Spectrum and Analytic Functional Calculus in Real and Quaternionic Frameworks

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Abstract

We present an approach to the spectrum and analytic functional calculus for quaternionic linear operators, following the corresponding results concerning the real linear operators. In fact, the construction of the analytic functional calculus for real linear operators can be refined to get a similar construction for quaternionic linear ones, in a classical manner, using a Riesz-Dunford-Gelfand type kernel, and considering spectra in the complex plane. A quaternionic joint spectrum for pairs of operators is also discussed, and an analytic functional calculus is constructed, via a Martinelli type kernel in two variables.

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1 Introduction

In this text we consider $\mathbb{R}$-, $\mathbb{C}$-, and $\mathbb{H}$-linear operators, that is, real, complex and quaternionic linear operators, respectively.

While the spectrum of a linear operator is traditionally defined for complex linear operators, it is sometimes useful to have it also for real linear operators, as well as for quaternionic linear ones. The definition of the spectrum for a real linear operator goes seemingly back to Kaplansky (see [10]), and it can be stated as follows. If T is a real linear operator on the real vector space $\mathcal{V}$, a point $u+iv$ $(u,v\in\mathbb{R})$ is in the spectrum of T if the operator $(u-T)^2+v^2$ is not invertible on $\mathcal{V}$, where the scalars are identified with multiples of the identity on $\mathcal{V}$. Although this definition involves only operators acting in $\mathcal{V}$, the spectrum is, nevertheless, a subset of the complex plane. As a matter of fact, a motivation of this choice can be illustrated via the complexification of the space $\mathcal{V}$ (see Section 2).

The spectral theory for quaternionic linear operators is largely discussed in numerous work, in particular in the monographs [5] and [4], and in many of their references as well. In these works, the construction of an analytic functional calculus (called *S-analytic functional calculus*) means to associate to each function from the class of the so-called *slice hyperholomorphic* or *slice regular functions* a quaternionic linear operator, using a specific noncommutative kernel.

The idea of the present work is to replace the class of slice regular functions by a class holomorphic functions, using a commutative kernel of type Riesz-Dunford-Gelfand. These two classes are isomorphic via a Cauchy type transform (see [21]), and the image of the analytic functional calculus is the same, as one might expect (see Remark 8).

As in the case of real operators, the verbatim extension of the classical definition of the spectrum for quaternionic operators is not appropriate, and so a different definition using the squares of operators and real numbers was given, which can be found in [5] (see also [4]). We discuss this definition in our framework (see Definition 1), showing later that its "complex border" contains the most significant information, leading to the construction of an analytic functional calculus, equivalent to that obtained via the slice hyperholomorphic functions.

In fact, we first consider the spectrum for real operators on real Banach spaces, and sketch the construction of an analytic functional calculus for them, using some classical ideas (see Theorem 2). Then we extend this framework to a quaternionic one, showing that the approach from the real case can be easily adapted to the new situation.

As already mentioned, and unlike in [5] or [4], our functional calculus is obtained via a Riesz-Dunford-Gelfand formula, defined in a partially commutatative context, rather than the non-commutative Cauchy type formula used by previous authors. Our analytic functional calculus holds for a class of analytic operator valued functions, whose definition extends that of stem functions, and it applies, in particular, to a large family of quaternionic linear operators. Moreover, we can show that the analytic functional calculus obtained in this way is equivalent to the analytic functional calculus obtained in [5] or [4], in the sense that the images of these functional calculi coincide (see Remark 8).

We finally discuss the case of pairs of commuting real operators, in the spirit of [20], showing some connections with the quaternionic case. Specifically, we define a quaternionic spectrum for them and construct an analytic functional calculus using a Martinelli type formula, showing that for such a construction only a sort of "complex border" of the quaternionic spectrum should be used.

This work is just an introductory one. Hopefully, more contributions on this line will be presented in the future.

2 Spectrum and Conjugation

Let $\mathcal{A}$ be a unital real Banach algebra, not necessarily commutative. As mentioned in the Introduction, the (complex) spectrum of an element $a \in \mathcal{A}$ may be defined by the equality

$$\sigma_{\mathbb{C}}(a) = \{u + iv; (u - a)^2 + v^2 \text{ is not invertible}, u, v \in \mathbb{R}\}, \tag{1}$$

This set is *conjugate symmetric*, that is $u+iv \in \sigma_{\mathbb{C}}(a)$ if and only if $u-iv \in \sigma_{\mathbb{C}}(a)$. A known motivation of this definition comes from the following remark.

Fixing a unital real Banach algebra $\mathcal{A}$, we denote by $\mathcal{A}_{\mathbb{C}}$ the complexification of $\mathcal{A}$, which is given by $A_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, written simply as $\mathcal{A} + i\mathcal{A}$, where the sum is direct, identifying the element $1 \otimes a + i \otimes b$ with the element $a + ib$, for all $a, b \in \mathcal{A}$.

Then $\mathcal{A}_{\mathbb{C}}$ is unital complex algebra, which can be organized as a Banach algebra, with a (not necessarily unique) convenient norm. To fix the ideas, we recall that the product of two elements is given by $(a + ib)(c + id) = ac - bd + i(ad+bc)$ for all $a, b, c, d \in \mathcal{A}$, and the norm may be defind by $\|a+ib\| = \|a\|+\|b\|$, where $\| * \|$ is the norm of $\mathcal{A}$.

In the algebra $\mathcal{A}_{\mathbb{C}}$, the complex numbers commute with all elements of $\mathcal{A}$. Moreover, we have a *conjugation* given by

$$\mathcal{A}_{\mathbb{C}} \ni a + ib \mapsto a - ib \in \mathcal{A}_{\mathbb{C}},\ a, b \in \mathcal{A},$$

which is a unital conjugate-linear automorphism, whose square is the identity. In particular, an arbitrary element $a + ib$ is invertible if and only if $a - ib$ is invertible.

The usual spectrum, defined for each element $a \in \mathcal{A}_{\mathbb{C}}$, will be denoted by $\sigma(a)$. Regarding the algebra $\mathcal{A}$ as a real subalgebra of $\mathcal{A}_{\mathbb{C}}$, one has the following.

Lemma 1 *For every $a \in \mathcal{A}$ we have the equality $\sigma_{\mathbb{C}}(a) = \sigma(a)$.*

Proof. The result is well known but we give a short proof, because a similar idea will be later used.

Let $\lambda = u + iv$ with $u, v \in \mathbb{R}$ arbitrary. Assuming $\lambda - a$ invertible, we also have $\bar{\lambda} - a$ invertible. From the obvious identity

$$(u - a)^2 + v^2 = (u + iv - a)(u - iv - a),$$

we deduce that the element $(u - a)^2 + v^2$ is invertible, implying the inclusion $\sigma_{\mathbb{C}}(a) \subset \sigma(a)$.

Conversely, if $(u - a)^2 + v^2$ is invertible, then both $u + iv - a, u - iv - a$ are invertible via the decomposition from above, showing that we also have $\sigma_{\mathbb{C}}(a) \supset \sigma(a)$.

Remark 1 The spectrum $\sigma(a)$ with $a \in \mathcal{A}$ is always a conjugate symmetric set.

We are particularly interested to apply the discussion from above to the context of linear operators. The spectral theory for real linear operators is well known, and it is developed actually in the framework of linear relations (see [1]). Nevertheless, we present here a different approach, which can be applied, with minor changes, to the case of some quaternionic operators.

For a real or complex Banach space $\mathcal{V}$, we denote by $\mathcal{B}(\mathcal{V})$ the algebra of all bounded $\mathbb{R}$-(respectively $\mathbb{C}$-)linear operators on $\mathcal{V}$. As before, the multiples of the identity will be identified with the corresponding scalars.

Let $\mathcal{V}$ be a real Banach space, and let $\mathcal{V}_{\mathbb{C}}$ be its complexification, which, as above, is identified with the direct sum $\mathcal{V} + i\mathcal{V}$. Each operator $T \in \mathcal{B}(\mathcal{V})$ has a natural extension to an operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, given by $T_{\mathbb{C}}(x+iy) = Tx + iTy$, $x, y \in \mathcal{V}$. Moreover, the map $\mathcal{B}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is unital, $\mathbb{R}$-linear and multiplicative. In particular, $T \in \mathcal{B}(\mathcal{V})$ is invertible if and only if $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is invertible.

Fixing an operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we define the operator $S^{\flat} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ to be equal to CSC, where $C : \mathcal{V}_{\mathbb{C}} \mapsto \mathcal{V}_{\mathbb{C}}$ is the conjugation $x + iy \mapsto x - iy$, $x, y \in \mathcal{V}$. It is easily seen that the map $\mathcal{B}(\mathcal{V}_{\mathbb{C}}) \ni S \mapsto S^{\flat} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is a unital conjugate-linear automorphism, whose square is the identity on $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. Because $\mathcal{V} = \{u \in \mathcal{V}_{\mathbb{C}}; Cu = u\}$, we have $S^{\flat} = S$ if and only if $S(\mathcal{V}) \subset \mathcal{V}$. In particular, we have $T_{\mathbb{C}}^{\flat} = T_{\mathbb{C}}$. In fact, because of the representation

$$S = \frac{1}{2}(S + S^{\flat}) + i\frac{1}{2i}(S - S^{\flat}),\ S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}}),$$

where $(S + S^{\flat})(\mathcal{V}) \subset \mathcal{V}, i(S - S^{\flat})(\mathcal{V}) \subset \mathcal{V}$, the algebras $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$ and $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$ are isomorphic and they will be often identified, and $\mathcal{B}(\mathcal{V})$ will be regarded as a (real) subalgebra of $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In particular, if $S = U + iV$, with $U, V \in \mathcal{B}(\mathcal{V})$, we have $S^{\flat} = U - iV$, so the map $S \mapsto S^{\flat}$ is the conjugation of the complex algebra $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$ induced by the conjugation C of $\mathcal{V}_{\mathbb{C}}$.

For every operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we denote, as before, by $\sigma(S)$ its usual spectrum. As $\mathcal{B}(\mathcal{V})$ is a real algebra, the (complex) spectrum of an operator $T \in \mathcal{B}(\mathcal{V})$ is given by the equality (1):

$$\sigma_{\mathbb{C}}(T) = \{u + iv; (u - T)^2 + v^2 \text{ is not invertible}, u, v \in \mathbb{R}\}.$$

Corollary 1 *For every $T \in \mathcal{B}(\mathcal{V})$ we have the equality $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.*

3 Analytic Functional Calculus for Real Operators

Having a concept of spectrum for real operators, an important step for further development is the construction of an analytic functional calculus. Such a construction has been done actually in the context of real linear relations in [1]. In what follows we shall present a similar construction for real linear operators. Although the case of linear relations looks more general, unlike in [1], we perform our construction using a class of operator valued analytic functions insted of scalar valued analytic functions. Moreover, our arguments look simpler, and the construction is a model for a more general one, to get an analytic functional calculus for quaternionic linear operators.

If $\mathcal{V}$ is a real Banach space, and so each operator $T \in \mathcal{B}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$, which is compact and nonempty, one can use the classical

Riesz-Dunford functional calculus, in a slightly generalized form (that is, replacing the scalar-valued analytic functions by operator-valued analytic ones, which is a well known idea).

The use of vector versions of the Cauchy formula is simplified by adopting the following definition. Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a *Cauchy domain* (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented. Note that a Cauchy domain is bounded but not necessarily connected.

Remark 2 If $\mathcal{V}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{V})$, we have the usual analytic functional calculus for the operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ (see [6]). That is, in a slightly generalized form, and for later use, if $U \supset \sigma(T_{\mathbb{C}})$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic, we put

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain Δ containing $\sigma(T_{\mathbb{C}})$ in U. In fact, because $\sigma(T_{\mathbb{C}})$ is conjugate symmetric, we may and shall assume that both U and Γ are conjugate symmetric. Because the function $\zeta \mapsto F(\zeta)(\zeta - T_{\mathbb{C}})^{-1}$ is analytic in $U \setminus \sigma(T_{\mathbb{C}})$, the integral does not depend on the particular choice of the Cauchy domain Δ.

A natural question is to find an appropriate condition to we have $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$, which would imply the invariance of $\mathcal{V}$ under $F(T_{\mathbb{C}})$.

With the notation of Remark 2, we have the following.

Theorem 1 *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^{\flat} = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$.*

Proof. We use the notation from Remark 2, assuming, in addition, that Γ is conjugate symmetric as well. We put $\Gamma_{\pm} := \Gamma \cap \mathbb{C}_{\pm}$, where $\mathbb{C}_{+}$ (resp. $\mathbb{C}_{-}$) equals to $\{\lambda \in \mathbb{C}; \Im\lambda \geq 0\}$ (resp. $\{\lambda \in \mathbb{C}; \Im\lambda \leq 0\}$). We write $\Gamma_{+} = \cup_{j=1}^{m} \Gamma_{j+}$, where Γ_{j+} are the connected components of Γ_{+}. Similarly, we write $\Gamma_{-} = \cup_{j=1}^{m} \Gamma_{j-}$, where Γ_{j-} are the connected components of Γ_{-}, and Γ_{j-} is the reflexion of Γ_{j+} with respect of the real axis.

As Γ is a finite union of Jordan piecewise smooth closed curves, for each index j we have a parametrization $\phi_j : [0,1] \mapsto \mathbb{C}$, positively oriented, such that $\phi_j([0,1]) = \Gamma_{j+}$. Taking into account that the function $t \mapsto \overline{\phi_j(t)}$ is a parametrization of Γ_{j-} negatively oriented, and setting $\Gamma_j = \Gamma_{j+} \cup \Gamma_{j-}$, we can write

$$F_j(T_{\mathbb{C}}) := \frac{1}{2\pi i} \int_{\Gamma_j} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta =$$

$$\frac{1}{2\pi i} \int_0^1 F(\phi_j(t))(\phi_j(t) - T_{\mathbb{C}})^{-1} \phi_j'(t) dt$$

$$-\frac{1}{2\pi i}\int_0^1 F(\overline{\phi_j(t)})(\overline{\phi_j(t)}-T_{\mathbb{C}})^{-1}\overline{\phi_j'(t)}dt.$$

Therefore,

$$F_j(T_{\mathbb{C}})^\flat = -\frac{1}{2\pi i}\int_0^1 F(\phi_j(t))^\flat(\overline{\phi_j(t)}-T_{\mathbb{C}})^{-1}\overline{\phi_j'(t)}dt$$

$$+\frac{1}{2\pi i}\int_0^1 F(\overline{\phi_j(t)})^\flat(\phi_j(t)-T_{\mathbb{C}})^{-1}\phi_j'(t)dt.$$

According to our assumption on the function F, we obtain $F_j(T_{\mathbb{C}}) = F_j(T_{\mathbb{C}})^\flat$ for all j, and therefore

$$F(T_{\mathbb{C}})^\flat = \sum_{j=1}^m F_j(T_{\mathbb{C}})^\flat = \sum_{j=1}^m F_j(T_{\mathbb{C}}) = F(T_{\mathbb{C}}),$$

which concludes the proof.

Remark 3 If $\mathcal{A}$ is a unital real Banach algebra, $\mathcal{A}_{\mathbb{C}}$ its complexification, and $U \subset \mathbb{C}$ is open, we denote by $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ the algebra of all analytic $\mathcal{A}_{\mathbb{C}}$-valued functions. If U is conjugate symmetric, and $\mathcal{A}_{\mathbb{C}} \ni a \mapsto \bar{a} \in \mathcal{A}_{\mathbb{C}}$ is its natural conjugation, we denote by $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ the real subalgebra of $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ consisting of those functions F with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$. Adapting a well known terminology, such functions will be called ($\mathcal{A}_{\mathbb{C}}$-*valued*) *stem functions*.

When $\mathcal{A} = \mathbb{R}$, so $\mathcal{A}_{\mathbb{C}} = \mathbb{C}$, the space $\mathcal{O}_s(U, \mathbb{C})$ will be denoted by $\mathcal{O}_s(U)$, which is a real algebra. Note that $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ is also a bilateral $\mathcal{O}_s(U)$-module.

In the next result, we identify the algebra $\mathcal{B}(\mathcal{V})$ with a subalgebra of $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In ths case, when $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, we may write

$$F(T) = \frac{1}{2\pi i}\int_\Gamma F(\zeta)(\zeta - T)^{-1}d\zeta,$$

because the right hand side of this formula belongs to $\mathcal{B}(\mathcal{V})$, via Theorem 1.

The properties of the map $F \mapsto F(T)$, which can be called the *analytic functional calculus of* T, are summarized by the following.

Theorem 2 *Let $\mathcal{V}$ be a real Banach space, let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then the assignment*

$$\mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{V})$$

is an $\mathbb{R}$-linear map, and the map

$$\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}(\mathcal{V})$$

is a unital real algebra morphism.

Moreover, the following properties are true:

(1) For all $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$, we have $(Ff)(T) = F(T)f(T)$.

(2) For every polynomial $P(\zeta) = \sum_{n=0}^m A_n\zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^m A_nT^n \in \mathcal{B}(\mathcal{V})$.

Proof. The arguments are more or less standard (see [6]). The $\mathbb{R}$-linearity of the maps

$$\mathcal{O}_s(U,\mathcal{B}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{V}),\ \mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}(\mathcal{V}),$$

is clear. The second one is actually multiplicative, which follows from the multiplicativiry of the usual analytic functional calculus of T.

In fact, we have a more general property, specifically $(Ff)(T) = F(T)f(T)$ for all $F \in \mathcal{O}_s(U,\mathcal{B}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$. This follows from the equalities,

$$(Ff)(T) = \frac{1}{2\pi i}\int_{\Gamma_0} F(\zeta)f(\zeta)(\zeta - T)^{-1}d\zeta =$$

$$\left(\frac{1}{2\pi i}\int_{\Gamma_0} F(\zeta)(\zeta - T)^{-1}d\zeta\right)\left(\frac{1}{2\pi i}\int_{\Gamma} f(\eta)(\eta - T)^{-1}d\eta\right) = F(T)f(T),$$

obtained as in the classical case (see [6], Section VII.3), which holds because f is $\mathbb{C}$-valued and commutes with the operators in $\mathcal{B}(\mathcal{V})$. Here Γ, Γ_0 are the boundaries of two Cauchy domains Δ, Δ_0 respectively, such that $\Delta \supset \bar{\Delta}_0$, and Δ_0 contains $\sigma(T)$.

Note that, in particular, for every polynomial $P(\zeta) = \sum_{n=0}^m A_n\zeta^n$ with $A_n \in \mathcal{B}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^m A_n q^n \in \mathcal{B}(\mathcal{V})$ for all $T \in \mathcal{B}(\mathcal{V})$.

Example 1 Let $\mathcal{V} = \mathbb{R}^2$, so $\mathcal{V}_{\mathbb{C}} = \mathbb{C}^2$, endowed with its natural Hilbert space structure. Let us first observe that we have

$$S = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \Longleftrightarrow S^\flat = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{a}_3 & \bar{a}_4 \end{pmatrix},$$

for all $a_1, a_2, a_3, a_4 \in \mathbb{C}$.

Next we consider the operator $T \in \mathcal{B}(\mathbb{R}^2)$ given by the matrix

$$T = \begin{pmatrix} u & v \\ -v & u \end{pmatrix},$$

where $u, v \in \mathbb{R}, v \neq 0$. The extension $T_{\mathbb{C}}$ of the operator T to $\mathbb{C}^2$, which is a normal operator, is given by the same formula. Note that

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; (\lambda - u)^2 + v^2 = 0\} = \{u \pm iv\} = \sigma(T_{\mathbb{C}}).$$

Note also that the vectors $\nu_\pm = (\sqrt{2})^{-1}(1, \pm i)$ are normalized eigenvectors for $T_{\mathbb{C}}$ corresponding to the eigenvalues $u \pm iv$, respectively. The spectral projections of $T_{\mathbb{C}}$ corresponding to these eigenvalues are given by

$$E_\pm(T_{\mathbb{C}})\mathbf{w} = \langle \mathbf{w}, \nu_\pm\rangle\nu_\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

for all $\mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$.

Let $U \subset \mathbb{C}$ be an open set with $U \supset \{u \pm iv\}$, and let $F : U \mapsto \mathcal{B}(\mathbb{C}^2)$ be analytic. We shall compute explicitly $F(T_{\mathbb{C}})$. Let Δ be a Cauchy domain contained in U with its boundary Γ, and containing the points $u \pm iv$. Assuming $v > 0$, we have

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta =$$

$$F(u+iv)E_+(T_{\mathbb{C}}) + F(u-iv)E_-(T_{\mathbb{C}}) =$$

$$\frac{1}{2}F(u+iv)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2}F(u-iv)\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Assume now that $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$. Then we must have

$$(F(u+iv) - F(u-iv)^\flat)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = (F(u+iv)^\flat - F(u-iv))\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

We also have the equalities

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 2\begin{pmatrix} 1 \\ i \end{pmatrix}, \ \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = 2\begin{pmatrix} 1 \\ -i \end{pmatrix}, \ \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 0,$$

Using these equalities, we finally deduce that

$$(F(u+iv) - F(u-iv)^\flat)\begin{pmatrix} 1 \\ i \end{pmatrix} = 0,$$

and

$$(F(u-iv) - F(u+iv)^\flat)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 0,$$

which are necessary conditions for the equality $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$. As a matter of fact, this example shows, in particular, that the condition $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, used in Theorem 1, is sufficient but it might not be always necessary.

4 Analytic Functional Calculus for Quaternionic Operators

4.1 Quaternionic Spectrum

We now recall some known definitions and elementary facts (see, for instance, [5], Section 4.6, and/or [21]).

Let $\mathbb{H}$ be the abstract algebra of quaternions, which is the four-dimensional $\mathbb{R}$-algebra with unit 1, generated by the "imaginary units" $\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$, which satisfy

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{l},\ \mathbf{kl} = -\mathbf{lk} = \mathbf{j},\ \mathbf{lj} = -\mathbf{jl} = \mathbf{k},\ \mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1.$$

We may assume that $\mathbb{H} \supset \mathbb{R}$ identifying every number $x \in \mathbb{R}$ with the element $x1 \in \mathbb{H}$.

The algebra $\mathbb{H}$ has a natural multiplicative norm given by

$$\|\mathbf{x}\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_0^2},\ \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l},\ x_0, x_1, x_2, x_3 \in \mathbb{R},$$

and a natural involution

$$\mathbb{H} \ni \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto \mathbf{x}^* = x_0 - x_1\mathbf{j} - x_2\mathbf{k} - x_3\mathbf{l} \in \mathbb{H}.$$

Note that $\mathbf{x}\mathbf{x}^* = \mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2$, implying, in particular, that every element $\mathbf{x} \in \mathbb{H} \setminus \{0\}$ is invertible, and $\mathbf{x}^{-1} = \|\mathbf{x}\|^{-2}\mathbf{x}^*$.

For an arbitrary quaternion $\mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l},\ x_0, x_1, x_2, x_3 \in \mathbb{R}$, we set $\Re\mathbf{x} = x_0 = (\mathbf{x} + \mathbf{x}^*)/2$, and $\Im\mathbf{x} = x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} = (\mathbf{x} - \mathbf{x}^*)/2$, that is, the *real* and *imaginary part* of $\mathbf{x}$, respectively.

We consider the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ of the $\mathbb{R}$-algebra $\mathbb{H}$ (see also [8]), which will be identified with the direct sum $\mathbb{M} = \mathbb{H} + i\mathbb{H}$. Of course, the algebra $\mathbb{M}$ contains the complex field $\mathbb{C}$. Moreover, in the algebra $\mathbb{M}$, the elements of $\mathbb{H}$ commute with all complex numbers. In particular, the "imaginary units" $\mathbf{j}, \mathbf{k}, \mathbf{l}$ of the algebra $\mathbb{H}$ are independent of and commute with the imaginary unit i of the complex plane $\mathbb{C}$.

In the algebra $\mathbb{M}$, there also exists a natural conjugation given by $\bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{a} = \mathbf{b} + i\mathbf{c}$ is arbitrary in $\mathbb{M}$, with $\mathbf{b}, \mathbf{c} \in \mathbb{H}$ (see also [8]). Note that $\overline{\mathbf{a} + \mathbf{b}} = \bar{\mathbf{a}} + \bar{\mathbf{b}}$, and $\overline{\mathbf{a}\mathbf{b}} = \bar{\mathbf{a}}\bar{\mathbf{b}}$, in particular $\overline{r\mathbf{a}} = r\bar{\mathbf{a}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{M}$, and $r \in \mathbb{R}$. Moreover, $\bar{\mathbf{a}} = \mathbf{a}$ if and only if $\mathbf{a} \in \mathbb{H}$, which is a useful characterization of the elements of $\mathbb{H}$ among those of $\mathbb{M}$.

Remark 4 In the algebra $\mathbb{M}$ we have the identities

$$(\lambda - \mathbf{x}^*)(\lambda - \mathbf{x}) = (\lambda - \mathbf{x})(\lambda - \mathbf{x}^*) = \lambda^2 - \lambda(\mathbf{x} + \mathbf{x}^*) + \|\mathbf{x}\|^2 \in \mathbb{C},$$

for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{H}$. If the complex number $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is nonnull, then both element $\lambda - \mathbf{x}^*$, $\lambda - \mathbf{x}$ are invertible. Conversely, if $\lambda - \mathbf{x}$ is invertible, we must have $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ nonnull; otherwise we would have $\lambda = \mathbf{x}^* \in \mathbb{R}$, so $\lambda = \mathbf{x} \in \mathbb{R}$, which is not possible. Therefore, the element $\lambda - \mathbf{x} \in \mathbb{M}$ is invertible if and only if the complex number $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is nonnull. Hence, the element $\lambda - \mathbf{x} \in \mathbb{M}$ is not invertible if and only if $\lambda = \Re\mathbf{x} \pm i\|\Im\mathbf{x}\|$. In this way, the *spectrum* of a quaternion $\mathbf{x} \in \mathbb{H}$ is given by the equality $\sigma(\mathbf{x}) = \{s_\pm(\mathbf{x})\}$, where $s_\pm(\mathbf{x}) = \Re\mathbf{x} \pm \mathbf{i}\|\Im\mathbf{x}\|$ are the *eigenvalues* of $\mathbf{x}$ (see also [20, 21]).

The polynomial $P_{\mathbf{x}}(\lambda) = \lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is the *minimal polynomial* of $\mathbf{x}$. In fact, the equality $\sigma(\mathbf{y}) = \sigma(\mathbf{x})$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{H}$ is an equivalence relation in the algebra $\mathbb{H}$, which holds if and only if $P_{\mathbf{x}} = P_{\mathbf{y}}$. In fact, setting $\mathbb{S} = \{\kappa \in \mathbb{H}; \Re\kappa = 0, \|\kappa\| = 1\}$ (that is the unit sphere of purely imaginary quaternions), representig an arbitrary quaternion $\mathbf{x}$ under the form $x_0 + y_0\kappa_0$, with $x_0, y_0 \in \mathbb{R}$ and $\kappa_0 \in \mathbb{S}$, a quaternion $\mathbf{y}$ is equivalent to $\mathbf{x}$ if anf only if it is of the form $x_0 + y_0\kappa$ for some $\kappa \in \mathbb{S}$ (see [3] or [21] for some details).

Remark 5 Following [5], a *right* $\mathbb{H}$-*vector space* $\mathcal{V}$ is a real vector space having a right multiplication with the elements of $\mathbb{H}$, such that $(x+y)\mathbf{q} = x\mathbf{q}+y\mathbf{q}$, $x(\mathbf{q}+\mathbf{s}) = x\mathbf{q} + x\mathbf{s}$, $x(\mathbf{q}\mathbf{s}) = (x\mathbf{q})\mathbf{s}$ for all $x, y \in \mathcal{V}$ and $\mathbf{q}, \mathbf{s} \in \mathbb{H}$.

If $\mathcal{V}$ is also a Banach space the operator $T \in \mathcal{B}(\mathcal{V})$ is *right* $\mathbb{H}$-*linear* if $T(x\mathbf{q}) = T(x)\mathbf{q}$ for all $x \in \mathcal{V}$ and $\mathbf{q} \in \mathbb{H}$. The set of right $\mathbb{H}$ linear operators will be denoted by $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is, in particular, a unital real algebra.

In a similar way, one defines the concept of a *left* $\mathbb{H}$-*vector space*. A real vector space $\mathcal{V}$ will be said to be an $\mathbb{H}$-*vector space* if it is simultaneously a right $\mathbb{H}$- and a left $\mathbb{H}$-vector space. As noticed in [5], it is the framework of $\mathbb{H}$-vector spaces an appropriate one for the study of right $\mathbb{H}$-linear operators.

If $\mathcal{V}$ is $\mathbb{H}$-vector space which is also a Banach space, then $\mathcal{V}$ is said to be a *Banach* $\mathbb{H}$-*space*. In this case, we also assume that $R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$, and the map $\mathbb{H} \ni \mathbf{q} \mapsto R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ is norm continuous, where $R_{\mathbf{q}}$ be the right multiplication of the elements of $\mathcal{V}$ by a given quaternion $\mathbf{q} \in \mathbb{H}$. Similarly, if $L_{\mathbf{q}}$ is the left multiplication of the elements of $\mathcal{V}$ by the quaternion $\mathbf{q} \in \mathbb{H}$, we assume that $L_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ for all $\mathbf{q} \in \mathbb{H}$, and that the map $\mathbb{H} \ni \mathbf{q} \mapsto L_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ is norm continuous. Note also that

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}) = \{T \in \mathcal{B}(\mathcal{V}); TR_{\mathbf{q}} = R_{\mathbf{q}}T,\, \mathbf{q} \in \mathbb{H}\}.$$

To adapt the discussion regarding the real algebras to this case, we first consider the complexification $\mathcal{V}_{\mathbb{C}}$ of $\mathcal{V}$. Because $\mathcal{V}$ is an $\mathbb{H}$-bimodule, the space $\mathcal{V}_{\mathbb{C}}$ is actually an $\mathbb{M}$-bimodule, via the multiplications

$$(\mathbf{q}+i\mathbf{s})(x+iy) = \mathbf{q}x - \mathbf{s}y + i(\mathbf{q}y+\mathbf{s}x), (x+iy)(\mathbf{q}+i\mathbf{s}) = x\mathbf{q} - y\mathbf{s} + i(y\mathbf{q}+x\mathbf{s}),$$

for all $\mathbf{q}+i\mathbf{s} \in \mathbb{M}$, $\mathbf{q}, \mathbf{s} \in \mathbb{H}$, $x+iy \in \mathcal{V}_{\mathbb{C}}$, $x, y \in \mathcal{V}$. Moreover, the operator $T_{\mathbb{C}}$ is right $\mathbb{M}$-linear, that is $T_{\mathbb{C}}((x+iy)(\mathbf{q}+i\mathbf{s})) = T_{\mathbb{C}}(x+iy)(\mathbf{q}+i\mathbf{s})$ for all $\mathbf{q}+i\mathbf{s} \in \mathbb{M}$, $x+iy \in \mathcal{V}_{\mathbb{C}}$, via a direct computation.

Let C be the conjugation of $\mathcal{V}_{\mathbb{C}}$. As in the real case, for every $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we put $S^{\flat} = CSC$. The left and right multiplication with the quaternion $\mathbf{q}$ on $\mathcal{V}_{\mathbb{C}}$ will be also denoted by $L_{\mathbf{q}}, R_{\mathbf{q}}$, respectively, as elements of $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We set

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}) = \{S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}}); SR_{\mathbf{q}} = R_{\mathbf{q}}S,\, \mathbf{q} \in \mathbb{H}\},$$

which is a unital complex algebra containing all operators $L_{\mathbf{q}}, \mathbf{q} \in \mathbb{H}$. Note that if $S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, then $S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$. Indeed, because $CR_{\mathbf{q}} = R_{\mathbf{q}}C$, we also have we have $S^{\flat}R_{\mathbf{q}} = R_{\mathbf{q}}S^{\flat}$. In fact, as we have $(S+S^{\flat})(\mathcal{V}) \subset \mathcal{V}$ and $i(S-S^{\flat})(\mathcal{V}) \subset \mathcal{V}$, it folows that the algebras $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ are isomorphic, and they will be often identified, where $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} = \mathcal{B}^{\mathrm{r}}(\mathcal{V}) + i\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is the complexification of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is also a unital complex Banach algebra.

Looking at the Definition 4.8.1 from [5] (see also [4]), we give the folowing.

Definition 1 For a given operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the set

$$\sigma_{\mathbb{H}}(T) := \{\mathbf{q} \in \mathbb{H}; T^2 - 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2) \text{ not invertible}\}$$

is called the *quaternionic spectrum* (or simply the *Q-spectrum*) of T.

The complement $\rho_{\mathbb{H}}(T) = \mathbb{H} \setminus \sigma_{\mathbb{H}}(T)$ is called the *quaternionic resolvent* (or simply the *Q-resolvent*) of T.

Note that, if $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$), then $\{\mathbf{s} \in \mathbb{H}; \sigma(\mathbf{s}) = \sigma(\mathbf{q})\} \subset \sigma_{\mathbb{H}}(T)$.

Assuming that $\mathcal{V}$ is a Banach $\mathbb{H}$-space, then $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is a unital real Banach $\mathbb{H}$-algebra (that is, a Banach algebra which also a Banach $\mathbb{H}$-space), via the algebraic operations $(\mathbf{q}T)(x) = \mathbf{q}T(x)$, and $(T\mathbf{q})(x) = T(\mathbf{q}x)$ for all $\mathbf{q} \in \mathbb{H}$ and $x \in \mathcal{V}$. Hence the complexification $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ is, in particular, a unital complex Banach algebra. Also note that the complex numbers, regarded as elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, commute with the elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$. For this reason, for each $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the resolvent set

$$\rho_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; (T^2 - 2(\Re\lambda)T + |\lambda|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})\} =$$

$$\{\lambda \in \mathbb{C}; (\lambda - T_{\mathbb{C}})^{-1} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})\} = \rho(T_{\mathbb{C}}),$$

and the associated spectrum $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.

Clearly, there exists a strong connexion between $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$. In fact, the set $\sigma_{\mathbb{C}}(T)$ looks like a "complex border" of the set $\sigma_{\mathbb{H}}(T)$. Specifically, we can prove the following.

Lemma 2 *For every $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the equalities*

$$\sigma_{\mathbb{H}}(T) = \{\mathbf{q} \in \mathbb{H}; \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{q}) \neq \emptyset\}. \tag{2}$$

and

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \sigma(\mathbf{q}); \mathbf{q} \in \sigma_{\mathbb{H}}(T)\}. \tag{3}$$

Proof. Let us prove (2). If $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$, and so the $T^2 - 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2$ is not invertible, choosing $\lambda \in \{\Re\mathbf{q} \pm i\|\Im\mathbf{q}\|\} = \sigma(\mathbf{q})$, we clearly have $T^2 - 2(\Re\lambda)T + |\lambda|^2$ not invertible, implying $\lambda \in \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{q}) \neq \emptyset$.

Conversely, if for some $\mathbf{q} \in \mathbb{H}$ there exists $\lambda \in \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{q})$, and so $T^2 - 2(\Re\lambda)T + |\lambda|^2 = T^2 - 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2$ is not invertible, implying $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$.

We now prove (3). Let $\lambda \in \sigma_{\mathbb{C}}(T)$, so the operator $T^2 - 2(\Re\lambda)T + |\lambda|^2$ is not invertible. Setting $\mathbf{q} = \Re(\lambda) + \|\Im\lambda\|\kappa$, with $\kappa \in \mathbb{S}$, we have $\lambda \in \sigma(\mathbf{q})$. Moreover, $T^2 + 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2$ is not invertible, and so $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$.

Conversely, if $\lambda \in \sigma(\mathbf{q})$ for some $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$, then $\lambda \in \{\Re\mathbf{q} \pm i\|\Im(\mathbf{q})\|\}$, showing that $T^2 - 2\Re(\lambda)T + |\lambda|^2 = T^2 + 2(\Re\mathbf{q})T + \|\mathbf{q}\|^2$ is not invertible.

Remark As expected, the set $\sigma_{\mathbb{H}}(T)$ is nonempty and bounded, which follows easily from Lemma 2. It is also compact, as a consequence of Definition 1, because the set of invertible elements in $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is open.

We recall that a subset $\Omega \subset \mathbb{H}$ is said to be *spectrally saturated* (see [20],[21]) if whenever $\sigma(\mathbf{h}) = \sigma(\mathbf{q})$ for some $\mathbf{h} \in \mathbb{H}$ and $\mathbf{q} \in \Omega$, we also have $\mathbf{h} \in \Omega$. As noticed in [20] and [21], this concept coincides with that of *axially symmetric set*, introduced in [5].

Note that the subset $\sigma_{\mathbb{H}}(T)$ spectrally saturated.

4.2 Analytic Functional Calculus

If $\mathcal{V}$ is a Banach $\mathbb{H}$-space, because $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ is real Banach space, each operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$. Therefore, applying the corresponding result for real operators, we may construct an analytic functional calculus using the classical Riesz-Dunford functional calculus, in a slightly generalized form. In this case, our basic complex algebra is $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, endowed with the conjugation $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} \ni S \mapsto S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$.

Theorem 3 *Let $U \subset \mathbb{C}$ be open and conjugate symmetric. If $F : U \mapsto \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^{\flat} = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$.*

Both the statement and the proof of Theorem 3 are similar to those of Theorem 1, and will be omitted.

As in the real case, we may identify the algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$. In ths case, when $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) = \{F \in \mathcal{O}(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}); F(\bar{\zeta}) = F(\zeta)^{\flat}\ \forall \zeta \in U\}$ (see also Remark 3), we can write, via the previous Theorem,

$$F(T) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - T)^{-1} d\zeta \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}),$$

for a suitable choice of Γ.

The next result provides an *analytic functional calculus* for operators from the real algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

Theorem 4 *Let $\mathcal{V}$ be a real Banach space, let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then the map*

$$\mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is $\mathbb{R}$-linear, and the map

$$\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is a unital real algebra morphism.

Moreover, the following properties are true:

(1) *For all $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$, we have $(Ff)(T) = F(T)f(T)$.*

(2) *For every polynomial $P(\zeta) = \sum_{n=0}^{m} A_n \zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^{m} A_n T^n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$.*

The proof of this result is similar to that of Theorem 2 and will be omitted.

Remark 6 The algebra $\mathbb{H}$ is, in particular, a Banach $\mathbb{H}$-space. As already noticed, the left multiplications $L_{\mathbf{q}}$, $\mathbf{q} \in \mathbb{H}$, are elements of $\mathcal{B}^{\mathrm{r}}(\mathbb{H})$. In fact, the map $\mathbb{H} \ni \mathbf{q} \mapsto L_{\mathbf{q}} \in \mathcal{B}^{\mathrm{r}}(\mathbb{H})$ is a injective morphism of real algebras allowing the identification of $\mathbb{H}$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathbb{H})$.

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) := \{\lambda \in \mathbb{C}, \exists \mathbf{q} \in \Omega, \lambda \in \sigma(\mathbf{q})\}$, which is open and conjugate symmetric (see [21]). Denotig by $f_{\mathbb{H}}$ the function $\Omega \ni \mathbf{q} \mapsto f(\mathbf{q}), \mathbf{q} \in \Omega$, for every $f \in \mathcal{O}_s(U)$, we set

$$\mathcal{R}(\Omega) := \{f_{\mathbb{H}}; f \in \mathcal{O}_s(U)\},$$

which is a commutative real algebra. Defining the $F_{\mathbb{H}}$ in a similar way for each $F \in \mathcal{O}_s(U, \mathbb{M})$, we set

$$\mathcal{R}(\Omega, \mathbb{H}) := \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U, \mathbb{M})\},$$

which, according to the next theorem, is a right $\mathcal{R}(\Omega)$-module.

The next result is an analytic functional calculus for quaternions (see [21], Theorem 5), obtained as a particular case of Theorem 4 (see also its predecessor in [5]).

Theorem 5 *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega)$. The space $\mathcal{R}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}(\Omega, \mathbb{H})$ is a right $\mathcal{R}(\Omega)$-module, the map*

$$\mathcal{O}_s(U, \mathbb{M}) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H})$$

is a right module isomorphism, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega)$$

is an $\mathbb{R}$-algebra isomorphism.

Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Most of the assertions of Theorem 5 can be obtained directly from Theorem 4. The injectivity of the map $\mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega)$, as well as an alternative complete proof, can be obtained as in the proof of Theorem 5 from [21].

Remark 7 That Theorems 3 and 4 have practically the same proof as Theorems 1 and 2 (respectively) is due to the fact that all of them can be obtained as particular cases of more general results. Indeed, considering a unital real Banach algebra $\mathcal{A}$, and its complexification $\mathcal{A}_{\mathbb{C}}$, identifying $\mathcal{A}$ with a real subalgebra of $\mathcal{A}_{\mathbb{C}}$, for a function $F \in \mathcal{O}_s(U, A_{\mathbb{C}})$, where $U \subset \mathbb{C}$ is open and conjugate symmetric, the element $F(b) \in \mathcal{A}$ for each $b \in \mathcal{A}$ with $\sigma_{\mathbb{C}}(b) \subset U$. The assertion follows as in the proof of Theorem 1. The other results also have their counterparts. We omit the details.

Remark 8 The space $\mathcal{R}(\Omega, \mathbb{H})$ can be independently defined, and it consists of the set of all $\mathbb{H}$-valued functions, which are *slice regular* in the sense of [5], Definition 4.1.1. They are used in [5] to define a quaternionic functional calculus for quaternionic linear operators (see also [4]). Roughly speaking, given a quaternionic linear operator, each regular quaternionic-valued function defined in a neighborhood Ω of its quaternionic spectrum is associated with another quaternionic linear operator, replacing formally the quaternionic variable with that operator. This constraction is largely explained in the fourth chapter of [5].

Our Theorem 4 constructs an analytic functional calculus with functions from $\mathcal{O}_s(U,\mathcal{B}^{\rm r}(\mathcal{V})_{\mathbb{C}})$, where U is a a neighborhood of the complex spectrum of a given quaternionic linear operator, leading to another quaternionic linear operator, replacing formally the complex variable with that operator. We can show that those functional calculi are equivalent. This is a consequence of the fact that the class of regular quaternionic-valued function used by the construction in [5] is isomorphic to the class of analytic functions used in our Theorem 5. The advantage of our approach is its simplicity and a stronger connection with the classical approach, using spectra defined in the complex plane, and Cauchy type kernels partially commutative.

Let us give an argument concerning the equivalence of those analytic functional calculi. For an operator $T \in \mathcal{B}^{\rm r}(\mathcal{V})$, the so-called *right S-resolvent* is defined via the formula

$$S_R^{-1}(\mathbf{s},T) = -(T-\mathbf{s}^*)(T^2 - 2\Re(\mathbf{s})T + \|\mathbf{s}\|)^{-1},\ \mathbf{s} \in \rho_{\mathbb{H}}(T) \tag{4}$$

(see [5], formula (4.27)). Fixing an element $\kappa \in \mathbb{S}$, and a spectrally saturated open set $\Omega \subset \mathbb{H}$, for $\Phi \in \mathcal{R}(\Omega,\mathbb{H})$ one sets

$$\Phi(T) = \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} \Phi(\mathbf{s})d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T), \tag{5}$$

where $\Sigma \subset \Omega$ is a spectrally saturated open set containing $\sigma_{\mathbb{H}}(T)$, such that $\Sigma_\kappa = \{u+v\kappa \in \Sigma; u,v \in \mathbb{R}\}$ is a subset whose boundary $\partial(\Sigma_\kappa)$ consists of a finite family of closed curves, piecewise smooth, positively oriented, and $d\mathbf{s}_\kappa = -\kappa du \wedge dv$. Formula (5) is a (right) quaternionic functional calculus, as defined in [5], Section 4.10.

Because the space $\mathcal{V}_{\mathbb{C}}$ is also an $\mathbb{H}$-space, we may extend these formulas to the operator $T_{\mathbb{C}} \in \mathcal{B}^{\rm r}(\mathcal{V}_{\mathbb{C}})$, extending the operator $T \in \mathcal{B}^{\rm r}(\mathcal{V})$, by replacing T by $T_{\mathbb{C}}$ in formulas (4) and (5). For the function $\Phi \in \mathcal{R}(\Omega,\mathbb{H})$ there exists a function $F \in \mathcal{O}_s(U,\mathcal{B}^{\rm r}(\mathcal{V}_{\mathbb{C}})$ such that $F_{\mathbb{H}} = \Phi$. Denoting by Γ_κ the boundary of a Cauchy domain in $\mathbb{C}$ containing the compact set $\cup\{\sigma(\mathbf{s}); \mathbf{s} \in \overline{\Sigma_\kappa}\}$, we can write

$$\Phi(T_{\mathbb{C}}) = \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}\left(\frac{1}{2\pi i}\int_{\Gamma_\kappa} F(\zeta)(\zeta-\mathbf{s})^{-1}d\zeta\right)d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) =$$

$$\frac{1}{2\pi i}\int_{\Gamma_\kappa} F(\zeta)\left(\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)}(\zeta-\mathbf{s})^{-1}d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s},T_{\mathbb{C}})\right)d\zeta.$$

It follows from the complex linearity of $S_R^{-1}(\mathbf{s},T_{\mathbb{C}})$, and from formula (4.49) in [5], that

$$(\zeta-\mathbf{s})S_R^{-1}(\mathbf{s},T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s},T_{\mathbb{C}})(\zeta - T_{\mathbb{C}}) - 1,$$

whence

$$(\zeta - \mathbf{s})^{-1} S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1} + (\zeta - \mathbf{s})^{-1}(\zeta - T_{\mathbb{C}})^{-1},$$

and therefore,

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} (\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}})(\zeta - T_{\mathbb{C}})^{-1} +$$

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} (\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa (\zeta - T_{\mathbb{C}})^{-1} = (\zeta - T_{\mathbb{C}})^{-1},$$

because

$$\frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} d\mathbf{s}_\kappa S_R^{-1}(\mathbf{s}, T_{\mathbb{C}}) = 1 \text{ and } \frac{1}{2\pi}\int_{\partial(\Sigma_\kappa)} (\zeta - \mathbf{s})^{-1} d\mathbf{s}_\kappa = 0,$$

as in Theorem 4.8.11 from [5], since the $\mathbb{M}$-valued function $\mathbf{s} \mapsto (\zeta - \mathbf{s})^{-1}$ is analytic in a neighborhood of the set $\overline{\Sigma_\kappa} \subset \mathbb{C}_\kappa$ for each $\zeta \in \Gamma_\kappa$, respectively. Therefore $\Phi(T_{\mathbb{C}}) = \Phi(T)_{\mathbb{C}} = F(T_{\mathbb{C}}) = F(T)_{\mathbb{C}}$, implying $\Phi(T) = F(T)$.

5 Some Examples

Example 2 One of the simplest Banach $\mathbb{H}$-space is the space $\mathbb{H}$ itself. As already noticed (see Remark 6), taking $\mathcal{V} = \mathbb{H}$, so $\mathcal{V}_{\mathbb{C}} = \mathbb{M}$, and fixing an element $\mathbf{q} \in \mathbb{H}$, we may consider the operator $L_{\mathbf{q}} \in \mathcal{B}^{\mathrm{r}}(\mathbb{H})$, whose complex spectrum is given by $\sigma_{\mathbb{C}}(L_{\mathbf{q}}) = \sigma(\mathbf{q}) = \{\Re\mathbf{q} \pm i\|\Im\mathbf{q}\|\}$. If $U \subset \mathbb{C}$ is conjugate symmetric open set containing $\sigma_{\mathbb{C}}(L_{\mathbf{q}})$, and $F \in \mathcal{O}_s(U, \mathbb{M})$, then we have

$$F(L_{\mathbf{q}}) = F(s_+(\mathbf{q}))\iota_+(\mathfrak{s}_{\tilde{\mathbf{q}}}) + F(s_-(\mathbf{q}))\iota_-(\mathfrak{s}_{\tilde{\mathbf{q}}}) \in \mathbb{M}, \quad (6)$$

where $s_\pm(\mathbf{q}) = \Re\mathbf{q} \pm i\|\Im\mathbf{q}\|$, $\tilde{\mathbf{q}} = \Im\mathbf{q}$, $\mathfrak{s}_{\tilde{\mathbf{q}}} = \tilde{\mathbf{q}}\|\tilde{\mathbf{q}}\|^{-\mathbf{1}}$, and $\iota_\pm(\mathfrak{s}_{\tilde{\mathbf{q}}}) = 2^{-1}(1 \mp i\mathfrak{s}_{\tilde{\mathbf{q}}})$ (see [21], Remark 3).

Example 3 Let $\mathfrak{X}$ be a topological compact space, and let $C(\mathfrak{X}, \mathbb{M})$ be the space of $\mathbb{M}$-valued continuous functions on $\mathfrak{X}$. Then $C(\mathfrak{X}, \mathbb{H})$ is the real subspace of $C(\mathfrak{X}, \mathbb{M})$ consisting of $\mathbb{H}$-valued functions, which is also a Banach $\mathbb{H}$-space with respect to the operations $(\mathbf{q}F)(x) = \mathbf{q}F(x)$ and $(F\mathbf{q})(x) = F(x)\mathbf{q}$ for all $F \in C(\mathfrak{X}, \mathbb{H})$ and $x \in \mathfrak{X}$. Moreover, $C(\mathfrak{X}, \mathbb{H})_{\mathbb{C}} = C(\mathfrak{X}, \mathbb{H}_{\mathbb{C}}) = C(\mathfrak{X}, \mathbb{M})$.

We fix a function $\Theta \in C(\mathfrak{X}, \mathbb{H})$ and define the operator $T \in \mathcal{B}(C(\mathfrak{X}, \mathbb{H}))$ by the relation $(TF)(x) = \Theta(x)F(x)$ for all $F \in C(\mathfrak{X}, \mathbb{H})$ and $x \in \mathfrak{X}$. Note that $(T(F\mathbf{q}))(x) = \Theta(x)F(x)\mathbf{q} = ((TF)\mathbf{q})(x)$ for all $F \in C(\mathfrak{X}, \mathbb{H}), \mathbf{q} \in \mathbb{H}$, and $x \in \mathfrak{X}$. In othe words, $T \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))$. Note also that the operator T is invertible if and only if the function Θ has no zero in $\mathfrak{X}$.

Let us compute the Q-spectrum of T. According to Definition 1, we have

$$\rho_{\mathbb{H}}(T) = \{\mathbf{q} \in \mathbb{H}; (T^2 - 2\Re\mathbf{q}\, T + \|\mathbf{q}\|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))\}.$$

Consequently, $\mathbf{q} \in \sigma_{\mathbb{H}}(T)$ if and only if zero is in the range of the function

$$\tau(\mathbf{q}, x) := \Theta(x)^2 - 2\Re\mathbf{q}\,\Theta(x) + \|\mathbf{q}\|^2,\ x \in \mathfrak{X}.$$

Similarly,

$$\rho_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; (T^2 - 2\Re\lambda\, T + \|\lambda\|^2)^{-1} \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))\},$$

and so $\lambda \in \sigma_{\mathbb{C}}(T)$ if and only if zero is in the range of the function

$$\tau(\lambda, x) := \Theta(x)^2 - 2\Re\lambda\, \Theta(x) + |\lambda|^2,\ x \in \mathfrak{X}.$$

Looking for solutions $u + iv, u, v \in \mathbb{R}$, of the equation $(u - \Theta(x))^2 + v^2 = 0$, a direct calculation shows that $u = \Re\Theta(x)$ and $v = \pm\|\Im\Theta(x)\|$. Hence

$$\sigma_{\mathbb{C}}(T) = \{\Re\Theta(x) \pm i\|\Im\Theta(x)\|; x \in \mathfrak{X}\} = \cup_{x\in\mathfrak{X}}\sigma(\Theta(x)).$$

Of course, for every open conjugate symmetric subset $U \subset \mathbb{C}$ containing $\sigma_{\mathbb{C}}(T)$, and for every function $\Phi \in \mathcal{O}_c(U, \mathcal{B}(C(\mathfrak{X}, \mathbb{M})))$, we may construct the operator $\Phi(T) \in \mathcal{B}^{\mathrm{r}}(C(\mathfrak{X}, \mathbb{H}))$, using Theorem 4.

6 Quaternionic Joint Spectrum of Paires

In many applications, it is more convenient to work with matrix quaternions rather than with abstract quaternions. Specifically, one considers the injective unital algebra morphism

$$\mathbb{H} \ni x_1 + y_1\mathbf{j} + x_2\mathbf{k} + y_2\mathbf{l} \mapsto \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & x_1 - iy_1 \end{pmatrix} \in \mathbb{M}_2,$$

with $x_1, y_1, x_2, y_2 \in \mathbb{R}$, where $\mathbb{M}_2$ is the complex algebra of 2×2-matrix, whose image, denoted by $\mathbb{H}_2$ is the real algebra of matrix quaternions. The elements of $\mathbb{H}_2$ can be also written as matrices of the form

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z_1} \end{pmatrix},\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

A strong connection between the spectral theory of pairs of commuting operators in a complex Hilbert space and the algebra of quaternions has been firstly noticed in [17]. Another connection will be presented in this section.

If $\mathcal{V}$ is an arbitrary vector space, we denote by $\mathcal{V}^2$ the Cartesian product $\mathcal{V} \times \mathcal{V}$.

Let $\mathcal{V}$ be a real Banach space, and let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators. The extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ also consists of commuting operators. For simplicity, we set

$$Q(\mathbf{T}_{\mathbb{C}}) := \begin{pmatrix} T_{1\mathbb{C}} & T_{2\mathbb{C}} \\ -T_{2\mathbb{C}} & T_{1\mathbb{C}} \end{pmatrix}$$

which acts on the complex Banach space $\mathcal{V}_{\mathbb{C}}^2$.

We now extend the definition of the quaternionic resolvent set and spectrum for a single operator to the case of a pair of operators.

Definition 2 Let $\mathcal{V}$ be a real Banach space. For a given pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ of commuting operators, the set of those $Q(\mathbf{z}) \in \mathbb{H}_2$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, such that the operator

$$T_1^2 + T_2^2 - 2\Re z_1 T_1 - 2\Re z_2 T_2 + |z_1|^2 + |z_2|^2$$

is invertible in $\mathcal{B}(\mathcal{V})$ is said to be the *quaternionic joint resolvent* (or simply the *Q-joint resolvent*) of $\mathbf{T}$, and is denoted by $\rho_{\mathbb{H}}(\mathbf{T})$.

The complement $\sigma_{\mathbb{H}}(\mathbf{T}) = \mathbb{H}_2 \setminus \rho_{\mathbb{H}}(\mathbf{T})$ is called the *quaternionic joint spectrum* (or simply the *Q-joint spectrum*) of $\mathbf{T}$.

For every pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ we put $\mathbf{T}_{\mathbb{C}}^c = (T_{1\mathbb{C}}, -T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$, and for every pair $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ we put $\mathbf{z}^c = (\bar{z}_1, -z_2) \in \mathbb{C}^2$

Lemma 3 *A matrix quaternion $Q(\mathbf{z})\,(\mathbf{z} \in \mathbb{C}^2)$ is in the set $\rho_{\mathbb{H}}(\mathbf{T})$ if and only if the operators $Q(\mathbf{T}_{\mathbb{C}}) - Q(\mathbf{z})$, $Q(\mathbf{T}_{\mathbb{C}}^c) - Q(\mathbf{z}^c)$ are invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}}^2)$.*

Proof The assertion follows from the equalities

$$\begin{pmatrix} T_{1\mathbb{C}} - z_1 & T_{2\mathbb{C}} - z_2 \\ -T_{2\mathbb{C}} + \bar{z}_2 & T_{1\mathbb{C}} - \bar{z}_1 \end{pmatrix} \begin{pmatrix} T_{1\mathbb{C}} - \bar{z}_1 & -T_{2\mathbb{C}} + z_2 \\ T_{2\mathbb{C}} - \bar{z}_2 & T_{1\mathbb{C}} - z_1 \end{pmatrix} =$$

$$\begin{pmatrix} T_{1\mathbb{C}} - \bar{z}_1 & -T_{2\mathbb{C}} + z_2 \\ T_{2\mathbb{C}} - \bar{z}_2 & T_{1\mathbb{C}} - z_1 \end{pmatrix} \begin{pmatrix} T_{1\mathbb{C}} - z_1 & T_{2\mathbb{C}} - z_2 \\ -T_{2\mathbb{C}} + \bar{z}_2 & T_{1\mathbb{C}} - \bar{z}_1 \end{pmatrix} =$$
$$[(T_{1\mathbb{C}} - z_1)(T_{1\mathbb{C}} - \bar{z}_1) + (T_{2\mathbb{C}} - z_2)(T_{2\mathbb{C}} - \bar{z}_2)]\mathbf{I}.$$

for all $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, where $\mathbf{I}$ is the identity. Consequently, the operators $Q(\mathbf{T}_{\mathbb{C}}) - Q(\mathbf{z})$, $Q(\mathbf{T}_{\mathbb{C}}^c) - Q(\mathbf{z}^c)$ are invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}}^2)$ if and only if the operator $(T_{1\mathbb{C}} - z_1)(T_{1\mathbb{C}} - \bar{z}_1) + (T_{2\mathbb{C}} - z_2)(T_{2\mathbb{C}} - \bar{z}_2)$ is invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$. Because we have

$$T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2 =$$
$$[T_1^2 + T_1^2 - 2\Re z_1 T_1 - 2\Re z_2 T_2 + |z_1|^2 + |z_2|^2]_{\mathbb{C}},$$

the operators $Q(\mathbf{T}_{\mathbb{C}}) - Q(\mathbf{z})$, $Q(\mathbf{T}_{\mathbb{C}}^c) - Q(\mathbf{z}^c)$ are invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}}^2)$ if and only if the operator $T_1^2 + T_1^2 - 2\Re z_1 T_1 - 2\Re z_2 T_2 + |z_1|^2 + |z_2|^2$ is invertible in $\mathcal{B}(\mathcal{V})$.

Lemma 3 shows that we have the property $Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T})$ if and only if $Q(z^c) \in \sigma_{\mathbb{H}}(\mathbf{T}^c)$. Putting

$$\sigma_{\mathbb{C}^2}(\mathbf{T}) := \{\mathbf{z} \in \mathbb{C}^2; Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T})\},$$

the set $\sigma_{\mathbb{C}^2}(\mathbf{T})$ has a similar property, specifically $\mathbf{z} \in \sigma_{\mathbb{C}^2}(\mathbf{T})$ if and only if $\mathbf{z^c} \in \sigma_{\mathbb{C}^2}(\mathbf{T^c})$. As in the quaternionic case, the set $\sigma_{\mathbb{C}^2}(\mathbf{T})$ looks like a "complex border" of the set $\sigma_{\mathbb{H}}(\mathbf{T})$.

Remark 9 For the extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in B(V_{\mathbb{C}})^2$ of the commuting pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})$ there is an interesting connexion with the *joint spectral theory* of J. L. Taylor (see [15, 16]; see also [19]). Namely, if the operator

$T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2$ is invertible, then the point $\mathbf{z} = (z_1, z_2)$ belongs to the joint resolvent of $\mathbf{T}_\mathbb{C}$. Indeed, setting

$$R_j(\mathbf{T}_\mathbb{C}, \mathbf{z}) = (T_{j\mathbb{C}} - \bar{z}_j)(T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2)^{-1},$$

$q = Q(\mathbf{z})$ for $j = 1, 2$, we clearly have

$$(T_{1\mathbb{C}} - z_1)R_1(\mathbf{T}_\mathbb{C}, \mathbf{z}) + (T_{2\mathbb{C}} - z_2)R_2(\mathbf{T}_\mathbb{C}, \mathbf{z}) = \mathbf{I},$$

which, according to [15], implies that $\mathbf{z}$ is in the joint resolvent of $\mathbf{T}_\mathbb{C}$. A similar argument shows that, in this case the point $\mathbf{z}^c$ belongs to the joint resolvent of $\mathbf{T}_\mathbb{C}^c$. In addition, if $\sigma(T_\mathbb{C})$ designates the Taylor spectrum of $T_\mathbb{C}$, we have the inclusion $\sigma(T_\mathbb{C}) \subset \sigma_{\mathbb{C}^2}(\mathbf{T})$. In particular, for every complex-valued function f analytic in a neighborhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, the operator $f(\mathbf{T}_\mathbb{C})$ can be computed via Taylor's analytic functional calculus. In fact, we have a Martinelli type formula for the analytic functional calculus:

Theorem 6 *Let $\mathcal{V}$ be a real Banach space, let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators, let $U \subset \mathbb{C}^2$ be an open set, let $D \subset U$ be a bounded domain containing $\sigma_{\mathbb{C}^2}(\mathbf{T})$, with piecewise-smooth boundary Σ, and let $f \in \mathcal{O}(U)$. Then we have*

$$f(\mathbf{T}_\mathbb{C}) = \frac{1}{(2\pi i)^2} \int_\Sigma f(\mathbf{z}))L(\mathbf{z}, \mathbf{T}_\mathbb{C})^{-2}(\bar{z}_1 - T_{1\mathbb{C}})d\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}})d\bar{z}_1]dz_1 dz_2,$$

where

$$L(\mathbf{z}, \mathbf{T}_\mathbb{C}) = T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2.$$

Proof. Theorem III.9.9 from [19] implies that the map $\mathcal{O}(U) \ni f \mapsto f(\mathbf{T}_\mathbb{C}) \in \mathcal{B}(\mathcal{V}_\mathbb{C})$, defined in terms of Taylor's analytic functional calculus, is unital, linear, multiplicative, and ordinary complex polynomials in $\mathbf{z}$ are transformed into polynomials in $\mathbf{T}_\mathbb{C}$ by simple substitution, where $\mathcal{O}(U)$ is the algebra of all analytic functions in the open set $U \subset \mathbb{C}^2$, provided $U \supset \sigma(\mathbf{T}_\mathbb{C})$.

The only thing to prove is that, when $U \supset \sigma_{\mathbb{C}^2}(\mathbf{T})$, Taylor's functional calculus is given by the stated (canonical) formula. In order to do that, we use an argument from the proof of Theorem III.8.1 in [19], to make explicit the integral III(9.2) from [19] (see also [12]).

We consider the exterior algebra

$$\Lambda[e_1, e_2, \bar{\xi}_1, \bar{\xi}_2, \mathcal{O}(U) \otimes \mathcal{V}_\mathbb{C}] = \Lambda[e_1, e_2, \bar{\xi}_1, \bar{\xi}_2] \otimes \mathcal{O}(U) \otimes \mathcal{V}_\mathbb{C},$$

where the indeterminates e_1, e_2 are to be associated with the pair $\mathbf{T}_\mathbb{C}$, we put $\bar{\xi}_j = d\bar{z}_j$, $j = 1, 2$, and consider the operators $\delta = (z_1 - T_{1\mathbb{C}}) \otimes e_1 + (z_2 - T_{2\mathbb{C}}) \otimes e_2$, $\bar{\partial} = (\partial/\partial\bar{z}_1) \otimes \bar{\xi}_1 + (\partial/\partial\bar{z}_2) \otimes \bar{\xi}_2$, acting naturally on this exterior algebra, via the calculus with exterior forms.

To simplify the computation, we omit the symbol $\otimes$, and the exterior product will be denoted simply par juxtaposition.

We fix the exterior form $\eta = \eta_2 = fye_1e_2$ for some $f \in \mathcal{O}(U)$ and $y \in \mathcal{X}_{\mathbb{C}}$, which clearly satisfy the equation $(\delta + \bar{\partial})\eta = 0$, and look for a solution θ of the equation $(\delta + \bar{\partial})\theta = \eta$. We write $\theta = \theta_0 + \theta_1$, where θ_0, θ_1 are of degree 0 and 1 in e_1, e_2, respectively. Then the equation $(\delta + \bar{\partial})\theta = \eta$ can be written under the form $\delta\theta_1 = \eta$, $\delta\theta_0 = -\bar{\partial}\theta_1$, and $\bar{\partial}\theta_0 = 0$. Note that

$$\theta_1 = fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-1}[(\bar{z}_1 - T_{1\mathbb{C}})ye_2 - (\bar{z}_2 - T_{2\mathbb{C}})]ye_1$$

is visibly a solution of the equation $\delta\theta_1 = \eta$. Further, we have

$$\bar{\partial}\theta_1 = fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}[(z_1 - T_{1\mathbb{C}})(\bar{z}_2 - T_{2\mathbb{C}})y\bar{\xi}_1e_1 - (z_1 - T_{1\mathbb{C}})(\bar{z}_1 - T_{1\mathbb{C}})y\bar{\xi}_2e_1 +$$

$$(z_2 - T_{2\mathbb{C}})(\bar{z}_2 - T_{2\mathbb{C}})y\bar{\xi}_1e_2 - (z_2 - T_{2\mathbb{C}})(\bar{z}_1 - T_{1\mathbb{C}})y\bar{\xi}_2e_2] =$$

$$\delta[fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}(\bar{z}_1 - T_{1\mathbb{C}})y\bar{\xi}_2 - fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}(\bar{z}_2 - T_{2\mathbb{C}})y\bar{\xi}_1],$$

so we may define

$$\theta_0 = -fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}(\bar{z}_1 - T_{1\mathbb{C}})y\bar{\xi}_2 + fL(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}(\bar{z}_2 - T_{2\mathbb{C}})y\bar{\xi}_1.$$

Formula III(8.5) from [19] shows that

$$f(\mathbf{T}_{\mathbb{C}})y = -\frac{1}{(2\pi i)^2}\int_U \bar{\partial}(\phi\theta_0)dz_1dz_2 =$$

$$\frac{1}{(2\pi i)^2}\int_\Sigma f(\mathbf{z}))L(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}[(\bar{z}_1 - T_{1\mathbb{C}})yd\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}})yd\bar{z}_1]dz_1dz_2,$$

for all $y \in \mathcal{X}_{\mathbb{C}}$, via Stokes's formula, where ϕ is a smooth function such that $\phi = 0$ in a neighborhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, $\phi = 1$ on Σ and the support of $1 - \phi$ is compact.

Remark 10 (1) We may extend the previous functional calculus to $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$-valued analytic functions, setting, for such a function F and with the notation from above,

$$F(\mathbf{T}_{\mathbb{C}}) = \frac{1}{(2\pi i)^2}\int_\Sigma F(\mathbf{z}))L(\mathbf{z}, \mathbf{T}_{\mathbb{C}})^{-2}(\bar{z}_1 - T_{1\mathbb{C}})d\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}})d\bar{z}_1]dz_1dz_2.$$

In particular, if $F(\mathbf{z}) = \sum_{j,k\geq 0} A_{jk\mathbb{C}}z_1^jz_2^k$, with $A_{j,k} \in \mathcal{B}(\mathcal{V})$, where the series is convergent in neighborhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, we obtain

$$F(\mathbf{T}) := F(\mathbf{T}_{\mathbb{C}})|\mathcal{V} = \sum_{j,k\geq 0} A_{jk}T_1^jT_2^k \in \mathcal{B}(\mathcal{V}).$$

(2) The connexion of the spectral theory of pairs with the algebra of quaternions is even stronger in the case of complex Hilbert spaces. Specifically, if $\mathcal{H}$ is a complex Hilbert space and $\mathbf{V} = (V_1, V_2)$ is a commuting pair of bounded linear operators on $\mathcal{H}$, a point $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ is in the joint resolvent of $\mathbf{V}$ if and only if the operator $Q(\mathbf{V}) - Q(\mathbf{z})$ is invertible in $\mathcal{H}^2$, where

$$Q(\mathbf{V}) = \begin{pmatrix} V_1 & V_2 \\ -V_2^* & V_1^* \end{pmatrix}.$$

(see [17] for details). In this case, there is also a Martinelli type formula which can be used to construct the associated analytic functional calculus (see [18],[19]). An approach to such a construction in Banach spaces, by using a so-called splitting joint spectrum, can be found in [14].

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