Moment Problems in Hereditary Function Spaces

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Outline

ABSTRACT

We introduce a concept of hereditary family of multi-indices, and consider vector spaces of functions generated by families associated to such sets of multi-indices, called hereditary function spaces. Then, integral representations of some square positive functionals on hereditary function spaces, in particular truncated moment problems on hereditary spaces of polynomials, are investigated.

Notation

Some standard notation: $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ are the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively.

For a fixed integer $n \in \mathbb{N}$, the Cartesian product $\mathbb{Z}_+^n$ is said to be the set of *multi-indices of lenght n*.

Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, and $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ be arbitrary. Then $\mathbf{t}^{\mathbf{k}}$ means the monomial $t_1^{k_1} \ldots t_n^{k_n}$, and $|\mathbf{k}| = k_1 + \ldots + k_n$.

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A Polynomial Moment Problem

Let $\mathfrak{B}(\mathbb{R}^n)$ denote the set of all Borel subsets of $\mathbb{R}^n$, let $\mathbb{K}$ be an arbitrary finite subset of $\mathbb{Z}_+^n$, and let $\mathcal{P}_{\mathbb{K}}^n$ be the complex vector space spanned by the set of monomials $\{\mathbf{t}^{\mathbf{k}} : \mathbf{k} \in \mathbb{K}\}$. Let also $\{\gamma_{\mathbf{k}}; \mathbf{k} \in \mathbb{K}\}$ be an arbitrary set of real numbers.

The $\mathbb{K}$*-truncated multidimensional moment problem* consists of finding a non-negative measure μ on $\mathfrak{B}(\mathbb{R}^n)$ such that each monomial $\mathbf{t}^{\mathbf{k}}$ is μ-integrable, and

$$\gamma_{\mathbf{k}} = \int \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}), \quad \mathbf{k} \in \mathbb{K}. \tag{1}$$

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A General Moment Problem

The moment problems can be stated in a more abstract context, for functions more general that polynomials.

Let $(\Omega, \mathfrak{S})$ be a *measurable space*, and let $\mathcal{F}$ be a vector space consisting of $\mathfrak{S}$-measurable complex-valued functions on Ω, invariant under complex conjugation.

Given a linear map $\Lambda : \mathcal{F} \mapsto \mathbb{C}$, we investigate the existence of a positive measure μ on Ω such that

$$\Lambda(f) = \int_\Omega f(\omega) d\mu(\omega), \quad f \in \mathcal{F}.$$

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A Remark on the Finite Dimensionality

Thanks to an argument due to Stochel, in many situations of interest we may restrict ourselves to the case when the space $\mathcal{F}$ is finite dimensional. The finite dimensionality of the space $\mathcal{F}$ leads to the possibility to replace an existing measure μ by another one consisting of a finite number of atoms, via an idea going back to Tchakaloff.

More Notation

In the set $\mathbb{Z}_+^n$ we consider the order relation "$\leq$" given by $\mathbf{k} \leq \mathbf{p}$ whenever $k_j \leq p_j$, $j = 1, \ldots, n$, where $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$.

We define the maps $S_j : \mathbb{Z}_+^n \mapsto \mathbb{Z}_+^n$ via the formulas

$$S_j(k_1, \ldots, k_j, \ldots, k_n) = (k_1, \ldots, k_j + 1, \ldots, k_n) \tag{2}$$

for all $(k_1, \ldots, k_j, \ldots, k_n) \in \mathbb{Z}_+^n$, and $j = 1, \ldots, n$, which are, in fact, mutually commuting shifts.

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Hereditary Sets

Definition 1 A subset $\mathbb{K} \subset \mathbb{Z}_+^n$ is said to *hereditary* if for every $\mathbf{k} \in \mathbb{K}$ and $\mathbf{r} \in \mathbb{Z}_+^n$ such that $\mathbf{r} \leq \mathbf{k}$, we have $\mathbf{r} \in \mathbb{K}$.

Examples

(1) Let $\mathbb{K} = \mathbb{K}_m = \{\mathbf{k} \in \mathbb{Z}_+^n : |\mathbf{k}| \leq m\}$, for some fixed $m \in \mathbb{N}$. Then $\mathbb{K}$ is hereditary.

(2) Let $\mathbb{K} = \mathbb{K}_{\mathbf{d}} = \{\mathbf{k} \in \mathbb{Z}_+^n : \mathbf{k} \leq \mathbf{d}\}$, where $\mathbf{d} \in \mathbb{Z}_+^n$ is fixed. Then $\mathbb{K}$ is hereditary.

(3) Let $\mathbf{k}_1, \ldots, \mathbf{k}_r$ be fixed elements of $\mathbb{Z}_+^n$. Then the set

$$\bigcup_{j=1}^{r} \{\mathbf{k} \in \mathbb{Z}_+^n : \mathbf{k} \leq \mathbf{k}_j\}$$

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More about Hereditary Sets

Lemma 1 Let $\mathbb{K}_j \subset \mathbb{Z}_+^n$ $(j = 1, 2)$ **be hereditary. Then** $\mathbb{K} = \mathbb{K}_1 + \mathbb{K}_2 \subset \mathbb{Z}_+^n$ **is also hereditary.**

Remark 1 Let $\mathbb{K} \subset \mathbb{Z}_+^n$ be a hereditary finite set. We define, by recurrence, the sets of indices $\mathbb{K}_r = \{\mathbf{S}^{\mathbf{p}}\mathbf{k} : |\mathbf{p}| \le r, \mathbf{k} \in \mathbb{K}\}, r \ge 0\}$, so $\mathbb{K}_0 = \mathbb{K}$, and $\mathbf{S} = (S_1, \dots, S_n)$ is given by formula (2).

Note that we have $\mathbb{K}_0 \subset \mathbb{K}_1 \subset \mathbb{K}_2 \subset \cdots$. In fact, $\mathbb{K}_r = \{\mathbf{S}^{\mathbf{p}}\mathbf{S}^{\mathbf{k}}\mathbf{0}, |\mathbf{p}| \le r, \mathbf{k} \in \mathbb{K}\}$ for all $r \ge 0$.

Moreover, the set $\mathbb{K}_\infty = \cup_{r \ge 0}\mathbb{K}_r$ is also hereditary.

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Function Spaces

Let $(\Omega, \mathfrak{S})$ be a measurable space, and let also $\mathcal{M}_{\mathfrak{S}}(\Omega)$ be the algebra of all complex-valued $\mathfrak{S}$-measurable functions on Ω

A vector subspace $\mathcal{F} \subset \mathcal{M}_{\mathfrak{S}}(\Omega)$ such that $1 \in \mathcal{F}$ and if $f \in \mathcal{F}$, then $\bar{f} \in \mathcal{F}$, is said to be a *function space*.

Fixing a function space $\mathcal{F}$, let $\mathcal{F}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{F}$, which is itself a function space. We have $\mathcal{F} \subset \mathcal{F}^{(2)}$, and $\mathcal{F} = \mathcal{F}^{(2)}$ when $\mathcal{F}$ is an algebra.

If $\mathcal{T} \subset \mathcal{F}$ is a function subspace, then $\mathcal{RT}$ designates the "real part" of $\mathcal{T}$, that is $\{f \in \mathcal{T}; f = \bar{f}\}$.

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Some Examples

Important examples of function spaces are derived from the space $\mathcal{P}^n$ of all polynomials in $n \geq 1$ real variables, denoted as above by $t_1, \ldots, t_n$, with complex coefficients.

For every integer $m \geq 0$, let $\mathcal{P}^n_m$ be the subspace of $\mathcal{P}^n$ consisting of all polynomials p with $\deg(p) \leq m$. Both $\mathcal{P}^n_m$ and $\mathcal{P}^n$ are function spaces on $\mathbb{R}^n$.

In fact, $\mathcal{P}^n_m = \mathcal{P}^n_{\mathbb{K}_m}$, with $\mathbb{K}_m$ as in Example (1).
Similarly, $\mathcal{P}^n_{\mathbf{d}} = \mathcal{P}^n_{\mathbb{K}_{\mathbf{d}}}$, with $\mathcal{P}^n_{\mathbb{K}_{\mathbf{d}}}$ as in Example (2) is also a function space.

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Unital Square Positive Functionals

Definition 2 Let $\mathcal{F}$ be a function space and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{F}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{F}$;
(3) $\Lambda(1) = 1$.
This is, a *unital square positive functional*, briefly a *uspf*.

An example of a uspf is given by a probability measure μ and a functions space $\mathcal{F}$ on $(\Omega, \mathfrak{S})$, consisting of square μ-integrable functions. Then the map $\mathcal{F}^{(2)} \ni f \mapsto \int_\Omega f d\mu \in \mathbb{C}$ is a uspf.

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An Associated Hilbert Space

Fixing a function space $\mathcal{F}$ and a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, we have a semi-inner product given by the equality

$$\langle f, g \rangle_0 = \Lambda(f\bar{g}), \;\; f, g \in \mathcal{F}.$$

Then we set

$$\mathcal{I}_{\mathcal{F}} = \{f \in \mathcal{F}; \langle f, f \rangle_0 = 0\} = \{f \in \mathcal{F}; \Lambda(|f|^2) = 0\},$$

which is a vector subspace of $\mathcal{F}$. Moreover, the quotient $\mathcal{H}_{\mathcal{F}} := \mathcal{F}/\mathcal{I}_{\mathcal{F}}$ is an inner product space, with the inner product given by

$$\langle \hat{f}, \hat{g} \rangle = \Lambda(f\bar{g}), \;\; \hat{f} = f + \mathcal{I}_{\mathcal{F}}, \, \hat{g} = g + \mathcal{I}_{\mathcal{F}}.$$

When the quotient $\mathcal{H}_{\mathcal{F}}$ is finite dimensional, it is actually a Hilbert space, which will be said to be *the Hilbert space associated to* $(\mathcal{F}, \Lambda)$.

The Moment Problem in this Context

Problem The *moment problem* for a given uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{F}$ is a fixed function space on $(\Omega, \mathfrak{S})$, means to find necessary and sufficient conditions insuring the existence of a probability measure μ, defined on $\mathfrak{S}$, such that $\mathcal{F}$ consists of square μ-integrable functions and $\Lambda(f) = \int_\Omega f d\mu$, $f \in \mathcal{F}^{(2)}$. When such a measure μ exists, it is said to be a *representing measure of* Λ *(with support) in* Ω.

Tchakaloff's Property

When $\mathcal{F}$ is finite dimensional, more generally if $\mathcal{H}_{\mathcal{F}}$ is finite dimensional, and the uspf Λ on $\mathcal{F}^{(2)}$ has an arbitrary representing measure, then one expects that this measure may be replaced by an atomic one. As previously mentioned, such a property goes back to Tchakaloff.

An Extreme Case

In an extreme case, the atomic representing measure is unique *provided it exists*:

Proposition 1 Let $\mathcal{F}$ be a function space on Ω, and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf. Assume that the associated Hilbert space $\mathcal{H}_{\mathcal{F}}$ is finite dimensional. Then there exists at most one d-atomic representing measure of the uspf Λ, with support in Ω, having $d := \dim \mathcal{H}_{\mathcal{F}}$ atoms.

Hereditary Function Spaces

Let $\mathcal{F}$ be a a function space on Ω. Let also $\mathbb{K} \subset \mathbb{Z}_+^n$ be a subset containing $\mathbf{0} = (0, \ldots, 0)$, and let $\theta = (\theta_1, \ldots, \theta_n)$ be an n-tuple of elements of $\mathcal{RF}$.

Definition 3 If the family $\{\theta^\alpha : \alpha \in \mathbb{K}\}$ spans the space $\mathcal{F}$, we say that the function space $\mathcal{F}$ is $\mathbb{K}$-*generated by* θ.
If the set $\mathbb{K}$ is hereditary, we say that the function space $\mathcal{F}$ is *hereditary*.

Note that if $\mathcal{F}$ is $\mathbb{K}$-generated by θ, then $\mathcal{F}^{(2)}$ is $\mathbb{K}_2$-generated by θ, where $\mathbb{K}_2 = \mathbb{K} + \mathbb{K}$.

Hereditary Function Spaces

Let $\mathcal{F}$ be a a function space on Ω. Let also $\mathbb{K} \subset \mathbb{Z}_+^n$ be a subset containing $\mathbf{0} = (0, \ldots, 0)$, and let $\theta = (\theta_1, \ldots, \theta_n)$ be an n-tuple of elements of $\mathcal{RF}$.

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A Structure Remark

Remark 2 If $\mathcal{F}$ is a function space on Ω which is $\mathbb{K}$-generated by an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RF}$, we must have the equality, $\mathcal{F} = \{p \circ \theta; p \in \mathcal{P}_{\mathbb{K}}^n\}$, where θ is regarded as a function from Ω into $\mathbb{R}^n$, where $\mathcal{P}_{\mathbb{K}}^n$ is the complex space of polynomials $\mathbb{K}$-generated by $\mathbf{t} = (t_1, \ldots, t_n)$.

Idempotents

We fix a function space $\mathcal{F}$ and a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, having a finite dimensional associated Hilbert space $\mathcal{H}_{\mathcal{F}}$, whose norm is denoted by $\| * \|$. We denote by $\mathcal{RH}_{\mathcal{F}}$ the real Hilbert space given by the quotient $\mathcal{RF}/\mathcal{RI}_{\mathcal{F}}$.

Definition 4 An element $\iota \in \mathcal{RH}_{\mathcal{F}}$ is said to be an *idempotent* (*associated to* Λ) if

$$\|\iota\|^2 = \langle \iota, \hat{1} \rangle. \tag{3}$$

Set $\mathcal{ID}(\mathcal{H}_{\mathcal{F}}) := \{\iota \in \mathcal{RH}_{\mathcal{F}}; \langle \iota, \hat{1} \rangle \neq 0\}$, that is, the family of all nonnull idempotents of $\mathcal{H}_{\mathcal{F}}$.

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Lemma 2 If $\{\eta_1, \ldots, \eta_d\} \subset \mathcal{RH}_{\mathcal{F}}$ is an orthonormal basis with $\langle \eta_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$, the set $\{\langle \eta_1, \hat{1} \rangle \eta_1, \ldots \langle \eta_d, \hat{1} \rangle \eta_d\}$ is an orthogonal basis of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotents. Moreover,

$$\langle \eta_1, \hat{1} \rangle \eta_1 + \cdots + \langle \eta_d, \hat{1} \rangle \eta_d = \hat{1},$$

where $d = \mathrm{dim}\mathcal{H}_{\mathcal{F}}$.

Corollary 1 There are functions $b_1, \ldots, b_d \in \mathcal{RF}$ such that $\|b_j\|_0^2 = \langle b_j, 1\rangle_0 > 0$, $\langle b_j, b_k\rangle_0 = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, and every $f \in \mathcal{F}$ can be uniquely represented as

$$f = \sum_{j=1}^{d} \langle b_j, 1\rangle_0^{-1} \langle f, b_j\rangle_0 b_j + f_0,$$

with $f_0 \in \mathcal{I}_{\mathcal{F}}$ and $d = \mathrm{dim}\mathcal{H}_{\mathcal{F}}$.

Definition 3 Let $\mathcal{F}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \ldots, \theta_n) \subset \mathcal{RF}$, endowed with a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$. Assume that the space $\mathcal{H}_\mathcal{F}$ is finite dimensional, and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis $\mathcal{H}_\mathcal{F}$ consisting of idempotent elements.
We say that the tuple θ is $\mathcal{B}$-*multiplicative* if

$$\Lambda(\theta^{\mathbf{p}} b_j)\Lambda(\theta^{\mathbf{q}} b_j) = \Lambda(b_j)\Lambda(\theta^{\mathbf{p}+\mathbf{q}} b_j), \tag{4}$$

whenever $\mathbf{p} + \mathbf{q} \in \mathbb{K}$, $j = 1, \ldots, d$.

Theorem 1

Let $\mathcal{F}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \dots, \theta_n) \subset \mathcal{R}\mathcal{F}$, and endowed with a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$. Assume that the space $\mathcal{H}_{\mathcal{F}}$ is finite dimensional.
The uspf Λ has a representing measure on Ω consisting of $d := \dim \mathcal{H}_{\mathcal{F}}$ atoms if and only if there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$, consisting of idempotent elements, such that θ is $\mathcal{B}$-multiplicative, and $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$.

A Consequence

Corollary 2 Let $\mathcal{F}$ be a function space on Ω, spanned by the *n*-tuple θ. A uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ has a representing measure on Ω consisting of $d := \dim \mathcal{H}_{\mathcal{F}}$ atoms if either
(1) there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}$ consisting of idempotent elements such that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, or
(2) $\theta(\Omega) = \mathbb{R}^n$.

A Natural Isometry

Remark 3 Let $\mathcal{F}$ be a function space, and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf. We assume that the quotient space $\mathcal{H}_{\mathcal{F}} = \mathcal{F}/\mathcal{I}_{\mathcal{F}}$ is finite dimensional, that is, it is a Hilbert space. Let also $\mathcal{G}$ be a function subspace of $\mathcal{F}$, so $\Lambda|\mathcal{G}^{(2)}$ is a uspf. If $\mathcal{I}_{\mathcal{G}}$ and $\mathcal{H}_{\mathcal{G}}$ are defined by replacing $\mathcal{F}$ by $\mathcal{G}$, we have an isometry

$$\mathcal{H}_{\mathcal{G}} \ni g + \mathcal{I}_{\mathcal{G}} \mapsto g + \mathcal{I}_{\mathcal{F}} \in \mathcal{H}_{\mathcal{F}}. \tag{5}$$

In particular, $\mathcal{H}_{\mathcal{G}}$ is also a Hilbert space, and $\dim \mathcal{H}_{\mathcal{G}} \leq \dim \mathcal{H}_{\mathcal{F}}$

Dimensional Stability

We use the previous notation.

Definition 4 We say that the uspf Λ is *dimensionally stable at* $\mathcal{G}$ if $\dim \mathcal{H}_{\mathcal{G}} = \dim \mathcal{H}_{\mathcal{F}}$. In this case, the isometry (5) is surjective, that is, (5) is a unitary transformation.

This is equivalent to the fact that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ such that $f - g \in \mathcal{I}_{\mathcal{F}}$. Note that if $f \in \mathcal{RF}$, we can choose $g \in \mathcal{RG}$ such that $f - g \in \mathcal{RI}_{\mathcal{F}}$, because $\mathcal{I}_{\mathcal{F}} = \mathcal{RI}_{\mathcal{F}} + i\mathcal{RI}_{\mathcal{F}}$.

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This is equivalent to the fact that for every $f \in \mathcal{F}$ there exists a $g \in \mathcal{G}$ such that $f - g \in \mathcal{I}_{\mathcal{F}}$. Note that if $f \in \mathcal{R}\mathcal{F}$, we can choose $g \in \mathcal{R}\mathcal{G}$ such that $f - g \in \mathcal{R}\mathcal{I}_{\mathcal{F}}$, because $\mathcal{I}_{\mathcal{F}} = \mathcal{R}\mathcal{I}_{\mathcal{F}} + i\mathcal{R}\mathcal{I}_{\mathcal{F}}$.

A Key Result

Lemma 3 Let $\mathcal{F}$ be a function space, let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf, and let $\theta = \{\theta_1, \ldots, \theta_n\}$ be in $\mathcal{RF}$. Let also $\mathcal{G}$ be a function subspace of $\mathcal{F}$ such that $\theta_j \mathcal{G} \subset \mathcal{F}$ for all $j = 1, \ldots, n$, and that Λ is dimensionally stable at $\mathcal{G}$. Then

$$(\sum_{j=1}^{n} \theta_j \mathcal{I}_{\mathcal{F}}) \cap \mathcal{F} \subset \mathcal{I}_{\mathcal{F}}.$$

In particular, $\theta_j \mathcal{I}_{\mathcal{G}} \subset \mathcal{I}_{\mathcal{F}}$ for all $j = 1, \ldots, n$.

Induced Multiplication Operators

Remark 3 Let $J : \mathcal{H}_\mathcal{G} \mapsto \mathcal{H}_\mathcal{F}$ be the unitary transformation given by (5). We define the operators $M_j : \mathcal{H}_\mathcal{G} \mapsto \mathcal{H}_\mathcal{F}$ by the equalities $M_j(g + \mathcal{I}_\mathcal{G}) = \theta_j g + \mathcal{I}_\mathcal{F}$ for all $j = 1, \ldots, m$ and $g \in \mathcal{G}$, which are correctly defined. Next, we consider on the Hilbert space $\mathcal{H}_\mathcal{F}$ the linear operators $T_j = M_j J^{-1}$ for all $j = 1, \ldots, n$.

Note that, fixing $f \in \mathcal{F}$ and choosing $g \in \mathcal{G}$ such that $f - g \in \mathcal{I}_\mathcal{F}$, we have $T_j(f + \mathcal{I}_\mathcal{F}) = \theta_j g + \mathcal{I}_\mathcal{F}$ for all j.

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Self-adjointness

Proposition 2 The linear maps T_j, $j = 1, \ldots, n$, are self-adjoint operators, and $T = (T_1, \ldots, T_n)$ is a commuting tuple on $\mathcal{H}_F$.

Consequence of Dimensional Stability

Theorem 2

Let $\mathcal{G}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \ldots, \theta_n) \subset \mathcal{RG}$, where $\mathbb{K} \subset \mathbb{Z}_+^n$ is finite. Let also $\mathcal{F} = \sum_{j=0}^{n} \theta_j \mathcal{G}$ ($\theta_0 = 1$), and let $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$ be a uspf such that Λ is dimensionally stable at $\mathcal{G}$. Then we have:
(1) there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}_{\mathcal{F}}$ consisting of idempotent elements such that $\theta = (\theta_1, \ldots, \theta_n)$ is $\mathcal{B}$-multiplicative;
(2) the uspf Λ has a d-atomic representing measure with support in Ω, where $d := \dim \mathcal{H}_{\mathcal{F}}$, if and only $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$;
(3) if the uspf Λ has an atomic representing measure with support in Ω, this atomic measure is uniquely determined.

A Sequence of Hereditary Spaces

Remark 4 Fixing a $\mathbb{K}$-generated space $\mathcal{G}$ by a family $\theta = (\theta_1, \ldots, \theta_n) \subset \mathcal{RG}$, we have a sequence of hereditary function spaces $\{\mathcal{F}_r : r \geq 0\}$ given by

$$\mathcal{F}_r = \sum_{j=0}^{n} \theta_j \mathcal{F}_{r-1} \, (\theta_0 = 1, \; r \geq 1),$$

where $\mathcal{F}_0 = \mathcal{G}$

Extension of a USPF

Theorem 3

Let $\mathcal{G}$ be a hereditary function space $\mathbb{K}$-generated by $\theta = (\theta_1, \ldots, \theta_n) \subset \mathcal{RG}$ in Ω, where $\mathbb{K} \subset \mathbb{Z}_+^n$ is finite. Let also $\mathcal{F}_r = \sum_{j=0}^n \theta_j \mathcal{F}_{r-1}$ ($\theta_0 = 1$, $r \geq 1$), where $\mathcal{F}_0 = \mathcal{G}$. We fix a uspf $\Lambda : \mathcal{F}^{(2)} \mapsto \mathbb{C}$, supposed to be dimensionally stable at $\mathcal{G}$, where $\mathcal{F} = \mathcal{F}_1$. Also set $\mathcal{F}_\infty$ to be the space $\cup_{r \geq 0} \mathcal{F}_r$. Then $\mathcal{F}_\infty$ is a function space with $\mathcal{F}_\infty^{(2)} = \mathcal{F}_\infty$, and the uspf Λ can be uniquely extended to a uspf $\Lambda_\infty : \mathcal{F}_\infty \mapsto \mathbb{C}$, having a d-atomic measure in Ω, where $d = \dim(\mathcal{H}_\mathcal{G})$.

A Final Remark

Remark 5 From the proof of the previous theorem, we deduce that $\mathcal{H}_r := \mathcal{F}_r/\mathcal{I}_r (r \geq 1)$ are unitarily equivalent Hilbert spaces. This assertion is true even for $r = \infty$.

Merci beaucoup pour votre attention !