MOMENTS, IDEMPOTENTS, AND INTERPOLATION

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Outline

Abstract

The aim of this talk is to present a new approach to truncated moment problems, based on the use of spaces of characters of certain associated finite dimensional commutative Banach algebras. The existence of representing measures for such functionals is characterized via some intrinsic conditions.

Numerous relatively recent contributions in this area are due to R. Curto and L. Fialkow. Other contributors are M. Putinar, M. Laurent, H. Möller, S. Burgdorf and I. Klep, etc.

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Truncated Moment Problems

Solving a truncated moment problems means, roughly speaking, that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a fixed length $n \ge 1$, and $m \ge 0$ is an integer, one looks for a positive measure μ on $\mathbb{R}^n$ (usually called a *representing measure* for γ) such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \le 2m$.

If such a measure exists, we may always assume it to be atomic, via Tchakaloff's theorem.

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If such a measure exists, we may always assume it to be atomic, via Tchakaloff's theorem.

Tchakaloff's Theorem

In 1957, V. Tchakaloff essentially proved the following cubature formula:

Theorem Let $F \subset \mathbb{R}^2$ be a bounded closed set. Fixing an integer $n \geq 1$ and setting $N = (n+1)(n+2)/2$, there are points $(u_k, v_k) \in \mathbb{R}^2$ and constants $a_k \geq 0$, $k = 1, \ldots, N$ such that

$$\int\int_F p(u,v)dudv = \sum_{k=1}^{N} a_k p(u_k, v_k)$$

for all polynomials p in two variables, of total degree $\leq n$.

This result was improved by several authors: R. Curto and L. Fialkow, M. Putinar, C. Bayer and J. Teichmann, etc.

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A Special Version of Tchakaloff's Theorem

In 2006, C. Bayer and J. Teichmann proved the following version of Tchakaloff's result:

Theorem Let μ be a positive Borel measure on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be measurable, with $\mu(\mathbb{R}^n \setminus A) = 0$. If

$$\int_{\mathbb{R}^n} (t_1^2 + \cdots + t_n^2)^{1/2} d\mu(t) < +\infty,$$

there exist an integer $k \leq n$, k points $\xi_1, \ldots, \xi_k \in A$, and weights $\lambda_1 > 0, \ldots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^n} p(t) d\mu(t) = \sum_{j=1}^{k} \lambda_j p(\xi_j),$$

for every polynomial p of total degree equal to 1.

We present, in the following, another version of Tchakaloff's theorem, obtained with our methods. We use a slightly stronger hypothesis but the consequences seem to improve older results of various authors.

A Consequence of our Results

Theorem Let μ be a positive Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} (t_1^2 + \cdots + t_n^2) d\mu(t) < +\infty,$$

Then there exist a subset $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and positive numbers $\lambda_1, \ldots, \lambda_d$, where $d \leq n + 1$, such that

$$\int_{\mathbb{R}^n} p(t) d\mu(t) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}), \ \ p \in \mathcal{P}_2.$$

Moreover, the weights $\lambda_1, \ldots, \lambda_d$, and the nodes $\xi^{(1)}, \ldots, \xi^{(d)}$ as well, are given by explicit formulas.

Some Notation

We fix an integer $n \geq 1$ associated with the euclidean space $\mathbb{R}^n$, and for every integer $m \geq 0$ we denote by $\mathcal{P}_m$ (resp. $\mathcal{RP}_m$) the vector space of all polynomials in n real variables, with complex (resp. real) coefficients, of total degree less or equal to m. The vector space of all polynomials in n real variables, with complex (resp. real) coefficients, will be denoted by $\mathcal{P}$ (resp. $\mathcal{RP}$).

Whenever it is necessary to specify the value of n, we write $\mathcal{P}_m^n = \mathcal{P}_m$, respectively $\mathcal{P}^n = \mathcal{P}$.

Square Positive Functionals

Let us fix an integer $m \geq 0$, and let us consider a linear map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with the properties
(1) $\Lambda(\bar{p}) = \overline{\Lambda(p)},\ p \in \mathcal{P}_{2m}$;
(2) $\Lambda(|p|^2) \geq 0,\ p \in \mathcal{P}_m$;
(3) $\Lambda(1) = 1$.

A map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with the properties (1)-(3) will be designated as a *unital square positive functional* (briefly, a *uspf*).

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Truncated Moment Problem Again

It seems to be more convenient to state the truncated moment problem in the following (equivalent) way.

Given a *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, the *truncated moment problem* means to find necessary and sufficient conditions for the existence of a finite set Ξ (usually in $\mathbb{R}^n$) and a probability measure μ on Ξ such that $\Lambda(p) = \int_\Xi p(\xi)d\mu(\xi)$ for all $p \in \mathcal{P}_{2m}$.

As before, the measure μ, when exists, is said to be a *representing measure* for Λ.

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Example

Let $\mathcal{P}_4^1$ be the space of polynomials in one real variable, denoted by t, of degree at most 4. We set $\Lambda(1) = \Lambda(t) = \Lambda(t^2) = \Lambda t^3) = 1$, $\Lambda(t^4) = 2$, and extend Λ to the space $\mathcal{P}_4^1$ by linearity. The properties (1) and (3) are obvious. Moreover, if $p(t) = x_0 + x_1 t + x_2 t^2 \in \mathcal{P}_2^1$, then

$$\Lambda(|p|^2) = |x_0 + x_1 + x_2|^2 + |x_2|^2 \geq 0,$$

showing that Λ also satisfies (2). Nevertheless, one can see that Λ has no representing measure.

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Square Positive Functionals Again

Let us remark that every *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ satisfies the

Cauchy-Schwarz inequality:

$$|\Lambda(pq)|^2 \leq \Lambda(|p|^2)\Lambda(|q|^2),\ p, q \in \mathcal{P}_m. \quad (1)$$

Setting

$$\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\},$$

the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace.

Basic Hilbert Space

The quotient space

$$\mathcal{H}_\Lambda = \mathcal{P}_m / \mathcal{I}_\Lambda$$

is a Hilbert space, whose scalar product is given by

$$\langle p + \mathcal{I}_\Lambda, q + \mathcal{I}_\Lambda \rangle = \Lambda(p\bar{q}), \;\; p, q \in \mathcal{P}_m. \tag{2}$$

The symbol $\mathcal{RH}_\Lambda$ designate the space $\{\hat{p} \in \mathcal{H}_\Lambda; p \in \mathcal{RP}_m\}$, which is a real Hilbert space. Fixing an element $\hat{p} \in \mathcal{RH}_\Lambda$, we always suppose that its representative p is in $\mathcal{RP}_m$.

Let us remark that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

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Idempotents with Respect to a USPF

Definition 1 An element $\hat{p} \in \mathcal{RH}_\Lambda$ is called Λ-*idempotent* (or simply *idempotent* when Λ is fixed) if it is a solution of the equation

$$\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle. \tag{3}$$

Remark 1 Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is an idempotent if and only if $\Lambda(p^2) = \Lambda(p)$.

Put

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{4}$$

which is a nonempty family because $\hat{1} \in \mathcal{ID}(\Lambda)$.

A Lemma

Lemma 1 (1) If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, and $\hat{q}$, $\hat{1} - \hat{q}$ are orthogonal.
(2) If $\{\hat{p}_1, \ldots, \hat{p}_d\} \subset \mathcal{ID}(\Lambda)$ are mutually orthogonal, then $\sum_{j=1}^d \hat{p}_j \in \mathcal{ID}(\Lambda)$.
(3) If $\{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ is a family of mutually orthogonal elements, this family is maximal with respect to the inclusion if and only if $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$.

Special Orthogonal Bases

Let $\mathfrak{S}_\Lambda = \{\hat{v} \in \mathcal{RH}_\Lambda; \|\hat{v}\| = 1\}$, and $\mathfrak{S}_\Lambda^1 = \{\hat{v} \in \mathfrak{S}_\Lambda; \langle \hat{v}, \hat{1} \rangle \neq 0\}$. The existence of orthogonal bases consisting of idempotents with respect to a fixed *uspf* Λ is given by the following.

Proposition 1 We have the properties:
(1) $\mathcal{ID}(\Lambda) = \{\langle \hat{v}, \hat{1} \rangle \hat{v}; \hat{v} \in \mathfrak{S}_\Lambda, \langle \hat{v}, \hat{1} \rangle \neq 0\} = \{\Lambda(v)\hat{v}; \hat{v} \in \mathfrak{S}_\Lambda, \Lambda(v) \neq 0\}$.
(2) The map

$$\mathfrak{S}_\Lambda^1 \ni \hat{v} \mapsto \langle \hat{v}, \hat{1} \rangle \hat{v} \in \mathcal{ID}(\Lambda) \tag{5}$$

is bijective.
(3) If $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathfrak{S}_\Lambda$ is an orthonormal basis in $\mathcal{H}_\Lambda$ with $\langle \hat{v}_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$, then $\{\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d\}$ is an orthogonal basis in $\mathcal{H}_\Lambda$ consisting of idempotents. Moreover,

$$\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1 + \cdots + \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d = \hat{1}.$$

Theorem 1

For every *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, the space $\mathcal{H}_\Lambda$ has othogonal bases consisting of idempotents.

Corollary 1 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ a *uspf*. There exist polynomials $b_1, \ldots, b_d \in \mathcal{RP}_m$ such that $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, and every $p \in \mathcal{P}_m$ has a unique representation of the form

$$p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0,$$

with $p_0 \in \mathcal{I}_\Lambda$ and $d = \mathrm{dim} \mathcal{H}_\Lambda$.

C^*–Algebra Structures

Given a *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, according to Theorem 1 we can choose an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$. With respect to the basis $\mathcal{B}$, we can define on $\mathcal{H}_\Lambda$ a structure of a unital commutative C^*–algebra.

If $\hat{p} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^{d} \beta_j \hat{b}_j$, are from $\mathcal{H}_\Lambda$, we put

$$\hat{p} \cdot \hat{q} = \sum_{j=1}^{d} \alpha_j \beta_j \hat{b}_j.$$

The involution and norm are given respectively by

$$\hat{p}^* = \sum_{j=1}^{d} \overline{\alpha_j} \hat{b}_j, \ \|\hat{p}\|_\infty = \max_{1 \le j \le d} |\alpha_j|.$$

To obtain the assertion, we also use the equality $\hat{1} = \sum_{j=1}^{d} \hat{b}_j$.

Characters

The C^*-algebra structure of $\mathcal{H}_\Lambda$ associated to the orthogonal basis $\mathcal{B}$ is referred to as the *C^*-algebra (structure of) $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$*.

The space of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, say $\Delta = \{\delta_1, \ldots, \delta_d\}$, coincides with the dual basis of $\mathcal{B}$. Using also the Hilbert space structure of $\mathcal{H}_\Lambda$, we obtain

$$\delta_j(\hat{p}) = \Lambda(b_j)^{-1}\langle \hat{p}, \hat{b}_j \rangle,\ \hat{p} \in \mathcal{H}_\Lambda,\ j = 1, \ldots, d.$$

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A First Integral Representation

Proposition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf*, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements, and that the set $\{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ generates this C^*-algebra. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal is $\dim \mathcal{H}_\Lambda$, and a linear map $\mathcal{P}_m \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, such that

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi), \;\; u \in \mathcal{P}_m.$$

where μ is a probability measure on Ξ.

Continuation

Moreover, the map $\mathcal{P}_m \ni u \mapsto u^\# \in C(\Xi)$ induces a $*$-isomorphism between the C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$.

If $r(\hat{t}_1, \ldots, \hat{t}_n) = 0$ for all $r \in \mathcal{I}_\Lambda$, $b_j(\xi^{(l)}) = 0$ for $j \neq l$ and $b_j(\xi^{(j)}) = 1$, $j, l = 1, \ldots, d$, then $u^\# = u|\Xi$ for all $u \in \mathcal{P}_m$.

An Interpolating Family

Remark 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf*, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by the orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$, consisting of idempotent elements. Also assume that the elements $\{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ generate the C^*-algebra $\mathcal{H}_\Lambda$. In particular, for each j there exists a polynomial $\pi_j \in \mathcal{P}$ such that $\hat{b}_j = \pi_j(\hat{t})$, $j = 1, \ldots, d$. If Δ is the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$, for every $\delta \in \Delta$ we have $\delta(\hat{b}_j) = \pi_j(\delta(\hat{t}))$, $j = 1, \ldots, d$, where $\hat{t} := (\hat{t}_1, \ldots, \hat{t}_n)$. This shows that $\{\pi_1, \ldots, \pi_d\}$ is an interpolating family of polynomials for the set $\{(\delta(\hat{t}_1), \ldots, \delta(\hat{t}_n)) \in \mathbb{R}^n; \delta \in \Delta\}$.

This assertion is a by-product of the proof of Proposition 2.

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Multiplicativity with Respect to a USPF

Our Main Theorem, which will be stated in the following, characterizes the existence of representing measures for a *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, having $d = \dim \mathcal{H}_\Lambda$ atoms, in terms of orthogonal bases of $\mathcal{H}_\Lambda$ consisting of idempotent elements. In other words, we use only intrinsic conditions.

Definition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* if

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j) \tag{6}$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

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The Main Result

Theorem 2

The *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a Λ-multiplicative basis of $\mathcal{H}_\Lambda$.

Corollary 2 The *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a family of polynomials $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the following properties:
(i) $\Lambda(b_j^2) = \Lambda(b_j) > 0,\ j = 1, \ldots, d;$
(ii) $\Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d, j \neq k;$
(iii) $\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$ whenever $0 \neq |\alpha| \leq |\beta|, |\alpha| + |\beta| \leq m, j = 1, \ldots, d.$

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Quadratic Moment Problems

Remark 3 Corollary 2 implies that all *uspf* $\Lambda : \mathcal{P}_2 \mapsto \mathbb{C}$ have representing measures in $\mathbb{R}^n$ with $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an arbitrary orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, then the condition

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

is automatically fulfilled when $|\alpha| + |\beta| \leq 1$, $j = 1, \ldots, d$.
In this case, we may write explicitly all representing measures of Λ. Indeed, with $b_1, \ldots, b_d$ as above, the support of the corresponding representing measure, say $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, is given by

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \ldots, d,$$

while the corresponding weights are $\Lambda(b_1), \ldots, \Lambda(b_d)$.

An Application

Remark 3 provides a proof of our version of Tchakaloff's theorem. Specifically, let ν be a positive Borel measure on $\mathbb{R}^n$ such that $\int(t_1^2 + \cdots + t_n^2)d\nu(t) < \infty$. We may assume $\nu(\mathbb{R}^n) = 1$. Then the map $\Lambda(p) = \int p d\nu$ is a *uspf* on $\mathcal{P}_2$. According to Remark 3, each orthogonal bases of $\mathcal{H}_\Lambda$ consisting of idempotents, and whose cardinal d is less or equal to $\dim\mathcal{P}_1 = n + 1$, is automatically Λ-multiplicative. Consequently, the subset $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$, and the positive numbers $\lambda_1, \ldots, \lambda_d$ are given by the corresponding representing measure of Λ.
The description of the weights $\lambda_1, \ldots, \lambda_d$, and that of the nodes $\xi^{(1)}, \ldots, \xi^{(d)}$ as well, is also given in Remark 3.

A Related System of Quadratic Equations

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf*. The Λ-multiplicativity can be used, at least in principle, to get a solution of the moment problem having a number of atoms equal to $\dim \mathcal{H}_\Lambda$. Specifically, according to Corollary 2, we must find a family of polynomials $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ with the properties $(i) - (iii)$.
Setting $b_j = \sum_\alpha x_{j\alpha} t^\alpha$, where $x_{j\alpha} = 0$ if $|\alpha| > m$, condition (i) of Corollary 2 means that

$$\sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{j\beta} = \sum_\alpha \gamma_\alpha x_{j\alpha}, \; j = 1, \ldots, d,$$

which is an **idempotent equation.**

Continuation

Condition (ii) is equivalent to

$$\sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{k\beta} = 0, \ \ j,k = 1,\dots,d, \, j < k,$$

which is an **orthogonality equation.**
Condition (iii) can be expressed as

$$\sum_{\xi,\eta} \gamma_{\alpha+\xi}\gamma_{\beta+\eta} x_{j\xi} x_{j\eta} = \sum_{\xi,\eta} \gamma_{\xi} \ \gamma_{\alpha+\beta+\eta} x_{j\xi} x_{j\eta},$$

$$0 \neq |\alpha| \leq |\beta|, \ \ |\alpha| + |\beta| \leq m, \, j = 1,\dots d,$$

which is a **Λ-multiplicativity equation.**

Finding a solution $\{x_{j\alpha}, j = 1, \ldots, d, |\alpha| \leq m\}$ of the equations from above, with $b_1, \ldots, b_d$ nonnull, provided it exists, means to solve the corresponding moment problem.
To find a general solution of these equation seems to be a difficult problem, but in some particular cases this is possible.

The 1-Atom Case

The case $d = 1$ is well known and easily obtained. We may approach this case from our point of view. We must have $\mathcal{H}_\Lambda = \mathbb{C}\hat{1}$, because $\dim \mathcal{H}_\Lambda = 1$ and $1 \notin \mathcal{I}_\Lambda$. For this reason, for each polynomial $p \in \mathcal{P}_m$ there exists a complex number θ_p such that $\hat{p} = \theta_p \hat{1}$, and we must have $\theta_p = \Lambda(p)$.
Clearly, $\mathcal{B} = \{\hat{1}\}$ is a basis of $\mathcal{H}_\Lambda$ consisting of one idempotent, which is Λ-multiplicative. According to Theorem 2, the *uspf* Λ must have a representing measure (clearly a Dirac measure) concentrated at the point $\xi := (\Lambda(t_1), \ldots, \Lambda(t_n)) \in \mathbb{R}^n$, because the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto \Lambda(p) \in \mathbb{C}$ is the only character of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B} = \{\hat{1}\}$.

Connection with Interpolation

The next result illustrates the strong connection between moment problems and polynomial interpolation.

Corollary 3 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf* with $\mathcal{I}_\Lambda = \{0\}$. Λ has a representing measure in $\mathbb{R}^n$ having $d = \dim \mathcal{P}_m$ atoms if and only if there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{H}_\Lambda = \mathcal{P}_m$ such that

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d, \ p \in \mathcal{P}_m,$$

where

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \ j = 1, \ldots, d.$$

Non-Singular Case

The next result characterizes the existence of representing measures in the nonsingular case.

Theorerm 3

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf* with $\mathcal{I}_\Lambda = \{0\}$, and let $\mathcal{B} = \{b_1, \ldots, b_d\} \subset \mathcal{H}_\Lambda = \mathcal{P}_m \, (d = \dim \mathcal{P}_m)$ be an orthogonal basis consisting of idempotent elements. Let also $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the dual basis of $\mathcal{B}$. Assume that $\mathcal{P}_m$ is endowed with the C^*-algebra structure induced by $\mathcal{B}$. The following conditions are equivalent.

Continuation

(i) $\mathcal{B}$ is Λ-multiplicative.
(ii) The polynomials $\{1, t_1, \ldots, t_n\}$ generate the C^*-algebra $\mathcal{P}_m$, and $\delta_k(b_j(t)) = 0$, $k \neq j$, $\delta_j(b_j(t)) = 1$, $j, k = 1, \ldots, d$.
(iii) The points

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \ j = 1, \ldots, d,$$

are distinct, and
$\delta_k(b_j(t)) = 0, k \neq j, \delta_j(b_j(t)) = 1, j, k = 1, \ldots, d.$

(*Here the elements $b_j(t)$ are computed in the C^*-algebra $\mathcal{P}_m$.*)

Recursiveness

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a *uspf* with $\mathcal{H}_\Lambda$ having a Λ-multiplicative basis. Then we have the property

$$p \in \mathcal{P}_{m-k} \cap \mathcal{I}_\Lambda, \ q \in \mathcal{P}_k \quad \Rightarrow \quad pq \in \mathcal{I}_\Lambda,$$

whenever $0 \leq k \leq m$ is an integer.

Hence, Λ-multiplicativity implies that the associated Hankel matrix is *recursively generated* in the sense of Curto and Fialkow. In addition, for $n = 1$, Λ-multiplicativity is equivalent (via Theorem 2) to the recursiveness property

$$p \in \mathcal{P}^1_{m-1} \cap \mathcal{I}_\Lambda \quad \Rightarrow \quad tp \in \mathcal{I}_\Lambda,$$

which is a necessary and sufficient condition for the existence of a representing measures in one variable.

Flatness

Let $M : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ be a *uspf*. Following Curto and Fialkow, we say that the uspf M is *flat* if $\mathcal{P}_m + \mathcal{I}_M = \mathcal{P}_{m+1}$. Setting $\Lambda = M|\mathcal{P}_{2m}$, the flatness of M is equivalent to saying that the natural isometry

$$\mathcal{H}_\Lambda \ni p + \mathcal{I}_\Lambda \mapsto p + \mathcal{I}_M \in \mathcal{H}_M$$

is a unitary operator. In particular, $d := \dim \mathcal{H}_\Lambda = \dim \mathcal{H}_M$. In our terms, the flatness of M is equivalent to the existence of an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H}_M$, consisting of idempotents, such that $b_1, \dots, b_d \in \mathcal{P}_m$.

Remark

Many results obtained by Curto and Fialkow have as a starting point the assumption of the existence of a flat extension for a given *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$. This hypothesis leads to the existence and the uniqueness of a representing measure of Λ. In particular, the flatness of a uspf Λ implies the existence of a Λ-multiplicative basis by Theorem 2. The converse is not true, in general, because the representing measure given by the Λ-multiplicativity is not necessarily unique.

Connection with K-Moment Problems

A closed subset $K \subset \mathbb{R}^n$ is said to be *semi-algebraic* if there exists a family $\mathcal{Q} := \{q_1, \ldots, q_s\} \subset \mathcal{RP}$ such that

$$K = K_{\mathcal{Q}} = \{t \in \mathbb{R}^n; q_j(t) \geq 0, j = 1, \ldots, s\}.$$

Given a *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ such that $\mathcal{H}_\Lambda$ has a Λ-multiplicative basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$, and assuming $\mathcal{Q} \subset \mathcal{RP}_m$, the representing measure μ given by Theorem 2 has support in K if and only if

$\Lambda(q_j b_k) \geq 0$ for all $j = 1, \ldots, s;\ k = 1, \ldots, d.$

Indeed, if $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ is the support of μ, we have

$$\Lambda(q_j b_k) = \int_\Xi q_j(t) b_k(t) d\mu(t) = \Lambda(b_k) q_j(\xi^{(k)})$$

for all $j = 1, \ldots, s$, $k = 1, \ldots, d$, implying our assertion.

Connection with the Full Moment Problem

We end this discussion with a characterization of the existence of representing measures for full moment problems, in terms of idempotent elements.

Theorem 4

A *uspf* $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ if and only if there exists an increasing sequence of nonnegative integers $\{m_k\}_{k\geq 1}$ such that every Hilbert space $\mathcal{H}_{\Lambda_k}$ has a Λ_k-multiplicative basis, where $\Lambda_k = \Lambda|\mathcal{P}_{2m_k}$, $k \geq 1$ an arbitrary integer.

The proof uses a weak compactenss argument due to J. Stochel.

Thank you for your attention !