Quaternionic Cayley Transform Revisited

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Introduction

The classical Cayley transform

$$\kappa(t) = \frac{t-i}{t+i} = \frac{t^2-1-2it}{1+t^2}$$

is a bijective map between the real line $\mathbb{R}$ and the set $\mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ is the unit circle in the complex plane $\mathbb{C}$. This formula can be extended to more general situations as, for instance, that of (not necessarily bounded) symmetric operators in Hilbert spaces, replacing formally the real variable by such an operator, which yields a homonymic transform whose construction is due to von Neumann [20] (see also [12]). A Cayley type transform may be actually defined for larger classes of operators, which are no longer symmetric, as well as for other objects, in particular for some linear relations (see for example [5]).

In order to find a formula of this type, valid for normal or formally normal operators (see [3]), one is leaded to consider a quaternionic framework. An attempt to extend this transform using the context of quaternions has been made in [18]. In the present paper, we modify the basic definitions from [18], which allows us to get (in a simpler way) the properties of the quaternionic Cayley transform directly from those of von Neumann's Cayley transform, and refine some results from the quoted work. Unlike in [18], our construction does not require densely defined operators, which might be useful for potential applications; moreover, it can be associated to larger classes of operators (in particular, to some differential operators having matrix coefficients, related to the so-called Dirac operator; see Example 2.2(2) as well as [9]).

We found it useful to include an approach to the quaternionic Cayley transform in the algebra of quaternions, which is the simplest yet significant case, for a better understanding of the general topics, exhibited in the first section. In addition, some computations from this section are later used.

In the second section of this paper, we revisit the construction of the Cayley transform for some operators, in the quaternionic context, as we already mentioned above. The main result from this section (Theorem 2.7) is an extended version of Theorem 2.14 from [18], valid for not necessarily densely defined operators.

We recall that the image of a (not necessarily bounded) self-adjoint operator by the usual Cayley transform is a unitary operator U with the property that $I-U$ is injective, where I is the identity. The converse is also true [12]. Inspired by this property, in the third section of this paper, we describe the unitary operators lying in the range of the quaternionic Cayley transform, which are images of some (not necessarily bounded) normal operators. As a matter of fact, it is Theorem 3.7 from this section the main result of the present paper. An example related to this result is given in the last section.

A characterization of those Hilbert space (bounded) operators having normal extensions (on a possibly larger Hilbert space) was given many years ago (see [6] and [2]). The corresponding problem, stated for unbounded operators (see [3], [15], [16], etc.), happened to be more resistant. Nevertheless, there are some criteria, more or less explicit, describing certain unbounded operators (or families of unbounded operators) having normal extensions (see [1], [3], [17], [18], etc.). In fact, the main motivation of the introduction of the quaternionic Cayley transform in [18] was precisely to try to give an answer to this extension problem, with applications to some moment problems. In the fourth section of this work, we deal again with this extension problem, trying to improve the corresponding results from [18]. In particular, we do not require the invariance of the domain of definition under the given operator, and get results for both densely defined operators (Theorem 4.7) and not necessarily densely defined ones (Corollary 4.8). An application of these results is Theorem 4.10, extending Theorem 3.8 from [18], using quite a mild commutativity condition (designated by (c)), and continuing a series of related results appearing in [11], [7], [8], [14], etc. Other applications are to be expected in future work.

Finally, the last section of this work exhibits an example related to Theorem 3.7, showing that some moment problems with constraints may be approached with our methods.

Let us briefly recall the strategy from [18] concerning the normal extensions (see also Remark 4.9). Let $\mathcal{D}$ be a dense subspace in a Hilbert space

$\mathcal{H}$. Let also T be a densely defined linear operator in $\mathcal{H}$, with the property that T and its adjoint T^* are both defined on $\mathcal{D}$. Writing $T = A + iB$, with $A = (T + T^*)/2$ and $B = (T - T^*)/2i$, and so A and B are symmetric operators on $\mathcal{D}$, we can associate the operator T with the matrix operator

$$Q_T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

It is known (see [18], Theorem 3.7) that T is normal in $\mathcal{H}$ if and only if if the operator Q_T is normal in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Because our techniques, based on a quaternionic Cayley transform, give conditions to insure the existence of a normal extension for a matrix operator resembling to Q_T, we can go back to the operator T, which satisfies only some verifiable conditions. In fact, we have such results actually for the case when A and B are symmetric operators, defined on a not necessarily dense domain in $\mathcal{H}$. More information in this respect will be given in the last section of this work.

Let us finally note that the quaternionic algebra is intimately related also to the spectral theory of pairs of commuting operators (see [19]).

1 Cayley transforms in the algebra of quaternions

In this section, we present an approach to the Cayley transform in the algebra of quaternions.

Consider the 2×2-matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hamilton algebra of quaternions $\mathbb{H}$ will be identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$ of 2×2-matrices with complex entries, generated by the matrices $\mathbf{I}$, $i\mathbf{J}$, $\mathbf{K}$ and $i\mathbf{L}$. The embedding $\mathbb{H} \subset \mathbb{M}_2$ allows us to regard the elements of $\mathbb{H}$ as matrices and to perform some operations in $\mathbb{M}_2$ rather than in $\mathbb{H}$. (The matrices $\mathbf{J}$, $-i\mathbf{K}$ and $\mathbf{L}$, which are called the *Pauli matrices* in mathematical physics, do not belong to $\mathbb{H}$. Nevertheless, the matrices $\mathbf{J}$ and $\mathbf{L}$ play an important role in our development.)

If we put

$$Q(z) = Q(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

for every $z = (z_1, z_2) \in \mathbb{C}^2$, the set $\{Q(z); z \in \mathbb{C}^2\}$ is precisely the algebra of quaternions, because of the decomposition

$$Q(z) = (\mathrm{Re} z_1)\mathbf{I} + i(\mathrm{Im} z_1)\mathbf{J} + (\mathrm{Re} z_2)\mathbf{K} + i(\mathrm{Im} z_2)\mathbf{L}.$$

Note that

$$\mathbf{J}^* = \mathbf{J},\ \mathbf{K}^* = -\mathbf{K},\ \mathbf{L}^* = \mathbf{L},\ \mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I},$$

$$\mathbf{JK} = \mathbf{L} = -\mathbf{KJ},\ \mathbf{KL} = \mathbf{J} = -\mathbf{LK},\ \mathbf{JL} = \mathbf{K} = -\mathbf{LJ},$$

where the adjoints are computed in the Hilbert space $\mathbb{C}^2$ (endowed with the usual Euclidean norm).

Note also that $Q(z)Q(z)^* = Q(z)^*Q(z) = \|z\|^2\mathbf{I}$ for all $z \in \mathbb{C}^2$, and so $Q(z)$ is normal for each $z \in \mathbb{C}^2$. Moreover, $\|Q(z)\| = \|z\|$ for all $z \in \mathbb{C}^2$ and $Q(z)^{-1} = \|z\|^{-2}Q(z)^*$ for all $z \in \mathbb{C}^2 \setminus \{0\}$. In other words, every nonnull element of the algebra $\mathbb{H}$ is invertible.

Setting $\mathbf{E} = i\mathbf{J}$, we have $\mathbf{E}^* = -\mathbf{E}$, $\mathbf{E^2} = -\mathbf{I}$ and

$$Q(z) = (\mathrm{Re} z_1)\mathbf{I} + (\mathrm{Re} z_2)\mathbf{K} + \mathbf{E}((\mathrm{Im} z_1)\mathbf{I} + (\mathrm{Im} z_2)\mathbf{K})$$

for every $z = (z_1, z_2) \in \mathbb{C}^2$.

Similarly, setting $\mathbf{F} = i\mathbf{L}$, we have $\mathbf{F}^* = -\mathbf{F}$, $\mathbf{F^2} = -\mathbf{I}$ and

$$Q(z) = (\mathrm{Re} z_1)\mathbf{I} + (\mathrm{Re} z_2)\mathbf{K} + ((\mathrm{Im} z_2)\mathbf{I} + (\mathrm{Im} z_1)\mathbf{K})\mathbf{F}$$

for every $z = (z_1, z_2) \in \mathbb{C}^2$.

Definition 1.1 Let $a, b, c \in \mathbb{R}$, and let

$$S = S_{a,b,c} = \begin{pmatrix} a & b + ic \\ -b + ic & a \end{pmatrix} = a\mathbf{I} + b\mathbf{K} + ic\mathbf{L}.$$

The $\mathbf{E}$–*Cayley transform* of S is the matrix

$$U = (S - \mathbf{E})(S + \mathbf{E})^{-1} \in \mathbb{H}.$$

Let again $a, b, c \in \mathbb{R}$, and let

$$T = T_{a,b,c} = \begin{pmatrix} a + ic & b \\ -b & a - ic \end{pmatrix} = a\mathbf{I} + b\mathbf{K} + ic\mathbf{J}.$$

The $\mathbf{F}$–*Cayley transform* of T is the matrix

$$V = (T - \mathbf{F})(T + \mathbf{F})^{-1} \in \mathbb{H}.$$

Remark 1.2 (1) The matrices U, V are well defined since $S \neq -\mathbf{E}$ and $T \neq -\mathbf{F}$.

(2) The concepts from Definition 1.1 have similar properties to those of the quaternionic Cayley transform from [18], defined for matrices of the form $S = a\mathbf{I} + b\mathbf{K}$, via the formula $(S - Q')(S + Q')^{-1}$, with $Q' = Q(i\sqrt{2}/2, i\sqrt{2}/2)$.

(3) Let $Q = a\mathbf{I} + ib\mathbf{J} + c\mathbf{K} + id\mathbf{L}$, with $a, b, c, d \in \mathbb{R}$. We have $b = 0$ if and only if $\mathbf{J}Q = Q^*\mathbf{J}$, and $d = 0$ if and only if $\mathbf{L}Q = Q^*\mathbf{L}$.

Proposition 1.3 *Let $a, b, c \in \mathbb{R}$, and let $S = S_{a,b,c}$. The matrix*

$$U = (S - \mathbf{E})(S + \mathbf{E})^{-1}$$

is unitary and $U \neq \mathbf{I}$.

Conversely, given a unitary matrix $U \in \mathbb{H}$ with $U \neq \mathbf{I}$, there are $a, b, c \in \mathbb{R}$ such that $S = S_{a,b,c}$, where

$$S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}.$$

Moreover, the $\mathbf{E}$–Cayley transform of the matrix S is the unitary matrix U.

Proof. The proof uses some properties of the Cayley transform for self-adjoint matrices in $\mathbb{M}_2$ (which can be easily derived from [12], 13.17-13.21). Let $S = S_{a,b,c}$, and let $U = (S - \mathbf{E})(S + \mathbf{E})^{-1}$. The matrix $A = \mathbf{J}S$ is self-adjoint, via Remark 1.2(3). Threfore, the matrix $W = (A - i\mathbf{I})(A + i\mathbf{I})^{-1}$, which is the Cayley transform of A, is unitary and $\mathbf{I} - W$ is invertible. But we have

$$W = (\mathbf{J}S - i\mathbf{J}^2)(\mathbf{J}S + i\mathbf{J}^2)^{-1} = \mathbf{J}(S - \mathbf{E})(S + \mathbf{E})^{-1}\mathbf{J}.$$

Consequently, $U = \mathbf{J}W\mathbf{J}$ is a unitary matrix. Moreover, $\mathbf{I} - U = \mathbf{J}(\mathbf{I} - W)\mathbf{J}$ is invertible, which in $\mathbb{H}$ is equivalent to $U \neq \mathbf{I}$.

Conversely, let $U \in \mathbb{H}$ be unitary, with $U \neq \mathbf{I}$. Set $W = \mathbf{J}U\mathbf{J}$, which is a unitary matrix with $\mathbf{I} - W$ invertible. Therefore, the matrix $A = i(\mathbf{I} + W)(\mathbf{I} - W)^{-1}$ is well defined and self-adjoint, as an inverse Cayley transform. Setting $S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$, we have $S \in \mathbb{H}$ and

$$\mathbf{J}S = \mathbf{J}(\mathbf{J}^2 + \mathbf{J}W\mathbf{J})(\mathbf{J}^2 - \mathbf{J}W\mathbf{J})^{-1}\mathbf{E} = A.$$

In particular, we have $S = S_{a,b,c}$ for some $a, b, c \in \mathbb{R}$, via Remark 1.2(3).

Finally, the equation

$$S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E} = (\mathbf{I} - U)^{-1}(\mathbf{I} + U)\mathbf{E}.$$

has a unique solution $U = (S - \mathbf{E})(S + \mathbf{E})^{-1}$, which is precisely the $\mathbf{E}$–Cayley transform of S.

Remark 1.4 Let $a, b, c \in \mathbb{R}$, and let $S = S_{a,b,c}$. A direct calculation shows that the $\mathbf{E}$–Cayley transform of S is given by

$$U = (a^2 + b^2 + c^2 + 1)^{-1}((a^2 + b^2 + c^2 - 1)\mathbf{I} - 2c\mathbf{K} - 2ai\mathbf{J} + 2bi\mathbf{L})$$

$$= \frac{1}{a^2 + b^2 + c^2 + 1}\begin{pmatrix} a^2 + b^2 + c^2 - 1 - 2ai & -2c + 2bi \\ 2c + 2bi & a^2 + b^2 + c^2 - 1 + 2ai \end{pmatrix}.$$

Conversely, we give a unitary matrix $U \in \mathbb{H}$ such that $\mathbf{I} \neq U$. In fact, a unitary matrix $U \in \mathbb{H}$ is necessarily of the form

$$U = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},$$

with $z_1, z_2 \in \mathbb{C}$ and $|z_1|^2 + |z_2|^2 = 1$. As we also have $\mathbf{I} \neq U$, and so $\mathrm{Re} z_1 \neq 1$, the matrix $S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$ is given by

$$S = \frac{1}{\mathrm{Re} z_1 - 1}\begin{pmatrix} \mathrm{Im} z_1 & i z_2 \\ i\bar{z}_2 & \mathrm{Im} z_1 \end{pmatrix}.$$

This shows, in particular, that $S = S_{a,b,c}$, with $a = (\mathrm{Re} z_1 - 1)^{-1}\mathrm{Im} z_1$, $b = -(\mathrm{Re} z_1 - 1)^{-1}\mathrm{Im} z_2$ and $c = (\mathrm{Re} z_1 - 1)^{-1}\mathrm{Re} z_2$.

In fact, the matrix $S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$ may be called the *inverse* $\mathbf{E}$–*Cayley transform* of the unitary matrix U (see also [18] for a similar concept).

For the $\mathbf{F}$–Cayley transform, we have the following.

Proposition 1.5 *Let $a, b, c \in \mathbb{R}$, and let $T = T_{a,b,c}$. The matrix*

$$V = (T - \mathbf{F})(T + \mathbf{F})^{-1}$$

is unitary and $V \neq \mathbf{I}$.

Conversely, given a unitary matrix $V \in \mathbb{K}$ with $V \neq \mathbf{I}$, there are $a, b, c \in \mathbb{R}$ such that $T = T_{a,b,c}$, where

$$T = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{F}.$$

Moreover, the $\mathbf{F}$–Cayley transform of the matrix T is the unitary matrix V.

The proof is similar to that of Proposition 1.3 and will be omitted.

Remark 1.6 (1) Let $\mathbb{R}[\mathbf{K}] = \{a\mathbf{I} + b\mathbf{K} \in \mathbb{H}; a, b \in \mathbb{R}\}$, which is a commutative and involutive $\mathbb{R}$–subalgebra of $\mathbb{H}$. In fact, the assignment

$$\mathbb{C} \ni a + ib \mapsto a\mathbf{I} + b\mathbf{K} \in \mathbb{R}[\mathbf{K}]$$

is an isometric $*$–isomorphism, allowing us to identify the complex field $\mathbb{C}$ with the subalgebra $\mathbb{R}[\mathbf{K}] \subset \mathbb{H}$. Moreover, for a matrix $Q \in \mathbb{H}$, we have $Q \in \mathbb{R}[\mathbf{K}]$ if and only if $Q = -\mathbf{K}Q\mathbf{K}$.

(2) Let also

$$\mathcal{U}_{\mathbf{K}}(\mathbb{H}) = \{U \in \mathbb{H}; U \neq \mathbf{I},\, U \,\text{unitary},\, U^* = -\mathbf{K}U\mathbf{K}\}.$$

As we have, $Q^* = -\mathbf{K}Q\mathbf{K}$ for an arbitrary $Q = a\mathbf{I} + ib\mathbf{J} + c\mathbf{K} + id\mathbf{L}$ with $a, b, c, d \in \mathbb{R}$ if and only if $c = 0$, it follows

$$\mathcal{U}_{\mathbf{K}}(\mathbb{H}) = \{Q \in \mathbb{H}; Q = a\mathbf{I} + ib\mathbf{J} + id\mathbf{L},\ a, b, d \in \mathbb{R},$$

$$a^2 + b^2 + d^2 = 1,\ a \neq 1\}.$$

Theorem 1.7 *The map $\kappa_{\mathbf{E}} : \mathbb{R}[\mathbf{K}] \mapsto \mathcal{U}_{\mathbf{K}}(\mathbb{H})$ given by the formula $\kappa_{\mathbf{E}}(S) = (S - \mathbf{E})(S + \mathbf{E})^{-1}$ is bijective.*

Similarly, the map $\kappa_{\mathbf{F}} : \mathbb{R}[\mathbf{K}] \mapsto \mathcal{U}_{\mathbf{K}}(\mathbb{H})$ given by $\kappa_{\mathbf{F}}(S) = (S-\mathbf{F})(S+\mathbf{F})^{-1}$ is bijective.

In addition, $\kappa_{\mathbf{F}}(S) = \kappa_{\mathbf{E}}(-S\mathbf{K})$ for all $S \in \mathbb{R}[\mathbf{K}]$.

Proof. If $S \in \mathbb{R}[\mathbf{K}]$, then $U = \kappa_{\mathbf{E}}(S)$ is unitary and $U \neq \mathbf{I}$, via Proposition 1.3. Moreover, as $S \in \mathbb{R}[\mathbf{K}]$, and so $S = -\mathbf{K}S\mathbf{K}$, we have:

$$U^* = U^{-1} = (-\mathbf{K}S\mathbf{K} + \mathbf{KEK})(-\mathbf{K}S\mathbf{K} - \mathbf{KEK})^{-1} = -\mathbf{K}U\mathbf{K}.$$

Conversely, if $U \in \mathbb{H}$, $U \neq \mathbf{I}$, U is unitary and $U^* = -\mathbf{K}U\mathbf{K}$, then the matrix $S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$ has the property

$$S = (U^{-1} + \mathbf{I})(U^{-1} - \mathbf{I})^{-1}\mathbf{E} = (-\mathbf{K}U\mathbf{K} - \mathbf{K}^2)(-\mathbf{K}U\mathbf{K} + \mathbf{K}^2)^{-1}\mathbf{E} =$$

$$-\mathbf{K}(\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{EK} = -\mathbf{K}S\mathbf{K}.$$

The similar properties of the map $\kappa_{\mathbf{F}} : \mathbb{R}[\mathbf{K}] \mapsto \mathcal{U}_{\mathbf{K}}(\mathbb{H})$, which follow from the Proposition 1.5, as well as the verification of the equality $\kappa_{\mathbf{F}}(S) = \kappa_{\mathbf{E}}(-S\mathbf{K})$ for all $S \in \mathbb{R}[\mathbf{K}]$ are left to the reader.

Remark 1.8 We have already noted in Remark 1.6(1) that the map $\mathbb{C} \ni a + ib \mapsto a\mathbf{I} + b\mathbf{K} \in \mathbb{R}[\mathbf{K}]$ is an isometric $*$–isomorphism, allowing us to identify the complex field $\mathbb{C}$ with the subalgebra $\mathbb{R}[\mathbf{K}] \subset \mathbb{H}$. Therefore, the **E**–Cayley transform may be regarded as a map from $\mathbb{C}$ into (the unit ball of) $\mathbb{H}$. Specifically, the **E**–Cayley transform of $\mathbb{C}$ into $\mathbb{H}$ may be defined as

$$\mathcal{K}(w) = \frac{1}{|w|^2 + 1}\left((|w|^2 - 1)\mathbf{I} - 2(\mathrm{Re}w)\mathbf{E} + 2(\mathrm{Im}w)\mathbf{F}\right),\ w \in \mathbb{C},$$

a formula derived from Remark 1.4. In particular, for $w = t \in \mathbb{R}$, we obtain

$$\mathcal{K}(t) = \frac{1}{t^2 + 1}\left((t^2 - 1)\mathbf{I} - 2it\mathbf{E}\right) = (t\mathbf{I} - \mathbf{E})(t\mathbf{I} + \mathbf{E})^{-1},$$

which allows us to recapture the classical Cayley transform, via the identification of $\mathbb{C}$ with $\{u\mathbf{I} + v\mathbf{E} \in \mathbb{H},\ u, v \in \mathbb{R}\}$.

Similar formulas hold if we use the **F**–Cayley transform instead of the **E**–Cayley transform.

Remark 1.9 Most of the results from this section can be easily extended by replacing the real numbers with bounded commuting self-adjoint operators in a Hilbert space. We omit the details. In fact, such results are also particular cases of the corresponding statements in the next section.

2 Quaternionic Cayley transform of unbounded operators revisited

In this section, we extend the quaternionic Cayley transform(s), defined in the previous section, to some class of unbounded operators, acting on the Cartesian product of two Hilbert spaces. We shall mainly deal with the extension of the **E**–Cayley transform, the properties of the corresponding extension of the **F**–Cayley transform being similar.

Let $\mathcal{H}$ be a complex Hilbert space, whose scalar product is denoted by $\langle *, * \rangle$, and whose norm is denoted by $\| * \|$. We especially work in the Hilbert space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$, whose scalar product, naturally induced by that from $\mathcal{H}$, is denoted by $\langle *, * \rangle_2$, and whose norm is denoted by $\| * \|_2$.

The matrices from $\mathbb{M}_2$ naturally act on $\mathcal{H}^2$ simply by replacing their entries with the corresponding multiples of the identity on $\mathcal{H}$. In particular, the matrices $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}, \mathbf{E}, \mathbf{F}$, defined in the previous section, naturally act on $\mathcal{H}^2$, and we still have the relations

$$\mathbf{J}^* = \mathbf{J},\ \mathbf{K}^* = -\mathbf{K},\ \mathbf{L}^* = \mathbf{L},\ \mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I},$$

$$\mathbf{J}\mathbf{K} = \mathbf{L} = -\mathbf{K}\mathbf{J},\ \mathbf{K}\mathbf{L} = \mathbf{J} = -\mathbf{L}\mathbf{K},\ \mathbf{J}\mathbf{L} = \mathbf{K} = -\mathbf{L}\mathbf{J},$$

$$\mathbf{E}^* = -\mathbf{E},\ \mathbf{E}^{\mathbf{2}} = -\mathbf{I},\ \mathbf{F}^* = -\mathbf{F},\ \mathbf{F}^{\mathbf{2}} = -\mathbf{I}.$$

We fix some notation and terminology for Hilbert space (always linear) operators. For an operator T acting in $\mathcal{H}$, we denote by $D(T)$ its domain of definition. The range of T is denoted by $R(T)$, while $N(T)$ stands for the kernel of T. If T is closable, the closure of T will be denoted by $\bar{T}$. If T is densely defined, let T^* be its adjoint. If T_2 extends T_1, we write $T_1 \subset T_2$.

Lemma 2.1 *Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$. Suppose that the operator $\mathbf{J}S$ is symmetric. Then we have*

$$\|(S \pm \mathbf{E})x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S).$$

If, in addition, $\mathbf{J}D(S) \subset D(S)$, we have

$$\|(S \pm \mathbf{E})\mathbf{E}x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S),$$

if and only if $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$.

Proof. Note that

$$\|(S \pm \mathbf{E})x\|_2^2 = \|Sx\|_2^2 + \langle Sx, \pm i\mathbf{J}x\rangle_2 + \langle \pm i\mathbf{J}x, Sx\rangle_2 + \| \pm \mathbf{E}x\|_2^2$$
$$= \|Sx\|_2^2 + \|x\|_2^2,$$

because $\mathbf{J}S$ is symmetric and $\mathbf{E}$ is unitary.

Now, if in addition we have $\mathbf{J}D(S) \subset D(S)$, and so $\mathbf{J}D(S) = D(S)$ because $\mathbf{J}^2 = \mathbf{I}$, we can write as above:

$$\|(S \pm \mathbf{E})\mathbf{E}x\|_2^2 = \|S\mathbf{E}x\|_2^2 + \langle S\mathbf{E}x, \pm i\mathbf{J}\mathbf{E}x\rangle_2 + \langle \pm i\mathbf{J}\mathbf{E}x, S\mathbf{E}x\rangle_2 + \|x\|_2^2$$
$$= \|S\mathbf{J}x\|_2^2 + \|x\|_2^2,$$

from which we derive easily the assertion.

Example 2.2 (1) Let $A, B : D \subset \mathcal{H} \mapsto \mathcal{H}$ be symmetric operators. We put $S = S_{A,B} = A\mathbf{I} + B\mathbf{K}$, which is an operator in $\mathcal{H}^2$, defined on $D(S) = D^2 = D \oplus D$. The operator $\mathbf{J}S$ is easily seen to be symmetric in $\mathcal{H}^2$. Therefore,

$$\|Ax + By \pm ix\|^2 + \| - Bx + Ay \mp iy\|^2$$
$$= \|Ax + By\|^2 + \| - Bx + Ay\|^2 + \|x\|^2 + \|y\|^2$$

for all $(x, y) \in D^2$, via Lemma 2.1.

(2) Let $\mathcal{H} = L^2(\mathbb{R})$ and let $D \subset L^2(\mathbb{R})$ be the subset of all continuously differentiable functions with compact support. Consider the operator

$$T = i\frac{d}{dt}\mathbf{I} + \sigma(t)\mathbf{K} + i\tau(t)\mathbf{L},$$

defined on D^2, with values in $\mathcal{H}^2$, where σ and τ are continuous real-valued functions on $\mathbb{R}$. It is known (and easily seen) that the operator $\mathbf{J}T$ is symmetric. Moreover, $\mathbf{J}T$ has a self-adjoint extension, which is called the *Dirac operator* (see, for instance, [9] for some details). Of course, Lemma 2.1 applies to this operator T too. In addition, the operator T (as well as the previous one) has an $\mathbf{E}$–Cayley transform (defined in the next Remark).

Remark 2.3 Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric. Lemma 2.1 allows us to correctly define the operator

$$V : R(S + \mathbf{E}) \mapsto R(S - \mathbf{E}), \quad V(S + \mathbf{E})x = (S - \mathbf{E})x, \quad x \in D(S),$$

which is a partial isometry. In other words, $V = (S - \mathbf{E})(S + \mathbf{E})^{-1}$, defined on $D(V) = R(S + \mathbf{E})$.

The operator V will be called the $\mathbf{E}$–*Cayley transform* of S.

Similarly, if $\mathbf{L}S$ is symmetric, the corresponding version of Lemma 2.1 leads to the definition of an operator

$$W : R(S + \mathbf{F}) \mapsto R(S - \mathbf{F}), \quad V(S + \mathbf{F})x = (S - \mathbf{F})x, \quad x \in D(S),$$

which is again a partial isometry, and $W = (S - \mathbf{F})(S + \mathbf{F})^{-1}$, defined on $D(W) = R(S + \mathbf{F})$.

The operator W is called the $\mathbf{F}$–*Cayley transform* of S.

A similar concept of a (quaternionic) Cayley transform has been defined in [18].

Because the two Cayley transforms defined above are alike, in the sequel we shall mainly deal with the $\mathbf{E}$–Cayley transform. For a symmetric operator, by *Cayley transform* we always mean the classical concept, as defined by von Neumann in [20] (see also [12]).

Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry. Then the inverse V^{-1} is well defined on the subspace $D(V^{-1}) = R(V)$.

Lemma 2.4 *Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric, and let V be the $\mathbf{E}$–Cayley transform of S. We have the following:*

(a) the operator V is closed if and only if the operator S is closed, and if and only if the spaces $R(S \pm \mathbf{E})$ are closed;

(b) the operator $\mathbf{I} - V$ is injective; moreover, the operator S is densely defined if and only if the space $R(\mathbf{I} - V)$ is dense in $\mathcal{H}^2$;

(c) if $S\mathbf{K} \subset \mathbf{K}S$, then $S\mathbf{K} = \mathbf{K}S$ and $V^{-1} = -\mathbf{K}V\mathbf{K}$;

(d) the operator $\mathbf{J}S$ is self-adjoint if and only if the operator V is unitary in $\mathcal{H}^2$.

Proof. The assertions are similar to some assertion in [18] (see especially Lemma 2.8 from [18], where one should replace Q' by $\mathbf{E}$). We give some indirect arguments, using the Cayley transform [12]. We use freely some results from [12], 13.17-13.21.

Let $A = \mathbf{J}S$, which is symmetric. Then its Cayley transform W is a partial isometry from $R(A + i\mathbf{I})$ onto $R(A - i\mathbf{I})$. Moreover, $S \pm \mathbf{E} = \mathbf{J}(A \pm i\mathbf{I})$. Therefore, $V = \mathbf{J}W\mathbf{J}$.

(a) From the properties of the Cayley transform, it follows that A is closed iff W is closed, and iff the spaces $R(A \pm i\mathbf{I})$ are closed. It is clear that S

is closed iff A is closed and $R(S \pm \mathbf{E})$ are closed iff $R(A \pm i\mathbf{I})$ are closed, implying the assertion.

(b) The equality $\mathbf{I} - V = 2\mathbf{E}(S + \mathbf{E})^{-1}$ on $D(V)$ shows that $\mathbf{I} - V$ is injective and that $R(\mathbf{I} - V) = \mathbf{E}(D(S))$. The latter equality implies that the operator S is densely defined if and only if the space $R(\mathbf{I} - V)$ is dense in $\mathcal{H}^2$, because $\mathbf{E}$ is unitary.

(c) If $S\mathbf{K} \subset \mathbf{K}S$, then $\mathbf{K}D(S) \subset D(S)$, implying $\mathbf{K}D(S) = D(S)$ because $\mathbf{K}^2 = -\mathbf{I}$. Consequently, $S\mathbf{K} = \mathbf{K}S$, implying $\mathbf{K}(S \pm \mathbf{E}) = (S \mp \mathbf{E})\mathbf{K}$. Hence

$$V^{-1} = -(S + \mathbf{E})\mathbf{K}^2(S - \mathbf{E})^{-1} = -\mathbf{K}(S - \mathbf{E})(S + \mathbf{E})^{-1}\mathbf{K} = -\mathbf{K}V\mathbf{K}.$$

(d) The operator $\mathbf{J}S$ is self-adjoint if and only if its Cayley transform W is unitary, an hence if and only if $V = \mathbf{J}W\mathbf{J}$ is unitary.

Remark 2.5 (1) Let $S_j : D(S_j) \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S_j$ is symmetric, and let V_j be the $\mathbf{E}$–Cayley transform of $S_j\,(j = 1, 2)$. We have $S_1 \subset S_2$ if and only if $V_1 \subset V_2$. In other words, the $\mathbf{E}$–Cayley transform is an order preserving map. This assertion follows as the similar one from Lemma 2.13 in [18]. We omit the details.

(2) Suppose that the operator $\mathbf{I} - V$ is injective. Then the operator $S : R(\mathbf{E}(V - \mathbf{I})) \mapsto \mathcal{H}^2$, given by $S(\mathbf{E}(V - \mathbf{I})x) = (V + \mathbf{I})x$, $x \in D(V)$, is well defined and will be called the *inverse* $\mathbf{E}$*–Cayley transform* of the partial isometry V. In other words, $S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}$ on $D(S) = \mathbf{E}R(\mathbf{I} - V)$.

Of course, we may define, in a similar way, the *inverse* $\mathbf{F}$*–Cayley transform.* These two (quaternionic) inverse Cayley transforms have similar properties, and so we shall mainly deal with the inverse $\mathbf{E}$–Cayley transform. See also [18], for another similar concept.

Lemma 2.6 *Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry. Suppose that the operator $\mathbf{I} - V$ is injective. Then the operator $S : R(\mathbf{E}(V - \mathbf{I})) \mapsto \mathcal{H}^2$, given by $S(\mathbf{E}(V - \mathbf{I})x) = (V + \mathbf{I})x$, $x \in D(V)$, has the following properties:*

(i) the operator $\mathbf{J}S$ is symmetric and the $\mathbf{E}$–Cayley transform of S is V;

(ii) we have $V^{-1} = -\mathbf{K}V\mathbf{K}$ if and only if $S\mathbf{K} = \mathbf{K}S$.

Proof. (i) Set $W = \mathbf{J}V\mathbf{J}$, which is a partial isometry with $\mathbf{I} - W$ injective. Therefore, the the operator $A = i(\mathbf{I} + W)(\mathbf{I} - W)^{-1}$ is well defined and symmetric, as the inverse Cayley transform of W. Setting $S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}$, we have

$$\mathbf{J}S = \mathbf{J}(\mathbf{J}^2 + \mathbf{J}W\mathbf{J})(\mathbf{J}^2 - \mathbf{J}W\mathbf{J})^{-1}\mathbf{E} = A.$$

Hence, the operator $\mathbf{J}S = A$ is symmetric. Moreover, its $\mathbf{E}$–Cayley transform is equal to V:

$$(S + \mathbf{E})(S - \mathbf{E})^{-1} = \mathbf{J}(A + i\mathbf{I})(A - i\mathbf{I})^{-1}\mathbf{J} = \mathbf{J}W\mathbf{J} = V.$$

(ii) If $V^{-1} = -\mathbf{K}V\mathbf{K}$, we have:

$$\begin{aligned} S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E} &= (V^{-1} + \mathbf{I})(V^{-1} - \mathbf{I})^{-1}\mathbf{E} \\ &= (-\mathbf{K}V\mathbf{K} - \mathbf{K}^2)(-\mathbf{K}V\mathbf{K} + \mathbf{K}^2)^{-1}\mathbf{E} \\ &= -\mathbf{K}(\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}\mathbf{K} = -\mathbf{K}S\mathbf{K}. \end{aligned}$$

Conversely, assuming that $\mathbf{K}S = S\mathbf{K}$, we have $V^{-1} = -\mathbf{K}V\mathbf{K}$ by Lemma 2.4.

We summarize the properties of the quaternionic Cayley transform in the following result, which generalizes (to not necessarily densely defined operators) Theorem 2.14 from [18].

Theorem 2.7 *The $\mathbf{E}$-Cayley transform is an order preserving bijective map assigning to each operator S with $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ and $\mathbf{J}S$ symmetric a partial isometry V in in $\mathcal{H}^2$ with $\mathbf{I} - V$ injective. Moreover:*

(1) *the operator V is closed if and only if the operator S is closed;*

(2) *the equality $V^{-1} = -\mathbf{K}V\mathbf{K}$ holds if and only if the equality $S\mathbf{K} = \mathbf{K}S$ holds;*

(3) *the operator $\mathbf{J}S$ is self-adjoint if and only if V is unitary on $\mathcal{H}^2$.*

Remark 2.8 As noticed in [18] in a similar situation, an interesting class of operators having an $\mathbf{E}$–Cayley transform consists of operators $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that $\mathbf{J}S$ is symmetric and $S\mathbf{K} \subset \mathbf{K}S$ (which implies $\mathbf{K}S = S\mathbf{K}$). This is equivalent to saying, with the terminology of [18], that S is $(\mathbf{J}, \mathbf{L})$–symmetric (i.e., $\mathbf{J}S$, $\mathbf{L}S$ are symmetric and $\mathbf{K}D(S) \subset D(S)$; note that in [18], the operator S is in addition supposed to be densely defined, a hypothesis not always necessary in the present context), which is easily seen. Even more interesting is the class of those $(\mathbf{J}, \mathbf{L})$–symmetric operators having a normal extension, which is the main motivation of the introduction of the quaternionic Cayley transform in [18]. This situation will be again dealt with in the next sections, from a different point of view.

3 Unitary operators and the inverse quaternionic Cayley transform

In this this section, we are particularly intrested in those unitary operators producing (unbounded) normal operators, via the inverse $\mathbf{E}$–Cayley transform.

Lemma 3.1 *Let U be a bounded operator on $\mathcal{H}^2$. The operator U is unitary and has the property $U^* = -\mathbf{K}U\mathbf{K}$ if and only if there are a bounded operator T and bounded self-adjoint operators A, B on $\mathcal{H}$ such that $TT^* + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$ and*

$$U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix},$$

where I the identity on $\mathcal{H}$.

Proof. If U is given by the matrix in the statement, it is easily checked that U is a unitary operator on $\mathcal{H}^2$ and one has $U^* = -\mathbf{K}U\mathbf{K}$.

Conversely, assuming

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

we easily infer that

$$-\mathbf{K}U\mathbf{K} = \begin{pmatrix} U_{22} & -U_{21} \\ -U_{12} & U_{11} \end{pmatrix}.$$

The equality $U^* = -\mathbf{K}U\mathbf{K}$ leads to the equations $U_{11}^* = U_{22}$, $U_{12}^* = -U_{12}$, and $U_{21}^* = -U_{21}$. Setting $T = U_{11}$, $U_{12} = iA$ and $U_{21} = iB$, with A, B self-adjoint, the equations $U^*U = \mathbf{I}$ and $UU^* = \mathbf{I}$ are equivalent to the equalities $TT^* + A^2 = I$, $T^*T + B^2 = I$ and $AT = TB$.

Example 3.2 Let T be a contraction on $\mathcal{H}$. Setting $D_{T^*} = (I - TT^*)^{1/2}$, $D_T = (I - T^*T)^{1/2}$ and

$$U = \begin{pmatrix} T & iD_{T^*} \\ iD_T & T^* \end{pmatrix},$$

the operator U is unitary on $\mathcal{H}^2$ and satisfies the equation $U^* = -\mathbf{K}U\mathbf{K}$.

Indeed, $A = D_{T^*}$ and $B = D_T$ satisfy all conditions from the previuos lemma.

Lemma 3.3 *Let U be a unitary operator on $\mathcal{H}^2$ such that $\mathbf{I}-U$ is injective. If we set $S = (\mathbf{I}+U)(\mathbf{I}-U)^{-1}\mathbf{E}$, we have that S is densely defined, closed and $S^* = \mathbf{E}(\mathbf{I}+U)(\mathbf{I}-U)^{-1}$.*

Proof. The operator S is the inverse $\mathbf{E}$–Cayley transform of the unitary operator U. Therefore, $\mathbf{J}S$ is self-adjoint, via Lemma 2.4(d). This implies that S densely defined and closed. Moreover, $(\mathbf{J}S)^* = \mathbf{J}S = S^*\mathbf{J}$, whence $S^* = \mathbf{J}S\mathbf{J} = \mathbf{E}(\mathbf{I}+U)(\mathbf{I}-U)^{-1}$.

Lemma 3.4 *Let U be an operator on $\mathcal{H}^2$ having the form*

$$U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix},$$

with T, $A = A^*$, $B = B^*$ bounded operators on $\mathcal{H}$, such that $TT^* + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$. We have the equality $(U+U^*)\mathbf{E} = \mathbf{E}(U+U^*)$ if and only if T is normal and $A = B$.

Proof. The operator U is unitary but we do not use this property.
Note that

$$\mathbf{J}(U+U^*) = \begin{pmatrix} T+T^* & i(A-B) \\ i(A-B) & -(T+T^*) \end{pmatrix}.$$

Similarly,

$$(U+U^*)\mathbf{J} = \begin{pmatrix} T+T^* & -i(A-B) \\ -i(A-B) & -(T+T^*) \end{pmatrix}.$$

The equality $(U+U^*)\mathbf{E} = \mathbf{E}(U+U^*)$ is equivalent to $A = B$, which clearly implies $T^*T = TT^*$.

Conversely, if U has the matrix representation from the statement with T normal and $A = B$, then

$$U + U^* = \begin{pmatrix} T+T^* & 0 \\ 0 & T+T^* \end{pmatrix},$$

and the relation $(U+U^*)\mathbf{E} = \mathbf{E}(U+U^*)$ is obvious.

Remark. It follows from Lemma 3.1 and Lemma 3.4 that an operator U on $\mathcal{H}^2$ has the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal, A self-adjoint, such that $TT^* + A^2 = I$ and $AT = TA$, if and only if U is unitary, $U^* = -\mathbf{K}U\mathbf{K}$ and $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.

Lemma 3.5 *Let V be a partial isometry such that $V^{-1} = -\mathbf{K}V\mathbf{K}$ and $\mathbf{I} - V$ is injective. Let S be the inverse $\mathbf{E}$–Cayley transform of V. We have $\mathbf{J}D(S) \subset D(S)$ and $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$ if and only if there exists a surjective isometry $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I}-V) = (\mathbf{I}-V)G$.*

Proof. A similar construction is implicitly done in [18] (see especially Remark 2.16(2) and the proof of Theorem 2.18).

Assume first that $\mathbf{J}D(S) \subset D(S)$ and $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$. The inclusion $\mathbf{J}D(S) \subset D(S)$ means the inclusion $R(\mathbf{I} - V) \subset \mathbf{E}R(\mathbf{I} - V)$, via Remark 2.5(2). But this inclusion is actually equality because $\mathbf{E}^2 = -\mathbf{I}$. Consequently, for every $u \in D(V)$ we can find a unique vector $v \in D(V)$ such that $(\mathbf{I} - V)v = \mathbf{E}(\mathbf{I} - V)u$. Setting $Gu = v$, we get a linear operator $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I}-V) = (\mathbf{I}-V)G$, which is clearly bijective. In fact, $G^{-1} = -(\mathbf{I} - V)^{-1}\mathbf{E}(\mathbf{I} - V) = -G$.

We have only to show that G is an isometry on $D(G) = D(V)$. Let $x \in D(S) = \mathbf{E}D(S)$ and set $u = (S+\mathbf{E})x \in D(V)$. Let also $v = (S+\mathbf{E})\mathbf{E}x \in D(V)$. As we clearly have $\mathbf{I} - V = 2\mathbf{E}(S + \mathbf{E})^{-1}$, we deduce that

$$(\mathbf{I} - V)Gu = \mathbf{E}(\mathbf{I} - V)u = 2\mathbf{E}^2(S + \mathbf{E})^{-1}u = -2x$$

$$= 2\mathbf{E}(S + \mathbf{E})^{-1}v = (\mathbf{I} - V)v,$$

whence $Gu = v$. Moreover,

$$\|Gu\|_2 = \|(S + \mathbf{E})\mathbf{E}x\|_2 = \|(S + \mathbf{E})x\|_2 = \|u\|_2,$$

via Lemma 2.1, showing that G is an isometry on $D(V)$.

Conversely, assuming that there exists a surjective isometry $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I}-V) = (\mathbf{I}-V)G$, we have, in particular, $D(S) \subset R(\mathbf{I}-V)$, implying $\mathbf{J}D(S) \subset D(S)$. In addition, with the notation from above,

$$\|(S + \mathbf{E})\mathbf{E}x\|_2 = \|Gu\|_2 = \|u\|_2 = \|(S + \mathbf{E})x\|_2,$$

because G is an isometry, showing that $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$, again via Lemma 2.1.

Corollary 3.6 *Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be inverse $\mathbf{E}$–Cayley transform of U. The operator S is normal if and only is there exists a unitary operator G_U on $\mathcal{H}^2$ such that $\mathbf{E}(\mathbf{I} - U) = (\mathbf{I} - U)G_U$ and $(G_U)^* = -G_U$.*

Proof. If S is normal, we must have $D(S) = D(S^*)$ and $\|Sx\|_2 = \|S^*x\|_2$ for all $x \in D(S)$ (for some details concerning unbounded normal operators see [4], Section XII.9). Moreover, $\mathbf{J}S$ is self-adjoint, by Lemma 2.4(d). In particular, $\mathbf{J}S^* = S\mathbf{J}$. Therefore, $\|S\mathbf{J}x\|_2 = \|\mathbf{J}S^*x\|_2 = \|S^*x\|_2 = \|Sx\|_2$ for all $x \in D(S)$. This allows us to apply the previous lemma, leading actually to a unitary operator G_U with the desired properties.

Conversely, assume that there exists a unitary operator G_U on $\mathcal{H}^2$ such that $\mathbf{E}(\mathbf{I} - U) = (\mathbf{I} - U)G_U$ and $(G_U)^* = -G_U$. The previous lemma shows that the operator S has the properties $\mathbf{J}D(S) \subset D(S)$ and $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$. Note also that $\mathbf{J}S$ is self-adjoint, by Theorem 2.7. Therefore, $(\mathbf{J}S)^* = \mathbf{J}S = S^*\mathbf{J}$, and so $D(S^*) = D(S)$. Moreover,

$$\|S^*x\|_2 = \|\mathbf{J}S\mathbf{J}x\|_2 = \|S\mathbf{J}x\|_2 = \|Sx\|_2, \quad x \in D(S),$$

showing that S is normal.

Theorem 3.7 *Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be the inverse $\mathbf{E}$–Cayley transform of U. The operator S is normal if and only if $(U+U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.*

Proof. We use the notation $\mathrm{Re}(T)$ ($\mathrm{Im}(T)$) to designate the real (resp. the imaginary) part of a bounded operator T.

Let U be a unitary operator on $\mathcal{H}^2$ such that $U^* = -\mathbf{K}U\mathbf{K}$, $\mathbf{I} - U$ is injective and $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$. In particular, U has the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal and A self-adjoint in $\mathcal{H}$, such that $TT^*+A^2 = I$ and $AT = TA$ (see the Remark after Lemma 3.4) .

Note that

$$(\mathbf{I} - U)(\mathbf{I} - U^*) = 2 \begin{pmatrix} I - \mathrm{Re}(T) & 0 \\ 0 & I - \mathrm{Re}(T) \end{pmatrix}.$$

Since both $\mathbf{I}-U$ and $\mathbf{I}-U^*$ are injective, this formula shows that the operator $I-\mathrm{Re}(T)$ is injective too. Therefore,

$$(\mathbf{I}-U)^{-1}=\frac{1}{2}(\mathbf{I}-U^*)\begin{pmatrix}(I-\mathrm{Re}(T))^{-1} & 0\\ 0 & (I-\mathrm{Re}(T))^{-1}\end{pmatrix},$$

which is defined on $(I-\mathrm{Re}(T))(\mathcal{H})\oplus(I-\mathrm{Re}(T))(\mathcal{H})$. Using this formula and that $(\mathbf{I}+U)(\mathbf{I}-U^*)=2i\mathrm{Im}(U)$, we obtain

$$\begin{aligned}S&=(\mathbf{I}+U)(\mathbf{I}-U)^{-1}\mathbf{E}\\&=i\mathrm{Im}(U)\begin{pmatrix}(I-\mathrm{Re}(T))^{-1} & 0\\ 0 & (I-\mathrm{Re}(T))^{-1}\end{pmatrix}\mathbf{E}\\&=-\begin{pmatrix}\mathrm{Im}(T)(I-\mathrm{Re}(T))^{-1} & -A(I-\mathrm{Re}(T))^{-1}\\ A(I-\mathrm{Re}(T))^{-1} & \mathrm{Im}(T)(I-\mathrm{Re}(T))^{-1}\end{pmatrix},\end{aligned}$$

defined on $(I-\mathrm{Re}(T))(\mathcal{H})\oplus(I-\mathrm{Re}(T))(\mathcal{H})$.

Similarly, using Lemma 3.3, we have:

$$\begin{aligned}S^*&=\mathbf{E}(\mathbf{I}+U)(\mathbf{I}-U)^{-1}\\&=i\mathbf{E}\mathrm{Im}(U)\begin{pmatrix}(I-\mathrm{Re}(T))^{-1} & 0\\ 0 & (I-\mathrm{Re}(T))^{-1}\end{pmatrix}\\&=-\begin{pmatrix}\mathrm{Im}(T)(I-\mathrm{Re}(T))^{-1} & A(I-\mathrm{Re}(T))^{-1}\\ -A(I-\mathrm{Re}(T))^{-1} & \mathrm{Im}(T)(I-\mathrm{Re}(T))^{-1}\end{pmatrix},\end{aligned}$$

defined on $(I-\mathrm{Re}(T))(\mathcal{H})\oplus(I-\mathrm{Re}(T))(\mathcal{H})$.

The explicit formulas from above giving S and S^* show that $D(S)=D(S^*)$.

To finish the proof that S is normal, let

$$x\oplus y=(I-\mathrm{Re}(T))u\oplus(I-\mathrm{Re}(T))v\in\mathcal{D}(S).$$

Then

$$S(x\oplus y)=-\begin{pmatrix}\mathrm{Im}(T) & -A\\ A & \mathrm{Im}(T)\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}.$$

Similarly,

$$S^*(x\oplus y)=-\begin{pmatrix}\mathrm{Im}(T) & A\\ -A & \mathrm{Im}(T)\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}.$$

Since $B = \mathrm{Im}(T)$ and A are commuting self-adjoint operators, a direct calculation shows that

$$\|Bu - Av\|^2 + \|Au + Bv\|^2 = \|Bu + Av\|^2 + \| - Au + Bv\|^2,$$

implying $\|S(x \oplus y)\|_2 = \|S^*(x \oplus y)\|_2$ for all $x \oplus y \in \mathcal{D}(S)$.

Conversely, assuming S normal, we should have $D(S) = D(S^*)$. Since S is the inverse $\mathbf{E}$–Cayley transform of U, the operator $\mathbf{J}S$ is self-adjoint, by Lemma 2.4(d). Therefore, $(\mathbf{J}S)^* = \mathbf{J}S = S^*\mathbf{J}$, and so $\mathbf{J}D(S) = \mathbf{E}D(S) = D(S^*) = D(S)$. Consequently, the operator $G = G_U$ given by Corollary 3.6 is unitary on $\mathcal{H}^2$. The same corollary asserts that $G^* = -G$. Therefore, $\mathbf{E}(\mathbf{I}-U)(\mathbf{I}-U^*) = (\mathbf{I}-U)(\mathbf{I}-U^*)\mathbf{E}$. Consequently, $(U+U^*)\mathbf{E} = \mathbf{E}(U+U^*)$, as direct consequence of the latter equation.

Let $\mathcal{U}(\mathcal{H}^2)$ be the set of all unitary operators in $\mathcal{H}^2$. We also set

$$\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \{U \in \mathcal{U}(\mathcal{H}^2); U^* = -\mathbf{K}U\mathbf{K}, N(\mathbf{I} - U) = \{0\},$$
$$(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)\},$$

that is, those unitary operators whose inverse $\mathbf{E}$–Cayley transform is a normal operator, via the previous theorem.

The next result gives a complete description of the unitary operator G_U, defined by Corollary 3.6. We keep the notation from the previous theorem.

Proposition 3.8 *Let $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$. Then the operator*

$$i\begin{pmatrix} (T^*T - \mathrm{Re}(T))\Theta_T^{-1} & -iA(I - T^*)\Theta_T^{-1} \\ iA(I - T)\Theta_T^{-1} & -(T^*T - \mathrm{Re}(T))\Theta_T^{-1} \end{pmatrix},$$

is a densely defined isometry, where $\Theta_T = I - \mathrm{Re}(T)$, and its extension to $\mathcal{H}^2$ equals the unitary operator G_U.

Proof. Note that

$$\mathbf{E}(\mathbf{I} - U) = \begin{pmatrix} i(I - T) & A \\ -A & -i(I - T^*) \end{pmatrix},$$

and so

$$(\mathbf{I} - U^*)\mathbf{E}(\mathbf{I} - U) = 2i\begin{pmatrix} T^*T - \mathrm{Re}(T) & -iA(I - T^*) \\ iA(I - T) & -T^*T + \mathrm{Re}(T) \end{pmatrix},$$

via an easy calculation and the equality $T^*T + A^2 = I$.

Let

$$D = \begin{pmatrix} T^*T - \mathrm{Re}(T) & -iA(I - T^*) \\ iA(I - T) & -T^*T + \mathrm{Re}(T) \end{pmatrix}.$$

The operator

$$C = iD \begin{pmatrix} \Theta_T^{-1} & 0 \\ 0 & \Theta_T^{-1} \end{pmatrix},$$

defined on $\Theta_T(\mathcal{H}) \oplus \Theta_T(\mathcal{H})$, is an isometry. Indeed, note that

$$2iD = (\mathbf{I} - U^*)\mathbf{E}(\mathbf{I} - U) = G_U(\mathbf{I} - U^*)(\mathbf{I} - U)$$

$$= 2G_U \begin{pmatrix} \Theta_T & 0 \\ 0 & \Theta_T \end{pmatrix},$$

via the proof of Theorem 3.7. This equality shows that C is the restriction of G_U to $\Theta_T(\mathcal{H}) \oplus \Theta_T(\mathcal{H})$, and so C is an isometry. Moreover, as the space $R(\Theta_T)$ is dense in $\mathcal{H}$, because the self-adjoint operator Θ_T is injective, the domain $D(C)$ is dense in $\mathcal{H}^2$. Therefore, G_U is the (unique) extension of the densely defined operator C, which is the stated assertion.

Remark 3.9 Let

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) = \{S : D(S) \subset \mathcal{H}^2 \to \mathcal{H})^2;$$

$$S \text{ normal, } (\mathbf{J}S)^* = \mathbf{J}S,\ \mathbf{K}S = S\mathbf{K}\}.$$

Theorems 2.7 and 3.7 show that the map

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) \ni S \mapsto (S - \mathbf{E})(S + \mathbf{E})^{-1} \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$$

is bijective. In addition, we have $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if S is a densely defined operator in $\mathcal{H}^2$ having the form

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B are commuting self-adjoint operators. The latter assertion follows from [18] (see especially Proposition 2.4 and Theorem 3.7 from [18]).

Example 3.10 Let us compute the operators U and G_U from Proposition 3.8, in an important particular case.

Let μ be a probability measure in the plane $\mathbb{R}^2$, having moments of all orders. In particular, if (s,t) is the variable in $\mathbb{R}^2$, the numbers $\gamma_{k,l} = \int s^k t^l d\mu$ are well defined for all integers $k, l \geq 0$. Let $\mathcal{P}$ be the algebra of all polynomials in s, t, with complex coefficients. The hypothesis implies that $\mathcal{P} \subset L^2(\mu)$. The linear operators given by $p(s,t) \mapsto sp(s,t), p(s,t) \mapsto tp(s,t), p \in \mathcal{P}$, are easily seen to be symmetric on $\mathcal{P}$. In fact, these operators have natural self-adjoint extensions in $\mathcal{H} = L^2(\mu)$, defined by similar formulas, whose joint domain of definition given by $D_0 = \{f \in L^2(\mu); sf, tf \in L^2(\mu)\}$, and these extensions commute (i.e. their spectral measures commute).

To simplify the notation, we identify in what follows the multiplication operators with the corresponding (matrices of) functions. For instance, the matrix

$$N = \begin{pmatrix} s & t \\ -t & s \end{pmatrix},$$

defined on $D = D_0 \oplus D_0$, is in the class $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ and is normal. The **E**-Cayley transform U of N will have the form

$$U = \frac{1}{s^2+t^2+1} \begin{pmatrix} (s-i)^2 + t^2 & 2ti \\ 2ti & (s+i)^2 + t^2 \end{pmatrix},$$

via Remark 1.4.

With the notation of Theorem 3.7 and the convention from above concerning the multiplication operators, we have

$$T = \frac{s^2+t^2-1-2si}{s^2+t^2+1}, \quad A = \frac{2t}{s^2+t^2+1}.$$

A direct calculation shows that

$$T^*T - \mathrm{Re}T = \frac{2(s^2-t^2+1)}{(s^2+t^2+1)^2}, \quad \Theta_T = \frac{2}{s^2+t^2+1}.$$

Therefore,

$$(T^*T - \mathrm{Re}T)\Theta_T^{-1} = \frac{s^2-t^2+1}{s^2+t^2+1}, \quad -iA(I-T^*)\Theta_T^{-1} = \frac{-2t(s+i)}{s^2+t^2+1},$$

as well as $iA(I-T^*)\Theta_T^{-1} = 2t(i-s)/(s^2+t^2+1)$. Consequently,

$$G_U = \frac{i}{s^2+t^2+1} \begin{pmatrix} s^2-t^2+1 & -2t(s+i) \\ -2t(s-i) & -s^2+t^2-1 \end{pmatrix}.$$

4 Normal extensions

Remark 4.1 Let $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$, with $D(T) = D_0 \oplus D_0$, $D_0 \subset \mathcal{H}$. According to [18], Lemma 1.2, the equality $D(T) = D_0 \oplus D_0$ is equivalent to the inclusions

(i) $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.

In order that T have a normal extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$, the following conditions are necessary:

(ii) $\mathbf{J}T$ is symmetric;

(iii) $T\mathbf{K} = \mathbf{K}T$;

(iv) $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ for all $x \in D(T)$.

Indeed, $\mathbf{J}T \subset \mathbf{J}S$ and $\mathbf{J}S$ self-adjoint imply (ii). Next, $S\mathbf{K} = \mathbf{K}S$ and the inclusion $\mathbf{K}D(T) \subset D(T)$ imply (iii). Condition (iv) also holds, as in the last part of the proof of Corollary 3.6 (see also the proof of Proposition 4.2).

We denote by $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ the set of those operators $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that (i)–(iv) hold.

Let also $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ be the set of those partial isometries $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that:

(a) $V^{-1} = -\mathbf{K}V\mathbf{K}$;

(b) $\mathbf{I} - V$ is injective;

(c) $\mathbf{E}R(\mathbf{I}-V) = R(\mathbf{I}-V)$ and $(\mathbf{I}-V)^{-1}\mathbf{E}(\mathbf{I}-V)$ is an isometry on $D(V)$.

It follows from Lemma 3.5 that the $\mathbf{E}$–Cayley transform is a bijective map from $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ onto $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$. Note also that $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2) \cap \mathcal{U}(\mathcal{H}^2)$ by Corollary 3.6 and Theorem 3.7.

The interesting question concerning the existence of an extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ of an operator $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ is equivalent to the description of those partial isometries in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ having extensions in the family $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$.

Proposition 4.2 *Let $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ and let $\mathcal{D} \subset \mathcal{H}^2$ be a closed subspace with the properties $\mathbf{K}U(\mathcal{D}) \subset \mathcal{D}$ and $\mathbf{E}(\mathbf{I} - U)(\mathcal{D}) \subset (\mathbf{I} - U)(\mathcal{D})$. If $V = U|\mathcal{D}$, $\mathcal{E} = \mathcal{D}^\perp$ and $W = U|\mathcal{E}$, then $U = V \oplus W$ and $V, W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$*

Proof. If $V = U|\mathcal{D}$, then V is a partial isometry from $D(V) = \mathcal{D}$ onto $R(V) = U(\mathcal{D})$. Moreover, $V^{-1} = U^{-1}|R(V)$. Therefore, $V^{-1} = -\mathbf{K}V\mathbf{K}$, because of the equality $U^{-1} = -\mathbf{K}U\mathbf{K}$, and inclusion $\mathbf{K}U(\mathcal{D}) \subset \mathcal{D}$.

The injectivity of $\mathbf{I} - U$ implies that $\mathbf{I} - V$ also injective. In addition, $\mathbf{E}(\mathbf{I} - V)(\mathcal{D}) \subset (\mathbf{I} - V)(\mathcal{D})$ as a direct consequence of the given inclusion $\mathbf{E}(\mathbf{I}-U)(\mathcal{D}) \subset (\mathbf{I}-U)(\mathcal{D})$. As $\mathbf{E}^2 = -\mathbf{I}$, we have in fact that $\mathbf{E}(\mathbf{I}-V)(\mathcal{D}) =$

$(\mathbf{I}-V)(\mathcal{D})$. Hence, if T be the inverse $\mathbf{E}$–Cayley transform of V, then $\mathbf{J}D(T)=\mathbf{E}D(T)=R(\mathbf{I}-V)=\mathbf{E}R(\mathbf{I}-V)=D(T)$

Let us show that $(\mathbf{I}-V)^{-1}E(\mathbf{I}-V)$ is an isometry on $D(V)$. Let S be the inverse $\mathbf{E}$–Cayley transform of U, which is an extension of T. Because $\mathbf{J}S$ is self-adjoint by Lemma 2.4(d) and S is normal by Theorem 3.7, we have

$$\|T\mathbf{J}x\|_2=\|S\mathbf{J}x\|_2=\|\mathbf{J}S^*x\|_2=\|S^*x\|_2=\|Sx\|_2=\|Tx\|_2,$$

for all $x\in D(T)$ (as in Corollary 3.6). Thus, $(\mathbf{I}-V)^{-1}\mathbf{E}(\mathbf{I}-V)$ is an isometry on $D(V)$, by Lemma 3.5.

The properties (a)–(c) from Remark 4.1 being verified, we have that $V\in\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Now, let $\mathcal{E}=\mathcal{D}^\perp$, and let $W=U|\mathcal{E}$. We also have the inclusion $\mathbf{K}U(\mathcal{E})\subset\mathcal{E}$, because $(\mathbf{K}U)^*=-\mathbf{K}U$, as well as the inclusion and $\mathbf{E}(\mathbf{I}-U)(\mathcal{E})\subset(\mathbf{I}-U)(\mathcal{E})$ because the operator $G_U=(\mathbf{I}-U)^{-1}\mathbf{E}(\mathbf{I}-U)$ has the property $(G_U)^*=-G_U$. Therefore, the operator W is also a (closed) partial isometry in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, by the first part of the proof.

The equality $U=V\oplus W$ is obvious.

Remark 4.3 Let V be a closed partial isometry in $\mathcal{H}^2$ such that $V^{-1}=-\mathbf{K}V\mathbf{K}$. Then there exists a decomposition of the (Hilbert) space $D(V)$ of the form $D(V)=D(V)^+\oplus D(V)^-$ such that $V|D(V)^\pm=\pm i\mathbf{K}|D(V)^\pm$. If we denote by $P_V^\pm$ the projection of $D(V)$ onto $D(V)^\pm$, we have $V=i\mathbf{K}P_V^+-i\mathbf{K}P_V^-$. The projections $P_V^\pm$ completely determine the operator V, and they are called the $\mathbf{K}$-*projections* of V. As a matter of fact, we have $P_V^\pm=2^{-1}(\mathbf{I}\pm i\mathbf{K}V)$. See [18] (especially Remark 2.16(1)) for more details.

Lemma 4.4 *Let V be a closed partial isometry in $\mathcal{H}^2$ such that $V^{-1}=-\mathbf{K}V\mathbf{K}$. The operator $\mathbf{I}-V$ is injective and there exists a surjective isometry $G:D(V)\mapsto D(V)$ such that $\mathbf{E}(\mathbf{I}-V)=(\mathbf{I}-V)G$ if and only if the operator $P_{\mathbf{I}}^+P_V^-+P_{\mathbf{I}}^-P_V^+$ is injective and*

$$\|P_V^\pm P_V\mathbf{E}P_{\mathbf{I}}^\pm x\|_2=\|P_V^\mp P_VP_{\mathbf{I}}^\pm x\|_2,\quad x\in\mathcal{H}^2,$$

where $P_V^\pm$ are the $\mathbf{K}$-projections of V, $P_{\mathbf{I}}^\pm$ are the $\mathbf{K}$-projections of $\mathbf{I}$, and P_V is the projection of $\mathcal{H}^2$ onto $D(V)$.

Proof. We follow some lines from the proofs of Lemma 2.17 and Theorem 2.18 in [18].

Assume first that $\mathbf{I}-V$ is injective and there exists a surjective isometry (in fact, a unitary operator) $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I}-V) = (\mathbf{I}-V)G$. Using the formula $\mathbf{I}-V = (\mathbf{I}-i\mathbf{K})P_V^+ + (\mathbf{I}+i\mathbf{K})P_V^-$, obtained via Remark 4.3, and because $P_{\mathbf{I}}^{\pm} = 2^{-1}(\mathbf{I} \pm i\mathbf{K})$, we infer that the operator $P_{\mathbf{I}}^+ P_V^- + P_{\mathbf{I}}^- P_V^+$ is injective. Moreover,

$$(\mathbf{I}+i\mathbf{K})(\mathbf{E}P_V^+ - P_V^- G) + (\mathbf{I}-i\mathbf{K})(\mathbf{E}P_V^- - P_V^+ G) = 0.$$

Because the spaces $R(\mathbf{I}+i\mathbf{K})$ and $R(\mathbf{I}-i\mathbf{K})$ are orthogonal, it follows $P_{\mathbf{I}}^{\pm}(\mathbf{E}P_V^{\pm} - P_V^{\mp}G) = 0$. Passing to adjoints, we deduce the equality $G^* P_V^{\mp} P_V P_{\mathbf{I}}^{\pm} = -P_{\mathbf{I}}^{\pm} P_V \mathbf{E} P_{\mathbf{I}}^{\pm}$, valid on $\mathcal{H}^2$. Using the fact that G^* is also an isometry on $D(V)$, we obtain the desired relation.

Conversely, assume that $P_{\mathbf{I}}^+ P_V^- + P_{\mathbf{I}}^- P_V^+$ is injective, and so $\mathbf{I}-V$ is injective, and that $\|P_V^{\pm} P_V \mathbf{E} P_{\mathbf{I}}^{\pm} x\|_2 = \|P_V^{\mp} P_V P_{\mathbf{I}}^{\pm} x\|_2$ for all $x \in \mathcal{H}^2$. In this case we may define an isometry G_* on the space

$$D(G_*) = R(P_V^- P_V P_{\mathbf{I}}^+) \oplus R(P_V^+ P_V P_{\mathbf{I}}^-),$$

whose range is the space

$$R(G_*) = R(P_V^+ P_V \mathbf{E} P_{\mathbf{I}}^+) \oplus R(P_V^- P_V \mathbf{E} P_{\mathbf{I}}^-),$$

by the formula

$$G_* P_V^{\mp} P_V P_{\mathbf{I}}^{\pm} x = -P_V^{\pm} P_V \mathbf{E} P_{\mathbf{I}}^{\pm} x,\ x \in \mathcal{H}^2.$$

Note that the orthogonal complement of $D(G_*)$ in D(V) is null. Indeed, if $x \in D(V)$ is orthogonal to $D(G_*)$, then we must have $(P_{\mathbf{I}}^+ P_V^- + P_{\mathbf{I}}^- P_V^+)x = 0$, which implies $x = 0$. In addition, if $x \in D(V)$ is orthogonal to $R(G_*)$, we infer that $P_I^{\pm} \mathbf{E} P_V^{\pm} x = 0 = \mathbf{E} P_I^{\mp} P_V^{\pm} x$, whence, as above, we have again $x = 0$.

Therefore, G_* has a bounded extension on $D(V)$ onto $D(V)$, also denoted by G_*, which is an isometry. The adjoint G of G_* satisfies the equation $P_{\mathbf{I}}^{\pm}(\mathbf{E}P_V^{\pm} - P_V^{\mp}G) = 0$, which is equivalent to the equality $\mathbf{E}(\mathbf{I}-V) = (\mathbf{I}-V)G$.

The next result is a geometric characterization of those closed subspaces of $\mathcal{H}^2$ which are domains of definitions of partial isometries from $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Proposition 4.5 *Let $\mathcal{D} \subset \mathcal{H}^2$ be a closed subspace. There exists a $V \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ with $D(V) = \mathcal{D}$ if and only if there are two orthogonal projection $P^{\pm}$ in $\mathcal{H}^2$ such that*

(1) $\mathcal{D} = P^+(\mathcal{H}^2) \oplus P^-(\mathcal{H}^2)$;
(2) $P^\pm(\mathcal{H}^2) \cap P^\pm_{\mathbf{I}}(\mathcal{H}^2) = \{0\}$;
(3) $(P^\pm + \mathbf{E}P^\mp\mathbf{E})(P^\mp_{\mathbf{I}}(\mathcal{H}^2)) \subset P^\pm_{\mathbf{I}}(\mathcal{H}^2)$.
In the affirmative case, we also have $P^\pm(\mathcal{H}^2) = \overline{P^\pm P^\mp_{\mathbf{I}}(\mathcal{H}^2)} = \overline{P^\pm P^\mp_{\mathbf{I}}\mathbf{E}(\mathcal{H}^2)}$.

Proof. If $V \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ and $\mathcal{D} = D(V)$, with the notation from Remark 4.3 and Lemma 4.4, and setting $P^\pm = P^\pm_V P_V$, which are precisely the projections of $\mathcal{H}^2$ onto $D(V)^\pm$, we clearly have (1).

Next, as in the previous proof, the operator $P^+_{\mathbf{I}}P^- + P^-_{\mathbf{I}}P^+$, restricted to $D(V)$, is just $2^{-1}(\mathbf{I} - V)$, which is injective. It is easily seen that the injectivity of this operator is equivalent to (2).

To prove (3), we note that the equality $\|P^\pm\mathbf{E}P^\pm_{\mathbf{I}}x\|_2 = \|P^\mp P^\pm_{\mathbf{I}}x\|_2$, $x \in \mathcal{H}^2$ is equivalent to the equality $P^\pm_{\mathbf{I}}P^\mp P^\pm_{\mathbf{I}} = -\mathbf{E}P^\mp_{\mathbf{I}}P^\pm P^\mp_{\mathbf{I}}\mathbf{E}$, which in turn is equivalent to (3).

Conversely, assuming that (1)-(3) hold, we define V as the restriction of the operator $i\mathbf{K}P^+ - i\mathbf{K}P^-$ to $\mathcal{D}$. Condition (2) implies, as above, the injectivity of $\mathbf{I} - V$. Replacing condition (3) by the equivalent condition noticed in the first part of this proof, we proceed as in the proof of Lemma 4.4 to construct a surjective isometry G on $D(V)$, satisfying the equation $\mathbf{E}(\mathbf{I} - V) = (\mathbf{I} - V)G$. The proof also shows that

$$D(V) = \overline{R(P^-P^+_{\mathbf{I}}) \oplus R(P^+P^-_{\mathbf{I}})} = \overline{R(P^-P^+_{\mathbf{I}}\mathbf{E}) \oplus R(P^+P^-_{\mathbf{I}}\mathbf{E})},$$

implying the equalities $P^\pm(\mathcal{H}^2) = \overline{P^\pm P^\mp_{\mathbf{I}}(\mathcal{H}^2)} = \overline{P^\pm P^\mp_{\mathbf{I}}\mathbf{E}(\mathcal{H}^2)}$.

Lemma 4.6 *Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be densely defined. Then T is closable and its closure $\bar{T} \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$.*

Proof. First of all, note that the operator T is closable. Indeed, as the operator $\mathbf{J}T$ is symmetric, assuming that $(x_n)_{n\geq 1}$ is a sequence from $D(T)$ such that $x_n \to 0$ and $Tx_n \to y$ as $n \to \infty$, for all $v \in D(T)$ we have

$$\langle y, v\rangle = \lim_{n\to\infty}\langle Tx_n, v\rangle = \lim_{n\to\infty}\langle x_n, \mathbf{J}T\mathbf{J}v\rangle = 0,$$

showing that the closure of the graph of T is a graph.

Let $\bar{T}$ be the closure of T, and let $\bar{x} \in D(\bar{T})$. Hence $\bar{x} = \lim_{n\to\infty} x_n$ and $\bar{T}\bar{x} = \lim_{n\to\infty} Tx_n$ for some sequence $(x_n)_{n\geq 1}$ from $D(T)$. Condition (iv) from Remark 4.1 shows that $(T\mathbf{J}x_n))_{n\geq 1}$ is a Cauchy sequence, implying that $\mathbf{J}\bar{x} \in D(\bar{T})$ and $\|\bar{T}\mathbf{J}\bar{x}\|_2 = \|\bar{T}\bar{x}\|_2$. In other words, $\mathbf{J}D(\bar{T}) \subset D(\bar{T})$ and condition (iv) from Remark 4.1 hold for $\bar{T}$. As we also have $T\mathbf{K}x_n = \mathbf{K}Tx_n$ for all $n \geq 1$, we infer $\mathbf{K}D(\bar{T}) \subset D(\bar{T})$ and $\bar{T}\mathbf{K} = \mathbf{K}\bar{T}$, the latter being condition (iii) from Remark 4.1.

Finally, let if $\bar{y} \in D(\bar{T})$ is another element with $\bar{y} = \lim_{n\to\infty} y_n$ and $\bar{T}\bar{y} = \lim_{n\to\infty} Ty_n$ for some sequence $(y_n)_{n\geq 1}$ from $D(T)$, then we have

$$\langle \mathbf{J}\bar{T}\bar{x}, \bar{y}\rangle = \lim_{n\to\infty} \langle \mathbf{J}Tx_n, y_n\rangle = \lim_{n\to\infty} \langle x_n, \mathbf{J}Ty_n\rangle = \langle \bar{x}, \mathbf{J}\bar{T}\bar{y}\rangle,$$

implying condition (iii) from Remark 4.1 for $\bar{T}$. Consequently, $\bar{T} \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$.

The next result is an equivalent version of Theorem 3.10 from [18].

Theorem 4.7 *Let $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$ be densely defined. The operator T has an extension in $\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with $D(W) = R(T+\mathbf{E})^\perp$.*

Proof. According to Lemma 4.6, with no loss of generality we may assume that T is closed. If V is the $\mathbf{E}$–Cayley transform of T, then, as noticed in Remark 4.1, we have $V \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$. Moreover, V is closed, by Lemma 2.4. In particular, $D(V) = R(T+\mathbf{E})$ and $R(V) = R(T-\mathbf{E})$ are closed in $\mathcal{H}^2$.

Assume first that there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with $D(W) = R(T+\mathbf{E})^\perp$. Hence $R(W) = \mathbf{K}D(W) = R(T-\mathbf{E})^\perp$. Put $U = V \oplus W$, which is a unitary operator on $\mathcal{H}^2$. We want to show that $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$.

Since $\mathbf{K}D(V) = R(V)$ and $\mathbf{K}D(W) = R(W)$, we clearly have $U^* = V^{-1} \oplus W^{-1} = -\mathbf{K}(V \oplus W)\mathbf{K} = -\mathbf{K}U\mathbf{K}$.

Next, let $G_V : D(V) \mapsto D(V)$ and $G_W : D(W) \mapsto D(W)$ be the surjective isometries given by $G_V = (\mathbf{I}-V)^{-1}\mathbf{E}(\mathbf{I}-V)$ and $G_W = (\mathbf{I}-W)^{-1}\mathbf{E}(\mathbf{I}-W)$. Then $G = G_V \oplus G_W$ is a unitary operator on $\mathcal{H}^2$. In addition, if $x \in D(V)$ and $y \in D(W)$ are arbitrary, then

$$\mathbf{E}(\mathbf{I}-U)(x \oplus y) = \mathbf{E}(x - Vx) + \mathbf{E}(y - Wy) =$$

$$(G_V x - VG_V x) + (G_W y - WG_W y) = (\mathbf{I}-U)G(x \oplus y).$$

As follows from Lemma 2.4, the space $R(\mathbf{I}-V)$ is dense in $\mathcal{H}^2$ because the operator T is densely defined. Therefore, $R(\mathbf{I}-U) \supset R(\mathbf{I}-V)$ is dense in $\mathcal{H}^2$, implying that $\mathbf{I}-U$ is injective. Consequently, $U \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ and, because U is unitary, we actually have $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$, via Corollary 3.6. Clearly, T has a normal extension in $\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$, which is the inverse $\mathbf{E}$–Cayley transform of U.

Conversely, if the operator T has a normal extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$, and if $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ is the $\mathbf{E}$–Cayley transform of S, to find the operator $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, we apply Proposition 4.2 to $\mathcal{D} = D(V)$, where V is the $\mathbf{E}$–Cayley transform of T.

The next assertion provides an extension result for not necessarily densely defined operators.

Corollary 4.8 *Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be closed and let V be the $\mathbf{E}$–Cayley transform of T. The operator T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with the properties $D(W) = R(T + \mathbf{E})^{\perp}$ and $R(\mathbf{I} - V) \cap R(\mathbf{I} - W) = \{0\}$.*

Proof. We keep the notation and proceed as in the previous proof to show that the unitary operator $U = V \oplus W$ is in $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$, where $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ has the stated properties. The only thing to be proved is that $\mathbf{I} - U$ is injective. This is true because if $v \in D(V)$ and $w \in D(W)$ have the property $v \oplus w = U(v \oplus w)$, we infer that

$$R(\mathbf{I} - V) \ni v - Vv = Ww - w \in R(\mathbf{I} - W),$$

implying $v = w = 0$, because both $\mathbf{I} - V, \mathbf{I} - W$ are injective.

Conversely, we proceed again as in the proof of Theorem 4.7 and find the operator $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ as the $\mathbf{E}$–Cayley transform of a normal extension of T, and $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, via Proposition 4.2. Since U is an $\mathbf{E}$–Cayley transform, then $\mathbf{I} - U$ is injective. Choosing a vector $u \in R(\mathbf{I} - V) \cap R(\mathbf{I} - W)$, we have $u = v - Vv = Ww - w$, with $v \in D(V)$ and $w \in D(W)$. Therefore, $v \oplus w = U(v \oplus w)$, implying $v = w = u = 0$, and so $R(\mathbf{I} - V) \cap R(\mathbf{I} - W) = \{0\}$.

Remark 4.9 It follows from Theorem 4.7 that if $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ is densely defined and the space $R(T + \mathbf{E})$ is dense in $\mathcal{H}^2$, then T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$. Indeed, in this case we may apply the theorem with $W = 0$. This remark can be applied in the following situation. Let A, B be a pair of linear operators having a joint domain of definition $D_0 \subset \mathcal{H}$. As in the Introduction, we associate this pair with a matrix operator

$$T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

defined on $D(T) = D_0 \oplus D_0 \in \mathcal{H}^2$. First of all, let us find equivalent conditions on A, B such that $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$. Clearly, $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.

It is easily seen that T is symmetric if and only if both A, B are symmetric. The equality $\mathbf{K}T = T\mathbf{K}$ is also easily verified. Finally, the equality $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ holds for all $x \in D(T)$ if and only if

$$\langle Au, Bv\rangle + \langle Bv, Au\rangle = \langle Bu, Av\rangle + \langle Av, Bu\rangle \tag{c}$$

for all $u, v \in D_0$, which is a weak commutativity condition. Consequently, if A, B are symmetric and condition (c) holds, then $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$. In that case, the $\mathbf{E}$–Cayley transform of T is in the class $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

As a direct consequence of Remark 3.9 and Theorem 4.7, we obtain the following assertion (see also Theorem 3.8 from [18]):

Theorem 4.10 *Let A, B be symmetric operators on a dense joint domain of definition $D_0 \subset \mathcal{H}$, satisfying condition* (c)*. If the space*

$$\{((A + iI)u + Bv) \oplus ((A - iI)v - Bu); u, v \in D_0\} \tag{d}$$

is dense in $\mathcal{H}^2$, then the operators A and B have commuting self-adjoint extensions.

The density of the space from (d) is precisely the density of $R(T + \mathbf{E})$ in $\mathcal{H}^2$, implying $R(T + \mathbf{E})^{\perp} = \{0\}$.

This result, giving a criterion of commutativity of self-adjoint extension of some pairs of symmetric operators, is also related to a series of similar results appearing in [11], [7], [8], [14], etc.

5 A Moment Problem with Constraints

In this section, we combine various techniques from this paper to give an answer to a certain moment problem with some constraints.

Example 5.1 Let (s, t, u) denote the variable in $\mathbb{R}^3$, and let $\mathcal{P}$ be the algebra of all polynomials in s, t, u, with complex coefficients.

Adopting the terminology from [10], we say that the linear map $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ is a *square positive functional* if $\Lambda(\bar{p}) = \overline{\Lambda(p)}$, and $\Lambda(|p|^2) \geq 0$ for all $p \in \mathcal{P}$. If, moreover, $\Lambda(1) = 1$, we say that Λ is *unital*.

A *representing measure* for the unital square positive functional $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ with *support* in the measurable subset $\Sigma \subset \mathbb{R}^3$ is a probability measure μ on Σ such that $\Lambda(p) = \int_{\Sigma} p d\mu$ all $p \in \mathcal{P}$.

Of course, finding a representing measure for such a Λ means, in fact, to solve a moment problem.

For a given polynomial $q \in \mathcal{P}$ and a map $\Lambda : \mathcal{P} \mapsto \mathbb{C}$, we put $\Lambda_q(p) = \Lambda(qp)$ for all $p \in \mathcal{P}$.

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{R}^3$, and let $\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{S}^3; 0 \le s \le 1\}$, which is a compact semi-algebraic set. Let also $\theta(s,t,u) = 1 - s^2 - t^2 - u^2$ and $\sigma(s) = s$. As we have

$$\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{R}^3; \theta(s,t,u) = 0, \sigma(s) \ge 0, (1-\sigma)(s) \ge 0\},$$

we obtain from Theorem 1 in [13] that the unital square positive functional $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure with support in $\mathbb{S}^3_+$ if and only if

$$\Lambda_\theta = 0, \text{ and } \Lambda_\sigma,\ \Lambda_{1-\sigma},\ \Lambda_{\sigma(1-\sigma)} \text{ are square positive functionals.} \tag{P}$$

A more complicated situation, which can be treated with our methods, occurs when we impose some constraints.

Problem *Characterize those unital square positive functionals Λ on $\mathcal{P}$ with the property* (S)*, which have a representing measure with support in the set $\mathbb{S}^3_{++} = \{(s,t,u) \in \mathbb{S}^3_+; 0 \le s < 1\}$, such that all functions $(1-s)^{-m}(m \ge 1$ an integer) are integrable.*

According to Theorem 1 from [13], the functional Λ with the property (P) has a representing measure, say ν, supported on $\mathbb{S}^3_+$. In particular, $\Lambda(q) = 0$ for each polynomial q with $q|\mathbb{S}^3_+ = 0$. Nevertheless, it is not clear that ν is a solution to the Problem. Indeed, if for instance the point $(1,0,0)$ happens to be an atom for ν, the functions $(1-s)^{-m}$ $(m \ge 1)$ are not integrable.

From now on, let $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ be a square positive functional with the property (P). As noticed above, $\Lambda(q) = 0$ for each polynomial q with $q|\mathbb{S}^3_+ = 0$. We denote by $\mathcal{P}(\mathbb{S}^3_+)$ the algebra consisting of all (classes of) functions of the form $p|\mathbb{S}^3_+$, $p \in \mathcal{P}$, modulo the ideal of those polynomials q with $q|\mathbb{S}^3_+ = 0$. The map induced by Λ on $\mathcal{P}(\mathbb{S}^3_+)$ will still be designated by Λ.

To give a solution to the Problem, we should first extend the map Λ to the algebra $\mathcal{R}(\mathbb{S}^3_{++})$ generated by the rational functions $s^j t^k u^l (1-s)^{-m}$ restricted to $\mathbb{S}^3_{++}$, where j,k,l,m are nonnegative integers.

First of all, we note the formula

$$\frac{1}{(1-s)^{m+1}} = \sum_{\alpha \ge m} \binom{\alpha}{m} s^{\alpha - m}, \tag{5.1}$$

valid for all integers $m \geq 0$, where the series is convergent at each point $s \in [0,1)$. This series suggests the following supplementary hypothesis on Λ:

Condition *Setting*

$$p_{m,n}(s) = \sum_{\alpha=m}^{n} \binom{\alpha}{m} s^{\alpha-m}, \tag{5.2}$$

for all nonnegative integers $m, n\,(n \geq m)$ and $s \in [0,1)$, we assume that

$$\lim_{n_1,n_2\to\infty} \Lambda(|p_{m,n_1} - p_{m,n_2}|^2) = 0 \tag{C}$$

for all $m \geq 0$.

Condition (C) is necessary. Indeed if μ is a probability measure on S^3_{++} such that all rational functions $(1-s)^{-m-1}(m \geq 0)$ are integrable, as the sequence $(p_{m,n})_{n\geq m}$ is pointwise convergent to $(1-s)^{-m-1}$ and $|p_{m,n}(u)|^2 \leq (1-s)^{-2m-2}$, then $(p_{m,n})_{n\geq m}$ is a Cauchy sequence in $L^2(\mu)$ by the Lebesgue theorem of dominated convergence, implying (C).

We now deal with the converse assertion.

Using (C), for each polynomial $p \in \mathcal{P}(\mathbb{S}^3_+)$ and every integer $m \geq 0$, we may define

$$\tilde{\Lambda}(pr_m) = \lim_{n\to\infty} \Lambda(pp_{m,n}), \tag{5.3}$$

where $r_m(s) = (1-s)^{-m-1}$. Note that the limit exists via the Cauchy-Schwarz inequality. Moreover,

$$\tilde{\Lambda}(pr_{m_1}) = \tilde{\Lambda}((1-\sigma)^{m_2-m_1} pr_{m_2}) \tag{5.4}$$

if $m_2 \geq m_1$. To prove (5.4), we use the relation

$$p_{m,n}(s) = p_{m+1,n+1}(s) - sp_{m+1,n}(s),$$

valid for all nonnegative integres m, n with $n \geq m+1$. This is a direct consequence of (5.2) and the equality

$$\binom{\alpha+1}{m+1} - \binom{\alpha}{m+1} = \binom{\alpha}{m},$$

which is true whenever $m+1 \leq \alpha \leq n$. Hence, for a fixed polynomial p,

$$\tilde{\Lambda}(pr_m) = \lim_{n\to\infty} \Lambda(pp_{m,n}) = \lim_{n\to\infty} \Lambda(p(p_{m+1,n+1} - \sigma p_{m+1,n})) =$$

$$\lim_{n\to\infty}\Lambda((1-\sigma)pp_{m+1,n})=\tilde{\Lambda}((1-\sigma)pr_{m+1}).$$

Using this computation, we derive (5.4) by recurrence.

Let now $p_1,p_2\in\mathcal{P}(\mathbb{S}^3_+)$, and let m_1,m_2 be nonnegative integers such that $r_{m_2}^{-1}p_1-r_{m_1}^{-1}p_2=q$, where $q|\mathbb{S}^3_+=0$. Assuming, with no loss of generality, that $m_2\ge m_1$, we infer $p_2=(1-\sigma)^{m_2-m_1}p_1-qr_{m_1}$. This relation also shows that qr_{m_1} is a polynomial, which is null on $\mathbb{S}^3_+$. Therefore

$$\lim_{n\to\infty}\Lambda(p_2p_{m_2,n})=\lim_{n\to\infty}\Lambda(((1-\sigma)^{m_2-m_1}p_1-qr_{m_1})p_{m_2,n})=$$

$$\lim_{n\to\infty}\Lambda(p_1p_{m_1,n}+p_1((1-\sigma)^{m_2-m_1}p_{m_2,n}-p_{m_1,n})-qr_{m_1}p_{m_2,n})=$$

$$\lim_{n\to\infty}\Lambda(p_1p_{m_1,n}),$$

because $\lim_{n\to\infty}\Lambda(p_1(1-\sigma)^{m_2-m_1}p_{m_2,n}-p_{m_1,n}))=0$ by (5.4), and $\Lambda(qr_{m_1}p_{m_2,n})=0$ as $qr_{m_1}p_{m_2,n}=0$ on $\mathbb{S}^3_+$. Consequently,

$$\tilde{\Lambda}(p_2r_{m_2})=\tilde{\Lambda}(p_1r_{m_1}). \tag{5.5}$$

Relation (5.5) shows that $\tilde{\Lambda}$ induces a map on the algebra of fractions $\mathcal{F}(\mathbb{S}^3_{++})$ build from the algebra $\mathcal{P}(\mathbb{S}^3_+)$, with denominators in the set $\mathcal{S}=\{(1-s)^m;m\ge 0\}$. This map will be denoted again by Λ. The map $\Lambda:\mathcal{F}(\mathbb{S}^3_{++})\mapsto\mathbb{C}$ is a unital square positive functional. Indeed, fixing $f=p/(1-\sigma)^m$, we have

$$\Lambda(\bar{f})=\lim_{n\to\infty}\Lambda(\bar{p}p_{m,n})=\overline{\Lambda(f)},\ \Lambda(|f|^2)=\lim_{n\to\infty}\Lambda(|f|^2p_{2m,n})\ge 0,$$

$$\Lambda_\sigma(|f|^2)=\lim_{n\to\infty}\Lambda(\sigma|f|^2p_{2m,n})\ge 0,\ \Lambda_{1-\sigma}(|f|^2)=\lim_{n\to\infty}\Lambda((1-\sigma)|f|^2p_{2m,n})\ge 0, \tag{5.6}$$

using the corresponding properties of $\Lambda:\mathcal{P}(\mathbb{S}^3_+)\mapsto\mathbb{C}$. In particular, the map $\Lambda:\mathcal{F}(\mathbb{S}^3_{++})\mapsto\mathbb{C}$ satisfies the Cauchy-Schwartz inequality, and so the set $\mathcal{I}_\Lambda=\{f\in\mathcal{F}(\mathbb{S}^3_{++});\Lambda(|f|^2)=0\}$ is an ideal in the algebra $\mathcal{F}(\mathbb{S}^3_{++})$. Moreover, the assignment $(f,g)\mapsto\Lambda(f\bar{g})$ induces an inner product on the quotient $D_0=\mathcal{F}(\mathbb{S}^3_{++})/\mathcal{I}_\Lambda$. The completion of this quotient with respect to this inner product is a Hilbert space denoted by $\mathcal{H}$.

We now consider in $\mathcal{H}$ the multiplication operators B_0,C_0 induced by the functions $-t/(1-s)$ and $u/(1-s)$, respectively, defined on D_0. The operators

B_0, C_0 leave invariant the space D_0 and commute. Moreover, for every pair $g_1, g_2 \in D_0$, the system of equations

$$\left(\frac{-t}{1-s}+i\right) f_1 + \frac{u}{1-s} f_2 = g_1$$
$$\tag{5.7}$$
$$\frac{-u}{1-s} f_1 + \left(\frac{-t}{1-s}-i\right) f_2 = g_2$$

has the solution $f_1 = -2^{-1}((t+i-is)g_1 + ug_2)$, $f_2 = 2^{-1}(ug_1 - (t-i+is)g_2)$, via the equality $s^2 + t^2 + u^2 = 1$, and so $f_1, f_2 \in D_0$.

Setting $S_0 = B_0\mathbf{I} + C_0\mathbf{K}$ on $D_0 \oplus D_0$, the system (5.7) is precisely the equation $(S_0 + \mathbf{E})(f_1 \oplus f_2) = g_1 \oplus g_2$, showing that $R(S_0 + \mathbf{E})$ is equal to $D_0 \oplus D_0$. Hence, denoting by U_0 the $\mathbf{E}$-Cayley transform of S_0, a direct computation (see Remark 1.4) shows that U_0 is the matrix multiplication operator

$$U_0 = \begin{pmatrix} s+it & iu \\ iu & s-it \end{pmatrix},$$

defined on $D_0 \oplus D_0$.

As in Remark 4.9, we clearly have $S_0 \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$. Then the closure S of S_0 also belongs to $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$, in virtue of Lemma 4.6. If U is the $\mathbf{E}$-Cayley transform of S, then U should be closed, by Theorem 2.7. As U extends U_0, U must be a unitary operator on $\mathcal{H}^2$. Specifically, $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ because U is a unitary operator in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ (see Remark 1.4). In particular, $\mathbf{I} - U$ is injective. Keeping the notation related to U from the proof of Theorem 3.7, we also have that $I - \mathrm{Re}(T)$ is injective.

In fact, the multiplication by $s + it$ on D_0 is extended by T, and the multiplicatin by u on D_0 is extended by A.

Let E be the joint spectral measure of the pair (T, A), which is concentrated on S^3_+. Indeed, if $\mathcal{A}$ is the unital (commutative) C^*-algebra generated by T and A, the equality $T^*T + A^2 = I$ shows that the joint spectrum of the pair (T, A) may be identified with a compact subset of the sphere $\mathbb{S}^3$. In addition, as $0 \le \mathrm{Re}(T) \le I$, which is implied by the properties of the square positive forms Λ_σ and $\Lambda_{1-\sigma}$ given by (5.6), it follows that the measure E is concentrated in the set $\mathbb{S}^3_+$. As the operator $I - \mathrm{Re}(T)$ is injective, it follows that $E(\{(1,0,0)\}) = 0$. Consequently, the measure E has support in the set $\mathbb{S}^3_{++}$.

Since $1 + \mathcal{I}_\Lambda = (I - \mathrm{Re}(T))^m((1-\sigma)^{-m} + \mathcal{I}_\Lambda)$, it follows that $1 + \mathcal{I}_\Lambda$ is in the domain of $(I - \mathrm{Re}(T))^{-m}$ for all integers $m \geq 1$. Therefore, setting $\mu(*) = \langle E(*)(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle$, we obtain

$$\Lambda(pr_m) = \langle pr_m + \mathcal{I}_\Lambda, 1 + \mathcal{I}_\Lambda\rangle =$$
$$\langle (p(\mathrm{Re}(T), \mathrm{Im}(T), A)(I - \mathrm{Re}(T))^{-m}(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle = \int_{S^3_{++}} pr_m d\mu,$$

for all $f = pr_m \in: \mathcal{F}(\mathbb{S}^3_{++})$, showing that μ is a representing measure for $\Lambda : \mathcal{F}(\mathbb{S}^3_{++}) \mapsto \mathbb{C}$. In addition

$$\int_{S^3_{++}} \left(\frac{1}{1-s}\right)^{2m} d\mu = \|(I - \mathrm{Re}(T))^{-2m}(1 + \mathcal{I}_\Lambda)\|^2 < \infty,$$

for all integers $m \geq 1$, which completes our assertion

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