $d = \dim \mathcal{H}_\Lambda$ atoms if and only if there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of idempotent elements which is Λ-multiplicative, and $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$.*

Proof. Using the fact that $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ is Λ-multiplicative, we have $\delta(\widehat{\theta^\alpha}) = \delta(\hat{\theta}^\alpha)$ whenever $|\alpha| \leq m$ and $\delta \in \Delta$, via Lemma 4. In particular, we must have the equality $\widehat{\theta^\alpha} = \hat{\theta}^\alpha$ whenever $|\alpha| \leq m$, because the algebra $\mathcal{H}_\Lambda$ is semi-simple. In addition, as the elements $\{\widehat{\theta^\alpha}; |\alpha| \leq m\}$ span the linear space $\mathcal{H}_\Lambda$, the elements $\hat{\theta}_1, \dots, \hat{\theta}_n$ have to generate the algebra $\mathcal{H}_\Lambda$. Therefore, the family $\{\hat{\theta}_1, \dots, \hat{\theta}_n\}$ separates the points of Δ (which coincides with the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$), and so the map

$$\Delta \ni \delta \mapsto (\delta(\hat{\theta}_1), \dots, \delta(\hat{\theta}_n)) \in \mathbb{R}^n$$

is injective. Writing $\Delta = \{\delta_1, \dots, \delta_d\}$, we set $\xi^{(j)} = (\delta_j(\hat{\theta}_1), \dots, \delta_j(\hat{\theta}_n))$, $j = 1, \dots, d$, $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\}$.

As in Proposition 1 (see formula (5)), we have $\hat{\theta}^\alpha = \sum_{j=1}^d (\xi^{(j)})^\alpha \hat{b}_j$. Therefore, $\widehat{\theta^\alpha} = \sum_{j=1}^d (\xi^{(j)})^\alpha \hat{b}_j$ whenever $|\alpha| \leq m$. If $f = \sum_{|\alpha| \leq m} c_\alpha \theta^\alpha \in \mathcal{S}$, then

$$\hat{f} = \sum_{j=1}^d \sum_{|\alpha| \leq m} c_\alpha (\xi^{(j)})^\alpha \hat{b}_j.$$

This shows that $f^\#(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$, as in the proof of Proposition 1.

The hypothesis $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, allows to find a point $\omega_j \in \Omega$ such that $\xi^{(j)} = \theta(\omega_j) = \delta_j(\hat{\theta})$ for each $j = 1, \ldots, d$. Therefore, with the notation from above,

$$f^{\#}(\xi^{(j)}) = \sum_{|\alpha| \leq m} c_\alpha \theta(\omega_j)^\alpha = f(\omega_j),\ j = 1, \ldots, d,$$

which leads to the equality

$$\Lambda(f) = \int_{\Omega_\Lambda} f(\omega) d\mu(\omega),\ f \in \mathcal{S},$$

where $\Omega_\Lambda := \{\omega_1, \ldots, \omega_d\}$ and μ is the measure with weights $\Lambda(b_j)$ at ω_j, $j = 1, \ldots, d$, via the integral representation formula given by Proposition 1.

Now, if $h \in \mathcal{S}^{(2)}$ have the form $h = \sum_{l \in L} f_l g_l$, with $f_l, g_l \in \mathcal{S}$ for all $l \in L$, L finite, proceeding as in the proof of Proposition 3, we infer that

$$\Lambda(h) = \sum_{l \in L} \sum_{j,k=1}^{d} f_l(\omega_j) g_l(\omega_k) \Lambda(b_j b_k) = \int_{\Omega_\Lambda} h(\omega) d\mu(\omega),$$

which provides a representing measure of Λ on Ω.

Conversely, assume that the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω with $d = \dim \mathcal{H}_\Lambda$ atoms. Using the notation and discussion in Remark 7, we know that the C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Omega_\Lambda)$ are isomorphic via the map $\mathcal{H}_\Lambda \ni \hat{f} \mapsto f|\Omega_\Lambda \in C(\Omega_\Lambda)$, which leads to the existence of an orthogonal basis $\mathcal{B}$ of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements. In addition, the maps $\delta_j(\hat{f}) = f(\omega_j)$, $j = 1, \ldots, d$, are the characters of $\mathcal{H}_\Lambda$ (see also Remark 7). Therefore,

$$\delta_j(\widehat{\theta^\alpha}) = \theta^\alpha(\omega_j) = (\theta_1(\omega_j)^{\alpha_1} \cdots (\theta_n(\omega_j)^{\alpha_n} = \delta_j(\hat{\theta}^\alpha),$$

whenever $|\alpha| \leq m$ and $j = 1, \ldots, d$, showing that $\mathcal{B}$ is a Λ-multiplicative basis, via Lemma 4. Moreover, we clearly have $\delta_j(\hat{\theta}) = \theta(\omega_j)$, $j = 1, \ldots, d$, which concludes the proof of Theorem 3. ∎

Corollary 2 *Let $\mathcal{S}$ be a function space on Ω which is m-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Let also $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$, consisting of idempotent elements, which induces on $\mathcal{H}_\Lambda$ a C^*-algebra structure. Assume that the basis $\mathcal{B}$ is Λ-multiplicative with respect to θ. Then there exists a finite set $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and a probability measure μ on Ξ such that*

$$\Lambda(h(\theta)) = \int_{\Xi} h(\xi) d\mu(\xi),\ h \in \mathcal{P}^n_{2m}.$$

Proof. For every polynomial $p \in \mathcal{P}^n$, we have defined the element $p(\hat{\theta})$, via the polynomial functional calculus. If $\Delta = \{\delta_1, \ldots, \delta_d\}$ is the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$, we put $\xi^{(j)} = (\delta_j(\hat{\theta}_1), \ldots, \delta_j(\hat{\theta}_n))$, $j = 1, \ldots, d$,

$\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. According to formula (5), we have $p(\hat{\theta}) = \sum_{j=1}^{d} p(\xi^{(j)})\hat{b}_j$. Therefore, $p(\hat{\theta})^{\#}(\xi) = p(\xi)$ for all $\xi \in \Xi$. Moreover, because $\mathcal{B}$ is Λ-multiplicative, and if $p \in \mathcal{P}_m^n$, so $p(\hat{\theta}) = \widehat{p(\theta)}$, we obtain

$$\Lambda(p(\theta)) = \int_{\Xi} p(\xi) d\mu(\xi),$$

where μ is a probability measure with support in Ξ.

Further, if $h \in \mathcal{P}_{2m}^n$ has the form $h = \sum_{l\in L} p_l q_l$, with $p_l, q_l \in \mathcal{P}_m^n$ for all $l \in L$, L finite, since we have $p_l(\theta) = \sum_{j=1}^{d} p_l(\xi^{(j)})b_j + r_j$, $q_l(\theta) = \sum_{k=1}^{d} q_l(\xi^{(k)})b_k + s_k$, with $r_j, s_k \in \mathcal{I}_\Lambda$, $j, k = 1, \ldots, d$, we obtain

$$\Lambda(h(\theta)) = \sum_{l\in L} \sum_{j,k=1}^{d} p_l(\xi^{(j)}) q_l(\xi^{(j)}) \Lambda(b_j b_k) = \int_{\Xi} h(\xi) d\mu(\xi),$$

providing an integral representation of all functions of the form $h \circ \theta$, with $h \in \mathcal{P}_{2m}^n$. ■

Theorem 2 from [15] can be obtained as a consequence of Theorem 3:

Corollary 3 *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a Λ-multiplicative basis of $\mathcal{H}_\Lambda$.*

Proof. If $\mathcal{S} = \mathcal{P}_m$ for some $m \geq 0$, we have $\Omega = \mathbb{R}^n$, and $\mathcal{P}_m$ is m-generated by the n-tuple $t = (t_1, \ldots, t_n)$ consisting of the independent variables on $\mathbb{R}^n$. Then $\delta_j(\hat{t}) = \omega_j = \hat{t}(\omega_j)$, $j = 1, \ldots, n$, showing that inclusion condition appearing in the statement of Theorem 3 is automatically fulfilled. ■

Corollary 4 *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a family of polynomials $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the following properties:*

(i) $\Lambda(b_j^2) = \Lambda(b_j) > 0,\ j = 1, \ldots, d$;

(ii) $\Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k$;

(iii)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $0 \neq |\alpha| \leq |\beta|$, $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

This statement appears as Corolloary 3 in [15], and it follows directly from Corollary 3. We omit the details.

5 Results Related to Tchakaloff's Theorem

For an arbitrary Hausdorff topological space Ω, we denote by $\mathcal{B}(\Omega)$ the space of all complex-valued Borel functions on Ω, and by $\mathrm{Bor}(\Omega)$ the family of all Borel subsets of Ω.

Theorem 3 is particularly interesting when applied to a function space which is 1-generated by a given tuple. This happens because, in this case, condition (7) is automatically fulfilled.

Corollary 5 *Let $\mathcal{S}$ be a function space on Ω, which is 1-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let also $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. If either*

(1) *there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of idempotent elements such that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$,*

or

(2) $\theta(\Omega) = \mathbb{R}^n$,

the uspf Λ has a representing measure in Ω with $d = \dim \mathcal{H}_\Lambda$ atoms.

Proof. Assertion (1) follows directly from Theorem 3, because condition (7) is automatically fulfilled.

Assertion (2) is a particular case of assertion (1), but it is valid for any orthogonal basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of idempotent elements.

Example 4 The previous corollary provides an atomic representing measure for a large class of 1-generated function spaces. Here is an example. Let $r > n$, r, n be positive integers, let $\Omega' \subset \mathbb{R}^{r-n}$ be a Borel set, and let $\Omega = \mathbb{R}^n \times \Omega'$. We consider on Ω the functions $\theta_j(t, t') = t_j + \psi_j(t')$, $j = 1, \ldots, n$, where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $t' = (t'_1, \ldots, t'_{r-n}) \in \Omega'$, and $\psi_1, \ldots \psi_n$ are Borel functions. Let $\mathcal{S}$ be the function space on Ω spanned by $\theta := (\theta_1, \ldots, \theta_n)$ and $\theta_0 = 1$. Then every uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has an atomic representing measure in Ω. This follows from the previous corollary, because we clearly have $\theta(\Omega) = \mathbb{R}^n$.

Remark 8 (1) We can give explicit formulas related to Corollary 5. Let $\mathcal{S}$ be a function space on Ω, which is 1-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let also $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. We fix a basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents, which is automatically Λ-multiplicative with respect to θ, where $d = \dim(\mathcal{H}_\Lambda)$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the dual basis. We set $\xi^{(j)} = \delta_j(\hat{\theta}) \in \mathbb{R}^n$, $j = 1, \ldots, d$, and $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Then, as in Remark 2, if $p \in \mathcal{P}_2^n$, we have $\Lambda(p(\theta)) = \int_\Xi p(\xi) d\mu(\xi)$, where μ is probability measure with weights $\lambda_j := \Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \ldots, d$.

If $\delta_j(\hat{\theta}) = \theta(\omega_j)$, $j = 1, \ldots, d$, we actually have $\Lambda(h) = \sum_{j=1}^d \lambda_j h(\omega_j)$ for every $h \in \mathcal{S}^{(2)}$, as in the proof of Theorem 3. In fact,

$$\theta(\omega_j) = (\Lambda(b_j)^{-1}\Lambda(\theta_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(\theta_n b_j) \in \mathbb{R}^n, \ j = 1, \ldots, d.$$

Finally, when $\theta(\Omega) = \mathbb{R}^n$, the existence of the points ω_j, $j = 1, \ldots, d$ is insured for any basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents.

(2) Corollary 5(2) can be used to give a direct proof to our Theorem C, which is a version of Tchakaloff's theorem (see [15] for details).

Theorem 4 *Let $\mathcal{S}$ be a function space on Ω, m-generaterd by the n-tuple $\theta = (\theta_1, \dots, \theta_n)$, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. If Λ has a representing measure then it has an atomic representing measure.*

Proof. We consider the set $\mathbb{Z}_+^{n,m} := \{\alpha \in \mathbb{Z}_+^n; |\alpha| \le m\}$ endowed with the lexicographic order. In addition, we assign to each integer $j \in \{0, 1, 2, \dots, n_m\}$, where $n_m + 1$ is the cardinal of $\mathbb{Z}_+^{n,m}$, a multi-index $\alpha(j) \in \mathbb{Z}_+^{n,m}$ with $j \le k$ iff $\alpha(j) \le \alpha(k)$, and $\alpha(0) = 0$. In this way we have a map $\phi : \Omega \mapsto \mathbb{R}^{n_m}$ given by $\phi(\omega) = (\theta^{\alpha(1)}(\omega), \dots, \theta^{\alpha(n_m)}(\omega)) \in \mathbb{R}^{n_m}$.

Now assume that Λ has a representing measure, so it has form $\Lambda(h) = \int_\Omega h d\mu$, $h \in \mathcal{S}^{(2)}$, where μ is a positive Borel measure on Ω, with $\mu(\Omega) = 1$. Let ν be the measure induced by the measure μ and the Borel map ϕ. Note that

$$\int_{\mathbb{R}^{n_m}} |x_j| d\nu(x) = \int_\Omega |x_j \circ \phi| d\mu \le \int_\Omega |\theta^{\alpha(j)}| d\mu \le \Lambda(\theta^{2\alpha(j)})^{1/2} < \infty,$$

for all $j = 1, \dots, n_m$. This shows that we may apply Corollary 2 from [1] and deduce the existence of a positive integer $d \le n_m$, a set of points $\omega_1, \dots, \omega_d$ in the support of the measure μ, and positive numbers $\lambda_1, \dots, \lambda_d$ such that

$$\int_\Omega \theta^\alpha d\mu = \sum_{j=1}^d \lambda_j \theta^\alpha(\omega_j),\ \ |\alpha| \le 2m.$$

As we clearly have $\sum_{j=1}^d \lambda_j = 1$, it follows that Λ has an atomic representing measure. ■

Remark 9 Let $\mathcal{S}$ be a finite dimensional function space on Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf having a representing measure which may be supposed to be atomic, via Theorem 4. Let also $\mathcal{Q} := \{q_1, \dots, q_s\} \subset \mathcal{RS}$, and let

$$\Omega_{\mathcal{Q}} = \{\omega \in \Omega; q_j(\omega) \ge 0, j = 1, \dots, s\}.$$

Let $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents, constructed as in the first part of the proof of Proposition 3. The measure μ has support in $\Omega_{\mathcal{Q}}$ if and only if $\Lambda(q_j b_k) \ge 0$ for all $j = 1, \dots, s$; $k = 1, \dots, d$. Indeed, if $\Omega_{\mathcal{Q}} = \{\omega_1, \dots, \omega_d\}$ is the support of μ, we have

$$\Lambda(q_j b_k) = \int_{\Omega_{\mathcal{Q}}} q_j(t) b_k(t) d\mu(t) = \Lambda(b_k) q_j(\omega_k),\ j = 1, \dots, s,\ k = 1, \dots, d,$$

implying our assertion.

This remark may be applied, in particular, to spaces of functions consistng of polynomials, restricted to semi-algebraic sets.

6 Function Spaces and Point Evaluations

In this section we continue the discussion from Subsection 1.5.

Remark 10 (1) It follows from Lemma 3 that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

(2) Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf with the property $\mathcal{Z}_\Lambda \neq \emptyset$. With the notation from Remark 3(2), the set $\{\hat{\delta}_\omega; \omega \in \mathcal{Z}_\Lambda\}$ is a subset in the dual of the Hilbert space $\mathcal{H}_\Lambda$. Therefore, for every $\omega \in \mathcal{Z}_\Lambda$ there exists a vector $\hat{v}_\omega \in \mathcal{H}_\Lambda$ such that $\hat{\delta}_\omega(\hat{f}) = \langle \hat{f}, \hat{v}_\omega \rangle = \Lambda(f v_\omega) = f(\omega)$ for all $f \in \mathcal{S}$.

Set $\mathcal{V}_\Lambda = \{\hat{v}_\omega; \omega \in \mathcal{Z}_\Lambda\}$. The elements of $\mathcal{V}_\Lambda$ are not necessarily distinct. If $\omega' \neq \omega''$, we have $\hat{v}_{\omega'} \neq \hat{v}_{\omega''}$ iff the space $\mathcal{S}$ separates the points of Ω.

The next result, which extends Theorem 4 from [15], is an approach to truncated moment problems when the number of the atoms of the representing measures is not necessarily equal to the maximal cardinal of a family of orthogonal idempotents. The basic elements are in this case projections of idempotents.

Theorem 5 *Let $\mathcal{S}$ be a finite dimensional function space separating the points of Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ with $\mathcal{Z}_\Lambda$ nonempty. The uspf Λ has a representing measure in $\mathbb{R}^n$ consisting of d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{R}\mathcal{H}_\Lambda$ of distinct elements such that*

$$\Lambda(v_j) > 0, \quad \hat{v}_j / \Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d, \tag{8}$$

$$\hat{f} = \Lambda(v_1)^{-1} \Lambda(f v_1) \hat{v}_1 + \cdots + \Lambda(v_d)^{-1} \Lambda(f v_d) \hat{v}_d, \ f \in \mathcal{S}, \tag{9}$$

and

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \Lambda(v_j)^{-1} \Lambda(v_j v_k) \Lambda(v_j v_l), \ k, l = 1, \ldots, d. \tag{10}$$

Proof. Assume first that $\mu = \sum_{j=1}^{d} \lambda_j \delta_{\omega_j}$ is a representing measure for Λ, with $\lambda_j > 0$ for all $j = 1, \ldots, d$, with $\omega_1, \ldots, \omega_d$ distinct, and $\sum_{j=1}^{d} \lambda_j = 1$. The set $\Omega_\Lambda := \{\omega_1, \ldots, \omega_d\}$, which is exactly the support of μ, is a subset of $\mathcal{Z}_\Lambda$, by Lemma 3.

We now proceed as in the proof of Proposition 3. Note that $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; f|\Omega_\Lambda = 0\}$, This shows that there exists a map $\rho : \mathcal{H}_\Lambda \mapsto C(\Omega_\Lambda)$ given by $\hat{f} \mapsto f|\Omega_\Lambda$, which is correctly defined, linear and injective (but not necessarily surjective). Endoved with the norm $f \mapsto (\int_{\Omega_\Lambda} |f|^2 d\mu)^{1/2}$, the space $C(\Omega_\Lambda)$ becomes a Hilbert space, denoted by $L^2(\Omega_\Lambda, \mu)$, and the map ρ becomes an isomatry. This allows us to regard $\mathcal{H}_\Lambda$ as closed subspace of $L^2(\Omega_\Lambda, \mu)$.

Let $\chi_k \in L^2(\Omega_\Lambda, \mu)$ be the characteristic function of the set $\{\omega_k\}$. Clearly, the family $\{\chi_k;\ k = 1, \ldots, d\}$ consists of Λ_Ω-idempotents in the function space $L^2(\Omega_\Lambda, \mu)$, where Λ_Ω is the natural extension of Λ to $L^2(\Omega_\Lambda, \mu)$. In fact, $\{\chi_k;\ k = 1, \ldots, d\}$ is an orthogonal basis of $L^2(\Omega_\Lambda, \mu)$ consisting of idempotents.

Let P denote the orthogonal projection of the $L^2(\Omega_\Lambda, \mu)$ onto $\mathcal{H}_\Lambda$. In particular, $P\hat{1} = \hat{1}$. Let also Let $\hat{v}_j = P\chi_j$, $j = 1, \ldots, d$. Then

$$\Lambda(fv_j) = \langle \hat{f}, P\chi_j \rangle = \langle \hat{f}, \chi_j \rangle = \lambda_j f(\omega_j),\ f \in \mathcal{S},\ j = 1, \ldots, d,$$

so $\lambda_j := \Lambda_\Omega(\chi_j) = \langle \hat{1}, \chi_j \rangle = \Lambda(v_j) > 0$, and $v_j/\lambda_j = v_{\omega_j}$, which is precisely (8). In addition, as $\{\chi_1, \ldots, \chi_d\}$ is an orthogonal basis of $L^2(\Omega_\Lambda, \mu)$,

$$\hat{f} = P\hat{f} = P(\lambda_j^{-1} \langle \hat{f}, \chi_1 \rangle \chi_1 + \cdots + \lambda_d^{-1} \langle \hat{f}, \chi_d \rangle \chi_d) =$$

$$\Lambda(v_1)^{-1}\Lambda(fv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(v_d)\hat{v}_d,$$

for all $f \in \mathcal{S}$, showing that (9) holds.

Note also that

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \lambda_j (v_k v_l)(\omega_j) = \sum_{j=1}^{d} \lambda_j^{-1} \Lambda(v_k v_j)\Lambda(v_l v_j),\ k, l = 1, \ldots, d,$$

because

$$(v_k v_l)(\omega_j) = \Lambda(v_k v_{\omega_j})\Lambda(v_l v_{\omega_j}) = \lambda_j^{-2} \Lambda(v_k v_j)\Lambda(v_l v_j)$$

for all $k, l = 1, \ldots, d$, proving that (10) also holds.

Conversely, assume that there exists a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{RH}_\Lambda$ such that (8), (9), (10) hold. We must have $v_j/\lambda_j = v_{\omega_j}$ for a certain $\omega_j \in \mathcal{Z}_\Lambda$, with $\lambda_j = \Lambda(v_j) > 0$ for all $j = 1, \ldots, d$.

Set $\Omega_\Lambda := \{\omega_1, \ldots, \omega_d\} \subset \mathcal{Z}_\Lambda$, and consider the map $\mathcal{H}_\Lambda \ni \hat{f} \mapsto f|\Omega_\Lambda \in C(\Omega_\Lambda)$. Note that this map is correctly defined because the equality $\hat{f}_1 = \hat{f}_2$, which is equivalent to $f_1 - f_2 \in \mathcal{I}_\Lambda$, implies $f_1|\Omega_\Lambda = f_2|\Omega_\Lambda$, by Lemma 2.

Moreover, the map is injective because $f(\omega_j) = \lambda_j^{-1}\Lambda(fv_j) = 0$ for all $j = 1, \ldots, d$ implies $\hat{f} = 0$, via (9).

Since, in virtue of (9),

$$\Lambda(f) = \langle \hat{f}, \hat{v}_1 \rangle + \cdots + \langle \hat{f}, \hat{v}_d \rangle = \lambda_1 p(\omega_1) + \cdots + \lambda_d p(\omega_d),$$

for all $f \in \mathcal{S}$, the map $\Lambda|\mathcal{S}$ admits the extension $M(g) = \sum_{j=1}^{d} \lambda_j g(\omega_j)$, $g \in C(\Omega_\Lambda)$, which provides an integral representation for $\Lambda|\mathcal{S}$.

We want to show that the map M also extends Λ. For, let $h = \sum_{j \in J} f_j g_j$, with $f_j, g_j \in \mathcal{S}$ for all $j \in J$, where J is a finite set of indices.

Note first that

$$h(\omega_k) = \sum_{j \in J} f_j(\omega_k) q_j(\omega_k) = \lambda_k^{-2} \sum_{j \in J} \Lambda(f_j v_k)\Lambda(g_j v_k), \tag{11}$$

for all $k = 1, \ldots, d$. Then, on one hand,

$$M(h) = \sum_{k=1}^{d} \lambda_k f(\omega_k) = \sum_{k=1}^{d} \lambda_k \sum_{j \in J} f_j(\omega_k) g_j(\omega_k),$$

so that, using (11),

$$M(h) = \sum_{k=1}^{d} {\lambda_k}^{-1} \sum_{j\in J} \Lambda(f_j v_k)\Lambda(g_j v_k). \tag{12}$$

On the other hand, writing by (9)

$$\hat{f}_j = \sum_{l=1}^{d} {\lambda_l}^{-1}\Lambda(p_j v_l)\hat{v}_l,\ \hat{g}_j = \sum_{s=1}^{d} {\lambda_s}^{-1}\Lambda(q_j v_s)\hat{v}_s$$

for all $j \in J$, we have

$$h - \sum_{j\in J}\sum_{l,s=1}^{d} {\lambda_l}^{-1}{\lambda_s}^{-1}\Lambda(p_j v_l)\Lambda(q_j v_s)v_l v_s \in \ker(\Lambda),$$

so

$$\Lambda(h) = \sum_{j\in J}\sum_{l,s=1}^{d} {\lambda_l}^{-1}{\lambda_s}^{-1}\Lambda(f_j v_l)\Lambda(g_j v_s)\Lambda(v_l v_s) =$$

$$\sum_{j\in J}\sum_{l,s=1}^{d} {\lambda_l}^{-1}{\lambda_s}^{-1}\Lambda(f_j v_l)\Lambda(g_j v_s)\sum_{k=1}^{d}\lambda_k^{-1}\Lambda(v_k v_l)\Lambda(v_k v_s) =$$

$$\sum_{k=1}^{d}{\lambda_k}^{-1}\sum_{j\in J}\sum_{l=1}^{d}{\lambda_l}^{-1}\Lambda(f_j v_l)\Lambda(v_l v_k)\sum_{s=1}^{d}{\lambda_s}^{-1}\Lambda(g_j v_s)\Lambda(v_s v_k) =$$

$$\sum_{k=1}^{d}{\lambda_k}^{-1}\sum_{j\in J}\Lambda(f_j v_k)\Lambda(g_j v_k),$$

via (10), because of the equalities

$$\Lambda(f_j v_k) = \sum_{l=1}^{d}{\lambda_l}^{-1}\Lambda(f_j v_l)\Lambda(v_l v_k),\ \Lambda(g_j v_k) = \sum_{s=1}^{d}{\lambda_s}^{-1}\Lambda(g_j v_s)\Lambda(v_s v_k),$$

derived from (9). This computation leads to the equality $M(h) = \Lambda(h)$, for each p of the given form. Formula (12) shows that, in fact, the equality $M(p) = \Lambda(p)$ does not depend on the particular representation of h as a finite sum of the form $\sum_{j\in J} f_j g_j$, with $f_j, g_j \in \mathcal{S}$, and so $M(h) = \Lambda(h)$ holds for all $h \in \mathcal{S}^{(2)}$. ■

Remark 11 Condition $d \geq \dim \mathcal{H}_\Lambda$, appearing in the previous statement, is a necessary one, as follows from [2], Corollary 3.7.

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