An Idempotent Approach to Truncated Moment Problems

F.-H. Vasilescu

Department of Mathematics
University of Lille 1, France

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Truncated Moment Problems

The study of truncated moment problems means, roughly speaking, that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a given length $n \ge 1$ and $m \ge 0$ is an integer, one looks for a positive measure μ on $\mathbb{R}^n$ such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \le 2m$. As Tchakaloff firstly proved, if such a measure exists, we may always assume it to be atomic.

Some Contributors

The mathematical literature dedicated to truncated moment problems of the last fifteen years contains many contributions of R. Curto and L. Fialkow. In most of the papers by Curto and Fialkow, the approach to truncated moment problems is based on an associated moment matrix, whose positivity and *flatness* lead to a certain description of the solutions. The use of the *Riesz functional* to solve various moment problems and related topics appears in several works S. Burgdorf and I. Klep, L. Fialkow and J. Nie, H. M. Möller, M. Putinar as well as the author of this text.
Introducing a concept of *idempotent element* with respect to a unital square positive functional, we attempt, in the following, to give a new approach to truncated moment problems.

Framework: Function Spaces

Let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on a topological space Ω. We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space* (on Ω). Occasionally, we use the notation $\mathcal{RS}$ to designate the "real part" of $\mathcal{S}$, that is $\{f \in \mathcal{S}; f = \bar{f}\}$.

Let also $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

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Framework: Unital Square Positive Functionals

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$.
(3) $\Lambda(1) = 1$. has been dominated by the A linear map Λ with

the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.
When $\mathcal{S}$ is an algebra, conditions (2) and (3) imply condition (1). In this case, a map Λ with the property (2) is usually said to be *positive (semi)definite*.
Condition (3) may be replaced by $\Lambda(1) > 1$ but (looking for probability measures representing such a functional) we always assume (3) in the stated form, without loss of generality.

Elementary Properties of USPF's

If $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ p, q \in \mathcal{S}. \tag{1}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle f + \mathcal{I}_\Lambda, g + \mathcal{I}_\Lambda \rangle = \Lambda(f\bar{g}). \tag{2}$$

In fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\, \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$.
If $\mathcal{S}$ is finite dimensional, then $\mathcal{H}_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

FrIntegral Representations of Arbitrary Linear Functionalsamework Again

Let $n \geq 1$ will be a fixed integer. We freely use multi-indices from $\mathbb{Z}_+^n$ and the standard notation related to them.
The symbol $\mathcal{P}$ will designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients.
For every integer $m \geq 1$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where deg(p) is the total degree of p. Note that $\mathcal{P}_m^{(2)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

Giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$, $\gamma_0 = 1$, we associate it with a map $\Lambda_\gamma : \mathcal{P}_{2m} \mapsto \mathbb{C}$ given by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, extended to $\mathcal{P}_{2m}$ by linearity. The map Λ_γ is called the *Riesz functional associated to* γ.

We clearly have $\Lambda_\gamma(1) = 1$ and $\Lambda_\gamma(\bar{p}) = \overline{\Lambda_\gamma(p)}$ for all $p \in \mathcal{P}_{2m}$. If, moreover, $\Lambda_\gamma(|p|^2) \ge 0$ for all $p \in \mathcal{P}_m$, then Λ_γ is a uspf. In this case, we say that γ itself is *square positive*.

Conversely, if $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf, setting $\gamma_\alpha = \Lambda(t^\alpha)$, $|\alpha| \le 2m$, we have $\Lambda = \Lambda_\gamma$, as above. The multi-sequence γ is said to be the *multi-sequence associated to the uspf* Λ.

Introducing Idempotents

Let $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and let $C(\Xi)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ξ, endowed with the sup-norm. For every integer $m \geq 0$ we have the restriction map $\mathcal{P}_m \ni p \mapsto p|\Xi \in C(\Xi)$. Let us fix an integer m for which this map is surjective. (Such an m always exists via the Lagrange or other interpolation polynomials.) Let also $\mu = \sum_{j=1}^d \lambda_j \delta_{\xi^{(j)}}$, with $\lambda_j > 0$, $\delta_{\xi^{(j)}}$ the Dirac measure at $\xi^{(j)}$, $j = 1, \ldots, d$ and $\sum_{j=1}^d \lambda_j = 1$. We put $\Lambda(p) = \int_\Xi p d\mu$ for all $p \in \mathcal{P}_{2m}$, which is a uspf, for which μ is a representing measure.

Let now $f \in C(\Xi)$ be an idempotent, that is, the caracteristic function of a subset of Ξ. Then there exists a polynomial $p \in \mathcal{P}_m$, supposed to have real coefficients, such that $p|\Xi = f$. Consequently, $\Lambda(p^2) = \int_\Xi p^2 d\mu = \int_\Xi p d\mu = \Lambda(p)$. This shows that the solutions the equation $\Lambda(p^2) = \Lambda(p)$, which can be expressed only in terms of Λ, play an important role when trying to reconstruct the representing measure μ.

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Integral Representations of Arbitrary Functionals

Let $\mathcal{V}$ be a complex vector space, and let $\mathcal{V}^*$ be its (algebraic) dual.

Definition 1 We say that $\phi \in \mathcal{V}^*$ has has been dominated by thean *integral representation* on a subset $\Delta \subset \mathcal{V}^*$ if there exists a probability measure μ on Δ such that

$$\phi(x) = \int_\Delta \delta(x) d\mu(\delta), \; x \in \mathcal{V}.$$

The measure μ is said to be a *representing measure* for the functional ϕ, and it is *d-atomic* if the support of μ consists of *d* distinct points in Δ.

When $\mathcal{V}$ is a function space on Ω, we may identify Ω with a subset of $\mathcal{V}^*$. In this case, we may speak about representing measures with support in Ω.

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An Integral Representation Theorem

Integral representations are automatic for functionals on finite dimensional vector spaces. Specifically, we prove the following.

Theorem 1 If $\mathcal{V}$ is a finite dimensional complex vector space, and let $\phi \in \mathcal{V}^*$ be nonnull. Then we have:
(1) The functional ϕ induces C^*-algebra structures on the space $\mathcal{V}$.
(2) The functional ϕ has a *d*-atomic integral representation, where *d* is the dimension of $\mathcal{V}$.

Sketch of Proof of Theorem 1

The proof uses the following.

Lemma 1 Let $d = \dim \mathcal{V}$, and let $\iota \in \mathcal{V}$ be such that $\phi(\iota) = 1$. There exists a basis $\{b_1, \ldots, b_d\}$ of $\mathcal{V}$ such that $\phi(b_j) > 0$ for all $j = 1, \ldots, d$, and $\iota = b_1 + \cdots + b_d$.

Sketch of Proof of Theorem 1. Let $\{b_1, \ldots, b_d\}$ be a basis of $\mathcal{V}$ given by Lemma 1 for a fixed $\iota \in \mathcal{V}$ with $\phi(\iota) = 1$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\} \subset \mathcal{V}^*$ be the dual basis, so $\delta_j(b_k) = 1$ if $j = k$ and $= 0$ if $j \neq k$. Using the representation of every element $x \in \mathcal{V}$ as a sum $x = \sum_{j=1}^d \delta_j(x) b_j$, we get a bijective linear map

$$\mathcal{V} \ni x \mapsto x^\# \in C(\Delta),\ x^\#(\delta) = \delta(x),\ \delta \in \Delta,$$

and we may carry the C^*-algebra structure of $C(\Delta)$ onto $\mathcal{V}$.

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More Notation

Theorem 1 shows that every linear functional on a finite dimensional space has an integral representation via a probability measure, for some C^*-algebra structure of the ambient space, depending upon the given functional. We can refine the previous construction, relating it to a preexistent multiplicative structure.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\}$, and let $\mathcal{H}_\Lambda = \mathcal{P}_m/\mathcal{I}_\Lambda$, which has a Hilbert space structure induced by Λ. We denote $\langle *, * \rangle$, $\| * \|$, the inner product and the norm induced on $\mathcal{H}_\Lambda$ by Λ, respectively. For every $p \in \mathcal{P}_m$, we put $\hat{p} = p + \mathcal{I}_\Lambda \in \mathcal{H}_\Lambda$. When $\hat{p} \in \mathcal{H}_\Lambda$, we freely choose a fixed representative p.

Definition of Idempotents

The symbol $\mathcal{RH}_\Lambda$ will designate the set $\{\hat{p} \in \mathcal{RH}_\Lambda; p - \bar{p} \in \mathcal{I}_\Lambda\}$, that is, the set of "real" elements from $\mathcal{RH}_\Lambda$. If $\hat{p} \in \mathcal{RH}_\Lambda$, we always choose $p \in \mathcal{RP}_m$.

Definition 2 An element $\hat{p} \in \mathcal{RH}_\Lambda$ is said to be *idempotent* if it is a solution of the equation $\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle$.

Remark Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is idempotent if and only if $\Lambda(p^2) = \Lambda(p)$, via relation (2). Set

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{3}$$

which the family of nonnull idempotent elements from $\mathcal{RH}_\Lambda$. This family is nonempty because $\hat{1} \in \mathcal{ID}(\Lambda)$.

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Some Lemmas

Note that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

Lemma 2 (1) If $\hat{p}, \hat{q}, \hat{p} - \hat{q} \in \mathcal{ID}(\Lambda)$, then $\hat{q}$ and $\hat{p} - \hat{q}$ are orthogonal.
(2) If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, and $\hat{q}$, $\hat{1} - \hat{q}$ are orthogonal.
(3) If $\{\hat{p}_1, \ldots, \hat{p}_d\} \subset \mathcal{ID}(\Lambda)$ are mutually orthogonal, then $\sum_{j=1}^{d} \hat{p}_j \in \mathcal{ID}(\Lambda)$.

Integral Representations of Arbitrary Linear Functionals
Lemma 3 Let $\{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$, consistig of mutually orthogonal elements. The family $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is maximal with respect to the inclusion if and only if $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$.

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An Abstract Idempotent Equation

We are interested in the existence of orthogonal families of idempotents with respect to a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$.
It is easily checked that $p \in \mathcal{RP}_m$, $p = \sum_{|\xi| \le m} c_\xi t^\xi$, is a solution of the equation $\Lambda(p^2) = \Lambda(p)$ if and only if

$$\sum_{|\xi|,|\eta| \le m} \gamma_{\xi+\eta} c_\xi c_\eta - \sum_{|\xi| \le m} \gamma_\xi c_\xi = 0,$$

where $\gamma = (\gamma_\xi)_{|\xi| \le 2m}$ is the finite multi-sequence associated to the uspf Λ.
To study the existence of solutions for such an equation, it is convenient to use at the beginning an abstract framework.

Let $N \geq 1$ be an arbitrary integer, let $A = (a_{jk})_{j,k=1}^{N}$ be a matrix with real entries, that is positive on $\mathbb{C}^N$ (endowed with the standard scalar product denoted by $(*|*)$, and associated norm $|||*|||$), and let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$. We look for necessary and sufficient conditions insuring the existence of a solution $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ of the equation

$$(Ax|x) - 2(b|x) = 0. \tag{4}$$

The particular case which interests us will be dealt with in the following.
The range and the kernel of A, regarded as an operator on $\mathbb{C}^N$, will be denoted by $R(A)$, $N(A)$, respectively. Note also that $R(A) = R(B)$, and $N(A) = N(B)$, where $B = A^{1/2}$

The Idempotent Equation of a USPF

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$ be the multi-sequence associated to Λ. Then $A_\Lambda = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \le m}$ is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n ; |\xi| \le m\}$. In fact, by identifying the space $\mathcal{P}_m$ with $\mathbb{C}^N$ via the isomorphism

$$\mathcal{P}_m \ni p_x = \sum_{|\alpha| \le m} x_\alpha t^\alpha \mapsto x = (x_\alpha)_{|\alpha| \le m} \in \mathbb{C}^N, \qquad (5)$$

then $A = A_\Lambda$ is the operator with the property $(Ax|y) = \Lambda(p_x \bar{p}_y)$ for all $x, y \in \mathbb{C}^N$. The operator A will be occasionally called the *Hankel operator* of the uspf Λ. Note that $\mathcal{I}_\Lambda$ is isomorphic to $N(A)$, and $\mathcal{H}_\Lambda$ is isomorphic to $R(A)$, via the isomprphism (5). Note also that the elements $\hat{p}_x, \hat{p}_y$ are orthogonal in $\mathcal{H}_\Lambda$ if and only if $(Ax|y) = (Bx|By) = 0$.

Let us deal with equation (4) in this particular context. Set $2b = (\gamma_\xi)_{|\xi| \le m} \in \mathbb{R}^N$. In fact, $2b = A\iota$, where $\iota = (1, 0, \ldots, 0) \in \mathbb{R}^N$. With this notation, equation (4) will be called the *idempotent equation* of the uspf Λ. Namely,

$$(Ax|x) - (A\iota|x) = 0. \tag{6}$$

Because $\Lambda(p_x^2) = (Ax|x) = 0$ implies $\Lambda(p_x) = (A\iota|x) = 0$, we are intrested only in solutions $x = x^{(1)} \in R(A) = R(A_1)$, where $A_1 = A|R(A)$. Note also that $b = b^{(1)} \in R(A)$. Obviously, the vector ι is always a nonnull solution of the idempotent equation.

Orthogonal Families of Idempotents

Proposition 4 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $A : \mathbb{C}^N \mapsto \mathbb{C}^N$ be the associated Hankel operator.

The nonnull solutions of the idempotent equation of Λ in $R(A) \cap \mathbb{R}^n$ are given by

$$x^{(1)} = B_1^{-1}(y^{(1)} + B_1^{-1}b), \; y^{(1)} \in R(A) \cap \mathbb{R}^N, \; |||y^{(1)}||| = |||B_1^{-1}b|||,$$

except for $y^{(1)} = -B_1^{-1}b$. In addition, tha assignment $y^{(1)} \mapsto x^{(1)}$ is one-to one.

The idempotent equation of Λ has only one nonnull solution in $R(A) \cap \mathbb{R}^n$ if and only if $\dim_{\mathbb{R}} R(A) \cap \mathbb{R}^n = 1$.

If $d := \dim_{\mathbb{R}} R(A) \cap \mathbb{R}^n > 1$, there exists a family $\{x_1^{(1)}, \ldots, x_d^{(1)}\}$ of nonnull solutions in $R(A) \cap \mathbb{R}^n$ of the idempotent equation of Λ such that the vectors $\{B_1 x_1^{(1)}, \ldots, B_1 x_d^{(1)}\}$ are mutually orthogonal in $R(A)$.

Some Corollaries

Corollary 1 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. Then there are polynomials $b_1, \ldots, b_d \in \mathcal{RP}_m$ such that $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, and every $p \in \mathcal{P}_m$ can be uniquely represented as

$$p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0,$$

with $p_0 \in \mathcal{I}_\Lambda$ and $d = \dim \mathcal{H}_\Lambda$.

Corollary 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf such that the associated Hankel operator A is invertible, so $\mathcal{H}_\Lambda = \mathcal{P}_m$, $\mathcal{I}_\Lambda = \{0\}$, and $d = N = \dim\mathcal{P}_m$. The nontrivial solutions of the idempotent equation of Λ in $\mathbb{R}^N$ are given by

$$x = B^{-1}y + \frac{1}{2}\iota,\ \ y \in \mathbb{R}^N,\ \ |||y||| = \frac{1}{2}|||B\iota|||,$$

except for $y = -\frac{1}{2}B\iota$, where $B = A^{1/2}$. Moreover, if $\{v_1, \ldots, v_d\}$ is an orthogonal family in the sphere $\{v \in \mathcal{R}\mathcal{P}_m; |||v||| = \frac{1}{2}|||B\iota|||\}$, then the vectors

$$x^{(j)} = 4(A\iota|\iota)^{-1}(B\iota|v_j)B^{-1}v_j,\ j = 1, \ldots, d$$

are nonnull solution of the idempotent equation with $\{Bx^{(1)}, \ldots, Bx^{(d)}\}$ orthogonal, provided $(B\iota|v_j) \neq 0$, $\forall j$.

Theorem 2 For every $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ uspf there exist orthogonal bases of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements.

In fact, there exists a one-to-one correspondence between orthogonal families $\{v_1, \ldots, v_l\}$ in $S_r \subset R(A) \cap \mathbb{R}^N$, with $r = |||B_1^{-1}b|||$ and $y^{(0)} := B_1^{-1}b \notin \cup_{j=1}^{l}\{v_j\}^\perp$, and the families of nonnull solutions $\{x_1^{(1)}, \ldots, x_l^{(1)}\}$ in $R(A) \cap \mathbb{R}^n$ of the idempotent equation of Λ, with $B_1x_1^{(1)}, \ldots, B_1x_l^{(1)}$ mutually orthogonal. Specifically, we have

$$x_1^{(j)} = 2r^{-2}(y^{(0)}|v_j)B_1^{-1}v_j, \; j = 1\ldots, l.$$

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Example 1

Curto and Fialkow considered the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

acting as an operator on $\mathbb{C}^3$, which is positive. Unlike them, we are interested in the solutions of the idempotent equation $(A\mathbf{x}|\mathbf{x}) = (A\iota|\mathbf{x})$, where $\iota = (1,0,0)$. Looking only for solutions $(y,y,z) \in R(A)$ of the idempotent equation, we must have

$$(A(y,y,z)|(y,y,z)) = ((1,1,1))|(y,y,z)),$$

because $A\iota = (1,1,1) \in R(A)$. This is equivalent to the equality

$$4y^2 + 4yz + 2z^2 - 2y - z = 0,$$

which represents an ellipse passing through the origin.

$\mathcal{H}_\Lambda$ as a C^*-Algebra

Remark According to Theorem 2, the space $\mathcal{H}_\Lambda$ has orthogonal bases consisting of idempotent elements. If $\mathcal{B}$ is such a basis, we may speak about the C^*-algebra structure of $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, in the spirit of Theorem 1. More generally, if $\mathcal{B} \subset \mathcal{ID}(\Lambda)$ is a collection of mutually orthogonal elements whose sum is $\hat{1}$, and if $\mathcal{H}_\mathcal{B}$ is the complex vector space generated by $\mathcal{B}$ in $\mathcal{H}_\Lambda$, we may speak about the C^*-algebra structure of $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$. Using the basis $\mathcal{B}$ of the space $\mathcal{H}_\mathcal{B}$, we deduce that there exist a multiplication, an involution, and a norm on $\mathcal{H}_\mathcal{B}$, making it a unital, commutative, finite dimensional C^*-algebra.

In fact, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ with $\hat{1} = \sum_{j=1}^{d} \hat{b}_j$, and if $\hat{p} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^{d} \beta_j \hat{b}_j$, are elements from $\mathcal{H}_{\mathcal{B}}$, their product is given by $\hat{p} \cdot \hat{q} = \sum_{j=1}^{d} \alpha_j \beta_j \hat{b}_j$. The involution and the norm are given by $\hat{p}^* = \sum_{j=1}^{d} \bar{\alpha}_j \hat{b}_j$, $\|\hat{p}\|_\infty = \max_{1 \le j \le d} |\alpha_j|$, respectively, if $\hat{p} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$.

In addition, the dual basis of $\mathcal{B}$ is also the space of characters of the C^*- algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$.

In fact, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ with $\hat{1} = \sum_{j=1}^{d} \hat{b}_j$, and if $\hat{p} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^{d} \beta_j \hat{b}_j$, are elements from $\mathcal{H}_{\mathcal{B}}$, their product is given by $\hat{p} \cdot \hat{q} = \sum_{j=1}^{d} \alpha_j \beta_j \hat{b}_j$. The involution and the norm are given by $\hat{p}^* = \sum_{j=1}^{d} \bar{\alpha}_j \hat{b}_j$, $\|\hat{p}\|_\infty = \max_{1 \leq j \leq d} |\alpha_j|$, respectively, if $\hat{p} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$.

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Integral Representations for USPF's

Proposition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be a collection of mutually orthogonal elements with $\hat{1} = \sum_{j=1}^{d} \hat{b}_j$, and let $\mathcal{H}_{\mathcal{B}}$ be the complex vector space generated by $\mathcal{B}$ in $\mathcal{H}_{\Lambda}$.
Let Δ be the space of characters of the C^*-algebra $\mathcal{H}_{\mathcal{B}}$, induced by $\mathcal{B}$. If $\mathcal{S}_{\mathcal{B}} = \{p \in \mathcal{P}_m; \hat{p} \in \mathcal{H}_{\mathcal{B}}\}$, there exists a linear map $\mathcal{S}_{\mathcal{B}} \ni p \mapsto p^{\#} \in C(\Delta)$, whose kernel is $\mathcal{I}_{\Lambda}$, such that

$$\Lambda(p) = \int_{\Delta} p^{\#}(\delta) d\mu(\delta), \; p \in \mathcal{S}_{\mathcal{B}},$$

where μ is a d-atomic probability measure on Δ.

Proposition 3

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements. Let also $\mathcal{H}_\mathcal{C}$ be the sub-C^*-algebra generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ in $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal is $\leq \dim \mathcal{H}_\Lambda$, and a linear map $\mathcal{S}_\mathcal{C} \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, such that

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi), \ u \in \mathcal{S}_\mathcal{C},$$

where $\mathcal{S}_\mathcal{C} = \{u \in \mathcal{P}_m; \hat{u} \in \mathcal{H}_\mathcal{C}\}$, and μ is a probability measure on Ξ.

Proposition 4

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements such that $\{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ generates the algebra $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal equals $\dim \mathcal{H}_\Lambda$, and a surjective linear map $\mathcal{P}_m \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, with the property

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi),\ \ u \in \mathcal{P}_m,$$

where μ is a probability measure on Ξ.
Moreover, the C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$ are $*$-isomorphic.
If $r(\hat{t}_1, \ldots, \hat{t}_n) = 0$ for all $r \in \mathcal{I}_\Lambda$, then $u^\# = u|\Xi$ for all $u \in \mathcal{P}_m$.

Remarks

(1) Assume that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ given by

$$\Lambda(p) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ \ p \in \mathcal{P}_{2m},$$

with $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^{d} \lambda_j = 1$, where $d = \mathrm{dim}\mathcal{H}_\Lambda$.
Let $r \geq m$ be an integer such that $\mathcal{P}_r$ contains interpolating polynomials for the family of points $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Setting $\Lambda_\mu(p) = \int_\Xi p d\mu,\ p \in \mathcal{P}_{2r}$, we have $\Lambda_\mu | \mathcal{P}_{2m} = \Lambda$, and $\mathcal{I}_{\Lambda_\mu} = \{p \in \mathcal{P}_r; p|\Xi = 0\}$.

Moreover, the space $\mathcal{H}_r := \mathcal{P}_r/\mathcal{I}_{\Lambda_\mu}$ is at least linearly isomorphic to $C(\Xi)$, where $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, via the map $\mathcal{H}_r \ni p + \mathcal{I}_{\Lambda_\mu} \mapsto p|\Xi \in C(\Xi)$.
As $\mathcal{H}_\Lambda$ may be regarded as a subspace of $\mathcal{H}_r$, and $\dim\mathcal{H}_\Lambda = \dim C(\Xi)$, $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in C(\Xi)$ is a linear isomorphism. Let $\chi_k \in C(\Xi)$ be the characteristic function of the set $\{\xi^{(k)}\}$ and let $\hat{b}_k \in \mathcal{H}_\Lambda$ be the element with $b_k|\Xi = \chi_k$, $k = 1, \ldots, d$. The integral representation leads to $\Lambda(b_k^2) = \Lambda(b_k)$, $\Lambda(b_k b_l) = 0$ for all $k, l = 1, \ldots, d$, $k \neq l$. This shows that $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is a basis of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents.

Consequently, if $\mathcal{H}_\Lambda$ is given the C^*-algebra structure induced by $\{\hat{b}_1, \ldots, \hat{b}_d\}$, then $\mathcal{H}_\Lambda$ and $C(\Xi)$ are isomorphic as C^*-algebras. Note also that $\Lambda(b_j) = \lambda_j$ for all $j = 1, \ldots, d$, and that if $\hat{p} = \alpha_1 \hat{b}_1 + \cdots + \alpha_d \hat{b}_d \in \mathcal{H}_\Lambda$ is arbitrary, then $\alpha_j = \lambda_j^{-1} \Lambda(p b_j) = p(\xi^{(j)})$ for all $j = 1, \ldots, d$.

(2) Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf such that $\mathcal{I}_\Lambda = \{0\}$; therefore, $\mathcal{P}_m = \mathcal{H}_\Lambda$. Let $\mathcal{B} = \{b_1, \ldots, b_d\}$ be an orthogonal basis of $\mathcal{P}_m$ consisting of idempotents, where $d = \dim \mathcal{P}_m$. Assume that the family $\{1, t_1, \ldots, t_d\}$ generates the C^*-algebra $\mathcal{P}_m$ induced by $\mathcal{B}$. In particular, considering the set $\Xi = \{\xi^{(1)} \ldots, \xi^{(d)}\}$, we obtain the equality $\mathcal{P}_m = \{\sum_{j=1}^d p(\xi^{(j)}) b_j;\ p \in \mathcal{P}\}$. As a matter of fact, because the linear map $\mathcal{P}_m \ni u \mapsto u^\# \in C(\Xi)$ is a $*$-isomorphism, and the last condition from Proposition 4 is automatically fulfilled, we must have $p = \sum_{j=1}^d p(\xi^{(j)}) b_j;\ p \in \mathcal{P}_m$, and so, $\Lambda|\mathcal{P}_m$ has a d-atomic representing measure μ on Ξ given by

$$\Lambda(p) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ p \in \mathcal{P}_m = \int_\Xi p(t) d\mu(t),$$

with $\lambda_j = \Lambda(b_j)$, $j = 1, \ldots, d$.

(3) Note that the monomial $\hat{t}^\alpha$ is an element of the algebra $\mathcal{H}_\mathcal{C}$, not necessarily equal to $\widehat{t^\alpha} = t^\alpha + \mathcal{I}_\Lambda \in \mathcal{H}_\Lambda$.

Definition Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\{b_1, \ldots, b_d\}$ be a subset of $\mathcal{RP}_m$ such that $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* if

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j) \tag{7}$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

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Main Result

Theorem 3 The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a Λ-multiplicative basis of $\mathcal{H}_\Lambda$.

Remark For the proof of Theorem 3, we need the following. Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. The basis $\mathcal{B}$ is Λ-multiplicative if and only if $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \leq m$ and δ is a character of the C^*-algebra $\mathcal{H}_\Lambda$ associated to $\mathcal{B}$.

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Corollary 3

The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a family of polynomials $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the following properties:

(i) $\Lambda(b_j^2) = \Lambda(b_j) > 0, \ j = 1, \ldots, d$;

(ii) $\Lambda(b_j b_k) = 0, \ j, k = 1, \ldots, d, \ j \neq k$;

(iii)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Corollary 4

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with invertible Hankel operator. The uspf Λ has a representing measure in $\mathbb{R}^n$ having $d = \dim \mathcal{P}_m$ atoms if and only if there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{H}_\Lambda = \mathcal{P}_m$ such that

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d, \ p \in \mathcal{P}_m,$$

where

$$\xi^{(j)} = (\Lambda(b_1)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_d)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \ j = 1, \ldots, d.$$

Theorem 4

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with invertible Hankel operator, and let $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{H}_\Lambda = \mathcal{P}_m\,(d = \dim \mathcal{P}_m)$ be an orthogonal basis consisting of idempotent elements. Assume that $\mathcal{P}_m$ is endowed with the C^*-algebra structure induced by $\mathcal{B}$. The following conditions are equivalent.

(i) $\mathcal{B}$ is Λ-multiplicative.

(ii) The polynomials $\{1, t_1, \ldots, t_n\}$ generate the C^*-algebra $\mathcal{P}_m$.

(iii) The points

$$\xi^{(j)} = (\Lambda(b_1)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_d)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d,$$

are distinct.

Example 2 The matrix A from Example 1 is the Hankel matrix associated to the uspf $\Lambda : \mathcal{P}_4^1$, where $\mathcal{P}_4^1$ is the space of of polynomials in one real variable t, with complex coefficients, of degree ≤ 4, and Λ is the Riesz functional associated to the sequence $\gamma = (\gamma_k)_{0\leq k\leq 4}$, $\gamma_0 = \cdots = \gamma_3 = 1$, $\gamma_4 = 2$. Note that $\mathcal{I}_\Lambda = \{p(t) = a - at; a \in \mathbb{C}\}$, and $\mathcal{H}_\Lambda = \{\hat{p}\,;\, p(t) = a + at + (a+b)t^2,\ a, b \in \mathbb{C}\}$. Setting $p_0(t) = 1/2 - t/2$, $p_1(t) = 1/2 + t/2$, we have $1 = p_0 + p_1$ and $t = p_1 - p_0$. But $p_0 \in \mathcal{I}_\Lambda$, and so $\hat{t} = \hat{1}$. Consequently, for any choice of an othogonal basis $\mathcal{H}_\Lambda$ consisting of idempotents, we cannot have $\widehat{t^2} = \hat{t}^2$ because $\hat{t}^2 = \hat{t} = \hat{1}$, while $\widehat{t^2} = t^2 + \mathcal{I}_\Lambda \neq \hat{1}$. This shows that Λ has no representing measure consisting of two atoms. As a matter of fact, the element $\hat{t}$ does not separate the points of the space of characters of $\mathcal{H}_\Lambda$ for any choice of an orthogonal basis $\{\hat{b}_1, \hat{b}_2\}$ consisiting of idempotent elements.

Example 3 Corollary 3 implies that all uspf $\Lambda : \mathcal{P}_2 \mapsto \mathbb{C}$ have representing measures in $\mathbb{R}^n$ having $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an arbitrary orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, then the condition

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

is automatically fulfilled when $|\alpha| + |\beta| \le 1$, $j = 1, \ldots, d$

Remark

Using a definition of Curto and Fialkow, we say that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a *flat extension* if there exists a uspf $M : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ extending Λ such that $\mathcal{P}_m + \mathcal{I}_M = \mathcal{P}_{m+1}$. This implies that the natural map

$$\mathcal{H}_\Lambda \ni p + \mathcal{I}_\Lambda \mapsto p + \mathcal{I}_M \in \mathcal{H}_M \tag{8}$$

is a unitary operator. In particular, $\dim \mathcal{H}_\Lambda = \dim \mathcal{H}_M$ and every orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents can be associated with an orthogonal basis of $\mathcal{H}_M$ consisting of idempotents.
Previous work leads to the existence (and uniqueness) of a basis $\mathcal{B}$ of $\mathcal{H}_M$, which is Λ-multiplicative.

Indeed, there exists a tuple $A = (A_1, \ldots, A_n)$ on $\mathcal{H}_M$ consisting of commuting self-adjoint operators, induced by the multiplications with the independent variables $t_1, \ldots, t_n$. Further, the Hilbert space $\mathcal{H}_M$ is linearly isomorphic with the C^*-algebra $\mathcal{C}_A := \{p(A); p \in \mathcal{P}\}$, via the map $\mathcal{H}_M \ni p + \mathcal{I}_M \mapsto p(A) \in \mathcal{C}_A$. The C^*-algebra structure of $\mathcal{H}_M$ is exactly that inherited from $\mathcal{C}_A$, having a basis consisting of idempotent elements $\{E_1, \ldots, E_d\}$, where $d = \dim \mathcal{H}_M = \dim \mathcal{C}_A$, which is, in fact, provided by the spectral measure of A. Choosing the polynomials b_j in $\mathcal{P}_{m+1}$ such that $b_j(A) = E_j$ $(j = 1, \ldots, d)$, we obtain the desired basis of $\mathcal{B}$ of $\mathcal{H}_M$.

Full Moment Problems

We now present a characterization of the existence of representing measures for full moment problems, in terms of idempotent elements.

Theorem 4 A uspf $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ if and only if there exists an increasing sequence of nonnegative integers $\{m_k\}_{k\geq 1}$ such that every Hilbert space $\mathcal{H}_{\Lambda_k}$ has a Λ_k-multiplicative basis.

The proof is based on a result by Stochel.

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Continuous Point Evaluations

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. For every point $\xi \in \mathbb{R}^n$, we denote by δ_ξ the point evaluation at ξ, that is, $\delta_\xi(p) = p(\xi)$, for every polynomial $p \in \mathcal{P}$. As in the Introduction, we set $\mathcal{I}_\Lambda = \{f \in \mathcal{P}_m; \Lambda(|f|^2) = 0\}$, while $\mathcal{H}_\Lambda$ is the finite dimensional Hilbert space $\mathcal{P}_m/\mathcal{I}_\Lambda$.

Definition The point evaluation δ_ξ is said to be Λ-*continuous* if there exists a constant $c_\xi > 0$ such that

$$|\delta_\xi(p)| \leq c_\xi \Lambda(|p|^2)^{1/2}, \ p \in \mathcal{P}_m.$$

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$$|\delta_\xi(p)| \leq c_\xi \Lambda(|p|^2)^{1/2}, \;\; p \in \mathcal{P}_m.$$

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\xi \in \mathbb{R}^n$ such that δ_ξ is Λ-continuous. For every polynomial p let us denote by $\mathcal{Z}(p)$ the set of its zeros.

Lemma We have the equality $\mathcal{Z}_\Lambda = \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p)$.

RemarkThe previous lemma shows that the set $\mathcal{Z}_\Lambda$ coincides with the algebraic variety of the moment sequence associated to Λ (introduced by Curto & Fialkow).

Lemma (Curto & Fialkow) Suppose that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has an atomic representing measure μ. Then $\mathrm{supp}(\mu) \subset \mathcal{Z}_\Lambda$.

Remark It follows from previous Lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\xi \in \mathbb{R}^n$ such that δ_ξ is Λ-continuous. For every polynomial p let us denote by $\mathcal{Z}(p)$ the set of its zeros.

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Remark It follows from previous Lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with the property $\mathcal{Z}_\Lambda \neq \emptyset$. As previously noted, the set $\{\delta_\xi^\Lambda; \xi \in \mathcal{Z}_\Lambda\}$ is a subset in the dual of the Hilbert space $\mathcal{H}_\Lambda$. Therefore, for every $\xi \in \mathcal{Z}_\Lambda$ there exists a unique vector $\hat{v}_\xi \in \mathcal{H}_\Lambda$ such that $\delta_\xi^\Lambda(\hat{p}) = \langle \hat{p}, \hat{v}_\xi \rangle = \Lambda(pv_\xi) = p(\xi)$ for all $p \in \mathcal{P}_m$. Let $\mathcal{V}_\Lambda = \{\hat{v}_\xi; \xi \in \mathcal{Z}_\Lambda\}$. We may and shall always assume that a chosen representative v_ξ from the equivalence class $\hat{v}_\xi$ is a polynomial with real coefficients.

Theorem 5

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ ($m \geq 1$) with $\mathcal{Z}_\Lambda$ nonempty. The uspf Λ has a representing measure in $\mathbb{R}^n$ consisting of d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{H}_\Lambda$ such that

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d,$$

$$\hat{p} = \Lambda(v_1)^{-1}\Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(pv_d)\hat{v}_d,\ p \in \mathcal{P}_m,$$

and

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \Lambda(v_j)^{-1}\Lambda(v_j v_k)\Lambda(v_j v_l),\ k, l = 1, \ldots, d.$$

Summary

The existence of orthogonal bases consisting of idempotents in the Hilbert spaces associated to uspf's provides a new approach to truncated moment problems. The efficiency of this approach was proved by some examples. In addition, representing measures in a larger sense exist in abundance. The main question in the actual framework is to determine explicitly which are the privileged bases having the Λ-multiplicativity property from Theorem 3.

Thank you !