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SPECTRAL MEASURES AND MOMENT PROBLEMS

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Dedicated to Professor Ion Colojoară on the occasion of his 70^{th} birthday

ABSTRACT. In this expository paper we try to emphasize some connections between functional analysis, in particular operator theory, and moment problems. A central rôle in this discussion is played by the operator-valued positive measures, in particular the spectral measures, which are mathematical objects related to the spectral decompositions of linear operators, domain in which I. Colojoară has important contributions (see, for instance, the monograph [Col]).

Part I. MOMENTS ON SEMI-ALGEBRAIC COMPACT SETS

I.1. INTRODUCTION

This chapter is a revisited and expanded version of the work [Vas2] (see also [Cla], [Dem2], [PuVa1], [Vas7] etc.). Results as Theorems I.3.1, I.4.8 and Corollaries I.4.9 and I.4.10 have been subsequently obtained by the author.

Let $t = (t_1, \ldots, t_n)$ denote the variable in the real Euclidean space $\mathbb{R}^n$, and let $P(\mathbb{R}^n)$ be the algebra of real polynomial functions in $t_1, \ldots, t_n$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is an arbitrary multi-index, set $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$.

For technical reasons, we often use polynomial with complex coefficients. We denote by $P_{\mathbb{C}}(\mathbb{R}^n)$ the space $P(\mathbb{R}^n) \otimes \mathbb{C}$, i.e., the space of all polynomials on $\mathbb{R}^n$ having complex coefficients.

Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ be an n-sequence of real numbers. We set

$$L_\gamma(t^\alpha) = \gamma_\alpha, \quad \alpha \in \mathbb{Z}_+^n, \tag{I.1.1}$$

and extend L_γ to $P_{\mathbb{C}}(\mathbb{R}^n)$ by linearity.

The n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ is said to be *positive semi-definite* if L_γ is positive semi-definite (that is, $L_\gamma(p\bar{p}) \geq 0$ for all $p \in P_{\mathbb{C}}(\mathbb{R}^n)$).

Let $K \subset \mathbb{R}^n$ be a closed set. The n-sequence γ is said to be a *K-moment sequence* if there exists a positive Borel measure μ on K such that $t^\alpha \in L^1(\mu)$ and $\gamma_\alpha = \int_K t^\alpha d\mu(t)$

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for all $\alpha \in \mathbb{Z}_+^n$. When such a measure μ exists, then it is called a *representing measure* of the sequence γ.

To solve the K*-moment problem* ([Berg]) means to characterize those n-sequences of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ ($\gamma_0 > 0$) which possess a representing measure on K.

Let $P(K) = \{p|K \ : \ p \in P(\mathbb{R}^n)\}$, and let $P_+(K) = \{p \in P(K) \ : \ p|K \geq 0\}$.

If the sequence γ possesses a representing measure μ on K, then the linear functional L_γ satisfies the condition

$$L_\gamma(p) \geq 0, \quad p \in P_+(K), \tag{I.1.2}$$

and $L_\gamma(1) > 0$.

Condition (I.1.2) is also sufficient ([Hav]). Nevertheless, except for the cases when $P_+(K)$ can be completely described, this condition is, in practice, difficult to verify. For this reason, to solve the K-moment problem, one possibility is to seek a "test subset" $\Theta \subset P_+(K)$, expressed as explicitly as possible in terms of K, such that condition (I.1.2) restricted to Θ imply the existence of a representing measure on K.

Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$, and let

$$K_{\mathcal{P}} = \{s \in \mathbb{R}^n; p_j(s) \geq 0, j = 1, \ldots, m\}. \tag{I.1.3}$$

A closed subset $K \subset \mathbb{R}^n$ will be called (in this text) *semi-algebraic* if there exists a family $\mathcal{P}$ such that $K = K_{\mathcal{P}}$.

In the next section we shall construct a fairly explicit "test set" for every semi-algebraic compact set K, which in turn will lead to an explicit solution to the K-moment problem.

I.2. Moments on semi-algebraic compact sets

Remark I.2.1. Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$. Suppose that $K = K_{\mathcal{P}}$ is compact. We attach to the family $\mathcal{P}$ a family $\hat{\mathcal{P}}$ constructed in the following way.

As we clearly have $m_j = \sup_{t \in K} p_j(t) < \infty$, we set $\hat{p}_j(t) = m_j^{-1} p_j(t)$, $t \in \mathbb{R}^n$, if $m_j > 0$, and $\hat{p}_j = p_j$ if $m_j = 0$, $j = 1, \ldots, m$.

We define $\hat{\mathcal{P}} = \{0, 1, \hat{p}_1, \ldots, \hat{p}_m\}$.

Note that $K = K_{\mathcal{P}} = K_{\hat{\mathcal{P}}}$, and that $0 \leq \hat{p}(t) \leq 1$ for all $t \in K$ and $\hat{p} \in \hat{\mathcal{P}}$.

We denote by $\Delta_{\mathcal{P}}$ the set of all products of the form

$$q_1 \cdots q_k (1 - r_1) \cdots (1 - r_l)$$

for polynomials $q_1, \ldots, q_k, r_1, \ldots, r_l \in \hat{\mathcal{P}}$ and integers $k, l \geq 1$.

We clearly have $p|K \geq 0$ for all $p \in \Delta_{\mathcal{P}}$. Note also that the set $\Delta_{\mathcal{P}}$ is explicitly constructed in terms of $\mathcal{P}$.

One can prove the following assertion (practically contained in the proof of Theorem 2.3 from [Vas2]).

Theorem I.2.2. *Let $\mathcal{P} = \{p_0 = 1, p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$. Suppose that $K = K_{\mathcal{P}}$ is compact and that the family $\mathcal{P}$ generates the algebra $P(\mathbb{R}^n)$.*

An n-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence if and only if the linear form L_γ is nonnegative on the set $\Delta_{\mathcal{P}}$.

In addition, the representing measure of a K-moment sequence is uniquely determined.

Remark I.2.3. An alternate proof of Theorem I.2.2 was given in [PuVa1]. Another proof can be derived from the representation theorem for real algebras given in [BeSc] (see also [Kri1] and [Kri2]; see [PrDe] for general representation theorems, leading to solutions to various moment problems, and [BCR] for other connexions). We are indebted to E. Becker and A. Prestel for some discussions concerning the actual state-of-the-art.

We shall give in the sequel a proof of Theorem I.2.2, following the lines of Theorem 2.3 from [Vas2].

Remark I.2.4. Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$ such that $K = K_{\mathcal{P}}$ be compact. Let also $\hat{\mathcal{P}}$ be constructed as in Remark I.2.1.

Let $a_k = a_k(K) = \inf\{t_k \ : \ t \in K\}$ and $b_k = b_k(K) = \sup\{t_k \ : \ t \in K\}$. Then we set $\hat{p}_{m+k}(t) = (t_k - a_k)/(b_k - a_k)$ if $b_k > a_k$, $\hat{p}^{\pm}_{m+k}(t) = \pm(t_k - a_k)$, if $b_k = a_k$, $k = 1, \ldots, n$.

We define $\tilde{\mathcal{P}} = \{0, 1, \hat{p}_1, \ldots, \hat{p}_m, \hat{p}_{m+k} \ : \ a_k < b_k,\ k = 1, \ldots, n\}$, and $\tilde{\mathcal{P}}_0 = \{1, \hat{p}^{\pm}_{m+k} \ : \ a_k = b_k,\ k = 1, \ldots, n\}$.

Notice that $K = K_{\mathcal{P}} = K_{\tilde{\mathcal{P}}}$, and that $0 \le \hat{p}(t) \le 1$ for all $t \in K$ and $\hat{p} \in \tilde{\mathcal{P}}$.

Definition I.2.5. Let $\mathcal{P} = \{p_1, \ldots, p_m\} \subset P(\mathbb{R}^n)$ be such that $K = K_{\mathcal{P}}$ be compact, and let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}_0$ be as in Remark I.2.4. We denote by $\tilde{\Delta}_{\mathcal{P}}$ the set of all products of the form

$$q_1 \cdots q_k (1 - r_1) \cdots (1 - r_l) h_1 \cdots h_u$$

for polynomials $q_1, \ldots, q_k, r_1, \ldots, r_l \in \tilde{\mathcal{P}}$, $h_1, \ldots, h_u \in \tilde{\mathcal{P}}_0$ and integers $k, l, u \ge 1$.

The next result is a version of Theorem I.2.2, holding for a set $\mathcal{P} = \{p_1, \ldots, p_m\}$ in $P(\mathbb{R}^n)$ not necessarily containing a family of generators.

Theorem I.2.6. *Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$ such that $K = K_{\mathcal{P}}$ be compact. An n-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$, $\gamma_0 > 0$, is a K-moment sequence if and only if the linear form L_γ is nonnegative on the set $\tilde{\Delta}_{\mathcal{P}}$.*

Proof. We shall show that Theorem I.2.6 can be obtained as a consequence of Theorem I.2.2.

We keep the notation from the Remark I.2.4. As the condition from the statement is clearly necessary, we shall deal only with its sufficiency.

If $\{k : a_k = b_k\} \neq \emptyset$, without loss of generality we may suppose the existence of an integer $d \in \{0, 1, \ldots, n-1\}$ such that $a_k = b_k$ for all $k \geq d+1$. We first discuss the case $d \geq 1$.

Let $u = (u_1, \ldots, u_d)$ be the variable of $\mathbb{R}^d$. Let also $\kappa \colon \mathbb{R}^d \to \mathbb{R}^n$ be given by $\kappa(u) = (u, a)$, where $a = (a_{d+1}, \ldots, a_n) \in \mathbb{R}^{n-d}$. If $\mathcal{Q} = \{p \circ \kappa : p \in \mathcal{P}\}$, and $K_d = \kappa^{-1}(K)$, then K_d is compact (in fact, $K = K_d \times \{a\}$) and $K_d = K_{\mathcal{Q}}$. Moreover, $\tilde{\Delta}_{\mathcal{Q}} = \{p \circ \kappa : p \in \tilde{\Delta}_{\mathcal{P}}\}$, as one can easily check.

Let us denote, for simplicity, $r_k(t) = (t_k - a_k)/(b_k - a_k)$, $k = 1, \ldots, d$, $r_k(t) = t_k - a_k$, $k = d+1, \ldots, n$. As $r_k \in \Delta_{\mathcal{P}}$ for $k = d+1, \ldots, n$, it follows that

$$L_\gamma(r_1^{k_1} \cdots r_d^{k_d} r_{d+1}^{k_{d+1}} \cdots r_n^{k_n}) = 0$$

for all integers $k_1 \geq 0, \ldots, k_n \geq 0$, provided $k_{d+1} + \cdots + k_n \neq 0$. Because the algebra $P(\mathbb{R}^n)$ is generated by $1, r_1, \ldots, r_n$, we derive the formula

$$L_\gamma(t^{(\xi,\eta)}) = L_\gamma(t^{(\xi,0)}) a^\eta, \quad \xi \in \mathbb{Z}_+^d, \eta \in \mathbb{Z}_+^{n-d}.$$

Setting $\delta_\xi = L_\gamma(t^{(\xi,0)})$, $\delta = (\delta_\xi)_{\xi \in \mathbb{Z}_+^d}$, we infer that

$$L_\gamma(p) = L_\delta(p \circ \kappa), \quad p \in P(\mathbb{R}^n).$$

In particular, L_δ is positive on $\tilde{\Delta}_{\mathcal{Q}}$, and the latter contains d linearly independent polynomials of first degree. In fact, $K_d \subset \prod_{j=1}^{d} [a_j, b_j]$ and we may apply Theorem I.2.2. Consequently, there exists a positive measure ν on K_d such that $\delta_\xi = \int_{K_d} u^\xi \, d\nu(u)$, $\xi \in \mathbb{Z}_+^d$. If θ_a is the Dirac measure concentrated at a, then $\mu = \nu \otimes \theta_a$ is a representing measure for γ.

The case $d = 0$ is obtained in a similar manner, and a representing measure for γ is $\gamma_0 \theta_a$. $\square$

We shall prepare the proof of Theorem I.2.2, which needs some auxiliary results.

Definition I.2.7. Let $K = K_{\mathcal{P}}$ be a semi-algebraic compact set. We denote by $\pi(\mathcal{P})$ the family of all linear mappings $L : P(\mathbb{R}^n) \to \mathbb{R}$ such that $L(1) = 1$ and $L(r) \geq 0$ for all $r \in \Delta_{\mathcal{P}}$. Clearly, $\pi(\mathcal{P})$ is a convex set.

Lemma I.2.8. *For all $L \in \pi(\mathcal{P})$ and $r \in \Delta_{\mathcal{P}}$ one has $0 \leq L(r) \leq 1$.*

Proof. (See also [Cas], Lemme 2.) Let $r = r_1 \cdots r_k \in \Delta_{\mathcal{P}}$, where either $r_j \in \hat{\mathcal{P}}$ or $1 - r_j \in \hat{\mathcal{P}}$ for all j. As we have

$$1 - r_1 \cdots r_n = (1 - r_1) + r_1(1 - r_2) + \cdots + r_1 \cdots r_{k-1}(1 - r_k),$$

one obtains $L(1 - r) \geq 0$, whence $L(r) \leq 1$. $\square$

Remark I.2.9. (1) Let $\Gamma_+(K)$ be the positive cone generated by $\Delta_{\mathcal{P}}$. The previous proof shows that if $r \in \Delta_{\mathcal{P}}$, then $1 - r \in \Gamma_+(K)$. A similar argument also shows that if $p, q \in \Delta_{\mathcal{P}}$, then $(1-p)q \in \Gamma_+(K)$. In particular, if $L\colon P(\mathbb{R}^n) \to \mathbb{R}$ is positive on $\Delta_{\mathcal{P}}$, then $L((1-p)q) \geq 0$ for all $p, q \in \Delta_{\mathcal{P}}$. If, in addition, $L(1) = 0$, then $L = 0$.

(2) As in [Cas], we identify the set $\pi(\mathcal{P})$ with a subset of the compact space $X = [0,1]^{\Delta_{\mathcal{P}}}$, which is possible via the fact that the unital algebra generated by $\hat{\mathcal{P}}$ in $P(\mathbb{R}^n)$ coincides with $P(\mathbb{R}^n)$, and by Lemma I.2.8. This subset is also closed, for if L_0 is a cluster point of $\pi(\mathcal{P})$ in X, then $L_0(r) \geq 0, r \in \Delta_{\mathcal{P}}, L_0(1) = 1$, and L_0 can be extended by linearity to $P(\mathbb{R}^n)$. Therefore, in the algebraic dual of $P(\mathbb{R}^n)$, endowed with the weak topology induced by $\Delta_{\mathcal{P}}$, the set $\pi(\mathcal{P})$ is convex and compact.

Lemma I.2.10. *Let L be an extreme point of $\pi(\mathcal{P})$. Then L is multiplicative on $P(\mathbb{R}^n)$.*

Proof. We proceed as in the proof of [Cas], Lemme 3. Let $p \in \Delta_{\mathcal{P}}$ be fixed. It suffices to prove that $L(pq) = L(p)L(q)$ for all $q \in \Delta_{\mathcal{P}}$. Let $\alpha = L(p)$. We have three possibilities:

If $0 < \alpha < 1$, we consider the linear functionals $L_1(r) = \alpha^{-1}L(pr)$ and $L_2(r) = (1-\alpha)^{-1}L((1-p)r), r \in P(\mathbb{R}^n)$. It is easily seen that $L_1, L_2 \in \pi(\mathcal{P})$, via Remark I.2.9 (1). As we have $L = \alpha L_1 + (1-\alpha)L_2$, and L is an extreme point of $\pi(\mathcal{P})$, we must have $L = L_1$, whence $L(pq) = L(p)L(q)$.

If $\alpha = 0$, then $L_0(r) = L(pr)$ is positive on $\Delta_{\mathcal{P}}$, and $L_0(1) = 0$, whence $L_0 = 0$, by Remark I.2.9 (1). This implies that $L(pq) = 0 = L(p)L(q)$.

If $\alpha = 1$, we apply the previous argument to the functional $L_1(r) = L((1-p)r)$, and obtain $L(pq) = L(q) = L(p)L(q)$. □

Lemma I.2.11. *For every $L \in \pi(\mathcal{P})$ there exists a uniquely determined probability measure μ on K such that $L(p) = \int_K p d\mu$ for all $p \in P(K)$.*

Proof. This assertion coincides with [Cas], Théorème 1. Here is a different proof.

Let $L_0 \in \pi(\mathcal{P})$ be an extreme point. Then L_0 is multiplicative on $P(\mathbb{R}^n)$, by Lemma I.2.10. Thus, if $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is given by $c_j = L_0(t_j)$, then we have $L_0(p) = p(c)$ for all $p \in P(\mathbb{R}^n)$. Since $0 \leq L_0(p) \leq 1, p \in \Delta_{\mathcal{P}}$, by Lemma I.2.8, we obtain that $c \in K$, and that

$$|L_0(p)| = |p(c)| \leq \|p\|_K = \sup_{t \in K} |p(t)|, \quad p \in P(\mathbb{R}^n).$$

If $L \in \pi(\mathcal{P})$ is of the form $L = \sum_{j \in J} \lambda_j L_j$, where $\lambda_j \geq 0$, $\sum_{j \in J} \lambda_j = 1$, L_j an extreme point of $\pi(\mathcal{P})$, J finite, then

$$|L(p)| \leq \sum_{j \in J} \lambda_j |L_j(p)| \leq \sum_{j \in J} \lambda_j \|p\|_K = \|p\|_K, \quad p \in P(\mathbb{R}^n),$$

by the first part of the proof.

Since the set $\pi(\mathcal{P})$ is convex and compact (see Remark I.2.9(2)), by virtue of the Krein-Milman theorem we have that every $L \in \pi(\mathcal{P})$ is in the closure of the convex hull of the set of extreme points of $\pi(\mathcal{P})$, whence we deduce that

$$|L(p)| \leq \|p\|_K, \qquad L \in \pi(\mathcal{P}),\ p \in P(\mathbb{R}^n),$$

by the previous similar estimates. This implies the existence and uniqueness of the measure μ for each $L \in \pi(\mathcal{P})$, by the Riesz representation theorem, via the density of $P(K)$ in the space of all real-valued continuous functions on K, given by the Weierstrass approximation theorem. $\square$

Proof of Theorem I.2.2. Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ be a K-moment sequence. Then the linear form $L = \gamma_0^{-1} L_\gamma$ is an element of $\pi(\mathcal{P})$, and the conclusion follows from Lemma I.2.11. $\square$

The representing measure previously obtained is always uniquely determined, as a consequence of the theorem of Weierstrass asserting the density of the space $P(K)$ in the space of all continuous functions on K.

Remark I.2.12. (1) For $n = 1$, if $\mathcal{P} = \{p_1, \ldots, p_m\} \subset P(\mathbb{R})$ and if $K = K_{\mathcal{P}}$ is compact, a positive semi-definite n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+}$ is a K-moment sequence if and only if each sequences $p_k(S)\gamma$ are positive semi-definite for all $k = 1, \ldots, m$, where $(S\gamma)_\alpha = \gamma_{\alpha+1}$ for all $\alpha \in \mathbb{Z}_+$, as shown in [BeMa].

(2) Let $K = K_{\mathcal{P}}$, and let $\Sigma_{\mathcal{P}}$ be the family of all polynomial functions in $P_+(K)$ of the form $q_1^2 + q_2^2 p_{j_1} \cdots p_{j_k}$, where $q_1, q_2 \in P(\mathbb{R}^n)$ are arbitrary, and $\{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$. It is shown in [Sch1] that if $K = K_{\mathcal{P}}$ is compact, then a sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence if and only if $L_\gamma(p) \geq 0$ for all $p \in \Sigma_{\mathcal{P}}$.

(3) A certain "test set" is constructed in [Cas], in a rather intricate manner (using extremal points of some spaces of polynomial functions), for an arbitrary compact set with nonempty interior. Although that "test set" is hard to describe in explicit terms, it is the method of [Cas] which has been adapted to obtain a proof for Theorem I.2.2 above.

(4) The case of an arbitrary compact set is treated in [Dem2], by extending the techniques from [Vas2].

For some applications, it is useful to have a better localization of the support of the representing measure of a moment sequence. In this respect, we have the following (see Theorem 2.9 from [Vas2]; see also [McG]).

Theorem I.2.13. *Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$ such that $K = K_{\mathcal{P}}$ is compact. Let also $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ be a K-moment sequence, and let μ be the representing measure of γ. Assume that there exists an $r \in P(\mathbb{R}^n)$ such that $L_\gamma(rp) \geq 0$ for all $p \in \tilde{\Delta}_{\mathcal{P}}$. Then*

$$\mathrm{supp}(\mu) \subset \{s \in K\ :\ r(s) \geq 0\}.$$

If $L_\gamma(rp) = 0$ for some $r \in P(K)$ and for all $p \in \tilde{\Delta}_{\mathcal{P}}$, then

$$\operatorname{supp}(\mu) \subset \{s \in K \;:\; r(s) = 0\}.$$

Proof. Assume first that $L_\gamma(r) > 0$. Then the sequence $\gamma'_\alpha = L_\gamma(t^\alpha r), \alpha \in \mathbb{Z}_+^n$, is a K-moment sequence, by Theorem I.2.2. Let μ' be the representing measure of $\gamma' = (\gamma'_\alpha)_{\alpha\in\mathbb{Z}_+^n}$. Since $\int p\, d\mu' = \int pr\, d\mu$ for all $p \in P(K)$, we obtain the equality $\int f\, d\mu' = \int fr\, d\mu$ for all continuous functions f on K, by the Weierstrass approximation theorem. This implies that $\mu' = r\mu$. As μ' is a positive measure, an easy measure theoretic argument shows that $\mu(B) = 0$, where $B = \{s \in K \;:\; r(s) < 0\}$.

Assume now that $L_\gamma(r) = 0$. We have two cases:

a) There is a $q_0 \in \Delta_{\mathcal{P}}$ such that $L_\gamma(q_0 r) > 0$. As $L_\gamma((q_0 + \epsilon)r) = L_\gamma(q_0 r) > 0$ for any $\epsilon > 0$, we may assume that $q_0(s) > 0$ for all $s \in K$. Then, by the first part of the proof,

$$\operatorname{supp}(\mu) \subset \{s \in K \;:\; q_0(s)r(s) \geq 0\} = \{s \in K \;:\; r(s) \geq 0\}.$$

b) If $L_\gamma(pr) = 0$ for all $p \in \Delta_{\mathcal{P}}$, then the measure $r\mu = 0$, whence, again by a measure theoretic argument, we deduce that $\operatorname{supp}(\mu) \subset \{s \in K \;:\; r(s) = 0\}$. □

Remark I.2.14. With the notation of Theorem I.2.13, assuming the existence of a polynomial $r \in P_{\mathbb{C}}(\mathbb{R}^n)$ such that $L_\gamma(rp) \geq 0$ for all $p \in \tilde{\Delta}_{\mathcal{P}}$, writing $r = r' + \mathrm{i}r''$, with $r', r'' \in P(\mathbb{R}^n)$, we obtain $L_\gamma(r'p) \geq 0$ and $L_\gamma(r''p) = 0$ for all $p \in \Delta_{\mathcal{P}}$. Consequently,

$$\operatorname{supp}(\mu) \subset \{s \in K \;:\; r'(s) \geq 0, r''(t) = 0\},$$

via Theorem I.2.13.

Example I.2.15. (1) Let $K = [0,1]^n \subset \mathbb{R}^n$. If $p_j(t) = t_j, p_{n+j}(t) = 1 - t_j, j = 1,\ldots,n$, and $\mathcal{P} = \{1, p_1, \ldots, p_n\}$, then $\hat{\mathcal{P}} = \{0, 1, p_1, \ldots, p_n\}$ and $K = K_{\mathcal{P}}$. It is easily seen that the condition $L_\gamma(r) \geq 0\,(r \in \Delta_{\mathcal{P}})$ from our Theorem I.2.2 is equivalent to the condition (13) from Theorem 1 in [HiSc] (stated in n dimensions). Therefore, Theorem I.2.2 provides, in particular, a new proof for the existence of a solution to the Hausdorff moment problem in several variables. In other words, $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence, with $K = [0,1]^n \subset \mathbb{R}^n$, if and only if

$$L_\gamma(t_1^{k_1} \cdots t_n^{k_n}(1 - t_1)^{\ell_1} \cdots (1 - t_n)^{\ell_n}) \geq 0,$$

for all integers $k_1 \geq 0, \ldots, k_n \geq 0, \ell_1 \geq 0, \ldots, \ell_n \geq 0$. This condition is equivalent to

$$\sum_{0\leq\xi\leq\beta} (-1)^{|\xi|} \binom{\beta}{\xi} \gamma_{\alpha+\xi} \geq 0,\;\; \alpha, \beta \in \mathbb{Z}_+^n.$$

Conversely, using the above mentioned result from [HiSc], one can give a different proof of Theorem I.2.2 (see [PuVa1] for details).

(2) Let $K = \{t \in \mathbb{R}^n \;:\; t_1^2 + \cdots + t_n^2 \leq 1\}$, i.e., the unit ball in $\mathbb{R}^n$. Let $p_j(t) = (1 + t_j)/2, p_{n+j}(t) = (1 - t_j)/2,\, j = 1,\ldots,n,\, p_{2n+1}(t) = 1 - t_1^2 - \cdots - t_n^2$,

and $p_{2n+2}(t) = t_1^2 + \cdots + t_n^2$. If $\mathcal{P} = \{1, p_1, \ldots, p_n, p_{2n+1}\}$, then $K = K_{\mathcal{P}}$, and $\hat{\mathcal{P}} = \{0, 1, p_1, \ldots, p_{2n+1}\}$.

By virtue of Theorem I.2.2, $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence, with $K = \{t \in \mathbb{R}^n; t_1^2 + \cdots + t_n^2 \leq 1\}$, if and only if

$$L_\gamma(p_1^{k_1} \cdots p_{2n+2}^{k_{2n+2}}) \geq 0,$$

for all integers $k_1 \geq 0, \ldots, k_{2n+2} \geq 0$.

Using Theorem I.2.13, we obtain that $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a K-moment sequence, with $K = \{t \in \mathbb{R}^n \ : \ t_1^2 + \cdots + t_n^2 = 1\}$, if and only if

$$L_\gamma(p_1^{k_1} \cdots p_{2n+2}^{k_{2n+2}}) \geq 0,$$

for all integers $k_1 \geq 0, \ldots, k_{2n} \geq 0, k_{2n+1} = 0, k_{2n+2} \geq 0$, and

$$L\gamma(p_1^{k_1} \cdots p_{2n+2}^{k_{2n+2}}) = 0,$$

for all integers $k_1 \geq 0, \ldots, k_{2n} \geq 0, k_{2n+1} > 0, k_{2n+2} \geq 0$.

Remark I.2.16. The methods developed in this section also allow us to approach some so-called "complex moment problems", that is, moment problems in the complex Euclidean space $\mathbb{C}^n$. Identifying the space $\mathbb{C}^n$ with the space $\mathbb{R}^{2n}$, we may derive easily some useful assertions, stated only in terms of complex variables. We shall briefly present the necessary changes for this transfer of information.

If $z = (z_1, \ldots, z_n)$ is the variable in the complex Euclidean space $\mathbb{C}^n$, we denote by $P(\mathbb{C}^n)$ the algebra of all complex polynomial functions in $\bar{z}_1, \ldots, \bar{z}_n, z_1, \ldots, z_n$. The algebra $P(\mathbb{C}^n)$ can be identified with the algebra $P_{\mathbb{C}}(\mathbb{R}^{2n})$.

Let $\gamma = (\gamma_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+^n}$ be an $2n$-sequence of complex numbers. We set

$$L_\gamma(\bar{z}^\alpha z^\beta) = \gamma_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{Z}_+^n, \tag{I.2.1}$$

and extend L_γ to $P(\mathbb{C}^n)$ by linearity.

The concepts of *K-moment sequence* ($K \subset \mathbb{C}^n$ a compact subset) and of *positive semi-definiteness* are defined as for real sequences.

Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{C}^n)$ such that p_j is a real valued function on $\mathbb{C}^n$ for all $j = 1, \ldots, n$. We set $K_{\mathcal{P}} := \{w \in \mathbb{R}^n \ : \ p_j(w) \geq 0, j = 1, \ldots, m\}$. Suppose that $K = K_{\mathcal{P}}$ is compact. As in Remarks I.2.1 and I.2.4, we attach to the family $\mathcal{P}$ a family $\hat{\mathcal{P}}$ constructed in the following way.

Because we have have $m_j = \sup_{w\in K} p_j(w) < \infty$, we set $\hat{p}_j(z) = m_j^{-1} p_j(z), z \in \mathbb{C}^n$, if $m_j > 0$, and $\hat{p}_j = p_j$ if $m_j = 0, j = 1, \ldots, m$.

Now, let $a_k = \inf\{\Re z_k \ : \ t \in K\}$, $b_k = \sup\{\Re z_k \ : \ t \in K\}$, $c_k = \inf\{\Im z_k \ : \ t \in K\}$, $d_k = \sup\{\Im z_k \ : \ t \in K\}$. We put $\hat{p}_{m+k}(\bar{z}, z) = (\Re z_k - a_k)/(b_k - a_k)$ if $b_k > a_k$, and $\hat{p}_{m+k}(\bar{z}, z) = (\Re z_k - a_k)$ if $b_k = a_k$, $k = 1, \ldots, n$. Similarly $\hat{p}_{m+n+k}(\bar{z}, z) =$

$(\Im z_k - c_k)/(d_k - c_k)$ if $d_k > c_k$, and $\hat{p}_{m+n+k}(\bar{z}, z) = (\Im z_k - c_k)$ if $d_k = c_k$, $k = 1, \dots, n$. Define

$$\hat{\mathcal{P}} = \{0, 1, \hat{p}_1, \dots, \hat{p}_m\},$$

$$\tilde{\mathcal{P}} = \{0, 1, \hat{p}_1, \dots, \hat{p}_m, \hat{p}_{m+j}, \hat{p}_{m+n+k} \;:\; a_j < b_j, c_k < d_k, j, k = 1, \dots, n\}$$

and

$$\tilde{\mathcal{P}}_0 = \{1, \pm\hat{p}_{m+j}, \pm\hat{p}_{m+n+k} \;:\; a_j = b_j, c_k = d_k, j, k = 1, \dots, n\}.$$

We have $K = K_{\mathcal{P}} = K_{\hat{\mathcal{P}}}$ and $0 \le \hat{p}(\bar{w}, w) \le 1$ for all $w \in K$ and $\hat{p} \in \hat{\mathcal{P}}$.

As in Remark I.2.1, we denote by $\Delta_{\mathcal{P}}$ the set of all products of the form

$$q_1 \cdots q_k (1 - r_1) \cdots (1 - r_l)$$

for polynomials $q_1, \dots, q_k, r_1, \dots, r_l \in \hat{\mathcal{P}}$ and integers $k, l \ge 1$. We also denote by $\tilde{\Delta}_{\mathcal{P}}$ the set of all products of the form

$$q_1 \cdots q_k (1 - r_1) \cdots (1 - r_l) h_1 \cdots h_u$$

for polynomials $q_1, \dots, q_k, r_1, \dots, r_l \in \tilde{\mathcal{P}}$, $h_1, \dots, h_u \in \tilde{\mathcal{P}}_0$ and integers $k, l, u \ge 1$.

Theorems I.2.2 and I.2.13 lead to the following assertion (not explicitly mentioned in [Vas2]; see also [Vas7]).

Theorem I.2.17. *Let $\mathcal{P} = \{p_1, \dots, p_m\}$ be a finite family in $P(\mathbb{C}^n)$ such that p_j is a real valued function on $\mathbb{C}^n$ for all $j = 1, \dots, n$. Suppose that $K = K_{\mathcal{P}}$ is compact.*

(i) *A $2n$-sequence of complex numbers $\gamma = (\gamma_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}_+^n}$ with $\bar{\gamma}_{\alpha,\beta} = \gamma_{\beta,\alpha}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$ and $\gamma_{0,0} > 0$ is a K-moment sequence if and only if the linear form L_γ is nonnegative on the set $\tilde{\Delta}_{\mathcal{P}}$.*

   When the family $\{1, p_1, \dots, p_m\}$ generates the algebra $P(\mathbb{C}^n)$, the set $\tilde{\Delta}_{\mathcal{P}}$ may be replaced by $\Delta_{\mathcal{P}}$.

(ii) *Assume that $\gamma = (\gamma_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}_+^n}$ $(\gamma_{0,0} > 0)$ is a K-moment sequence, and let μ be the representing measure of γ. Also assume that there exists a real valued polynomial $r \in P(\mathbb{C}^n)$ such that $L_\gamma(rp) \ge 0$ for all $p \in \Delta_{\mathcal{P}}$. Then*

$$\operatorname{supp}(\mu) \subset \{w \in K \;:\; r(\bar{w}, w) \ge 0\}.$$

   If $L_\gamma(rp) = 0$ for some $r \in P(\mathbb{C}^n)$ and for all $p \in \Delta_{\mathcal{P}}$, then

$$\operatorname{supp}(\mu) \subset \{w \in K \;:\; r(\bar{w}, w) = 0\}.$$

Proof. (i) Writing $z_j = t_j + \mathrm{i}s_j$, i.e., $t_j = \Re z_j$, $s_j = \Im z_j$, $j = 1, \dots, n$, we have a natural map, say θ, given by

$$\mathbb{C}^n \ni z = (z_1, \dots, z_n) \to (t_1, \dots, t_n, s_1, \dots, s_n) = (t, s) \in \mathbb{R}^n \times \mathbb{R}^n,$$

which allows the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

The hypothesis on γ implies the equality $L_\gamma(\bar{p}) = \overline{L_\gamma(p)}$ for all $p \in P(\mathbb{C}^n)$. Therefore, $L_\gamma(p)$ is a real number whenever $\bar{p} = p$. In particular, setting $\tilde{\gamma}_{\alpha,\beta} :=$

$L_\gamma(t^\alpha s^\beta)$, $\alpha,\beta \in \mathbb{Z}_+^n$, and $\tilde\gamma = (\tilde\gamma_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+^n}$, we get a $2n$-sequence of real numbers with $\tilde\gamma_{0,0} > 0$. We have in fact $L_{\tilde\gamma}(q) = L_\gamma(q\circ\theta^{-1})$ for all $q \in P_{\mathbb{C}}(\mathbb{R}^{2n})$.

If $\tilde{\mathcal{P}} := \{p\circ\theta^{-1} \ : \ p \in \mathcal{P}\}$, then $\Delta_{\tilde{\mathcal{P}}} = \{r\circ\theta^{-1} \ : \ r \in \Delta_{\mathcal{P}}\}$ and $\tilde K := \theta(K) = (\theta(K))_{\tilde{\mathcal{P}}}$. Therefore, $\tilde\gamma$ is a $\tilde K$-moment sequence if and only if γ is a K-moment sequence. Moreover, $\tilde\mu$ is a representing measure for $\tilde\gamma$ if and only if $\mu := \tilde\mu\circ\theta^{-1}$ is a representing measure for γ. Consequently, the assertion follows directly from Theorem I.2.2.

(ii) Using the discussion from (i), we infer easily the assertion, as a consequence of Theorems I.2.2 and I.2.13. We only note that, without loss of generality, the polynomial $r \in P(\mathbb{C}^n)$ may be assumed to be a real-valued function. $\square$

Example I.2.18. Let $\mathbb{D}^n = \{z \in \mathbb{C}^n \ : \ |z_j| \le 1, j = 1,\dots,n\}$, i.e. the unit polydisc in $\mathbb{C}^n$. We consider the following polynomials:

$$\begin{array}{ll} p_{1,j}(\bar z, z) = (1+\Re z_j)/2, & p_{2,j}(\bar z, z) = (1+\Im z_j)/2, \\ p_{3,j}(\bar z, z) = (1-\Re z_j)/2, & p_{4,j}(\bar z, z) = (1-\Im z_j)/2, \\ p_{5,j}(\bar z, z) = 1-|z_j|^2, & p_{6,j}(\bar z, z) = |z_j|^2, \end{array}$$

for all $j = 1,\dots,n$. Note that, if $\mathcal{P} = \{1, p_{1,j}, p_{2,j}, p_{5,j}, \ : \ j = 1,\dots,n\}$, then $\hat{\mathcal{P}} = \{0, 1, p_{1,j}, p_{2,j}, p_{5,j} \ : \ j = 1,\dots,n\}$, and $\mathbb{D}^n = K_{\mathcal{P}}$.

Let $\gamma = (\gamma_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+^n}$ be a $2n$-sequence of complex numbers with $\bar\gamma_{\alpha,\beta} = \gamma_{\beta,\alpha}$ for all $\alpha,\beta \in \mathbb{Z}_+^n$ and $\gamma_{0,0} > 0$. By Theorem I.2.17, γ is a K-moment sequence, with $K = \mathbb{D}^n$ if and only if

$$L_\gamma(p_{1,1}^{k_{1,1}} p_{1,2}^{k_{1,2}} \cdots p_{1,n}^{k_{1,n}} p_{2,1}^{k_{2,1}} \cdots p_{6,n}^{k_{6,n}}) \ge 0$$

for all integers $k_{1,1} \ge 0, k_{1,2} \ge 0, \dots, k_{6,n} \ge 0$. As we have the obvious identity $2 + z_j + \overline{z}_j = |z_j+1|^2 + 1 - |z_j|^2$, it follows that each expression $p_{1,1}^{k_{1,1}} p_{1,2}^{k_{1,2}} \cdots p_{6,n}^{k_{6,n}}$ can be written as a linear combination with positive coefficients of expressions of the form $(1-|z_1|^2)^{\eta_1}\cdots(1-|z_n|^2)^{\eta_n}|p(z)|^2$, where $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}_+^n$ and p is an analytic polynomial. Consequently, if

$$L_\gamma((1-|z_1|^2)^{\eta_1}\cdots(1-|z_n|^2)^{\eta_n})|p(z)|^2) \ge 0 \tag{I.2.2}$$

for all $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{Z}_+^n$ and all $p \in P_{\mathrm{a}}(\mathbb{C}^n)$, then γ is a K-moment sequence, where $P_{\mathrm{a}}(\mathbb{C}^n)$ is the algebra of analytic polynomials on $\mathbb{C}^n$. Condition (I.2.2) is equivalent to

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n}\sum_{\xi\le\eta}(-1)^{|\xi|}\binom{\eta}{\xi} c_\alpha \overline{c}_\beta \gamma_{\alpha+\xi,\beta+\xi} \ge 0 \tag{I.2.3}$$

for all $\eta \in \mathbb{Z}_+^n$ and all sequences of complex numbers $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ with only finitely many nonzero terms.

Condition (I.2.2) is also necessary.

For $n = 1$, a different characterization is given in [Atz], where Hilbert space methods are used.

Example I.2.19. Let $\mathbb{T}^n = \{z \in \mathbb{C}^n \ : \ |z_j| = 1, j = 1, \ldots, n\}$, i.e., the torus in $\mathbb{C}^n$. It follows from Theorem I.2.17 (ii), as well as from the previous example, that γ is a K-moment sequence, with $K = \mathbb{T}^n$ if and only if

$$L_\gamma(|p(z)|^2) \geq 0 \tag{I.2.4}$$

and

$$L_\gamma((1 - |z_1|^2)^{\eta_1} \cdots (1 - |z_n|^2)^{\eta_n})|p(z)|^2) = 0, \tag{I.2.5}$$

for all $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}_+^n$, $\eta \neq 0$, and all $p \in P_{\mathrm{a}}(\mathbb{C}^n)$. Condition (I.2.5) can be considerably simplified. Namely, it can be replaced by the condition

$$L_\gamma((1 - |z_1|^2)^{\eta_1} \cdots (1 - |z_n|^2)^{\eta_n})|p(z)|^2) = 0, \tag{I.2.6}$$

for all $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{Z}_+^n$, $|\eta| = 1$, and all $p \in P_{\mathrm{a}}(\mathbb{C}^n)$. Indeed, from (I.2.6) we infer easily that $L_\gamma(|z_1|^{2k_1} \cdots |z_n|^{2k_n}|p(z)|^2) = L_\gamma(|p(z)|^2)$ for all integers $k_1 \geq 0, \ldots, k_n \geq 0$, and this last equality leads us easily to (I.2.5).

Conditions (I.2.4) and (I.2.6) can be used to recapture the classical solutions to trigonometric moment problems in one or several variables. For instance, if $c = (c_\alpha)_{\alpha \in \mathbb{Z}^n}$ is an n-sequence of complex numbers with $c_0 > 0$ and $c_{-\alpha} = \bar{c}_\alpha$ for all $\alpha \in \mathbb{Z}^n$, we define the $2n$-sequence $\gamma = (\gamma_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}_+^n}$ by $\gamma_{\alpha,\beta} = c_{\alpha-\beta}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$. The $2n$-sequence γ has a representing measure concentrated in $\mathbb{T}^n$ (which clearly implies the existence of a representing measure for c on $\mathbb{T}^n$) if (i) $L_\gamma(|p(z)|^2) \geq 0$ and (ii) $L_\gamma((1-|z_j|^2)|p(z)|^2) = 0$ for all $j = 1, \ldots, n$ and $p \in P_{\mathrm{a}}(\mathbb{C})$. Condition (i) is equivalent to the classical Carathéodory-Féjer condition

$$\sum_{\alpha,\beta} c_{\alpha-\beta} \lambda_\alpha \bar{\lambda}_\beta \geq 0$$

for each finite family $(\lambda_\alpha)_\alpha$ of complex numbers, while (ii) follows from the fact that $\gamma_{\alpha+e_j,\beta+e_j} = \gamma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$, with $e_j = (\delta_{1j}, \ldots, \delta_{nj})$ and δ_{kj} the Kronecker symbol.

I.3. Decomposition of positive polynomials

In this section we give an application of Theorem I.2.2, which describes the structure of those polynomials that are positive on a semi-algebraic compact set.

Let $K = K_{\mathcal{P}}$ be a semi-algebraic compact set. As in the previous section, we denote by $\Gamma_+(K)$ the positive cone generated by $\Delta_{\mathcal{P}}$ in $P(\mathbb{R}^n)$. Then we have the following decomposition theorem (a version of which is stated in [Vas2], without proof, as Theorem 2.10):

Theorem I.3.1. *Let $\mathcal{P} = \{p_0 = 1, p_1, \ldots, p_m\}$ be a finite family in $P(\mathbb{R}^n)$. Suppose that $K = K_{\mathcal{P}}$ is compact and that the family $\mathcal{P}$ generates the algebra $P(\mathbb{R}^n)$.*

If $p \in P(\mathbb{R}^n)$ is strictly positive on K, then $p \in \Gamma_+(K)$.

The proof of Theorem I.3.1 is based on Lemma I.3.2 below, which is a version of Lemma 4 from [AmVa].

The linear space $\mathcal{A} = P(\mathbb{R}^n)$ will be endowed with the finest locally convex topology. A basis of the topology of the space $\mathcal{A}$ consists of all convex, absorbent and symmetric subsets of $\mathcal{A}$.

Lemma I.3.2. *With the conditions of Theorem I.3.1, the constant polynomial* $1 \in \mathcal{A}$ *belongs to the interior of* $\Gamma_+(K)$.

Proof. Let $\mathcal{A}' \subset \mathcal{A}$ consist of all $p \in \mathcal{A}$ for which there is an $\epsilon = \epsilon_p > 0$ with $1 + \lambda p \in \Gamma_+(K)$ for any $\lambda \in (-\epsilon, \epsilon)$. The set $\mathcal{A}'$ is a linear space. Indeed, $1 + \lambda(p + q) = (1/2)(1 + 2\lambda p + 1 + 2\lambda q) \in \mathcal{A}_+$, if $|\lambda| < \min\{\epsilon_p/2, \epsilon_q/2\}$, $p, q \in \mathcal{A}'$. Similarly, $1 + \lambda cp \in \mathcal{A}_+$ for all $c \in \mathbb{R}$ if $|\lambda|$ is sufficiently small.

Note that $\Delta_{\mathcal{P}}$ generates the linear space $\mathcal{A}$, as a direct consequence of the hypothesis. In particular, for every fixed multi-index $\alpha \in \mathbb{Z}_+^n$ there exists a finite family $(g_j)_{j\in J}$ in $\Delta_{\mathcal{P}} \subset \Gamma_+(K)$ such that

$$t^\alpha = \sum_{j\in J} c_j g_j = \sum_{j\in J_+} c_j g_j + \sum_{j\in J_-} c_j g_j,$$

where $\pm c_j > 0$ for $j \in J_\pm$. Set $\epsilon = \Big(\sum_{j\in J} |c_j|\Big)^{-1}$. If $0 \le \lambda \le \epsilon$, then

$$1 + \lambda t^\alpha = \sum_{j\in J_+} c_j g_j + 1 + \sum_{j\in J_-} \lambda c_j + \sum_{j\in J_-} (-\lambda c_j)(1 - g_j),$$

showing that $1 + \lambda t^\alpha \in \Gamma_+(K)$, since $1 - g_j \in \Gamma_+(K)$ for all $j \in J$, by Remark I.2.9(1). Similarly, $1 + \lambda t^\alpha \in \Gamma_+(K)$ if $-\epsilon \le \lambda \le 0$. Hence any monomial $t^\alpha \in \mathcal{A}'$. Consequently, $\mathcal{A}' = \mathcal{A}$.

Set $\mathcal{A}_+ = \Gamma_+(K)$ and $U = (\mathcal{A}_+ - 1) \cap (1 - \mathcal{A}_+)$, which is a convex set containing zero. Let $f \in \mathcal{A}$. Then $1 + \lambda f \in \mathcal{A}_+$ for $|\lambda| < \epsilon$. Therefore, $\lambda f \in \mathcal{A}_+ - 1, -\lambda f \in 1 - \mathcal{A}_+$, and so $\lambda f \in U$ for all $|\lambda| < \epsilon$. In other words, U is absorbent. Since U is clearly symmetric, it follows that U is a neighbourhood of the origin. Hence $V = U + 1 \subset \mathcal{A}_+$ is a neighbourhood of 1 in $\mathcal{A}$. $\square$

Corollary I.3.3. *If* $p \notin \Gamma_+(K)$, *there exists a linear functional* $L : \mathcal{A} \to \mathbb{R}$ *such that* $L(p) \le 0$ *and* $L|\Gamma_+(K) \ge 0$.

Proof. As the interior of $\mathcal{A}_+ = \Gamma_+(K)$ is nonempty, Mazur's theorem implies the existence of a linear functional $L\colon \mathcal{S} \to \mathbb{R}$ such that $L(p) \le \inf_{x\in\mathcal{A}_+} L(x)$. Since $\mathcal{A}_+$ is a cone, we cannot have $\inf_{x\in\mathcal{A}_+} L(x) < 0$. Therefore $L|\Gamma_+(K) \ge 0$. Moreover, $\inf_{x\in\mathcal{A}_+} L(x) \le \inf_{\epsilon>0} L(\epsilon 1) = 0$, whence $L(p) \le 0$. $\square$

Proof of Theorem I.3.1. Let $p \in P(\mathbb{R}^n)$ be such that $p(t) > 0$ for all $t \in K$. Assuming $p \notin \Gamma_+(K)$, from Corollary I.3.3 we derive the existence of a linear functional $L\colon \mathcal{S} \to \mathbb{R}$ such that $L(p) \leq 0$ and $L|\Gamma_+(K) \geq 0$. As $\Delta_{\mathcal{P}} \subset \Gamma_+(K)$, Theorem I.2.2 implies the existence of a positive measure μ on K such that $L(q) = \int_K q\, d\mu$ for all $q \in P(\mathbb{R}^n)$. Therefore, $0 < \int_K p\, d\mu = L(p) \leq 0$, which is a contradiction, and completes the proof of Theorem I.3.1. □

Remark I.3.4. A decomposition theorem via positive linear functions on compact convex polyhedra was given in [Han]. Theorem I.3.1 above seems to give an answer to a question from [Han]. A particular case of Theorem I.3.1 is stated in [Kri2]. See also [Cas], [Put], [Sch1] etc. for related results.

I.4. Moments and subnormality

That there exists a strong connection between the moment problem and subnormality has been known for a longtime (see [SzN1], [Emb], [Atz], [Agl2], [Ath], [AtPe], to quote only a few).

In this section we apply the results from the second section to refine some results from [AtPe].

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$.

If $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ is a commuting multioperator (briefly, a *c.m.*), then for every $p \in P(\mathbb{C}^n)$, $p(\overline{z}, z) = \sum_{\alpha,\beta} c_{\alpha,\beta}\overline{z}^\alpha z^\beta$, we set

$$p(T^*, T) = \sum_{\alpha,\beta} c_{\alpha,\beta} T^{*\alpha} T^\beta \tag{I.4.1}$$

(with $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$; this is part of the so-called "hereditary functional calculus", considered in [Agl1] and [Ath]).

Formula (I.4.1) can be expressed in a slightly different manner, at least for polynomial functions from $P_{\mathbb{C}}(\mathbb{R}^n) = P(\mathbb{R}^n) \otimes \mathbb{C}$.

Let $M_{T_j}\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be the operator $X \to T_j^* X T_j$, $X \in \mathcal{L}(\mathcal{H})$, $j = 1, \ldots, n$, and let $M_T = (M_{T_1}, \ldots, M_{T_n})$, which is a c.m. on $\mathcal{L}(\mathcal{H})$.

For every $p \in P_{\mathbb{C}}(\mathbb{R}^n)$, $p(t) = \sum_\alpha a_\alpha t^\alpha$, we define

$$p(M_T) = \sum_\alpha a_\alpha M_T^\alpha, \tag{I.4.2}$$

which is, in fact, a unital algebra homomorphism. Note also that

$$\hat{p}(T^*, T) = p(M_T)(1), \quad p \in P_{\mathbb{C}}(\mathbb{R}^n), \tag{I.4.3}$$

where $\hat{p}(\overline{z}, z) = \sum_\alpha a_\alpha \overline{z}^\alpha z^\alpha$, and 1 is the identity on $\mathcal{H}$.

We recall that a c.m. $T \in \mathcal{L}(\mathcal{H})^n$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a c.m. $N \in \mathcal{L}(\mathcal{K})^n$ consisting of normal operators (which is called a *normal extension* of T) such that $T_j = N_j|\mathcal{H}$, $j = 1, \dots, n$. Among all normal extensions of a subnormal c.m. T there exists a minimal one, which is unique up to unitary equivalence. In that case one also have $\|T_j\| = \|N_j\|$, $j = 1, \dots, n$ (see [Ito] for details).

Let $K = K_{\mathcal{P}}$ be a semi-algebraic compact subset of $\mathbb{R}^n$. Let also τ be the mapping

(I.4.4) $$\mathbb{C}^n \ni z = (z_1, \dots, z_n) \to (|z_1|^2, \dots, |z_n|^2) \in \mathbb{R}^n.$$

Note that the set $\tau^{-1}(K) \subset \mathbb{C}^n$ is also compact.

With this notation, we have the following (see Theorem 3.1 from [Vas2]):

Theorem I.4.1. *The commuting multioperator $T \in \mathcal{L}(\mathcal{H})^n$ has a normal extension $N \in \mathcal{L}(\mathcal{K})^n$ $(\mathcal{K} \supset \mathcal{H})$, whose joint spectrum lies in $\tau^{-1}(K)$, if and only if $(p \circ \tau)(T^*, T) \geq 0$ for all $p \in \tilde{\Delta}_{\mathcal{P}}$.*

Proof. If N is a normal extension of T, and if E is the spectral measure of N whose support lies in $L = \tau^{-1}(K)$, then for all $p \in \Delta_{\mathcal{P}}$, $p(t) = \sum_\alpha a_\alpha t^\alpha$, and $x \in \mathcal{H}$ we have:

$$\begin{aligned}
\langle (p \circ \tau)(T^*, T)x, x\rangle &= \sum_\alpha a_\alpha \|T^\alpha x\|^2 \\
&= \sum_\alpha a_\alpha \|N^\alpha x\|^2 \\
&= \int_L \sum_\alpha a_\alpha \overline{z}^\alpha z^\alpha \, d\langle E(z)x, x\rangle \\
&= \int_K p(t)\, d\mu(t) \geq 0,
\end{aligned}$$

since $p|K \geq 0$, where $\mu(A) = \langle E(\tau^{-1}(A))x, x\rangle$ for all Borel sets $A \subset \mathbb{R}^n$.

Conversely, we proceed as in [Ath], Theorem 4.1. The major change is the use of Theorem I.2.6 instead of the corresponding result from [HiSc].

Let $x \in \mathcal{H}$, $x \neq 0$, and let

$$\gamma_\alpha = \langle T^{*\alpha} T^\alpha x, x\rangle, \quad \alpha \in \mathbb{Z}_+^n.$$

Then

$$L_\gamma(p) = \langle (p \circ \tau)(T^*, T)x, x\rangle, \quad p \in P(K).$$

In particular, $L_\gamma(p) \geq 0$ if $p \in \Delta_{\mathcal{P}}$, and $\gamma_0 = \|x\|^2 > 0$. According to Theorem I.2.6, there exists a positive Borel measure μ_x on K such that

$$\langle T^{*\alpha} T^\alpha x, x\rangle = \int_K t^\alpha \, d\mu_x(t).$$

The support of μ_x is, in fact, contained in $K \cap \mathbb{R}^n_+$ for all $x \in \mathcal{H}$, since

$$L_\gamma(t_j p) = \langle (p \circ \tau)(T^*, T)T_j x, T_j x\rangle \geq 0, \quad j = 1, \dots, n,$$

implying the desired inclusion, via Theorem I.2.13.

A standard polarization argument, the uniqueness of the representing measure and an obvious change of variable implies the existence of a positive operator-valued measure F_T on $K \cap \mathbb{R}^n_+$ such that

$$T^{*\alpha}T^\alpha = \int\limits_{K\cap\mathbb{R}^n_+} t^{2\alpha} dF_T(t), \quad \alpha \in \mathbb{Z}^n_+. \tag{I.4.5}$$

Then the assertion follows from [Lub], Theorem 3.2. □

Remark I.4.2. As noticed in [AtPe] (referring to the proof of [Lub], Theorem 3.2), if E is the spectral measure of the minimal normal extension $N \in \mathcal{L}(\mathcal{K})^n$ of the subnormal c.m. $T \in \mathcal{L}(\mathcal{H})^n$, then one has the equality

$$F_T(A) = P_{\mathcal{H}} E(\tau^{-1}(A))|\mathcal{H} \tag{I.4.6}$$

for all Borel subsets $A \subset K$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, and F_T is as in (I.4.5).

Moreover, if $E'(A) = E(\tau^{-1}(A))$, $A \subset K$ a Borel subset, then the measures E' and F_T have the same support.

The uniquely determined positive operator-valued measure that satisfies (I.4.5) is called the *representing measure* of the subnormal c.m. $T \in \mathcal{L}(\mathcal{H})^n$.

The next result is a enlarged version of Theorem 3.4 from [Vas2].

Theorem I.4.3. *Let $\Gamma = (\Gamma_\alpha)_{\alpha\in\mathbb{Z}^n_+}$ be a sequence of bounded self-adjoint operators on $\mathcal{H}$, with $\Gamma_0 = 1$. Let also $L_\Gamma : P(\mathbb{R}^n) \to \mathcal{L}(\mathcal{H})$ be the mapping*

$$L_\Gamma(p) = \sum_\alpha c_\alpha \Gamma_\alpha \quad \text{if } p(t) = \sum_\alpha c_\alpha t^\alpha.$$

Assume that $K = K_{\mathcal{P}}$ is a semi-algebraic compact subset of $\mathbb{R}^n$. Then there exists a uniquely determined positive operator-valued measure F_Γ on K such that $L_\Gamma(p) = \int_K p dF_\Gamma$ for all $p \in P(K)$ if and only if $L_\Gamma(p) \geq 0$ for all $p \in \tilde{\Delta}_{\mathcal{P}}$.

In the affirmative case, assume, moreover, that there exists an $r \in P(\mathbb{R}^n)$ such that $L_\Gamma(rp) \geq 0$ for all $p \in \tilde{\Delta}_{\mathcal{P}}$. Then

$$\operatorname{supp}(F_\Gamma) \subset \{s \in K \,:\, r(s) \geq 0\}.$$

If $L_\Gamma(rp) = 0$ for some $r \in P(\mathbb{R}^n)$ and for all $p \in \tilde{\Delta}_{\mathcal{P}}$, then

$$\operatorname{supp}(F_\Gamma) \subset \{s \in K \,:\, r(s) = 0\}.$$

Proof. The construction of the measure F_Γ is performed as in the first part of the proof of Theorem I.3.1, using Theorem I.2.6. The uniqueness of the measure F_Γ follows from the uniqueness of each measure $\mu_x = \langle F_\Gamma(*)x, x\rangle$, $x \in \mathcal{H}$.

The remaining assertions are obtained via Theorem I.2.13. □

Remark I.4.4. Theorem I.4.3 contains, as a particular case, the following classical result of [SzN1]:

> A sequence $(\Gamma_k)_{k\in\mathbb{Z}_+}$ in $\mathcal{L}(H)$ can be represented under the form $\Gamma_k = \int_0^1 t^k\, dF(t)$, $k \geq 0$, for a certain operator-valued positive measure on $[0,1]$ if and only if we have $\sum\limits_{j=0}^{m}(-1)^j\binom{m}{j}\Gamma_{j+k} \geq 0$ for all integers $m, k \geq 0$.

See also [MaN] for further connections.

A consequence of Theorem I.4.3 is the following fact (see also Theorem 3.4 from [Vas2]).

Corollary I.4.5. *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a subnormal c.m., and assume that the support of the representing measure F_T de T is contained in the semi-algebraic compact set $K = K_\mathcal{P}$.*

If there exists an $r \in P(\mathbb{R}^n)$ such that $((pr)\circ\tau)(T^*, T) \geq 0$ for all $p \in \tilde{\Delta}_\mathcal{P}$, then

$$\mathrm{supp}(F_T) \subset \{s \in K \;:\; r(s) \geq 0\}.$$

If there exists $r \in P_\mathbb{C}(K)$ such that $(r\circ\tau)(T^*, T) = 0$, then one also has $(r\circ\tau)(N^*, N) = 0$, where N is the minimal normal extension of T.

Proof. The inclusion $\mathrm{supp}\,(F_T) \subset \{t \in K \;:\; r(t) \geq 0\}$ follows from Theorem I.4.3.

Assume now that $(r\circ\tau)(T^*, T) = 0$. Then for every $p \in \tilde{\Delta}_\mathcal{P}$ we have

$$((pr)\circ\tau)(T^*, T) = (pr)(M_T)(1) = p(M_T)r(M_T)(1) = 0,$$

because of the relation $(r\circ\tau)(T^*, T) = r(M_T)(1) = 0$. From Theorem I.4.3 (see also Remark I.2.14), we deduce that $\mathrm{supp}\,(F_T) \subset \{s \;:\; r(s) = 0\}$. According to Remark I.4.2, $\mathrm{supp}\,(E') = \mathrm{supp}\,(F_T)$. This shows that $(r\circ\tau)(\overline{z}, z) = 0$ for all $z \in \mathrm{supp}\,(E)$, and so $(r\circ\tau)(N^*, N) = 0$. □

Most of the assertions from [AtPe] (Propositions 4–8 ; see also [McG] and [Lem2]) are now consequences of our Theorem I.4.3.

Example I.4.6. Let us consider (as in [Vas2]) a family of polynomials $\{p_1, \ldots, p_m\}$, where

$$p_j(t) = 1 - \sum_{k=1}^{n} c_{jk}t_k, \quad j = 1, \ldots, m,$$

with the following properties:

(i) $c_{jk} \geq 0$ for all indices j, k;

(ii) for every $k \in \{1, \dots, n\}$ there exists a $j \in \{1, \dots, m\}$ such that $c_{jk} \neq 0$;
(iii) $p_j \neq 1$ for all $j \in \{1, \dots, n\}$.

Let $\mathcal{P} = \{1, p_1, \dots, p_m, p_{m+1}, \dots, p_{m+n}\}$, where $p_{m+k}(t) = t_k, k = 1, \dots, n$. It is easily seen that $K = K_{\mathcal{P}}$ is compact. From Theorem I.2.2 we derive the following assertion (see Proposition 3.6 from [Vas2]):

Proposition I.4.7. *Let $T \in \mathcal{L}(\mathcal{H})^n$ be a c.m. and let $K = K_{\mathcal{P}}$. Then T has a normal extension $N \in \mathcal{L}(\mathcal{K})^n$ ($\mathcal{K} \supset \mathcal{H}$), whose joint spectrum lies in $\tau^{-1}(K)$, if and only if*

$$p^\alpha(M_T)(1) := p_1(M_T)^{\alpha_1} \cdots p_m(M_T)^{\alpha_m}(1) \geq 0$$

for all $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$.

Proof. It is easy to check that every polynomial function from $\Delta_{\mathcal{P}}$ is a linear combination with positive coefficients of expressions of the form $t^\beta p_1(t)^{\alpha_1} \cdots p_m(t)^{\alpha_m}$ for all $\alpha \in \mathbb{Z}_+^m, \beta \in \mathbb{Z}_+^n$.

Since M_T^β is positive on the space $\mathcal{L}(\mathcal{H})$ for all β, the hypothesis implies that $M_T^\beta p(M_T)^\alpha(1) \geq 0$, $\alpha \in \mathbb{Z}_+^m$, $\beta \in \mathbb{Z}_+^n$, which is equivalent to the condition $(r \circ \tau)(T^*, T) \geq 0$ for all $r \in \Delta_{\mathcal{P}}$ via the above remark. Hence the assertion is a consequence of Theorem I.4.1. □

The next result is not explicitly stated in [Vas2].

Theorem I.4.8. *Let $\mathcal{P} = \{p_1, \dots, p_m\}$ be a finite family in $P(\mathbb{C}^n)$ such that p_j is a real valued function on $\mathbb{C}^n$ for all $j = 1, \dots, n$. Suppose that $K = K_{\mathcal{P}}$ is compact.*

Let also $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}_+^n}$ be a sequence of bounded operators acting on $\mathcal{H}$, such that $\Gamma_{\alpha,\beta}^* = \Gamma_{\beta,\alpha}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$, and $\Gamma_{0,0} = 1$. Set

$$L_\Gamma(\overline{z}^\alpha z^\beta) := \Gamma_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{Z}_+^n,$$

and extend L_Γ to $P(\mathbb{C}^n)$ by linearity.

(i) *The $2n$-sequence $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}_+^n}$ can be represented as*

$$\Gamma_{\alpha,\beta} = \int \bar{z}^\alpha z\beta \, dF_\Gamma(z), \quad \alpha, \beta \in \mathbb{Z}_+^n,$$

where F_Γ is an operator-valued positive measure with compact support in $\mathbb{C}^n$, if and only if L_Γ is nonnegative on the set $\Delta_{\mathcal{P}}$.

(ii) *Assume that Γ has a representing measure F_Γ. Also assume that there exists a real valued polynomial $r \in P(\mathbb{C}^n)$ such that $L_\gamma(rp) \geq 0$ for all $p \in \Delta_{\mathcal{P}}$. Then*

$$\operatorname{supp}(\mu) \subset \{w \in K \,:\, r(\bar{w}, w) \geq 0\}.$$

If $L_\gamma(rp) = 0$ for some $r \in P(\mathbb{C}^n)$ and for all $p \in \Delta_{\mathcal{P}}$, then

$$\operatorname{supp}(\mu) \subset \{w \in K \,:\, r(\bar{w}, w) = 0\}.$$

Proof. We proceed as in the proof of Theorem I.4.3, replacing the use of Theorems I.2.6 and I.2.13 by that of Theorem I.2.17. □

The next result is a characterization of completely monotonic (multi-)sequences of operators (i.e., sequences satisfying condition (I.4.7) below) (see Theorem 3.7 from [Vas2]; see also [SzN1] and [MaN]).

Corollary I.4.9. *Let* $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+^n}$ *be a sequence of bounded operators acting on* $\mathcal{H}$*, such that* $\Gamma_{\alpha,\beta}^* = \Gamma_{\beta,\alpha}$ *for all* $\alpha, \beta \in \mathbb{Z}_+^n$*, and* $\Gamma_{0,0} = 1$*. There exists an operator-valued positive measure* F_Γ *on* $\mathbb{D}^n = \{z \in \mathbb{C}^n \,:\, |z_j| \le 1, j = 1, \dots, n\}$ *such that*

$$\Gamma_{\alpha,\beta} = \int\limits_{\mathbb{D}^n} \overline{z}^\alpha z^\beta \, dF_\Gamma(z), \quad \alpha, \beta \in \mathbb{Z}_+^n$$

if and only if

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} \sum_{\xi\le\eta} (-1)^{|\xi|} \binom{\eta}{\xi} c_\alpha \overline{c}_\beta \Gamma_{\alpha+\xi,\beta+\xi} \ge 0 \tag{I.4.7}$$

for all $\eta \in \mathbb{Z}_+^n$ *and all sequences of complex numbers* $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ *with only finitely many nonzero terms.*

Proof. As in Example I.2.18, condition (I.4.7) is equivalent to

$$L_\Gamma((1 - |z_1|^2)^{\eta_1} \cdots (1 - |z_n|^2)^{\eta_n} |p(z)|^2) \ge 0 \tag{I.4.8}$$

for all $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}_+^n$ and all $p \in P_{\mathrm{a}}(\mathbb{C}^n)$. In particular, this shows that (I.4.7) is necessary.

Condition (I.4.7) is also sufficient. An argument from Example I.2.18 shows that (I.4.7) implies the positivity of L_Γ on $\Delta_{\mathcal{P}}$. Then the existence of an operator valued positive measure F_Γ on $\mathbb{D}^n$ with the desired properties follows by Theorem I.4.8. □

Then next two results are seemingly new.

Corollary I.4.10. *Let* $\Theta = (\Theta_\alpha)_{\alpha\in\mathbb{Z}^n}$ *be a sequence of bounded operators acting on* $\mathcal{H}$*, such that* $\Theta_\alpha^* = \Theta_{-\alpha}$ *for all* $\alpha \in \mathbb{Z}^n$*, and* $\Theta_0 = 1$*. There exists an operator-valued positive measure* F_Γ *on* $\mathbb{T}^n$ *such that*

$$\Theta_\alpha = \int\limits_{\mathbb{T}^n} z^\alpha \, dF_\Gamma(z), \quad \alpha \in \mathbb{Z}^n$$

if and only if

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} c_\alpha \overline{c}_\beta \Theta_{\alpha-\beta} \ge 0 \tag{I.4.9}$$

for all sequences of complex numbers $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ *with only finitely many nonzero terms.*

Proof. Let $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}^n_+}$ be given by $\Gamma_{\alpha,\beta} = \Theta_{\alpha-\beta}$. Note that $\Gamma^*_{\alpha,\beta} = \Gamma_{\beta,\alpha}$ for all $\alpha, \beta \in \mathbb{Z}^n_+$, and $\Gamma_{0,0} = 1$. As in Example I.2.19, there exists an operator-valued positive measure F_Γ on $\mathbb{T}^n$ such that

$$\Gamma_{\alpha,\beta} = \int_{\mathbb{T}^n} \overline{z}^\alpha z^\beta \, dF_\Gamma(z), \quad \alpha, \beta \in \mathbb{Z}^n_+$$

if and only if

$$\sum_{\alpha,\beta\in\mathbb{Z}^n_+} c_\alpha \overline{c}_\beta \Gamma_{\alpha,\beta} \geq 0 \tag{I.4.10}$$

and

$$\sum_{\alpha,\beta\in\mathbb{Z}^n_+} \sum_{\xi\leq\eta} (-1)^{|\xi|} \binom{\eta}{\xi} c_\alpha \overline{c}_\beta \Gamma_{\alpha+\xi,\beta+\xi} \geq 0 \tag{I.4.11}$$

for all $\eta \in \mathbb{Z}^n_+$ with $|\eta| = 1$, and all sequences of complex numbers $(c_\alpha)_{\alpha\in\mathbb{Z}^n_+}$ with only finitely many nonzero terms. It is easily seen that condition (I.4.10) is equivalent to condition (I.4.9), while condition (I.4.11) follows from the fact that $\Gamma_{\alpha+e_j,\beta+e_j} = \Gamma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n_+$ and $j = 1, \ldots, n$. □

Corollary I.4.11. *The multioperator $T = (T_1, \ldots, T_n)$ consisting of commuting bounded operators in $\mathcal{H}$ has a unitary dilation if and only if there exists an n-sequence $\Theta = (\Theta_\alpha)_{\alpha\in\mathbb{Z}^n}$ of bounded operators in $\mathcal{H}$, such that:*

(1) $T^\alpha = \Theta_\alpha$ *for all* $\alpha \in \mathbb{Z}^n_+$*;*
(2) $\Theta^*_\alpha = \Theta_{-\alpha}$ *for all* $\alpha \in \mathbb{Z}^n$*, and* $\Theta_0 = 1$*;*
(3) $\sum_{\alpha,\beta\in\mathbb{Z}^n_+} c_\alpha \overline{c}_\beta \Theta_{\alpha-\beta} \geq 0$ *for all of the sequences of complex numbers* $(c_\alpha)_{\alpha\in\mathbb{Z}^n_+}$ *with only finitely many nonzero terms.*

Proof. The assertion is a consequence of the previous corollary, via the Naimark dilation theorem (see [Nai]). □

Remark I.4.12. The positivity condition (I.4.9) (as well as condition (3) from Corollary I.4.11) may be checked on a considerably smaller family of sequences $(c_\alpha)_{\alpha\in\mathbb{Z}^n_+}$ (see [Vas7] for details).

Part II. **MOMENTS ON UNBOUNDED SEMI-ALGEBRAIC SETS**

II.1. Introduction

This chapter contains a synthesis of the results from [Vas6], [Vas5] and [PuVa4]. Some results (e.g., Theorems II.2.3 and II.2.4) appear in a more general form.

Let $\mathcal{R}$ be an algebra of complex-valued functions, defined on the Euclidean space $\mathbb{R}^n$, such that the constant function $1 \in \mathcal{R}$, and if $f \in \mathcal{R}$ then $\bar{f} \in \mathcal{R}$. We recall that a linear map $L\colon \mathcal{R} \to \mathbb{C}$ is said to be *of positive type* if $L(f\bar{f}) \geq 0$ for all $f \in \mathcal{R}$. If L is of positive type on $\mathcal{R}$, we shall always assume that $L(1) > 0$ (i.e., L is not degenerate).

A linear map $L\colon \mathcal{R} \to \mathbb{C}$ is said to be a *moment form* if there exists a finite positive Borel measure μ on $\mathbb{R}^n$ such that $L(f) = \int_{\mathbb{R}^n} f d\mu$, $f \in \mathcal{R}$. In that case, μ is called a *representing measure* for L.

Every moment form is obviously of positive type.

Let $\mathcal{R}$ be an algebra as above, and let $L : \mathcal{R} \to \mathbb{C}$ be of positive type. As it is well known (see the classical paper [GeNa]), this pair can be associated, in a canonical way, with a Hilbert space which will be called here the *associated GN-space*.

We shall be particularly interested in the sequel by the following case. Let $\mathcal{P}_n = P_{\mathbb{C}}(\mathbb{R}^n)$ be the algebra of all polynomial functions on $\mathbb{R}^n$, with complex coefficients.

An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ is said to be *positive semi-definite* if the *associated linear map* $L_\gamma : \mathcal{P}_n \to \mathbb{C}$ (see the first chapter) is positive semi-definite, where $L_\gamma(t^\alpha) = \gamma_\alpha, \alpha \in \mathbb{Z}_+^n$.

An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ is said to be a *moment sequence* when it is a $\mathbb{R}^n$-moment sequence (in the sense of the first chapter). This is equivalent to saying that the form L_γ is a moment form.

We shall describe in the following some solutions to the (determined) moment problem for the algebra $\mathcal{P}_n$, with support in a not necessarily compact semi-algebraic set, in particular solutions to what is usually called the *Hamburger moment problem* (in several variables), via extended sequences. Let us explain what we mean by this in the case of the plane. We obtain, as a particular case, the following result:

A 2-sequence $\gamma = (\gamma_{m_1,m_2})_{m_1,m_2 \in \mathbf{Z}_+}$ $(\gamma_{0,0} > 0)$ is a moment sequence if and only if there exists a positive semi-definite 4-sequence $\delta = (\delta_{m_1,m_2,m_3,m_4})_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_+}$ with the following properties:

(1) $\gamma_{m_1,m_2} = \delta_{m_1,m_2,0,0}$;
(2′) $\delta_{m_1,m_2,m_3,m_4} = \delta_{(m_1,m_2,m_3+1,m_4)} + \delta_{m_1+2,m_2,m_3+1,m_4}$;
(2″) $\delta_{m_1,m_2,m_3,m_4} = \delta_{(m_1,m_2,m_3,m_4+1)} + \delta_{m_1,m_2+2,m_3,m_4+1}$

for all m_1, m_2, m_3, $m_4 \in \mathbb{Z}_+$ It is clear that the positive semi-definite 4-sequences δ, having the properties $(2')$, $(2'')$, are completely determined, and the 2-sequences γ have a representing measure if and only if they are restrictions of such 4-sequences.

The passage from a 2-sequence to an extended one is partially motivated by the fact that there are 2-sequences which are positive semi-definite and which are not moment sequences (see, for instance, [Fug]). Therefore, some new parameters must be introduced. In addition, when the moment problem has several solutions, a parameterization of all solutions is also of interest. The existence of ”optimal” choices for such parameters is still to be investigated.

When one seeks, in this context, representing measures whose support is concentrated in $\mathbb{R}_+^n$, then the corresponding moment problem is called the *Stieltjes moment problem* (in several variables).

Solutions to Hamburger and Stieltjes moment problems, in several variables, by extended sequences, are provided, in particular, in the next sections. For a thorough discussion concerning these problems in one variable, as well as for historical remarks, we refer to the monographs [Akh] and [ShTa] (see also [BCR], [Dev], [Esk], [Fug], [Hav], [KoMi], [Sch2], etc. for various solutions in several variables).

A solution to the Hausdorff moment problem with operator data is recorded as early as 1952 (see [SzN1]). Since then, there have occurred many other contributions in this area. See [AtPe], [Fri], [MaN], [Nar], [Sch2], [StSz2], [Vas2], [Vas4] etc. for further development.

In the third section of this chapter we present operator versions of the results from [Vas6] and [PuVa4], which extend the corresponding results concerning the solutions of the moment problems of Hamburger and Stieltjes type, in several variables, to the case of operator data. Such assertions were already discussed in [Vas5], and they were obtained replacing the numerical data by sequences of hermitian (or even sesquilinear) forms as moment data. Characterizations of unbounded subnormal tuples of operators, also developed in [Vas5], will be discussed in the last section of this chapter.

II.2. Scalar moment problems in unbounded sets

In the first part of this section we present extensions of some results from [Vas6] (see also [Dem5]).

The next result is well known.

Lemma II.2.1. *Assume that S is a symmetric densely defined operator in the Hilbert space $\mathcal{H}$. If the sets $R(S \pm \mathrm{i})$ are dense in $\mathcal{H}$, then the closure of S is a self-adjoint operator.*

Proof. Let A be the closure of S, which is also a symmetric operator. From the classical identity
$$\|(A \pm \mathrm{i})x\|^2 = \|Ax\|^2 + \|x\|^2, \quad x \in D(A),$$
it follows that $R(A \pm \mathrm{i})$ are closed subspaces of $\mathcal{H}$. As we have $R(A \pm \mathrm{i}) \supset R(S \pm \mathrm{i})$, our hypothesis implies $R(A \pm \mathrm{i}) = \mathcal{H}$.

Let V be the Cayley transform of A (see [Rud], 13.17). Since $D(V) = R(A + \mathrm{i})$, $R(V) = R(A - \mathrm{i})$, the operator V is unitary, and so A must be selfadjoint (via [Rud], 13.19). $\square$

The next assertion is an extension of [Vas6], Lemma 2.2.

Lemma II.2.2. *Let $\theta_j(t) = (1 + t_j^2)^{-1}$, $1 \le j \le n$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. Let also $\{p_1, \dots, p_m\}$ be a finite subset in $\mathcal{P}_n$ consisting of polynomial functions with real coefficients. We set $\theta_j(t) = (1 + p_j(t)^2)^{-1}$, $n+1 \le j \le n+m$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, and let $\theta = (\theta_j)_{1 \le j \le n+m}$. Denote by $\mathcal{R}_\theta$ the complex algebra generated by $\mathcal{P}_n$ and by $(\theta_j)_{1 \le j \le n+m}$. Let $\rho\colon \mathcal{P}_{2n+m} \to \mathcal{R}_\theta$ be given by $\rho\colon p(t,s) \to p(t, \theta(t))$. Then ρ is a surjective unital algebras homomorphism, whose kernel is the ideal generated by the* *polynomials* $\sigma_j(t,s) = s_j(1+t_j^2) - 1$, $1 \le j \le n$, $\sigma_j(t,s) = s_j(1+p_j(t)^2) - 1$, $n+1 \le j \le n+m$.

Proof. That ρ is a surjective unital algebra homomorphism is obvious. We have only to determine the kernel of ρ.

Let $p \in \mathcal{P}_{2n+m}$ be a polynomial with the property $p(t,\theta(t)) = 0$, $t \in \mathbb{R}^n$. We write $p(t,s) = \sum_{\beta \in \mathbb{Z}_+^{n+m}} p_\beta(t)s^\beta$, with $p_\beta \in \mathcal{P}_n \setminus \{0\}$ only for a finite number of indices β. Then we have

$$\begin{aligned} p(t,s) &= p(t,s) - p(t,\theta(t)) \\ &= \sum_{\beta \neq 0} p_\beta(t)(s^\beta - \theta(t)^\beta) \\ &= \sum_{1 \le j \le n+m} (s_j - \theta_j(t))\ell_j(t,s,\theta(t)), \end{aligned}$$

where ℓ_j are polynomials.

Let $a_j = \max\{\beta_j \,:\, p_\beta \neq 0\}$, $1 \le j \le n$, and let

$$\tau(t) = \prod_{1 \le j \le n+m} (1+\zeta_j(t)^2)^{a_j},$$

where $\zeta_j(t) = t_j$, $j = 1,\dots,n$, $\zeta_j(t) = p_j(t)$, $j = n+1,\dots,n+m$. Then, from the above calculation, we deduce the equation

$$\tau(t)p(t,s) = \sum_{1 \le j \le n+m} (s_j(1+\zeta_j(t)^2) - 1)q_j(t,s), \tag{II.2.1}$$

with $q_j \in \mathcal{P}_{2n+m}$ for all indices j.

If $a_j = 0$ for all j, then $p(t,s) = p_0(t) = p(t,\theta(t)) = 0$. Therefore, with no loss of generality, we may assume $a_j \neq 0$ for some indices j.

It is easily seen that the polynomials $\tau, \sigma_j, 1 \le j \le n+m$, have no common zero in $\mathbb{C}^{2n+m}$. By a special case of Hilbert's Nullstellensatz (see, for instance, [Wae], Section 16.5), there are polynomials $\tilde{\tau}$, $(\tilde{\sigma}_j)_{1 \le j \le n+m}$ in $\mathcal{P}_{2n+m}$ such that

$$\tau\tilde{\tau} + \sum_{1 \le j \le n+m} \sigma_j\tilde{\sigma}_j = 1. \tag{II.2.2}$$

If we multiply (II.2.2) by p, and use (II.2.1), we obtain the relation

$$p = \sum_{1 \le j \le n+m} \sigma_j(q_j\tilde{\tau} + \tilde{\sigma}_j p),$$

which is precisely our assertion. □

In the next statement, the algebra $\mathcal{R}_\theta$ will have the meaning from Lemma II.2.2. This statement does not appear explicitly in [Vas6], and it extends Remark 2.4 from that paper (see also Theorem 2.5 from [PuVa4]).

Theorem II.2.3. *Let $L\colon \mathcal{R}_\theta \to \mathbb{C}$ be a linear map of positive type such that $L(p_k|r|^2) \geq 0$, $r \in \mathcal{R}_\theta$, $k = 1,\dots,m$. Then L has a uniquely determined representing measure whose support lies in the set $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$.*

If μ is the representing measure of L, then the algebra $\mathcal{R}_\theta$ is dense in $L^2(\mu)$.

Proof. The pair $\mathcal{R}_\theta, L$ can be associated with a GN-space $\mathcal{H}$, obtained as a completion of the quotient $\mathcal{R}_\theta/\mathcal{N}$, with $\mathcal{N} = \{r \in \mathcal{R}_\theta; \Lambda(r\bar{r}) = 0\}$.

We have in $\mathcal{H}$ the operators

$$\begin{aligned} T_j(r+\mathcal{N}) &= t_j r + \mathcal{N}, \quad r \in \mathcal{R}_\theta,\ j = 1,\dots,n, \\ S_k(r+\mathcal{N}) &= p_k r + \mathcal{N}, \quad r \in \mathcal{R}_\theta,\ k = 1,\dots,m, \end{aligned} \tag{II.2.3}$$

which are symmetric and densely defined, with $D(T_j) = D(S_k) = \mathcal{R}_\theta/\mathcal{N}$ for all j,k. We note that T_j satisfies the conditions of Lemma II.2.1 for each j. Indeed, if $r \in \mathcal{R}_\theta$ is arbitrary, then the functions $u_\pm(t) = (t_j \mp \mathrm{i})\theta_j(t) r(t)$ are solutions in $\mathcal{R}_\theta$ of the equations $(t_j \pm \mathrm{i})u_\pm(t) = r(t)$. This implies the equalities $R(T_j \pm \mathrm{i}) = D(T_j)$, and therefore Lemma II.2.1 applies to T_j. Hence T_j is essentially selfadjoint, and let A_j be the closure of T_j.

Similarly, S_k is essentially self-adjoint, and let B_k be the closure of S_k, $k = 1,\dots,m$.

We shall show that the operators

$$(\mathrm{i}-A_1)^{-1},\dots,(\mathrm{i}-A_n)^{-1},(\mathrm{i}-S_1)^{-1},\dots,(\mathrm{i}-S_m)^{-1}$$

mutually commute. Indeed, the previous argument shows that the maps $(\mathrm{i}-T_j)^{-1}$ and $(\mathrm{i}-S_k)^{-1}$ are well defined on $D = D(T_j) = D(S_k)$, and leave this space invariant, for all j,k. Moreover, the maps $(\mathrm{i}-T_1)^{-1},\dots,(\mathrm{i}-T_n)^{-1},(\mathrm{i}-S_1)^{-1},\dots,(\mathrm{i}-S_m)^{-1}$ mutually commute on D. Since A_j extends T_j, we clearly have

$$(\mathrm{i}-A_j)((\mathrm{i}-A_j)^{-1}-(\mathrm{i}-T_j)^{-1})\xi = 0, \quad \xi \in D,$$

implying $(\mathrm{i}-A_j)^{-1}|D = (\mathrm{i}-T_j)^{-1}$. Similarly, $(\mathrm{i}-B_k)^{-1}|D = (\mathrm{i}-S_k)^{-1}$. Therefore, for all $j,l = 1,\dots,n$, $j \neq l$, we have

$$\begin{aligned} (\mathrm{i}-A_j)^{-1}(\mathrm{i}-A_l)^{-1}\xi &= (\mathrm{i}-T_j)^{-1}(\mathrm{i}-T_l)^{-1}\xi \\ &= (\mathrm{i}-T_l)^{-1}(\mathrm{i}-T_j)^{-1}\xi \\ &= (\mathrm{i}-A_l)^{-1}(\mathrm{i}-A_j)^{-1}\xi \end{aligned}$$

where $\xi \in D$ is arbitrary. Similarly,

$$(\mathrm{i}-B_k)^{-1}(\mathrm{i}-B_l)^{-1}\xi = (\mathrm{i}-B_l)^{-1}(\mathrm{i}-B_k)^{-1}\xi$$

for all $k,l = 1,\dots,m$, $k \neq l$, and

$$(\mathrm{i}-A_j)^{-1}(\mathrm{i}-B_k)^{-1}\xi = (\mathrm{i}-B_k)^{-1}(\mathrm{i}-A_j)^{-1}\xi$$

for all $j = 1,\dots,n$, $k = 1,\dots,m$.

Since $(\mathrm{i}-A_1)^{-1},\dots,(\mathrm{i}-A_n)^{-1},(\mathrm{i}-B_1)^{-1},\dots,(\mathrm{i}-B_m)^{-1}$ are bounded and D is dense, this implies that they mutually commute. In particular, the selfadjoint operators $A_1,\dots,A_n$ have a joint spectral measure (see, for instance, [Vas1]). If E is the joint spectral measure of $A_1,\dots,A_n$, then $\mu(*) := \langle E(*)(1+\mathcal{N}), 1+\mathcal{N}\rangle$ is a representing measure for L. In other words, we have the equality

(II.2.4) $$L(r) = \int\limits_{\mathbb{R}^n} r(t)\,d\mu(t), \quad r\in\mathcal{R}_\theta.$$

Indeed, if $r(T)$ is the linear map on D given by $r(T)(f+\mathcal{N}) = rf+\mathcal{N}$, for all $r,f\in\mathcal{R}_\theta$, then we have $\theta(A)^\beta \supset \theta(T)^\beta$, for all $\beta\in\mathbb{Z}_+^{n+m}$, where $\theta(A)^\beta$ is given by the functional calculus of A. This follows from the obvious relations $\theta(A)^{-\beta}\supset\theta(T)^{-\beta}$, and $\theta(A)^{-\beta}(\theta(A)^\beta-\theta(T)^\beta)=0$. Therefore:

$$\begin{aligned}\langle T^\alpha\theta(T)^\beta(1+\mathcal{N}),1+\mathcal{N}\rangle &= \langle A^\alpha\theta(A)^\beta(1+\mathcal{N}),1+\mathcal{N}\rangle\\ &= \int\limits_{\mathbb{R}^n} t^\alpha\theta(t)^\beta d\langle E(t)(1+\mathcal{N}),1+\mathcal{N}\rangle.\end{aligned}$$

We prove now the assertion concerning the support of the representing measure. Note that $\bar{B}_k = p_k(A)$, $k=1,\dots,m$, where $p_k(A)$ is given by the functional calculus of A. Indeed, as we clearly have $S_k\subset p_k(A)$, and S_k is essentially selfadjoint, we must have $\bar{S}_k = p_k(A)$ for all k. Condition $\Lambda(p_k|r|^2)\ge 0$, $r\in\mathcal{R}_{\theta_{\mathrm{p}}}$, $k=1,\dots,m$, implies that S_k is positive for all k. Therefore, $p_k(A)$ is positive for all k. The spectral measure F_k of $p_k(A)$ is given by $F_k(B) = E(p_k^{-1}(B))$ for all Borel sets $B\subset\mathbb{R}$. Since the spectral measure F_k must be concentrated in $\mathbb{R}_+$ for all k, it follows that the spectral measure E of A is concentrated in the set $\bigcap\limits_{k=1}^m p_k^{-1}(\mathbb{R}_+)$, which implies that the representing measure of L itself is concentrated in the same set $\bigcap\limits_{k=1}^m p_k^{-1}(\mathbb{R}_+)$.

We have only to discuss the uniqueness of the representing measure of L.

Let ν be an arbitrary representing measure of L. Then the space $\mathcal{H}$ can be identified with a subspace of $L^2(\nu)$. Indeed, we must have $\langle r_1,r_2\rangle_\theta = \int r_1\bar{r}_2\,d\nu$ for all $r_1,r_2\in\mathcal{R}_\theta$. Therefore, as the functions from $\mathcal{N}$ are null ν-almost everywhere, the space $\mathcal{H}$ is identified with the closure of $\mathcal{R}_\theta$ in $L^2(\nu)$.

We proceed now as in [Fug], Theorem 7. The operators $(H_jf)(t) = t_jf(t)$, $t=(t_1,\dots,t_n)\in\mathbb{R}^n$, $f\in D(H_j) = \{g\in L^2(\nu)\ :\ t_jg\in L^2(\nu)\}$, $j=1,\dots,n$, are commuting selfadjoint in $L^2(\nu)$. Clearly, $H_j\supset T_j$, and so $H_j\supset A_j$ for all j. Therefore, since $(A_j+\mathrm{i}u)^{-1} = (H_j+\mathrm{i}u)^{-1}|\mathcal{H}$ for all $u\in\mathbb{R}$, it follows that the spectral measure E_j of H_j leaves invariant the space $\mathcal{H}$, as a consequence of [DuSc], Theorem XII.2.10, for all j. If E_H is the joint spectral measure of $H=(H_1,\dots,H_n)$, then $E_H(B_1\times\cdots\times B_n) = E_1(B_1)\cdots E_n(B_n)$ for all Borel sets $B_1,\dots,B_n$ in $\mathbb{R}$. This implies that the space $\mathcal{H}$ is invariant under E_H. Hence, $\chi_B = E_H(B)1\in\mathcal{H}$ for all Borel subsets B of $\mathbb{R}^n$, where χ_B is the characteristic function of B. This shows that $L^2(\nu)=\mathcal{H}$, since the simple functions form a dense subspace of $L^2(\nu)$. In particular, we have the equalities $H_j = A_j, j = 1, \ldots, n$. Therefore, with μ and E as above, $\mu(B) = \langle E(B)1, 1\rangle = \langle E_H(B)1, 1\rangle = \int \chi_B d\nu$, for all Borel sets B. Consequently, $\mu = \nu$, showing that the representing measure is unique.

Finally, since the space $\mathcal{H}$ is identified with the closure of $\mathcal{R}_\theta$ in $L^2(\mu)$, the previous discussion shows that $\mathcal{R}_\theta$ must be dense in $L^2(\mu)$. □

For the sake of simplicity, for any integer $N \geq 1$, we denote by $e_j \in \mathbb{Z}_+^N, j = 1, \ldots, N$, the multi-index whose coordinates are null except for the j-th coordinate (where $j \leq N$), which is equal to one.

A solution to the multivariate moment problem in a not necessarily bounded semi-algebraic set is given by the following, which is a new version of Theorem 2.7 from [PuVa4].

Theorem II.2.4. *Let $\{p_1, \ldots, p_m\}$ a finite subset in $\mathcal{P}_n$ consisting of polynomial functions with real coefficients. Write $p_k(t) = \sum\limits_\alpha c_{k\alpha} t^\alpha$, $t \in \mathbb{R}^n$, $k = 1, \ldots, m$.*

An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ $(\gamma_0 > 0)$ is a moment sequence and has a representing measure whose support lies in the set $\bigcap\limits_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a positive semi-definite $(2n+m)$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^{n+m}}$ with the following properties:

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$;
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_j)} + \delta_{(\alpha+2e_j,\beta+e_j)}$ *for all* $(\alpha,\beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^{n+m}$, $1 \leq j \leq n$, *and* $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_k)} + \sum_{\xi,\eta} c_{k\xi} c_{k\eta} \delta_{(\alpha+\xi+\eta,\beta+e_k)}$ *for all* $(\alpha,\beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^{n+m}$, $n+1 \leq k \leq n+m$;
(3) $\sum\limits_{\alpha,\xi,\xi',\eta,\eta'} c_{k\alpha} a_{\xi\eta} \bar{a}_{\xi'\eta'} \delta_{(\alpha+\xi+\xi',\eta+\eta')} \geq 0$ *for all* $k = 1, \ldots, m$ *and all finite collections of complex numbers* $(a_{\xi\eta})_{\xi,\eta}$.

In the affirmative case, the n-sequence γ has a uniquely determined representing measure in $\mathbb{R}^n$ if and only if the $(2n+m)$-sequence δ is unique.

Proof. We prove first that conditions (1), (2) and (3) are necessary. Assume that the sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ has a representing measure μ. Define

$$\delta_{(\alpha,\beta)} = \int\limits_{\mathbb{R}^n} t^\alpha \theta(t)^\beta \, d\mu(t), \quad (\alpha,\beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^{n+m},$$

where $\theta_j(t) = (1+\zeta_j(t)^2)^{-1}$, $1 \leq j \leq n+m$, with $\zeta_j(t) = t_j$, $1 \leq j \leq n$, $\zeta_j(t) = p_j(t)$, $n+1 \leq j \leq n+m$, and $\theta = (\theta_j)_{1\leq j\leq n+m}$. Clearly, $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^{n+m}}$ is a positive semi-definite $(2n+m)$-sequence, satisfying (1). Since

$$\int\limits_{\mathbb{R}^n} (\theta_j(t)(1+\zeta_j(t)^2)-1)t^\alpha\theta(t)^\beta\, d\mu(t) = 0$$

for all $(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^{n+m}$, $1\le j\le n+m$, we also have (2). Moreover, as the support of μ lies in $\bigcap\limits_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$, we have

$$\int\limits_{\mathbb{R}^n} p_k(t)\Big|\sum_{\xi,\eta} a_{\xi\eta}t^\xi\theta(t)^\eta\Big|^2 d\mu(t)\ge 0$$

for all $k=1,\dots,m$ and all polynomials $\sum\limits_{\xi,\eta} a_{\xi\eta}t^\xi s^\eta\in\mathcal{P}_{2n+m}$, showing that (3) also holds.

Conversely, assume that the $2n+m$-sequence $\delta=(\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ exists. Let θ_j be as above, and let $\mathcal{R}_\theta$ be the algebra generated by $\mathcal{P}_n$, and $\theta=(\theta_j)_{1\le j\le n+m}$. We shall define a positive semi-definite map Λ on $\mathcal{R}_\theta$, via the equality

$$\Lambda(r)=L_\delta(p),\quad r\in\mathcal{R}_\theta,$$

where $L_\delta\colon\mathcal{P}_{2n}\to\mathbb{C}$ is the linear map associated with δ, and $p\in\mathcal{P}_{2n+m}$ satisfies $r(t)=p(t,\theta(t))$, $t\in\mathbb{R}^n$.

Notice first that Λ is correctly defined. Indeed, by virtue of Lemma II.2.2, the algebra $\mathcal{R}_\theta$ is isomorphic to the quotient $\mathcal{P}_{2n+m}/\mathcal{I}_\sigma$, where $\mathcal{I}_\sigma$ is the ideal generated in $\mathcal{P}_{2n}$ by the polynomials $\sigma_j(t,s)=s_j(1+\zeta_j(t)^2)-1, 1\le j\le n+m$. Note that condition (2) implies $L_\delta|\mathcal{I}_\sigma=0$. Therefore, the map Λ, which can be identified with the map induced by L_δ on the quotient $\mathcal{P}_{2n+m}/\mathcal{I}_\sigma$, is correctly defined, and positive semi-definite as well, on $\mathcal{R}_\theta$. By virtue of Theorem II.2.3, there exists a uniquely determined representing measure μ for Λ. In particular

$$\gamma_\alpha=\delta_{(\alpha,0)}=\int t^\alpha d\mu(t),\quad \alpha\in\mathbb{Z}_+^n,$$

showing that γ has a representing measure.

We have only to discuss the uniqueness of the representing measure of γ.

If δ is uniquely determined, and if μ',μ'' are two representing measures for γ, then we must have

$$\int\limits_{\mathbb{R}^n} t^\alpha\theta(t)^\beta d\mu'(t)=\int\limits_{\mathbb{R}^n} t^\alpha\theta(t)^\beta d\mu''(t)$$

by the uniqueness of δ. Therefore $\int_{\mathbb{R}^n} r(t)d\mu'(t)=\int_r(t)d\mu''(t)$ for all $r\in\mathcal{R}_\theta$, implying $\mu'=\mu''$, by Theorem II.2.3.

Conversely, if the representing measure μ of γ is unique, and if the sequences δ',δ'' satisfy (1), (2), (3), then we have $\delta'_{\alpha,\beta}=\int_{\mathbb{R}^n} t^\alpha\theta(t)^\beta d\mu(t)=\delta''_{\alpha,\beta}$ for all indices α,β, which completes the proof of the theorem. □

The main result of [Vas6] is now a consequence of Theorem II.2.4.

Corollary II.2.5. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a moment n-sequence if and only if there exists a positive semi-definite $2n$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_j)} + \delta_{(\alpha+2e_j,\beta+e_j)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+^n, 1 \le j \le n$.

In the affirmative case, the n-sequence γ has a uniquely determined representing measure in $\mathbb{R}^n$ if and only if the $2n$-sequence δ is unique.

This is a particular case of Theorem II.2.4, obtained for $p_1 = \cdots = p_m = 0$.

Corollary II.2.6. *A sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+}$ $(\gamma_0 > 0)$ is a moment sequence, and has a uniquely determined representing measure, if and only if there exists a uniquely determined positive semi-definite 2-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^2}$, with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2,\beta+1)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+$.

The next result is a solution to the Stieltjes moment problem in several variables (see also Theorem 2.6 from [Vas6]).

Corollary II.2.7. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a moment sequence, and has a representing measure in $\mathbb{R}_+^n$, if and only if there exists a positive semi-definite $2n$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+e_j)} + \delta_{(\alpha+2e_j,\beta+e_j)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+^n, 1 \le j \le n$.
(3) $(\delta_{(\alpha+e_j,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ *is a positive semi-definite $2n$-sequence for* $j = 1, \dots, n$.

In the affirmative case, the n-sequence γ has a uniquely determined representing measure in $\mathbb{R}_+^n$ if and only if the $2n$-sequence δ is unique.

This is a particular case of Theorem II.2.4, obtained for $p_k(t) = t_k, k = 1, \dots, n$.

Corollary II.2.8. *A sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+}$ $(\gamma_0 > 0)$ is a moment sequence, and has a uniquely determined representing measure in $\mathbb{R}_+$, if and only if there exists a uniquely determined positive semi-definite 2-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+}$, with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2,\beta+1)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+$.
(3) *The sequence* $(\delta_{(\alpha+1,\beta)})_{\alpha,\beta\in\mathbb{Z}_+}$ *is positive semi-definite.*

There is an alternate approach developed in [PuVa4]. We shall shortly present it in the remaining part of this section. We start with Proposition 2.1 from [PuVa4].

Theorem II.2.9. *Let $T_1, \dots, T_n$ be symmetric operators in $\mathcal{H}$. Assume that there exists a dense linear space $D \subset \bigcap_{j,k=1}^{n} D(T_j T_k)$ such that $T_j T_k x = T_k T_j x,\ x \in D,\ j \neq k$; $j, k = 1, \dots, n$. If the operator $(T_1^2 + \cdots + T_n^2)|D$ is essentially self-adjoint, then the operators $T_1, \dots, T_n$ are essentially self-adjoint, and their closures $\bar{T}_1, \dots, \bar{T}_n$ commute.*

The proof of this result, stated for $n = 2$, can be found in [Nel], Corollary 9.2. For a different approach and an arbitrary n see [EsVa], Theorem 3.2.

Lemma II.2.10. *If A is a positive densely defined operator in $\mathcal{H}$, then A is essentially self-adjoint if and only if the range of $I + A$ is dense in $\mathcal{H}$.*

The proof can be found in [StZs], Lemma 9.5.

Lemma II.2.11. *Let $\mathbf{p} = (p_1, \dots, p_m)$ be a given m-tuple of real polynomials from $\mathcal{P}_n$, and let*

$$\theta_{\mathbf{p}}(t) = (1 + t_1^2 + \cdots + t_n^2 + p_1(t)^2 + \cdots + p_m(t)^2)^{-1}, \quad t \in \mathbb{R}^n.$$

Denote by $\mathcal{R}_{\theta_{\mathbf{p}}}$ the $\mathbb{C}$-algebra generated by $\mathcal{P}_n$ and $\theta_{\mathbf{p}}$. Let $\rho\colon \mathcal{P}_{n+1} \to \mathcal{R}_{\theta_{\mathbf{p}}}$ be given by $\rho : p(t,s) \mapsto p(t, \theta_{\mathbf{p}}(t))$. Then ρ is a surjective unital algebra homomorphism, whose kernel is the ideal generated by the polynomial $\sigma(t,s) = s(1 + t_1^2 + \cdots + t_n^2 + p_1(t)^2 + \cdots + p_m(t)^2) - 1$.

The result above is precisely Lemma 2.3 from [PuVa4].

Remark II.2.12. Set $\theta(t) = (1 + t_1^2 + \cdots + t_n^2)^{-1}$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, and let $\mathcal{R}_\theta$ be the $\mathbb{C}$-algebra of rational functions generated by $\mathcal{P}_n$ and θ. Let $\rho\colon \mathcal{P}_{n+1} \to \mathcal{R}_\theta$ be given by $\rho : p(t,s) \mapsto p(t, \theta(t))$. Then ρ is a surjective unital algebra homomorphism, whose kernel is the ideal generated by the polynomial $\sigma(t,s) = s(1 + t_1^2 + \cdots + t_n^2) - 1$. This is a particular case of the previous lemma, obtained for $\mathbf{p} = (0)$.

A key result in this approach is the following (see [PuVa4], Theorem 2.5).

Theorem II.2.13. *Let $\mathbf{p} = (p_1, \dots, p_m)$ be a given m-tuple of real polynomials from $\mathcal{P}_n$, and let*

$$\theta_{\mathbf{p}}(t) = (1 + t_1^2 + \cdots + t_n^2 + p_1(t)^2 + \cdots + p_m(t)^2)^{-1}, \quad t \in \mathbb{R}^n.$$

Denote by $\mathcal{R}_{\theta_{\mathbf{p}}}$ the $\mathbb{C}$-algebra generated by $\mathcal{P}_n$ and $\theta_{\mathbf{p}}$. Let Λ be a positive type map on $\mathcal{R}_{\theta_{\mathbf{p}}}$ such that $\Lambda(p_k |r|^2) \geq 0$, $r \in \mathcal{S}_{\theta_{\mathbf{p}}}$, $k = 1, \dots, m$. Then Λ has a uniquely determined representing measure whose support is in the set $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$. Moreover, the algebra $\mathcal{R}_{\theta_{\mathbf{p}}}$ is dense in $L^2(\mu)$.

The proof follows the lines of the proof of Theorem II.2.3, using Theorem II.2.9 instead of Lemma II.2.1. We omit the details.

Corollary II.2.14. *Let $\mathcal{R}_{\theta_{\mathbf{p}}}$ be the $\mathbb{C}$-algebra generated by $\mathcal{P}_n$ and $\theta(t) = (1+t_1^2+\cdots+t_n^2)^{-1}$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, and let $\Lambda\colon \mathcal{S}_\theta \to \mathbb{C}$ be an arbitrary positive semi-definite map. Then Λ has a uniquely determined representing measure μ in $\mathbb{R}^n$, and the algebra $\mathcal{R}_\theta$ is dense in $L^2(\mu)$.*

Moreover, if $\Lambda(t_j|r|^2) \geq 0$, $r \in \mathcal{R}_{\theta_{\mathbf{p}}}$, $j = 1, \ldots, n$, then the support of μ is contained in $\mathbb{R}_+^n$.

We apply the previous theorem with $\mathbf{p} = (0)$.

The next result is another general moment theorem, holding on arbitrary semi-algebraic sets (see [PuVa4], Theorem 2.7).

Theorem II.2.15. *Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ be an n-sequence of real numbers, and let $\mathbf{p} = (p_1, \ldots, p_m) \in \mathcal{P}_n^m$, where $p_k(t) = \sum_{\xi\in I_k} a_{k\xi}t^\xi$, $k = 1, \ldots, m$, with $I_k \subset \mathbb{Z}_+^n$ finite for all k. Then γ is moment sequence, and it has a representing measure whose support is in the set $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$, if and only if there exists a positive semi-definite $(n+1)$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.

(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \sum_{j=1}^n \delta_{(\alpha+2e_j,\beta+1)} + \sum_{k=1}^m \sum_{\xi,\eta\in I_k} a_{k\xi}a_{k\eta}\delta_{(\alpha+\xi+\eta,\beta+1)}$ *for all* $\alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_+$.

(3) *The $(n+1)$-sequences $(\sum_{\xi\in I_k} a_{k\xi}\delta_{(\alpha+\xi,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ are positive semi-definite for all $k = 1, \ldots, n$.*

The n-sequence γ has a uniquely determined representing measure on $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if the $(n+1)$-sequence δ is unique.

The proof follows the lines of Theorem II.2.4, using Theorem II.2.13 instead of Theorem II.2.3. We omit the details.

Theorem II.2.15 shows that for a given n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ there exists a one-to-one correspondence between the convex set $M_{\gamma,\mathbf{p}}$ of all representing measures of γ, with support in $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$, and the convex set $E_{\gamma,\mathbf{p}}$ of all extensions $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ with the properties (1), (2), (3) from this theorem. This correspondence obviously preserves the extremal points. In addition, if $\epsilon\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ is given by $\epsilon(t) = (t, \theta_{\mathbf{p}}(t))$, $t \in \mathbb{R}^n$, then for every $\mu \in M_{\gamma,\mathbf{p}}$ the measure $\mu_\epsilon(B) = \mu(\epsilon^{-1}(B))$, B a Borel set in $\mathbb{R}^{n+1}$, is a representing measure for δ.

Theorem II.2.15 (as well as Theorem II.2.4) applies, in particular, for compact semi-algebraic sets, providing alternate solutions to the corresponding moment problems.

Another solution of the Hamburger moment problem in several variables is given by the following.

Corollary II.2.16. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a moment sequence if and only if there exists a positive semi-definite $(n+1)$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2e_1,\beta+1)} + \cdots + \delta_{(\alpha+2e_n,\beta+1)}$ *for all* $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+$.

The n-sequence γ has a uniquely determined representing measure in $\mathbb{R}^n$ if and only if the $(n+1)$-sequence δ is unique.

This is a consequence of Theorem II.2.15, with $\mathbf{p} = (0)$ (see [PuVa4], Theorem 2.8).

We also have an alternate solution to the Stieltjes moment problem in several variables.

Corollary II.2.17. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\gamma_0 > 0)$ is a moment sequence, and it has a representing measure in $\mathbb{R}_+^n$, if and only if there exists a positive semi-definite $(n+1)$-sequence $\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ with the following properties:*

(1) $\gamma_\alpha = \delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2e_1,\beta+1)} + \cdots + \delta_{(\alpha+2e_n,\beta+1)}$ *for all* $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+$.
(3) $(\delta_{(\alpha+e_j,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$ *is a positive semi-definite $(n+1)$-sequence for all* $j = 1,\ldots,n$.

The n-sequence γ has a uniquely determined representing measure in $\mathbb{R}_+^n$ if and only if the $(n+1)$-sequence δ is unique.

Corollary II.2.17 is a particular case of Theorem II.2.15, with $\mathbf{p}(t) = (t_1,\ldots,t_n)$ (see [PuVa4], Theorem 2.9).

II.3. MORE ABOUT THE UNIQUENESS

The uniqueness of a representing measure of a moment n-sequence is characterized in Theorem II.2.4 by the uniqueness of the associated $(2n+m)$-sequence. Using some of the previous assertions and techniques, we shall discuss in this section an operator theoretic characterization of the uniqueness of the representing measure, as well as some related results, in the spirit of [Vas6], Section 3.

Definition II.3.1. Let $S = (S_1,\ldots,S_n)$ be a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$. We say that S has a *smallest selfadjoint extension* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a tuple $A = (A_1,\ldots,A_n)$ consisting of commuting selfadjoint operators in $\mathcal{K}$ with the following properties:

(1) $A_j \supset S_j$, $j = 1,\ldots,n$;
(2) if $B = (B_1,\ldots,B_n)$ is a tuple consisting of commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$ such that $B_j \supset S_j$, $j = 1,\ldots,n$, then $\mathcal{L} \supset \mathcal{K}$ and $B_j \supset A_j$, $j = 1,\ldots,n$.

Remark II.3.2. (i) In the previous definition, we write $\mathcal{K} \supset \mathcal{H}$ when there exists a linear isometry from $\mathcal{H}$ into $\mathcal{K}$, which allows the identification of $\mathcal{H}$ with a closed subspace of $\mathcal{K}$. In particular, the smallest selfadjoint extension, when exists, is uniquely determined.

(ii) If $S = (S_1, \ldots, S_n)$ is a tuple consisting of symmetric operators in a Hilbert space $\mathcal{H}$ such that the closures $A_1, \ldots, A_n$ of $S_1, \ldots, S_n$ are commuting selfadjoint operators, then $A = (A_1, \ldots, A_n)$ is the smallest selfadjoint extension of S.

If $n = 1$, and the deficiency indices are equal, this condition is also necessary. Indeed, if $S = S_1$ is a closed symmetric operator whose deficiency indices are equal, then $D(S)$ equals the intersection of the domains of all selfadjoint extensions of S, as proved in the Appendix of [Dev]. Assuming that S has a smallest selfadjoint extension $A = A_1$, we infer readily that $S = A$.

For $n > 1$, the smallest selfadjoint extension, whose structure is not yet well understood, may have unexpected properties. For instance, it follows from Theorem 4.4 of [BeTh] (see also Theorem II.3.4 below) that, for some tuples of symmetric operators, the smallest selfadjoint extension may exist in a Hilbert space strictly larger than the given one.

The next result is Theorem 3.3 from [Vas6].

Theorem II.3.3. *Let $S_1, \ldots, S_n$ be symmetric operators in a Hilbert space $\mathcal{H}$, such that $D = D(S_1) = \cdots = D(S_n)$, is invariant under $S_1, \ldots, S_n$. Let also $A_1, \ldots, A_n$ be commuting selfadjoint operators in a Hilbert space $\mathcal{K} \supset \mathcal{H}$, with $A_j \supset S_j$, $j = 1, \ldots, n$. Let*

$$\mathcal{K}_0 = \{(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x \ : \ x \in D, m \in \mathbb{Z}_+\},$$

which is a linear subspace of $\mathcal{K}$ invariant under $A_1, \ldots, A_n$.

The tuple $A = (A_1, \ldots, A_n)$ is a smallest selfadjoint extension of the tuple $S = (S_1, \ldots, S_n)$ if and only if

(1) *the subspace $\mathcal{K}_0$ is dense in $\mathcal{K}$;*
(2) *if $B_1, \ldots, B_n$ are commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$, such that $B_j \supset S_j, \ j = 1, \ldots, n$, then*

$$\|(1 + B_1^2)^{-m} \cdots (1 + B_n^2)^{-m} x\| = \|(1 + A_1^2)^{-m} \cdots (1 + A_n^2)^{-m} x\|$$

for all $x \in D, m \in \mathbb{Z}_+$.

Proof. Since D is invariant under $S_1, \ldots, S_n$ and $A_j \supset S_j$ for all j, it is easily seen that $\mathcal{K}_0$ is a linear subspace of $\mathcal{K}$, invariant under $A_1, \ldots, A_n$.

Assume that A is the smallest selfadjoint extension of S. Let $\mathcal{G}$ be the closure of $\mathcal{K}_0$ in $\mathcal{K}$, and set $C_j = A_j|\mathcal{K}_0$, $j = 1, \ldots, n$. We shall show that the closures of $C_1, \ldots, C_n$ are commuting selfadjoint operators in $\mathcal{G}$.

Put $r_m(A) = (1+A_1^2)^{-m}\cdots(1+A_n^2)^{-m}$, $m \in \mathbb{Z}_+$, and let $y = r_m(A)x,\ x \in D$, be fixed. For every index j we have:

$$y = r_m(A)x = (C_j \pm \mathrm{i})r_{m+1}(A)\prod_{k\neq j}(1+S_k^2)(S_j \mp \mathrm{i})x.$$

This shows that $R(C_j \pm \mathrm{i}) = \mathcal{K}_0$. According to Lemma II.2.1, the closure $\bar{C}_j$ of the operator C_j is selfadjoint in $\mathcal{G}$. Clearly, $\bar{C}_j \subset A_j$, implying that $(\mathrm{i}-\bar{C}_j)^{-1} \subset (\mathrm{i}-A_j)^{-1}$ for all indices j. From commutation of $(\mathrm{i}-A_j)^{-1}$, $(\mathrm{i}-A_k)^{-1}$ we obtain the commutation of $(\mathrm{i}-\bar{C}_j)^{-1}$, $(\mathrm{i}-\bar{C}_k)^{-1}$ for all indices j,k.

The hypothesis on A implies that $\bar{C}_j = A_j$ for all j. Therefore, the closure of $A_j|\mathcal{K}_0$ coincides with A_j for all j. In particular, the subspace $\mathcal{K}_0$ is dense in $\mathcal{K}$, which is condition (1).

If $B_1,\ldots,B_n$ are commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$ such that $B_j \supset S_j$, $j = 1,\ldots,n$, then, by the hypothesis on A, one must have $\mathcal{L} \supset K$ and $B_j \supset A_j$ for all j. Hence $(1+B_1^2)^{-m}\cdots(1+B_n^2)^{-m}x = (1+A_1^2)^{-m}\cdots(1+A_n^2)^{-m}x$ for all $x \in D$ and $m \in \mathbb{Z}_+$, i.e., condition (2) also holds.

Conversely, suppose that conditions (1) and (2) are satisfied. We shall show that A is the smallest selfadjoint extension of S.

Let $B_1,\ldots,B_n$ be commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$ such that $B_j \supset S_j$, $j = 1,\ldots,n$. We may define a linear map from $\mathcal{K}_0$ into $\mathcal{L}$ via the formula

$$\begin{gathered}\mathcal{K}_0 \to \mathcal{L},\\ (1+A_1^2)^{-m}\cdots(1+A_n^2)^{-m}x \mapsto (1+B_1^2)^{-m}\cdots(1+B_n^2)^{-m}x.\end{gathered} \tag{II.3.1}$$

Condition (2) shows that the map (II.3.1) is well defined and isometric. Moreover, $\mathcal{K}_0$ is dense in $\mathcal{K}$ via condition (1). Therefore, the map (II.3.1) extends to a linear isometry from $\mathcal{K}$ into $\mathcal{L}$, and we may identify $\mathcal{K}$ with a closed subspace of $\mathcal{L}$. Note that $A_j|\mathcal{K}_0 = B_j|\mathcal{K}_0$ for all j, via this identification. Let us show that the closure of $A_j|\mathcal{K}_0$ is A_j. Indeed, assuming the existence of a pair $u \oplus A_j u$ in the graph of A_j orthogonal to all pairs $r_m(A)x \oplus A_j r_m(A)x$ with $x \in D$ and $m \geq 0$ arbitrary (see the notation above), we infer that $\langle u, (1+A_j^2)r_m(A)x\rangle = 0$. This implies $u = 0$ because of the equality $(1+A_j^2)\mathcal{K}_0 = \mathcal{K}_0$. Therefore, $B_j \supset A_j$, $j = 1,\ldots,n$, showing that A is the smallest selfadjoint extension of S. $\square$

Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be a positive semi-definite n-sequence and let $L_\gamma\colon \mathcal{P}_n \to \mathbb{C}$ be the associated linear map, given by $L_\gamma(t^\alpha) = \gamma_\alpha$, $\alpha \in \mathbb{Z}_+^n$. Then we have a GN-space $\mathcal{H}$ associated with the pair $(\mathcal{P}_n, L_\gamma)$, which is obtained as the completion of the quotient $\mathcal{P}_n/\mathcal{N}$, where $\mathcal{N} = \{p \in \mathcal{P}_n \ :\ L_\gamma(p\bar{p}) = 0\}$.

As in some previous discussions, we define in $\mathcal{H}$ the operators

$$T_j(p+\mathcal{N}) = t_j p + \mathcal{N}, \quad p \in \mathcal{P}_n,\ j = 1,\ldots,n, \tag{II.3.2}$$

which are symmetric and densely defined, with $D(T_j) = \mathcal{P}_n/\mathcal{N}$ for all j.

The next result is Theorem 3.4 from [Vas6].

Theorem II.3.4. *Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be a moment n-sequence. The representing measure of γ is unique if and only if the tuple $T = (T_1, \ldots, T_n)$ has a smallest selfadjoint extension.*

Proof. Suppose that the tuple $T = (T_1, \ldots, T_n)$ has a smallest selfadjoint extension $A = (A_1, \ldots, A_n)$, acting in a Hilbert space $\mathcal{K} \supset \mathcal{H} = \mathcal{H}_\gamma$. If E_A is the spectral measure of A, then $\mu(*) = \langle E_A(*)(1 + \mathcal{N}, 1 + \mathcal{N}\rangle$ is a representing measure for γ (see the proof of Theorem II.2.3).

Let ν be another representing measure for γ. Let $B_j f(t) = t_j f(t)$, $t \in \mathbb{R}^n$, $f \in D(B_j) = \{g \in L^2(\nu) \ : \ t_j g \in L^2(\nu)\}$, $j = 1, \ldots, n$. Since $\int |p|^2 d\mu = \int |p|^2 d\nu$ for all polynomials $p \in \mathcal{P}_n$, the space $\mathcal{H}$ may be regarded as a closed subspace of $L^2(\nu)$, and $B_j \supset T_j$ for all j. Moreover, $B_1, \ldots, B_n$ are commuting selfadjoint operators. The hypothesis implies that $L^2(\nu) \supset \mathcal{K}$ and $B_j \supset A_j$, $j = 1, \ldots, n$. Therefore, if E_B is the spectral measure of $B = (B_1, \ldots, B_n)$, then $E_A = E_B|\mathcal{K}$, and

$$\nu(*) = \langle E_B(*)1, 1\rangle = \langle E_A(*)(1 + \mathcal{N}), 1 + \mathcal{N}\rangle = \mu(*).$$

Conversely, suppose that γ has a unique representing measure μ. Then $\mathcal{P}_n \subset L^2(\mu)$, and let $A_j f(t) = t_j f(t)$, $t \in \mathbb{R}^n$, $f \in D(A_j) = \{g \in L^2(\mu) \ : \ t_j g \in L^2(\mu)\}$, $j = 1, \ldots, n$, and $A = (A_1, \ldots, A_n)$. We shall use Theorem II.3.3 to prove that A is the smallest selfadjoint extension of T.

First of all, note that the space $\mathcal{K}_0$ from Theorem II.3.3 is equal in this case to the space $\mathcal{R}_\theta$ (defined in Lemma II.2.2). Since $\mathcal{R}_\theta$ is dense in $L^2(\mu)$, condition (1) from Theorem II.3.3 is fulfilled.

Next, let $B = (B_1, \ldots, B_n)$ be a tuple consisting of commuting selfadjoint operators in a Hilbert space $\mathcal{L} \supset \mathcal{H}$, such that $B_j \supset T_j$, $j = 1, \ldots, n$. If E_B is the spectral measure of B, then $\nu(*) = \langle E_B(*)1, 1\rangle$ is a representing measure for γ, and we must have $\nu = \mu$.

Let $r \in \mathcal{R}_\theta$ be arbitrary. We have

$$\|r(A)1\|^2 = \int |r(t)|^2 d\mu(t) = \int |r(t)|^2 d\langle E_B(t)1, 1\rangle = \|r(B)1\|^2.$$

Particularly, if $r_m(t) = (1 + t_1^2)^{-m} \cdots (1 + t_n^2)^{-m}$ and $p \in \mathcal{P}_n$, we obtain the equalities

$$\|r_m(A)p\| = \|r_m(A)p(A)1\| = \|r_m(B)p(B)1\| = \|r_m(B)p\|,$$

showing that condition (2) from Theorem II.3.3 is also fulfilled. By virtue of this theorem, the tuple A is the smallest selfadjoint extension of T. $\square$

Corollary II.3.5. *A positive semi-definite sequence $\gamma = (\gamma_k)_{k\in\mathbb{Z}_+}$ has a uniquely determined representing measure, say μ, if and only if the operator T given by* (II.3.2) *is essentially selfadjoint. In this case, the space of polynomial functions is dense in $L^2(\mu)$.*

Proof. The fact that T is essentially selfadjoint is well-known (see, for instance, [Dev]). It can be obtain from Theorem II.3.4, via Remark II.3.2(ii) and the fact that the operator T commutes with the natural involution on $\mathcal{H}$, and so its deficiency indices are equal (see

[DuSc], Theorem XII.4.18). As for the last assertion, we identify the space $\mathcal{H}_\gamma$ with a closed subspace of $L^2(\mu)$. Since the closure of T, say A, is selfadjoint in $\mathcal{H}_\gamma$, we obtain that $(1 + A^2)^{-m}p \in \mathcal{H}_\gamma$ for all $p \in \mathcal{P}_1$ and all integers $m \geq 0$. But $(1 + A^2)^{-m}p = (1+t^2)^{-m}p$, implying that $\mathcal{R}_\theta$ is in $\mathcal{H}_\gamma$. The density of $\mathcal{R}_\theta$ in $L^2(\mu)$ concludes the proof (see also [Fug]). $\square$

Other uniqueness results, related to [Vas3], can be found in [Vas6] as well.

II.4. Operator moment problems in unbounded semi-algebraic sets

We recall the notation from Lemma II.2.2. Let $\mathcal{P}_n = \mathcal{P}(\mathbb{R}^n)$ be the complex algebra of all polynomial functions on $\mathbb{R}^n$. Let $\theta_j(t) = (1+t_j^2)^{-1}$, $1 \leq j \leq n$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Let also $\{p_1, \ldots, p_m\}$ be a finite subset in $\mathcal{P}_n$ consisting of polynomial functions with real coefficients. We set $\theta_j(t) = (1 + p_j(t)^2)^{-1}$, $n+1 \leq j \leq n+m$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, and let $\theta = (\theta_j)_{1\leq j\leq n+m}$. Denote by $\mathcal{R}_\theta$ the complex algebra generated by $\mathcal{P}_n$ and by $(\theta_j)_{1\leq j\leq n+m}$.

We fix a Hilbert space $\mathcal{H}$ and a dense linear subspace $\mathcal{D}$ in $\mathcal{H}$. Let also $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$.

As in [Vas5], a sesquilinear map Λ on $\mathcal{R}_\theta \otimes \mathcal{D}$ is said to be a *moment form* if there exists a finite positive $\mathcal{L}(\mathcal{H})$-valued measure F on $\mathbb{R}^n$ (see [Ber]) such that

$$\Lambda(\phi, \psi) = \sum_{j,k} \int_{\mathbb{R}^n} f_j(t)\bar{g}_k(t) d\langle F(t)x_j, y_k\rangle$$

for all $\phi = \sum_j f_j \otimes x_j, \psi = \sum_k g_k \otimes g_k \in \mathcal{R}_\theta \otimes \mathcal{D}$. In this case, F is a *representing measure* for Λ.

If Λ is a moment form, then Λ is *positive semi-definite*, i.e., $\Lambda(\phi, \phi) \geq 0$ for all $\phi \in \mathcal{R}_\theta \otimes \mathcal{D}$.

A sesquilinear form Λ on $\mathcal{R}_\theta \otimes \mathcal{D}$ is said to be *unital* (respectively $\mathcal{R}_\theta$*-symmetric*) if $\Lambda(1 \otimes x, 1 \otimes x) = \|x\|^2$, $x \in D$ (respectively $\Lambda(r \cdot \phi, \psi) = \Lambda(\phi, \bar{r} \cdot \psi)$, $r \in \mathcal{R}_\theta$, $\phi, \psi \in \mathcal{R}_\theta \otimes \mathcal{D}$); see [Vas5] for more details.

Of course, in the definitions above, one can replace the algebra $\mathcal{R}_\theta$ by another algebra of functions, in particular by $\mathcal{P}_n$.

The next assertion is an operator version of Theorem II.2.3 (see also Theorem 2.2 and Theorem 2.8 from [Vas5]).

Theorem II.4.1. *Let Λ be a positive semi-definite form on $\mathcal{R}_\theta \otimes \mathcal{D}$, which is unital and $\mathcal{R}_\theta$-symmetric. Then Λ is a moment form having a uniquely determined representing measure.*

If, moreover, $\Lambda(p_k\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{R}_\theta \otimes \mathcal{D}$ and $k = 1, \ldots, m$, then the support of the representing measure of Λ is concentrated in the set $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$.

One can follow the lines of the proof of Theorem II.2.3. We omit the details.

Remark II.4.2. Let $\Theta = (\Theta_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be an n-sequence consisting of sesquilinear forms on $\mathcal{D}$, with Θ_0 the restriction to $\mathcal{D}$ of the scalar product of $\mathcal{H}$. The sequence Θ can be associated with a sesquilinear form Λ_Θ given by

$$\Lambda_\Theta(\phi, \psi) = \sum_{\alpha,\beta} \Theta_{\alpha+\beta}(x_\alpha, y_\beta)$$

for all $\phi = \sum\limits_\alpha t^\alpha \otimes x_\alpha$, $\psi = \sum\limits_\beta t^\beta \otimes y_\beta \in \mathcal{P}_n \otimes \mathcal{D}$.

We say that Θ is of *positive type*, respectively a *moment sequence*, if Λ_Θ is of positive type, respectively Λ_Θ is a moment sequence.

Let $\Gamma = (\Gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be an n-sequence of self-adjoint operators in $\mathcal{L}(H)$, with Γ_0 the identity on $\mathcal{H}$. The sequence Γ is associated with the hermitian form

$$\Lambda_\Gamma(\varphi, \psi) = \sum_{\alpha,\beta\in\mathbb{Z}_+^n} \langle \Gamma_{\alpha+\beta} x_\alpha, y_\beta \rangle,$$

where $\varphi = \sum\limits_\alpha t^\alpha \otimes x_\alpha$, $\psi = \sum\limits_\beta t^\beta \otimes y_\beta$ are arbitrary elements from $\mathcal{P}_n \otimes \mathcal{H}$.

We say that Γ is of *positive type*, respectively a *moment sequence*, if Λ_Γ is of positive type, respectively Λ_Γ is a moment sequence.

Theorem II.4.3. *Let $\Theta = (\Theta_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be an n-sequence consisting of sesquilinear forms on $\mathcal{D}$, with Θ_0 the restriction to $\mathcal{D}$ of the scalar product of $\mathcal{H}$.*

The n-sequence Θ is a moment sequence if and only if there exists a $2n$-sequence $\Omega = (\Omega_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ consisting of sesquilinear forms on $\mathcal{D}$, which is of positive type, with the following properties:

(1) $\Theta_\alpha = \Omega_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.
(2) $\Omega_{(\alpha,\beta)} = \Omega_{(\alpha,\beta+e_j)} + \Omega_{(\alpha+2e_j,\beta+e_j)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+^n$, $1 \le j \le n$.

In the affirmative case, the n-sequence Θ has a uniquely determined representing measure in $\mathbb{R}^n$ if and only if the $2n$-sequence Ω is unique.

The n-sequence Θ has a representing measure whose support is concentrated in $\mathbb{R}_+^n$ if and only if there exists a $2n$-sequence Ω satisfying (1) and (2), and which, in addition, has the property $\Lambda_\Omega(t_j\varphi, \varphi) \ge 0$ for all $\varphi \in \mathcal{P}_{2n} \otimes \mathcal{D}$ and $j = 1, \dots, n$.

This is an operator version of Corollaries II.2.5 and II.2.7 (see also Theorem 2.4 and Theorem 2.9 from [Vas5]). It can be directly proved or obtained from an operator version of Theorem II.2.4. We omit the details.

Corollary II.4.4. *Let $\Gamma = (\Gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(\Gamma_0 = 1_\mathcal{H})$ be an n-sequence of self-adjoint operators in $\mathcal{L}(H)$.*

The n-sequence Γ is a moment sequence if and only if there exists $2n$-sequence $\Delta = (\Delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ consisting of self-adjoint operators in $\mathcal{L}(H)$, which is of positive type, with the following properties:

(1) $\Gamma_\alpha = \Delta_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.

(2) $\Delta_{(\alpha,\beta)} = \Delta_{(\alpha,\beta+e_j)} + \Delta_{(\alpha+2e_j,\beta+e_j)}$ *for all* $\alpha, \beta \in \mathbb{Z}_+^n$, $1 \le j \le n$.

In this case, the n-sequence Γ has a uniquely determined representing measure in $\mathbb{R}^n$ if and only if the $2n$-sequence Δ is unique.

The n-sequence Γ has a representing measure whose support is concentrated in $\mathbb{R}_+^n$ if and only if there exists a $2n$-sequence Δ satisfying (1) and (2), and which, in addition, has the property that the $2n$-sequence $(\Delta_{(\alpha+e_j,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n}$ is of positive type for all $j = 1, \dots, n$.

This assertion coincides with Corollary 2.6 from [Vas5].

Following some ideas from [PuVa4], we may discuss a slightly different point of view, which reduces the number of parameters and gives a good control of the support of the representing measures.

Let $\mathcal{R}_{\theta_{\mathbf{p}}}$ be as in Lemma II.2.11.

Theorem II.4.5. *Let Λ be a positive semi-definite form on $\mathcal{R}_{\theta_{\mathbf{p}}} \otimes \mathcal{D}$, which is unital and $\mathcal{R}_{\theta_{\mathbf{p}}}$-symmetric. Then Λ is a moment form having a uniquely determined representing measure.*

If, moreover, $\Lambda(p_k\varphi, \varphi) \ge 0$ for all $\varphi \in \mathcal{R}_{\theta_{\mathbf{p}}} \otimes \mathcal{D}$ and $k = 1, \dots, m$, then the support of the representing measure of Λ is concentrated in the set $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$.

The proof follows the line of Theorem II.2.13. We omit the details. See also Theorem 2.8 from [Vas5].

Theorem II.4.6. *Let $\Theta = (\Theta_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ be an n-sequence consisting of hermitian forms on $\mathcal{D}$, with Θ_0 the restriction to $\mathcal{D}$ of the scalar product of $\mathcal{H}$. Let also $\mathbf{p} = (p_1, \dots, p_m)$ be an m-tuple of real polynomials from $\mathcal{P}_n$, where $p_k(t) = \sum_{\xi\in I_k} a_{k\xi} t^\xi$, $k = 1, \dots, m$, with $I_k \subset \mathbb{Z}_+^n$ finite for all k.*

The n-sequence Θ is a moment sequence and has a representing measure whose support is in the set $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a $(n+1)$-sequence of positive type $\Omega = (\Omega_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Z}_+^n\times\mathbb{Z}_+}$, consisting of hermitian forms on $\mathcal{D}$, with the following properties:

(1) $\Theta_\alpha = \Omega_{(\alpha,0)}$ *for all* $\alpha \in \mathbb{Z}_+^n$.

(2) $\Omega_{(\alpha,\beta)} = \Omega_{(\alpha,\beta+1)} + \sum_{j=1}^{n} \Omega_{(\alpha+2e_j,\beta+1)} + \sum_{k=1}^{m} \sum_{\xi,\eta\in I_k} a_{k\xi} a_{k\eta} \Omega_{(\alpha+\xi+\eta,\beta+1)}$ *for all* $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+$.

(3) $\Lambda_\Omega(p_k\varphi, \varphi) \ge 0$ *for all* $\varphi \in \mathcal{P}_{n+1} \otimes \mathcal{D}$ *and* $k = 1, \dots, m$.

The n-sequence Θ has a uniquely determined representing measure on $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$ if and only if the $(n+1)$-sequence Ω is unique.

This is the operator version of Theorem II.2.15. See Theorem 2.9 from [Vas5].

II.5. SUBNORMAL MULTIOPERATORS

In this section, the techniques developed in the previous ones are used to describe the existence of normal extensions for some tuples of unbounded operators. We mainly discuss the results from [Vas5] (see also [StSz1], [StSz2], [Dem3], [Dem4] for similar topics).

Two operators $T_j \colon D(T_j) \subset \mathcal{H} \to \mathcal{H}$, $j = 1, 2$, are said to be *permutable* ([IoVa]) if

$$T_1T_2x = T_2T_1x, \quad x \in D(T_1T_2) \cap D(T_2T_1).$$

A multioperator $T = (T_1, \ldots, T_n)$ in $\mathcal{H}$ is said to be *permutable* ([IoVa]) if T_j, T_k are permutable for all $j, k = 1, \ldots, n$.

According to [IoVa], Corollary 3.4, every multioperator consisting of commuting self-adjoint operators is permutable.

Remark II.5.1. Let $S = (S_1, \ldots, S_m)$ be permutable. Denote by Π_m the group of permutations of the set $\{1, \ldots, m\}$. If

$$D = \bigcap_{\pi \in \Pi_m} D(S_{\pi(1)} \cdots S_{\pi(m)}),$$

then one can easily see that

$$S_1 \cdots S_m x = S_{\pi(1)} \cdots S_{\pi(m)} x, \quad x \in D.$$

Particularly, given a permutable multioperator $T = (T_1, \ldots, T_n)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we can define, unambiguously, the operator $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ by setting $S_1 = \cdots = S_{\alpha_1} = T_1$, $S_{\alpha_1+1} = \cdots = S_{\alpha_1+\alpha_2} = T_2$, $S_{\alpha_1+\cdots+\alpha_{n-1}+1} = \cdots = S_{\alpha_1+\cdots+\alpha_n} = T_n$, on the subspace

$$D^\alpha(T) = \bigcap_{\pi \in \Pi_m} D(S_{\pi(1)} \cdots S_{\pi(m)}),$$

with $m = |\alpha|$. We also set

$$D^\infty(T) = \bigcap_{\alpha \in \mathbb{Z}_+^n} D^\alpha(T).$$

The direct extension of the concept of bounded subnormal (multi)operator (see also [StSz1]) leads to the following.

Definition II.5.2. Let $T = (T_1, \ldots, T_n)$ be a multioperator in $\mathcal{H}$. We say that T is *subnormal* if there exists a Hilbert space $\mathcal{K} \supset H$ and a multioperator $N = (N_1, \ldots, N_n)$ in $\mathcal{K}$ consisting of commuting normal operators such that $T_j \subset N_j$ for all $j = 1, \ldots, n$. In this case, N is said to be a *normal extension* of T.

We shall identify in the following the complex Euclidean space $\mathbb{C}^n$ with the real Euclidean space $\mathbb{R}^{2n}$ via the map

$$\mathbb{R}^{2n} \ni (t_1, \ldots, t_n, s_1, \ldots, s_n) \to (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

where $z_j = t_j + \mathrm{i}s_j$, $j = 1, \ldots, n$. With this identification, the space $\mathcal{P}_{2n}$ is itself identified with the space of all finite sums of the form $\sum_{\alpha,\beta} c_{\alpha,\beta}\bar{z}^\alpha z^\beta$.

The function $z \to z_j$ will be denoted by z_j, $j = 1 \ldots, n$.

Remark II.5.3. Suppose that the multioperator $T = (T_1, \ldots, T_n)$ is subnormal in $\mathcal{H}$ and let $N = (N_1, \ldots, N_n)$ be a normal extension of T in $\mathcal{K}$. As mentioned before, the multioperator N is permutable. This implies that T is also permutable. Then we have $D^\alpha(T) \subset D^\alpha(N)$ and $T^\alpha x = N^\alpha x$ for all $x \in D^\alpha(T)$ and $\alpha \in \mathbb{Z}_+^n$. In particular, $T^\alpha T^\beta = T^{\alpha+\beta}$ on $D^\infty(T)$ for all multi-indices α, β.

Let $\mathcal{D} \subset D^\infty(T)$ be a linear subspace dense in $\mathcal{H}$. Then the equation

$$\Lambda_T(\bar{z}^\alpha z^\beta \otimes x, \bar{z}^\xi z^\eta \otimes y) = \langle T^{\beta+\xi}x, T^{\alpha+\eta}y\rangle, \quad \alpha, \beta, \xi, \eta \in \mathbb{Z}_+^n, \;\; x, y \in \mathcal{D},$$

(extended by linearity) defines a sesquilinear form on $\mathcal{P}_{2n} \otimes \mathcal{D}$. As a matter of fact, the form Λ_T is positive semi-definite. Indeed, if E be the spectral measure attached to N, because $\mathcal{P}_{2n} \subset L^2(E_{x,x})$ for each $x \in D^\infty(N)$, it follows that E is a representing measure for the corresponding form Λ_N on the space $\mathcal{P}_{2n} \otimes D^\infty(N)$. Therefore, Λ_N is positive semi-definite. Since Λ_T is the restriction of Λ_N to $\mathcal{P}_{2n} \otimes \mathcal{D}$, it follows that Λ_T must be positive semi-definite too.

Remark II.5.4. The identification of $\mathbb{R}^{2n+1}$ with $\mathbb{C}^n \times \mathbb{R}$ permits the identification of $\mathcal{P}_{2n+1}$ with the algebra generated by the family of (linearly independent) monomials

$$\mathcal{G} = \{\bar{z}^\alpha z^\beta u^k \;:\; z \in \mathbb{C}^n, \;\; u \in \mathbb{R}, \;\; \alpha, \beta \in \mathbb{Z}_+^n, \;\; k \in \mathbb{Z}_+\}.$$

The basis $\mathcal{G}$ is invariant under the involution

$$\mathbb{Z}_+^{2n+1} \ni (\alpha, \beta, k) \to (\beta, \alpha, k) \in \mathbb{Z}_+^{2n+1},$$

and under multiplication as well.

Let $\Omega = (\Omega_{(\alpha,\beta),k})_{\alpha,\beta\in\mathbb{Z}_+^n, k\in\mathbb{Z}_+}$ be a $(2n+1)$-sequence of sesquilinear forms on $\mathcal{D} \subset H$ such that $\Omega_{(0,0),0}$ is the restriction of the scalar product to $\mathcal{D}$. Setting

$$\Lambda_\Omega(\bar{z}^{\alpha'} z^{\beta'} u^{k'} \otimes x', \bar{z}^{\alpha''} z^{\beta''} u^{k''} \otimes x'') = \Omega_{(\alpha'+\beta'', \alpha''+\beta'), k'+k''}(x', x'')$$

for all $\alpha', \alpha'', \beta'\beta'' \in \mathbb{Z}_+^n$, $k', k'' \in \mathbb{Z}_+$, $x', x'' \in \mathcal{D}$, we obtain, by extension, a sesquilinear form on $\mathcal{P}_{2n+1} \otimes \mathcal{D}$. We say that the sequence Ω is of *positive semi-definite* is the form Λ_Ω is positive semi-definite. Note that if Ω is of positive type, then

$$\text{(II.5.1)} \qquad \Omega_{(\alpha,\beta),k}(x, y) = \overline{\Omega_{(\beta,\alpha),k}(y, x)}, \quad \alpha, \beta \in \mathbb{Z}_+^n, \;\; k \in \mathbb{Z}_+, \;\; x, y \in \mathcal{D},$$

via a standard argument of positive semi-definiteness.

Theorem II.5.5. *Let $T = (T_1, \ldots, T_n)$ be a multioperator in $\mathcal{H}$ and let $\mathcal{D} \subset D(T_1) \cap \cdots \cap D(T_n)$ be a linear subspace. Assume that $\mathcal{D}$ is dense in $\mathcal{H}$ and invariant under $T_1, \ldots, T_n$.*

Then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a multioperator $N = (N_1, \ldots, N_n)$ in $\mathcal{K}$ consisting of commuting normal operators such that $N_j \supset T_j|\mathcal{D}$ for all $j = 1, \ldots, n$ if and only if there exists a $(2n+1)$-sequence $\Omega = (\Omega_{(\alpha,\beta),k})_{\alpha,\beta\in\mathbb{Z}_+^n, k\in\mathbb{Z}_+}$ of sesquilinear forms on $\mathcal{D}$ with the following properties:

(1) *$\Omega_{(0,0),0}$ is the restriction of the scalar product to $\mathcal{D}$.*
(2) *Ω is of positive type.*
(3) *$\Omega_{(0,e_j),0}(x,y) = \langle T_j x, y\rangle$ for all $x, y \in \mathcal{D}$ and $j = 1, \ldots, n$.*
(4) *$\Omega_{(e_j,e_j),0}(x,x) = \|T_j x\|^2$, $j = 1, \ldots, n$, $x \in \mathcal{D}$.*
(5) *$\Omega_{(\alpha,\beta),k} = \Omega_{(\alpha,\beta),k+1} + \sum_{j=1}^{n} \Omega_{(\alpha+e_j,\beta+e_j),k+1}$ for all $\alpha, \beta \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$.*

In the affirmative case, if in addition the closure of $T_j|\mathcal{D}$ extends T_j for all $j = 1, \ldots, n$, then T is subnormal.

Sketch of proof. Assume first that there exists a multioperator N with the stated properties. Let also E be the spectral measure of N, which acts on the Borel subsets in $\mathbb{C}^n$. We define the maps

$$\begin{aligned}&\Omega_{(\alpha,\beta),k}(x,y)\\ &\quad= \Big\langle \int \bar{z}^{\alpha} z^{\beta} (1+\|z\|^2)^{-k}\, dE(z)x, y\Big\rangle, \quad \alpha,\beta \in \mathbb{Z}_+^n,\;\; k \in \mathbb{Z}_+,\;\; x, y \in \mathcal{D}.\end{aligned} \tag{II.5.2}$$

We shall verify that the sequence $\Omega = (\Omega_{(\alpha',\alpha''),k})_{\alpha',\alpha''\in\mathbb{Z}_+^n, k\in\mathbb{Z}_+}$ has properties (1)–(5).

Properties (1), (3) and (4) are easily verified.

Let us show that Ω is of positive type (see Remark II.5.4). Indeed, if $\tau(z) = (z, (1+\|z\|^2)^{-1})$, $z \in \mathbb{C}^n$, then $F^{\tau}(*) = PE(\tau^{-1}(*))|\mathcal{H}$ is a representing measure for the form Λ_Ω, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, which follows from (II.5.2).

Property (5) is a consequence of the identity

$$[(1+\|z\|^2)(1+\|z\|^2)^{-1} - 1]\bar{z}^{\alpha} z^{\beta} (1+\|z\|^2)^{-k} = 0.$$

Conversely, if the $(2n+1)$-sequence Ω is given, and if Λ_Ω is the associated positive semi-definite form on $\mathcal{P}_{2n+1} \otimes \mathcal{D}$ (see Remark II.5.4), we use the GN-procedure to get the result. $\square$

Remark II.5.6. (1) Although the previous theorem is not an explicit characterization of the power subnormal multioperators (i.e., a characterization only in terms of the given multioperator), it leads to explicit characterizations, provided one can construct directly the sequence Ω. Such cases do exist, as will be shown in some work in progress.

(2) Our methods permit the control of the support of the spectral measure of a normal extension of a subnormal multioperator. As in Theorem II.2.15, let $\mathbf{p} = (p_1, \ldots, p_m)$ be an m-tuple of real polynomials from $\mathcal{P}_{2n}$, where $p_k(\bar{z}, z) = \sum_{\xi,\eta \in I_k} a_{k\xi\eta} \bar{z}^\xi z^\eta$, $k = 1, \ldots, m$, with $I_k \subset \mathbb{Z}_+^n$ finite for all k. If we replace condition (5) from Theorem II.5.5 by the stronger condition

$$\begin{aligned}\Omega_{(\alpha,\beta),\ell} = \Omega_{(\alpha,\beta),\ell+1} + \sum_{j=1}^{n} \Omega_{(\alpha+e_j\beta+e_j),\ell+1} \\ + \sum_{k=1}^{m} \sum_{\xi,\eta,\lambda,\mu \in I_k} a_{k\xi\lambda} \bar{a}_{k\eta\mu} \Omega_{(\xi+\mu+\alpha,\eta+\lambda+\beta),\ell+1}\end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $\ell \in \mathbb{Z}_+$, and add the condition

(6) $\Lambda_\Omega(p_k\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{P}_{2n+1} \otimes \mathcal{D}$ and $k = 1, \ldots, m$,

then we obtain that the support of a normal extension of T may be chosen to lie in $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$.

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