Normal Extensions via Quaternionic Cayley Transforms

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Introduction

The classical Cayley transform
$\kappa(t) = (t-i)(t+i)^{-1} : \mathbb{R} \mapsto \mathbb{T} \setminus \{1\}$
is a bijective map. It can be extended to (not necessarily bounded) symmetric operators in Hilbert spaces, replacing formally the real variable by such an operator (von Neumann) or to some linear relations (Labrousse and collaborators), and to other situations as well.

We can slightly modify the former definitions (also due to the author), which allows us to get (in a simpler way) the properties of the quaternionic Cayley transform directly from those of von Neumann's Cayley transform, and refine some former results. This new construction does not require densely defined operators, which might be useful for potential applications; moreover, it applies to a larger class of operators (in particular, to some differential operators with 2×2 matrix coefficients, related to the so-called Dirac operator.

A strategy concerning the normal extensions: Let $\mathcal{D}$ be dense in a Hilbert space $\mathcal{H}$. Let also T be a densely defined linear operator in $\mathcal{H}$, such that T and T^* are defined on $\mathcal{D}$. Writing $T = A + iB$, with $A = (T + T^*)/2$ and $B = (T - T^*)/2i$, and so A and B are symmetric on $\mathcal{D}$, we consider

$$Q_T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

It is known that T is normal in $\mathcal{H}$ if and only if if Q_T is normal in $\mathcal{H} \oplus \mathcal{H}$. Using a quaternionic Cayley transform, we give conditions to insure the existence of a normal extension of Q_T, and we can go back to T. In fact, we have results for A and B symmetric, not necessarily densely defined in $\mathcal{H}$.

1 Cayley transforms in the algebra of quaternions

Consider the 2×2-matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hamilton algebra of quaternions $\mathbb{H}$ will be identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$ of 2×2-matrices with complex entries, generated by the matrices $\mathbf{I}$, $i\mathbf{J}$, $\mathbf{K}$ and $i\mathbf{L}$. The embedding $\mathbb{H} \subset \mathbb{M}_2$ allows us to regard the elements of $\mathbb{H}$ as matrices and to perform some operations in $\mathbb{M}_2$ rather than in $\mathbb{H}$.

We have $\mathbf{J}^* = \mathbf{J}$, $\mathbf{K}^* = -\mathbf{K}$, $\mathbf{L}^* = \mathbf{L}$, $\mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I}$, $\mathbf{JK} = \mathbf{L} = -\mathbf{KJ}$, $\mathbf{KL} = \mathbf{J} = -\mathbf{LK}$, $\mathbf{JL} = \mathbf{K} = -\mathbf{LJ}$, where the adjoints are computed in the Hilbert space $\mathbb{C}^2$.

We also put $\mathbf{E} = i\mathbf{J}$, $\mathbf{F} = i\mathbf{L}$, and we we have $\mathbf{E}^* = -\mathbf{E}$, $\mathbf{E^2} = -\mathbf{I}$, $\mathbf{F}^* = -\mathbf{F}$, $\mathbf{F^2} = -\mathbf{I}$

Definition 1.1 Let $a, b, c \in \mathbb{R}$, and let

$$S = S_{a,b,c} = a\mathbf{I} + b\mathbf{K} + ic\mathbf{L}.$$

The $\mathbf{E}$–*Cayley transform* of S is the matrix

$$U = (S - \mathbf{E})(S + \mathbf{E})^{-1} \in \mathbb{H}.$$

Let again $a, b, c \in \mathbb{R}$, and let

$$T = T_{a,b,c} = a\mathbf{I} + b\mathbf{K} + ic\mathbf{J}.$$

The $\mathbf{F}$–*Cayley transform* of T is the matrix

$$V = (T - \mathbf{F})(T + \mathbf{F})^{-1} \in \mathbb{H}.$$

Proposition 1.2 *Let $a, b, c \in \mathbb{R}$, and let $S = S_{a,b,c}$. The matrix*

$$U = (S - \mathbf{E})(S + \mathbf{E})^{-1}$$

is unitary and $U \neq \mathbf{I}$.

Conversely, given a unitary matrix $U \in \mathbb{H}$ with $U \neq \mathbf{I}$, there are $a, b, c \in \mathbb{R}$ such that $S = S_{a,b,c}$, where

$$S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}.$$

Moreover, the $\mathbf{E}$–Cayley transform of the matrix S is the unitary matrix U.

Remark 1.3 Let $a, b, c \in \mathbb{R}$, and let $S = S_{a,b,c}$. A direct calculation shows that the **E**–Cayley transform of S is given by

$$U = (a^2 + b^2 + c^2 + 1)^{-1}$$
$$\times((a^2+b^2+c^2-1)\mathbf{I}-2c\mathbf{K}-2ai\mathbf{J}+2bi\mathbf{L})$$

Conversely, if $U \in \mathbb{H}$ is unitary with $\mathbf{I} \neq U$, and so

$$U = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix},$$

with $z_1, z_2 \in \mathbb{C}$ and $|z_1|^2 + |z_2|^2 = 1$ and $\text{Re}z_1 \neq 1$, the matrix
$S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$ is given by

$$S = \frac{1}{\text{Re}z_1 - 1} \begin{pmatrix} \text{Im}z_1 & iz_2 \\ i\bar{z}_2 & \text{Im}z_1 \end{pmatrix},$$

which is *inverse* **E**–*Cayley transform* of the matrix U.

2 Quaternionic Cayley transform of unbounded operators revisited

The quaternionic Cayley transforms, may be extended to some classes of unbounded operators, acting on the Cartesian product of two Hilbert spaces.

Let $\mathcal{H}$ be a complex Hilbert space, whose scalar product is denoted by $\langle *, * \rangle$, and whose norm is denoted by $\| * \|$. We especially work in the Hilbert space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$, whose natural scalar product is denoted by $\langle *, * \rangle_2$, and whose norm is denoted by $\| * \|_2$.

The matrices from $\mathbb{M}_2$ naturally act on $\mathcal{H}^2$. In particular, the matrices $\mathbf{I}, \mathbf{J}, \mathbf{K}$, $\mathbf{L}, \mathbf{E}, \mathbf{F}$, act on $\mathcal{H}^2$, and we still have similar properties.

For an operator T acting in $\mathcal{H}$, we denote by $D(T), R(T), N(T)$ the domain of definition, the range the kernel, respectively. If T is closable, the closure of T will be denoted by $\bar{T}$. If T is densely defined, let T^* be its adjoint. If T_2 extends T_1, we write $T_1 \subset T_2$.

Lemma 2.1 *Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$. Suppose that the operator $\mathbf{J}S$ is symmetric. Then we have*

$$\|(S \pm \mathbf{E})x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S).$$

If, in addition, $\mathbf{J}D(S) \subset D(S)$, we have

$$\|(S \pm \mathbf{E})\mathbf{E}x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S),$$

if and only if $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$.

Example 2.2 (1) Let $A, B : D \subset \mathcal{H} \mapsto$

$\mathcal{H}$ be symmetric operators. We put $S = S_{A,B} = A\mathbf{I} + B\mathbf{K}$, which is an operator in $\mathcal{H}^2$, defined on $D(S) = D \oplus D$. The operator $\mathbf{J}S$ is easily seen to be symmetric in $\mathcal{H}^2$.

(2) Let $\mathcal{H} = L^2(\mathbb{R})$ and let $D \subset L^2(\mathbb{R})$ be the subset of all continuously differentiable functions with compact support. Consider the operator

$$T = i\frac{d}{dt}\mathbf{I} + \sigma(t)\mathbf{K} + i\tau(t)\mathbf{L},$$

defined on D^2, with values in $\mathcal{H}^2$, where σ and τ are continuous real-valued functions on $\mathbb{R}$. It is known that the operator $\mathbf{J}T$ is symmetric. Moreover, $\mathbf{J}T$ has a self-adjoint extension, which is called the *Dirac operator*.

Remark 2.3 Let $S : D(S) \subset \mathcal{H}^2 \mapsto$

$\mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric. Lemma 2.1 allows us to correctly define the operator $V : R(S + \mathbf{E}) \mapsto R(S - \mathbf{E})$, $V(S + \mathbf{E})x = (S - \mathbf{E})x, \quad x \in D(S)$, which is a partial isometry. In other words, $V = (S - \mathbf{E})(S + \mathbf{E})^{-1}$, defined on $D(V) = R(S + \mathbf{E})$.

The operator V will be called the $\mathbf{E}$–*Cayley transform* of S.

Similarly, if $\mathbf{L}S$ is symmetric, we can define the $\mathbf{F}$–*Cayley transform* of S.

Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry. Then the inverse V^{-1} is well defined on the subspace $D(V^{-1}) = R(V)$.

As the two transforms are alike, we mainly deal with the first one.

Remark 2.4 (1) Let $S_j : D(S_j) \mapsto$

$\mathcal{H}^2$ be such that $\mathbf{J}S_j$ is symmetric, and let V_j be the $\mathbf{E}$–Cayley transform of $S_j\,(j = 1, 2)$. We have $S_1 \subset S_2$ if and only if $V_1 \subset V_2$. In other words, the $\mathbf{E}$–Cayley transform is an order preserving map.

(2) Suppose that the operator $\mathbf{I} - V$ is injective. Then the operator
$S : R(\mathbf{E}(V - \mathbf{I})) \mapsto \mathcal{H}^2$, given by
$S(\mathbf{E}(V - \mathbf{I})x) = (V + \mathbf{I})x$, $x \in D(V)$,
is well defined and will be called the *inverse* $\mathbf{E}$*–Cayley transform* of the partial isometry V. In other words,
$S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}$ on
$D(S) = \mathbf{E}R(\mathbf{I} - V)$.

Of course, we may define, in a similar way, the *inverse* $\mathbf{F}$*–Cayley transform.*

The properties of the quaternionic Cayley transform are summarized in the following result.

Theorem 2.5 *The* $\mathbf{E}$-*Cayley transform is an order preserving bijective map assigning to each operator* S *with* $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ *and* $\mathbf{J}S$ *symmetric a partial isometry* V *in in* $\mathcal{H}^2$ *with* $\mathbf{I} - V$ *injective. Moreover:*

(1) *the operator* V *is closed if and only if the operator* S *is closed;*

(2) *the equality* $V^{-1} = -\mathbf{K}V\mathbf{K}$ *holds if and only if the equality* $S\mathbf{K} = \mathbf{K}S$ *holds;*

(3) *the operator* $\mathbf{J}S$ *is self-adjoint if and only if* V *is unitary on* $\mathcal{H}^2$.

3 Unitary operators and the inverse quaternionic Cayley transform

We are particularly intrested in those unitary operators producing (unbounded) normal operators, via the inverse **E**–Cayley transform.

Remark. We can prove that an operator U on $\mathcal{H}^2$ has the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal, A self-adjoint, such that $TT^* + A^2 = I$ and $AT = TA$, if and only if U is unitary, $U^* = -\mathbf{K}U\mathbf{K}$ and $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.

The class of these unitary operators will be denoted by $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$.

Lemma 3.1 *Let V be a partial isometry such that $V^{-1} = -\mathbf{K}V\mathbf{K}$ and $\mathbf{I} - V$ is injective. Let S be the inverse $\mathbf{E}$–Cayley transform of V. We have $\mathbf{J}D(S) \subset D(S)$ and $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$ if and only if there exists a surjective isometry $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I} - V) = (\mathbf{I} - V)G$.*

Corollary 3.2 *Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be inverse $\mathbf{E}$–Cayley transform of U. The operator S is normal if and only is there exists a unitary operator G_U on $\mathcal{H}^2$ with $\mathbf{E}(\mathbf{I} - U) = (\mathbf{I} - U)G_U$ and $(G_U)^* = -G_U$.*

Theorem 3.3 *Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be the inverse $\mathbf{E}$–Cayley transform of U. The operator S is normal if and only if $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.*

The next result gives a complete description of the unitary operator G_U,

Proposition 3.4 *Let $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$. Then the operator*

$$\begin{pmatrix} (iT^*T - \mathrm{Re}(T))\Theta_T^{-1} & A(I - T^*)\Theta_T^{-1} \\ -A(I - T)\Theta_T^{-1} & -i(T^*T - \mathrm{Re}(T))\Theta_T^{-1} \end{pmatrix}$$

is a densely defined isometry, where $\Theta_T = I - \mathrm{Re}(T)$, and its extension to $\mathcal{H}^2$ equals the unitary operator G_U.

Remark 3.5 Let

$$\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2) = \{S : D(S) \subset \mathcal{H}^2 \to \mathcal{H})^2;$$
$$S \text{ normal}, (\mathbf{J}S)^* = \mathbf{J}S, \mathbf{K}S = S\mathbf{K}\}.$$

The previous theorems show that the map

$$\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2) \ni S \mapsto (S-\mathbf{E})(S+\mathbf{E})^{-1} \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$$

is bijective. In addition, we have $S \in \mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$ if and only if S is a densely defined operator in $\mathcal{H}^2$ having the form

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B are commuting self-adjoint operators.

4 Normal extensions

Remark 4.1 Let $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that

(i) $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.

In order that T have a normal extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$, the following conditions are necessary:

(ii) $\mathbf{J}T$ is symmetric;

(iii) $T\mathbf{K} = \mathbf{K}T$;

(iv) $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ for all $x \in D(T)$.

We denote by $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ the set of those operators $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that (i)–(iv) hold.

Let also $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ be the set of those partial isometries $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that:

(a) $V^{-1} = -\mathbf{K}V\mathbf{K}$;

(b) $\mathbf{I} - V$ is injective;

(c) $\mathbf{E}R(\mathbf{I} - V) = R(\mathbf{I} - V)$ and $(\mathbf{I} - V)^{-1}\mathbf{E}(\mathbf{I} - V)$ is an isometry on $D(V)$.

The $\mathbf{E}$–Cayley transform is a bijective map from $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ onto $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$. Note also that $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) \subset \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

The interesting question concerning the existence of an extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ of an operator $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ is equivalent to the description of those partial isometries in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ having extensions in the family $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$.

Proposition 4.2 *Let* $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ *and let* $\mathcal{D} \subset \mathcal{H}^2$ *be a closed subspace with the properties* $\mathbf{K}U(\mathcal{D}) \subset \mathcal{D}$ *and* $\mathbf{E}(\mathbf{I} - U)(\mathcal{D}) \subset (\mathbf{I} - U)(\mathcal{D})$. *If* $V = U|\mathcal{D}$, $\mathcal{E} = \mathcal{D}^{\perp}$ *and* $W = U|\mathcal{E}$, *then* $U = V \oplus W$ *and* $V, W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$

We can characterize of those closed subspaces of $\mathcal{H}^2$ which are domains of definitions of partial isometries from $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Proposition 4.3 *Let* $\mathcal{D} \subset \mathcal{H}^2$ *be a closed subspace and let* $P_{\mathbf{I}}^{\pm} = 2^{-1}(\mathbf{I} \pm i\mathbf{K})$. *There exists a* $V \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ *with* $D(V) = \mathcal{D}$ *if and only if there are two orthogonal projection* $P^{\pm}$ *in* $\mathcal{H}^2$ *such that*

(1) $\mathcal{D} = P^{+}(\mathcal{H}^2) \oplus P^{-}(\mathcal{H}^2)$;

(2) $P^{\pm}(\mathcal{H}^2) \cap P_{\mathbf{I}}^{\pm}(\mathcal{H}^2) = \{0\}$;

(3) $(P^{\pm} + \mathbf{E}P^{\mp}\mathbf{E})(P_{\mathbf{I}}^{\mp}(\mathcal{H}^2)) \subset P_{\mathbf{I}}^{\pm}(\mathcal{H}^2)$.

Lemma 4.4 *Let* $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$ *be densely defined. Then* T *is closable and its closure* $\bar{T} \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$.

Theorem 4.5 *Let* $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$ *be densely defined. The operator* T *has an extension in* $\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$ *if and only if there exists a* $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, *with* $D(W) = R(T + \mathbf{E})^{\perp}$.

The next assertion concerns not necessarily densely defined operators.

Corollary 4.6 *Let* $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$ *be closed and let* V *be the* $\mathbf{E}$*–Cayley transform of* T. *The operator* T *has an extension in* $\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$ *if and only if there exists a* $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, *with the properties* $D(W) = R(T + \mathbf{E})^{\perp}$ *and* $R(\mathbf{I} - V) \cap R(\mathbf{I} - W) = \{0\}$.

Remark 4.7 The results stated above apply to a large class of linear operators in the Hilbert space $\mathcal{H}$. Specifically, let A, B be a pair of linear operators having a joint domain of definition $D_0 \subset \mathcal{H}$. As already discussed, we associate this pair with a matrix operator

$$T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

defined on $D(T) = D_0 \oplus D_0 \in \mathcal{H}^2$. We want to find equivalent conditions on A, B such that $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$.

Clearly, $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.

It is easily seen that T is symmetric if and only if both A, B are symmetric. The equality $\mathbf{K}T = T\mathbf{K}$ is also easily verified.

Finally, the equality $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ holds for all $x \in D(T)$ if and only if
(c)

$$\langle Au, Bv\rangle + \langle Bv, Au\rangle = \langle Bu, Av\rangle + \langle Av, Bu\rangle$$

for all $u, v \in D_0$, which is a weak commutativity condition. Consequently, if A, B are symmetric and condition (c) holds, then $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$. In that case, the $\mathbf{E}$–Cayley transform of T is in the class $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

As a direct consequence of the previous results, we obtain the following assertion:

Let A, B be symmetric operators on a dense joint domain of definition $D_0 \subset \mathcal{H}$, satisfying condition (c)*. If the space*

$$\{((A+iI)u+Bv)\oplus((A-iI)v-Bu); u, v \in D_0\}$$

is dense in $\mathcal{H}^2$, then the operators A and B have commuting self-adjoint extensions.

This result is, in fact, a version of a celebrated theorem of Nelson's concerning the commuting self-adjoint extensions of symmetric operators.