SPECTRAL THEORY IN QUATERNIONIC AND REAL CONTEXTS

F.-H. Vasilescu

Department of Mathematics
Faculty of Sciences and Technologies
University of Lille
France

Timişoara, July 2- 6, 2018

Outline

ABSTRACT

Regarding quaternions as normal matrices, we firstly characterize those 2×2 matrix-valued functions, defined on subsets of quaternions, whose values are quaternions. Then we investigate the regularity of quaternionic-valued functions, defined by the analytic functional calculus, eluding the non-commutativity. Constructions of analytic functional calculi for real linear operators, in particular for quaternionic linear ones, are finally discussed.

Motivation

The basic idea of the present paper to define the regularity (that is, a sort of holomorphy) of a quaternionic-valued function via the analytic functional calculus acting on quaternions. We have chosen to consider the algebra of quaternions not as an abstract object but as a real subalgebra of the complex algebra of 2×2 matrices with complex entries. Among the advantages of this representation is that we may view the quaternions as linear operators actually on complex spaces, commuting with the complex numbers. Another one is to regard each quaternion as a normal operator, having a spectrum which can be used to define various compatible functional calculi, including the analytic one.

References

There exists a large literature concerning the analysis of quaternions. We cite two monographs from a lage collection of works dedicated to this subject.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

References

There exists a large literature concerning the analysis of quaternions. We cite two monographs from a lage collection of works dedicated to this subject.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

References

There exists a large literature concerning the analysis of quaternions. We cite two monographs from a lage collection of works dedicated to this subject.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Hamilton's Algebra of Quaternions

Introduced in science by W. R. Hamilton as early as 1843, the quaternions form a unital non commutative division algebra, with numerous applications in mathematics and physics. In mathematics, the celebrated Frobenius theorem, proved in 1877, placed the algebra of quaternions among the only three finite dimensional division algebras over the real numbers, which is a remarkable feature shared with the real and complex fields.

Abstract Hamilton's algebra $\mathbb{H}_0$ is the 4-dimensional $\mathbb{R}$-algebra with unit 1, generated by $\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$, where $\mathbf{j}, \mathbf{k}, \mathbf{l}$ satisfy

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{l},\ \mathbf{kl} = -\mathbf{lk} = \mathbf{j},\ \mathbf{lj} = -\mathbf{jl} = \mathbf{k},\ \mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1.$$

Hamilton's Algebra of Quaternions

Introduced in science by W. R. Hamilton as early as 1843, the quaternions form a unital non commutative division algebra, with numerous applications in mathematics and physics. In mathematics, the celebrated Frobenius theorem, proved in 1877, placed the algebra of quaternions among the only three finite dimensional division algebras over the real numbers, which is a remarkable feature shared with the real and complex fields.

Abstract Hamilton's algebra $\mathbb{H}_0$ is the 4-dimensional $\mathbb{R}$-algebra with unit 1, generated by $\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$, where $\mathbf{j}, \mathbf{k}, \mathbf{l}$ satisfy

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{l},\ \mathbf{kl} = -\mathbf{lk} = \mathbf{j},\ \mathbf{lj} = -\mathbf{jl} = \mathbf{k},\ \mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1.$$

Hamilton's Algebra of Quaternions 2

In the algebra $\mathbb{M}_2$ of 2×2 complex matrices we set

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{L} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

with $i^2 = -1$. As we have

$$\mathbf{J}^2 = \mathbf{K}^2 = \mathbf{L}^2 = -\mathbf{I},$$

$$\mathbf{JK} = \mathbf{L} = -\mathbf{KJ}, \ \mathbf{KL} = \mathbf{J} = -\mathbf{LK}, \ \mathbf{LJ} = \mathbf{K} = -\mathbf{JL},$$

and the assignment

$$\mathbb{H}_0 \ni x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto x_0\mathbf{I} + x_1\mathbf{J} + x_2\mathbf{K} + x_3\mathbf{L} \in \mathbb{M}_2 \quad (1)$$

is an isometric unital $\mathbb{R}$-algebra morphism , from now on the algebra of quaternions, denoted by $\mathbb{H}$, is identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$, generated by the matrices **I**, **J**, **K** and **L**.

Hamilton's Algebra of Quaternions 3

The matrices from $\mathbb{M}_2$ act as linear maps on the space $\mathbb{C}^2$, endowed with the natural scalar product $\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$ and the associated norm $\|\mathbf{z}\|^2 = |z_1|^2 + |z_2|^2$, with $\mathbf{z} = (z_1, z_2), \mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$. We set

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \in \mathbb{H},\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

We have $\|Q(\mathbf{z})\| = \|\mathbf{z}\|, \mathbf{z} \in \mathbb{C}^2$, and the map $\mathbf{z} \mapsto Q(\mathbf{z})$, $\mathbb{C}^2 \mapsto \mathbb{H}$ is a well-defined $\mathbb{R}$-linear surjective isometry.
The algebra $\mathbb{H}$ has a natural involution, given by $Q(\mathbf{z})^* = Q(\mathbf{z}^*)$, with $\mathbf{z}^* = (\bar{z}_1, -z_2)$. We also have $Q(\mathbf{z})Q(\mathbf{z})^* = Q(\mathbf{z})^* Q(\mathbf{z}) = \|\mathbf{z}\|^2 \mathbf{I}$ for all $\mathbf{z} \in \mathbb{C}^2$, and so $Q(\mathbf{z})$ is a normal matrix for each $\mathbf{z} \in \mathbb{C}^2$, and every nonnull element of $\mathbb{H}$ is invertible.

Hamilton's Algebra of Quaternions 3

The matrices from $\mathbb{M}_2$ act as linear maps on the space $\mathbb{C}^2$, endowed with the natural scalar product $\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$ and the associated norm $\|\mathbf{z}\|^2 = |z_1|^2 + |z_2|^2$, with $\mathbf{z} = (z_1, z_2), \mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$. We set

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \in \mathbb{H},\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

We have $\|Q(\mathbf{z})\| = \|\mathbf{z}\|, \mathbf{z} \in \mathbb{C}^2$, and the map $\mathbf{z} \mapsto Q(\mathbf{z})$, $\mathbb{C}^2 \mapsto \mathbb{H}$ is a well-defined $\mathbb{R}$-linear surjective isometry.

The algebra $\mathbb{H}$ has a natural involution, given by $Q(\mathbf{z})^* = Q(\mathbf{z}^*)$, with $\mathbf{z}^* = (\bar{z}_1, -z_2)$. We also have $Q(\mathbf{z})Q(\mathbf{z})^* = Q(\mathbf{z})^* Q(\mathbf{z}) = \|\mathbf{z}\|^2 \mathbf{I}$ for all $\mathbf{z} \in \mathbb{C}^2$, and so $Q(\mathbf{z})$ is a normal matrix for each $\mathbf{z} \in \mathbb{C}^2$, and every nonnull element of $\mathbb{H}$ is invertible.

Hamilton's Algebra of Quaternions 3

The matrices from $\mathbb{M}_2$ act as linear maps on the space $\mathbb{C}^2$, endowed with the natural scalar product $\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$ and the associated norm $\|\mathbf{z}\|^2 = |z_1|^2 + |z_2|^2$, with $\mathbf{z} = (z_1, z_2), \mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$. We set

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \in \mathbb{H},\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

We have $\|Q(\mathbf{z})\| = \|\mathbf{z}\|, \mathbf{z} \in \mathbb{C}^2$, and the map $\mathbf{z} \mapsto Q(\mathbf{z})$, $\mathbb{C}^2 \mapsto \mathbb{H}$ is a well-defined $\mathbb{R}$-linear surjective isometry.
The algebra $\mathbb{H}$ has a natural involution, given by $Q(\mathbf{z})^* = Q(\mathbf{z}^*)$, with $\mathbf{z}^* = (\bar{z}_1, -z_2)$. We also have $Q(\mathbf{z})Q(\mathbf{z})^* = Q(\mathbf{z})^* Q(\mathbf{z}) = \|\mathbf{z}\|^2 \mathbf{I}$ for all $\mathbf{z} \in \mathbb{C}^2$, and so $Q(\mathbf{z})$ is a normal matrix for each $\mathbf{z} \in \mathbb{C}^2$, and every nonnull element of $\mathbb{H}$ is invertible.

Skew Complex Conjugation

On the algebra $\mathbb{M}_2$ we define what we will call a *skew complex conjugation*, setting

$$\mathbf{a}^{\sim} := \begin{pmatrix} \bar{a}_4 & -\bar{a}_3 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix},$$

for every

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbb{M}_2.$$

The map $\mathbf{a} \mapsto \mathbf{a}^{\sim}$ is conjugate homogeneous and additive, in particular $\mathbb{R}$-linear, multiplicative, unital, $(\mathbf{a}^{\sim})^{\sim} = \mathbf{a}$, and $(\mathbf{a}^*)^{\sim} = (\mathbf{a}^{\sim})^*$. In addition, $\mathbf{a} = \mathbf{a}^{\sim}$ if and only if $\mathbf{a}$ is a quaternion.

Being a $*$-automorphism $\mathbb{R}$-linear, the map $\mathbf{a} \mapsto \mathbf{a}^{\sim}$ must be an isometry.

Skew Complex Conjugation

On the algebra $\mathbb{M}_2$ we define what we will call a *skew complex conjugation*, setting

$$\mathbf{a}^{\sim} := \begin{pmatrix} \bar{a}_4 & -\bar{a}_3 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix},$$

for every

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbb{M}_2.$$

The map $\mathbf{a} \mapsto \mathbf{a}^{\sim}$ is conjugate homogeneous and additive, in particular $\mathbb{R}$-linear, multiplicative, unital, $(\mathbf{a}^{\sim})^{\sim} = \mathbf{a}$, and $(\mathbf{a}^{*})^{\sim} = (\mathbf{a}^{\sim})^{*}$. In addition, $\mathbf{a} = \mathbf{a}^{\sim}$ if and only if $\mathbf{a}$ is a quaternion.
Being a $*$-automorphism $\mathbb{R}$-linear, the map $\mathbf{a} \mapsto \mathbf{a}^{\sim}$ must be an isometry.

Skew Complex Conjugation 2

Note that

$$\mathbf{a} = \frac{\mathbf{a} + \mathbf{a}^{\sim}}{2} + i\frac{\mathbf{a} - \mathbf{a}^{\sim}}{2i}, \ \mathbf{a} \in \mathbb{M}_2,$$

with $\mathbf{a} + \mathbf{a}^{\sim}$, $i(\mathbf{a} - \mathbf{a}^{\sim}) \in \mathbb{H}$. In other words, $\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$. We also have $\mathbb{H} \cap i\mathbb{H} = \{0\}$. Indeed, if $q = ir$ with $q, r \in \mathbb{H}$, we have $q^{\sim} = q = (ir)^{\sim} = -ir = -q$, whence $q = 0$, showing that the decomposition

$$\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$$

is a direct sum.

Algebra $\mathbb{H}$ as a Matrix Subalgebra

Summarizing, the space $\mathbb{C}^2$ is endowed with its natural scalar product $\langle *, * \rangle$, and norm $\| * \|$. The $\mathbb{R}$-linear the map $\mathbb{C}^2 \ni \mathbf{z} \mapsto Q(\mathbf{z}) \in \mathbb{H}$ is a bijective isometry, so giving $q \in \mathbb{H}$ there is a unique $\mathbf{z}_q \in \mathbb{C}^2$ such that $q = Q(\mathbf{z}_q)$.

The algebra $\mathbb{H}$ will be regarded as an $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathbb{M}_2$. In particular, every element $q_{\mathbf{z}} = Q(\mathbf{z})$ is a normal operator on the Hilbert space $\mathbb{C}^2$.

Notation For every complex space Banach $\mathcal{X}$, and each Banach space operator T on $\mathcal{X}$, the symbol $\sigma(T)$ will designate the spectrum of T, and the symbol $\rho(T)$ will be resolvent set of the operator T.

Algebra $\mathbb{H}$ as a Matrix Subalgebra

Summarizing, the space $\mathbb{C}^2$ is endowed with its natural scalar product $\langle *, * \rangle$, and norm $\| * \|$. The $\mathbb{R}$-linear the map $\mathbb{C}^2 \ni \mathbf{z} \mapsto Q(\mathbf{z}) \in \mathbb{H}$ is a bijective isometry, so giving $q \in \mathbb{H}$ there is a unique $\mathbf{z}_q \in \mathbb{C}^2$ such that $q = Q(\mathbf{z}_q)$.

The algebra $\mathbb{H}$ will be regarded as an $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathbb{M}_2$. In particular, every element $q_{\mathbf{z}} = Q(\mathbf{z})$ is a normal operator on the Hilbert space $\mathbb{C}^2$.

Notation For every complex space Banach $\mathcal{X}$, and each Banach space operator T on $\mathcal{X}$, the symbol $\sigma(T)$ will designate the spectrum of T, and the symbol $\rho(T)$ will be resolvent set of the operator T.

Algebra $\mathbb{H}$ as a Matrix Subalgebra

Summarizing, the space $\mathbb{C}^2$ is endowed with its natural scalar product $\langle *, * \rangle$, and norm $\| * \|$. The $\mathbb{R}$-linear the map $\mathbb{C}^2 \ni \mathbf{z} \mapsto Q(\mathbf{z}) \in \mathbb{H}$ is a bijective isometry, so giving $q \in \mathbb{H}$ there is a unique $\mathbf{z}_q \in \mathbb{C}^2$ such that $q = Q(\mathbf{z}_q)$.

The algebra $\mathbb{H}$ will be regarded as an $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathbb{M}_2$. In particular, every element $q_{\mathbf{z}} = Q(\mathbf{z})$ is a normal operator on the Hilbert space $\mathbb{C}^2$.

Notation For every complex space Banach $\mathcal{X}$, and each Banach space operator T on $\mathcal{X}$, the symbol $\sigma(T)$ will designate the spectrum of T, and the symbol $\rho(T)$ will be resolvent set of the operator T.

Lemma 1

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ be fixed. The spectrum $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$ of the normal operator $Q(\mathbf{z})$ is given by

$$s_\pm(\mathbf{z}) = \Re z_1 \pm i\sqrt{(\Im z_1)^2 + |z_2|^2}. \quad (2)$$

We have $s_+(\mathbf{z}) = \overline{s_-(\mathbf{z})}$, and the points $s_+(\mathbf{z}), s_-(\mathbf{z})$ are distinct if and only if $Q(\mathbf{z}) \notin \mathbb{R}\mathbf{I}$. Moreover:
(a) if $z_2 \neq 0$, the elements

$$\nu_\pm(\mathbf{z}) = \frac{1}{\sqrt{|z_2|^2 + |s_\pm(\mathbf{z}) - z_1|^2}}(z_2, s_\pm(\mathbf{z}) - z_1) \in \mathbb{C}^2 \quad (3)$$

are the eigenvectors corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, and they form an orthonormal basis of the Hilbert space $\mathbb{C}^2$;

Lemma 1 cont.

(b) if $z_2 = 0$ but $\Im z_1 \neq 0$, we have $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, with $s_+(\mathbf{z}) = z_1, s_-(\mathbf{z}) = \bar{z}_1$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding the eigenvalues $z_1, \bar{z}_1$, respectively;
(c) if $z_2 = 0$ and $\mathbf{z} = (x, 0)$ with $x \in \mathbb{R}$, we have $\sigma(Q(\mathbf{z})) = \{x\}$, with $s_+(\mathbf{z}) = s_-(\mathbf{z}) = x$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding to the eigenvalue x.

The eigenvectors $\{\nu_\pm(\mathbf{z})\}$ of $Q(\mathbf{z})$ corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, will be called the **canonical eigenvectors** of $Q(\mathbf{z})$.

Other notation: When $q = Q(\mathbf{z})$ we put $\sigma(q) = \sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, $s_\pm(q) = s_\pm(\mathbf{z})$ and $\nu_\pm(q) = \nu_\pm(\mathbf{z})$.

Lemma 1 cont.

(b) if $z_2 = 0$ but $\Im z_1 \neq 0$, we have $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, with $s_+(\mathbf{z}) = z_1$, $s_-(\mathbf{z}) = \bar{z}_1$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding the eigenvalues $z_1, \bar{z}_1$, respectively;
(c) if $z_2 = 0$ and $\mathbf{z} = (x, 0)$ with $x \in \mathbb{R}$, we have $\sigma(Q(\mathbf{z})) = \{x\}$, with $s_+(\mathbf{z}) = s_-(\mathbf{z}) = x$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding to the eigenvalue x.

The eigenvectors $\{\nu_\pm(\mathbf{z})\}$ of $Q(\mathbf{z})$ corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, will be called the **canonical eigenvectors** of $Q(\mathbf{z})$.

Other notation: When $q = Q(\mathbf{z})$ we put $\sigma(q) = \sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, $s_\pm(q) = s_\pm(\mathbf{z})$ and $\nu_\pm(q) = \nu_\pm(\mathbf{z})$.

Lemma 1 cont.

(b) if $z_2 = 0$ but $\Im z_1 \neq 0$, we have $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, with $s_+(\mathbf{z}) = z_1, s_-(\mathbf{z}) = \bar{z}_1$, and $\nu_+(\mathbf{z}) = (1,0)$, $\nu_-(\mathbf{z}) = (0,1)$ are eigenvectors corresponding the eigenvalues $z_1, \bar{z}_1$, respectively;
(c) if $z_2 = 0$ and $\mathbf{z} = (x,0)$ with $x \in \mathbb{R}$, we have $\sigma(Q(\mathbf{z})) = \{x\}$, with $s_+(\mathbf{z}) = s_-(\mathbf{z}) = x$, and $\nu_+(\mathbf{z}) = (1,0)$, $\nu_-(\mathbf{z}) = (0,1)$ are eigenvectors corresponding to the eigenvalue x.

The eigenvectors $\{\nu_\pm(\mathbf{z})\}$ of $Q(\mathbf{z})$ corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, will be called the **canonical eigenvectors** of $Q(\mathbf{z})$.

Other notation: When $q = Q(\mathbf{z})$ we put $\sigma(q) = \sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, $s_\pm(q) = s_\pm(\mathbf{z})$ and $\nu_\pm(q) = \nu_\pm(\mathbf{z})$.

Example

Let $\mathbb{S} = \{\mathfrak{s} = x_1\mathbf{J} + x_2\mathbf{K} + x_3\mathbf{L}; x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$, that be the unit sphere of purely imaginary quaternions. Every quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ can be written as $q = x\mathbf{I} + y\mathfrak{s}$, for some $\mathfrak{s} \in \mathbb{S}$, where x, y are real numbers.

One can easily prove that for every $q = x\mathbf{I} + y\mathfrak{s}$, $x, y \in \mathbb{R}$, we have $\sigma(q) = \{x \pm iy\}$.

Note that the spectrum of q does not depend on $\mathfrak{s}$.

Lemma 2

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, and let $\nu_\pm(\mathbf{z}) = (\nu_{\pm 1}(\mathbf{z}), \nu_{\pm 2}(\mathbf{z})) \in \mathbb{C}^2$ be the canonical eigenvectors of $Q(\mathbf{z})$. Then we have

$$|\nu_{-1}(\mathbf{z})|^2 = |\nu_{+2}(\mathbf{z})|^2,\ |\nu_{-2}(\mathbf{z})|^2 = |\nu_{+1}(\mathbf{z})|^2,$$
$$\nu_{-1}(\mathbf{z})\overline{\nu_{-2}}(\mathbf{z}) + \nu_{+1}(\mathbf{z})\overline{\nu_{+2}}(\mathbf{z}) = 0 \qquad (*)$$

We note that equalities $(*)$ do not follow, in general, from the orthogonality of $\nu_+(\mathbf{z})$ and $\nu_-(\mathbf{z})$.

Lemma 2

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, and let $\nu_\pm(\mathbf{z}) = (\nu_{\pm 1}(\mathbf{z}), \nu_{\pm 2}(\mathbf{z})) \in \mathbb{C}^2$ be the canonical eigenvectors of $Q(\mathbf{z})$. Then we have

$$
\begin{gathered}
|\nu_{-1}(\mathbf{z})|^2 = |\nu_{+2}(\mathbf{z})|^2, \; |\nu_{-2}(\mathbf{z})|^2 = |\nu_{+1}(\mathbf{z})|^2, \\
\nu_{-1}(\mathbf{z})\overline{\nu_{-2}}(\mathbf{z}) + \nu_{+1}(\mathbf{z})\overline{\nu_{+2}}(\mathbf{z}) = 0
\end{gathered} \qquad (*)
$$

We note that equalities $(*)$ do not follow, in general, from the orthogonality of $\nu_+(\mathbf{z})$ and $\nu_-(\mathbf{z})$.

Remark 1

Given $\zeta \in \mathbb{C}$, we can determine all quaternions q with $\sigma(q) = \{\zeta, \bar{\zeta}\}$. Assuming $\Im\zeta \geq 0$, we look for the points $\mathbf{z} = (z_1, z_2) \in \mathbb{C}$ satisfying the equation

$$\zeta = s_+(\mathbf{z}) = \Re z_1 + i\sqrt{(\Im z_1)^2 + |z_2|^2},$$

so $\bar{\zeta} = s_-(\mathbf{z}) = \Re z_1 - i\sqrt{(\Im z_1)^2 + |z_2|^2}$. Setting $u = z_2$ as a parameter, we obtain $\Re z_1 = \Re\zeta$, and $(\Im z_1)^2 = (\Im\zeta)^2 - |u|^2$, provided $|u|^2 \leq (\Im\zeta)^2$. The solutions are given by the set

$$\{\mathbf{z} = (\Re\zeta \pm i\sqrt{(\Im\zeta)^2 - |u|^2}, u) \in \mathbb{C}^2,\ |u| \leq \Im\zeta\},$$

so we have, for every such a $\mathbf{z}$, $\sigma(Q(\mathbf{z})) = \{\zeta, \bar{\zeta}\}$ Lemma 1
If $\Im\zeta \leq 0$, we apply the previous discussion to $\bar{\zeta}$.

Remark 2(1)

(1) A subset $U \subset \mathbb{C}$ is said to be *conjugate symmetric* if $\zeta \in U$ if and only if $\bar{\zeta} \in U$.
For an arbitrary conjugate symmetric subset $U \subset \mathbb{C}$ we put $U_{\mathbb{H}} = \{q \in \mathbb{H}; \sigma(q) \subset U\}$. Note that, for every $\zeta \in U$ and $u \in \mathbb{C}$ with $|u| \leq |\Im\zeta|$, setting

$$q_\zeta^\pm(u) := (\Re\zeta \pm i\sqrt{(\Im\zeta)^2 - |u|^2}, u) \in \mathbb{C}^2,\ |u| \leq |\Im\zeta|,$$

we have

$$U_{\mathbb{H}} = \{Q(q_\zeta^\pm(u)); \zeta \in U,\ u \in \mathbb{C},\ |u| \leq |\Im\zeta|\},$$

via Remark 1.

Remark 2(2)

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if $\sigma(r) = \sigma(q)$ for some $r \in \mathbb{H}$ and $q \in A$ implies $r \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put
$\mathfrak{S}(A) = \{\zeta \in \mathbb{C}; \exists q \in A : \zeta \in \sigma(q)\}$.
A subset $A \subset \mathbb{H}$ is spectrally saturated if and only if there exists a conjugate symmetric subset $S \subset \mathbb{C}$ such that $A = S_{\mathbb{H}}$. In this case, $S = \mathfrak{S}(A)$.
If $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open (via the upper semi-continuity of the spectrum).
Conversely, if $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, one can prove that $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open.

Remark 2(2)

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if $\sigma(r) = \sigma(q)$ for some $r \in \mathbb{H}$ and $q \in A$ implies $r \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put
$\mathfrak{S}(A) = \{\zeta \in \mathbb{C}; \exists q \in A : \zeta \in \sigma(q)\}$.

A subset $A \subset \mathbb{H}$ is spectrally saturated if and only if there exists a conjugate symmetric subset $S \subset \mathbb{C}$ such that $A = S_{\mathbb{H}}$. In this case, $S = \mathfrak{S}(A)$.

If $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open (via the upper semi-continuity of the spectrum).

Conversely, if $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, one can prove that $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open.

Remark 2(2)

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if $\sigma(r) = \sigma(q)$ for some $r \in \mathbb{H}$ and $q \in A$ implies $r \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put
$\mathfrak{S}(A) = \{\zeta \in \mathbb{C}; \exists q \in A : \zeta \in \sigma(q)\}$.
A subset $A \subset \mathbb{H}$ is spectrally saturated if and only if there exists a conjugate symmetric subset $S \subset \mathbb{C}$ such that $A = S_{\mathbb{H}}$. In this case, $S = \mathfrak{S}(A)$.
If $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open (via the upper semi-continuity of the spectrum).
Conversely, if $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, one can prove that $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open.

Remark 2(2)

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if $\sigma(r) = \sigma(q)$ for some $r \in \mathbb{H}$ and $q \in A$ implies $r \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put
$\mathfrak{S}(A) = \{\zeta \in \mathbb{C}; \exists q \in A : \zeta \in \sigma(q)\}$.
A subset $A \subset \mathbb{H}$ is spectrally saturated if and only if there exists a conjugate symmetric subset $S \subset \mathbb{C}$ such that $A = S_{\mathbb{H}}$. In this case, $S = \mathfrak{S}(A)$.
If $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open (via the upper semi-continuity of the spectrum).
Conversely, if $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, one can prove that $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open.

Remark 2(3)

(3) We finally note that, for a given conjugate symmetric subset $U \subset \mathbb{C}$, the set $U_{\mathbb{H}}$ is precisely the *circularization* of U, so it is *axially symmetric*, by an existing terminology. Nevertheless, we continue to call such a set spectrally saturated, a name which better reflects our spectral approach.

An important particular case is when $U = \mathbb{D}_r := \{\zeta \in \mathbb{C}; |\zeta| < r\}$, for some $r > 0$. Because the norm of the normal operator induced by q on $\mathbb{C}^2$ is equal to its spectral radius, we must have $U_{\mathbb{H}} = \{q \in \mathbb{H}; \|q\| < r\}$.

Remark 2(3)

(3) We finally note that, for a given conjugate symmetric subset $U \subset \mathbb{C}$, the set $U_{\mathbb{H}}$ is precisely the *circularization* of U, so it is *axially symmetric*, by an existing terminology. Nevertheless, we continue to call such a set spectrally saturated, a name which better reflects our spectral approach.

An important particular case is when $U = \mathbb{D}_r := \{\zeta \in \mathbb{C}; |\zeta| < r\}$, for some $r > 0$. Because the norm of the normal operator induced by q on $\mathbb{C}^2$ is equal to its spectral radius, we must have $U_{\mathbb{H}} = \{q \in \mathbb{H}; \|q\| < r\}$.

Notation

Let $U \subset \mathbb{C}$ be conjugate symmetric, and let $F : U \mapsto \mathbb{M}_2$. We write

$$F(\zeta) = \begin{pmatrix} f_{11}(\zeta) & f_{12}(\zeta) \\ f_{21}(\zeta) & f_{22}(\zeta) \end{pmatrix}, \ \zeta \in U,$$

with $f_{mn} : U \mapsto \mathbb{C}$, $m, n \in \{1, 2\}$, and set

$$F^{\sim}(\zeta) = \begin{pmatrix} \overline{f_{22}(\overline{\zeta})} & -\overline{f_{21}(\overline{\zeta})} \\ -\overline{f_{12}(\overline{\zeta})} & \overline{f_{11}(\overline{\zeta})} \end{pmatrix}, \ \zeta \in U.$$

In other words, $F^{\sim}(\zeta) = (F(\zeta))^{\sim}$ for all $\zeta \in U$, where "$\sim$" designates the skew complex conjugation.

A Class of $\mathbb{M}_2$-Valued Functions

We temporarily say that F is *skew conjugate symmetric* if $F(\bar{\zeta}) = F^{\sim}(\zeta),\ \zeta \in U$.

Note that the function F is skew conjugate symmetric if and only if F has the form

$$F(\zeta) = \begin{pmatrix} \frac{f_1(\zeta)}{-f_2(\bar{\zeta})} & \frac{f_2(\zeta)}{f_1(\bar{\zeta})} \end{pmatrix},\ \zeta \in U,$$

for some functions $f_1, f_2 : U \mapsto \mathbb{C}$.
In fact, the class of skew conjugate symmetric functions coincides with the class known in the literature as that of **stem functions**.

A Class of $\mathbb{M}_2$-Valued Functions

We temporarily say that F is *skew conjugate symmetric* if $F(\bar{\zeta}) = F^{\sim}(\zeta)$, $\zeta \in U$.

Note that the function F is skew conjugate symmetric if and only if F has the form

$$F(\zeta) = \begin{pmatrix} \frac{f_1(\zeta)}{-f_2(\bar{\zeta})} & \frac{f_2(\zeta)}{f_1(\bar{\zeta})} \end{pmatrix}, \ \zeta \in U,$$

for some functions $f_1, f_2 : U \mapsto \mathbb{C}$.

In fact, the class of skew conjugate symmetric functions coincides with the class known in the literature as that of **stem functions**.

A Class of $\mathbb{M}_2$-Valued Functions

We temporarily say that F is *skew conjugate symmetric* if $F(\bar{\zeta}) = F^{\sim}(\zeta)$, $\zeta \in U$.

Note that the function F is skew conjugate symmetric if and only if F has the form

$$F(\zeta) = \begin{pmatrix} \frac{f_1(\zeta)}{-f_2(\bar{\zeta})} & \frac{f_2(\zeta)}{f_1(\bar{\zeta})} \end{pmatrix}, \quad \zeta \in U,$$

for some functions $f_1, f_2 : U \mapsto \mathbb{C}$.
In fact, the class of skew conjugate symmetric functions coincides with the class known in the literature as that of **stem functions.**

A Conjugation on $\mathbb{M}_2$

To recall the concept of stem function, let us remark that the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ may be identified with $\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$, which is a direct sum, via the the map

$$\mathbb{H} + i\mathbb{H} \ni \mathbf{b} + i\mathbf{c} \mapsto \mathbf{b} \otimes 1 + \mathbf{c} \otimes i \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C},$$

using the decomposition

$\mathbf{a} = (\mathbf{a} + \mathbf{a}^{\sim})/2 + i(\mathbf{a} - \mathbf{a}^{\sim})/2i, \ \mathbf{a} \in \mathbb{M}_2$, with $\mathbf{a} + \mathbf{a}^{\sim}, \ i(\mathbf{a} - \mathbf{a}^{\sim}) \in \mathbb{H}$.

The corresponding conjugation of $\mathbb{M}_2$ is in this case $\mathbf{a} = \mathbf{b} + i\mathbf{c} \mapsto \bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{b}, \mathbf{c} \in \mathbb{H}$ are uniquely determined by a given $\mathbf{a} \in \mathbb{M}_2$.

A Conjugation on $\mathbb{M}_2$

To recall the concept of stem function, let us remark that the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ may be identified with $\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$, which is a direct sum, via the the map

$$\mathbb{H} + i\mathbb{H} \ni \mathbf{b} + i\mathbf{c} \mapsto \mathbf{b} \otimes 1 + \mathbf{c} \otimes i \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C},$$

using the decomposition

$\mathbf{a} = (\mathbf{a} + \mathbf{a}^{\sim})/2 + i(\mathbf{a} - \mathbf{a}^{\sim})/2i, \ \mathbf{a} \in \mathbb{M}_2$, with

$\mathbf{a} + \mathbf{a}^{\sim}, \ i(\mathbf{a} - \mathbf{a}^{\sim}) \in \mathbb{H}$.

The corresponding conjugation of $\mathbb{M}_2$ is in this case $\mathbf{a} = \mathbf{b} + i\mathbf{c} \mapsto \bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{b}, \mathbf{c} \in \mathbb{H}$ are uniquely determined by a given $\mathbf{a} \in \mathbb{M}_2$.

Stem Functions

Usually, stem functions are defined on the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. With the identification from above, a stem function is a map $F : U \mapsto \mathbb{M}_2$, where $U \subset \mathbb{C}$ is conjugate symmetric, with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$.

One can show that a function $F : U \mapsto \mathbb{M}_2$ is skew conjugate symmetric if and only if it is a stem function.

As the term "stem function" is currently used in literature, from now on we shall designate a skew symmetric function as a stem function, using, nevertheless, our equivalent definition. Finally, note that a stem function is not necessarily $\mathbb{H}$-valued. It is $\mathbb{H}$-valued if and only if $f_1(\bar{\zeta}) = f_1(\zeta)$ and $f_2(\bar{\zeta}) = -f_2(\zeta)$ for all $\zeta \in U$.

Stem Functions

Usually, stem functions are defined on the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. With the identification from above, a stem function is a map $F : U \mapsto \mathbb{M}_2$, where $U \subset \mathbb{C}$ is conjugate symmetric, with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$.
One can show that a function $F : U \mapsto \mathbb{M}_2$ is skew conjugate symmetric if and only if it is a stem function.

As the term "stem function" is currently used in literature, from now on we shall designate a skew symmetric function as a stem function, using, nevertheless, our equivalent definition. Finally, note that a stem function is not necessarily $\mathbb{H}$-valued. It is $\mathbb{H}$-valued if and only if $f_1(\bar{\zeta}) = f_1(\zeta)$ and $f_2(\bar{\zeta}) = -f_2(\zeta)$ for all $\zeta \in U$.

Stem Functions

Usually, stem functions are defined on the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. With the identification from above, a stem function is a map $F : U \mapsto \mathbb{M}_2$, where $U \subset \mathbb{C}$ is conjugate symmetric, with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$.
One can show that a function $F : U \mapsto \mathbb{M}_2$ is skew conjugate symmetric if and only if it is a stem function.

As the term "stem function" is currently used in literature, from now on we shall designate a skew symmetric function as a stem function, using, nevertheless, our equivalent definition.
Finally, note that a stem function is not necessarily $\mathbb{H}$-valued. It is $\mathbb{H}$-valued if and only if $f_1(\bar{\zeta}) = f_1(\zeta)$ and $f_2(\bar{\zeta}) = -f_2(\zeta)$ for all $\zeta \in U$.

Stem Functions

Usually, stem functions are defined on the tensor product $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. With the identification from above, a stem function is a map $F : U \mapsto \mathbb{M}_2$, where $U \subset \mathbb{C}$ is conjugate symmetric, with the property $F(\bar{\zeta}) = \overline{F(\zeta)}$ for all $\zeta \in U$.
One can show that a function $F : U \mapsto \mathbb{M}_2$ is skew conjugate symmetric if and only if it is a stem function.

As the term ”stem function“ is currently used in literature, from now on we shall designate a skew symmetric function as a stem function, using, nevertheless, our equivalent definition.
Finally, note that a stem function is not necessarily $\mathbb{H}$-valued. It is $\mathbb{H}$-valued if and only if $f_1(\bar{\zeta}) = f_1(\zeta)$ and $f_2(\bar{\zeta}) = -f_2(\zeta)$ for all $\zeta \in U$.

Functional Calculus 1

For a fixed $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ and each function $f : \sigma(Q(\mathbf{z})) \mapsto \mathbb{C}$ we may define the operator

$$f(Q(\mathbf{z}))\mathbf{w} = f(s_+(\mathbf{z}))\langle \mathbf{w}, \nu_+(\mathbf{z})\rangle \nu_+(\mathbf{z}) + f(s_-(\mathbf{z}))\langle \mathbf{w}, \nu_-(\mathbf{z})\rangle \nu_-(\mathbf{z}),$$

where $\mathbf{w} \in \mathbb{C}^2$ is arbitrary, with a slight but traditional abuse of notation.
We note that this formula is a particular case of the functional calculus given by the spectral theorem for compact normal operators.

Functional Calculus 2

More generally, for a function $F : \sigma(Q(\mathbf{z})) \mapsto \mathbb{M}_2$, we may define a matrix (with respect to the canonical basis of $\mathbb{C}^2$) by the formula

$$F(Q(\mathbf{z}))\mathbf{w} = F(s_+(\mathbf{z}))\langle \mathbf{w}, \nu_+(\mathbf{z})\rangle \nu_+(\mathbf{z}) + F(s_-(\mathbf{z}))\langle \mathbf{w}, \nu_-(\mathbf{z})\rangle \nu_-(\mathbf{z}),$$

where $\mathbf{w} \in \mathbb{C}^2$ is arbitrary. In fact, if $U \subset \mathbb{C}$ is conjugate symmetric, the formula from above leads to a function $F : U_{\mathbb{H}} \mapsto \mathbb{M}_2$.

In particular when $q = s\mathbf{I}$, $s \in \mathbb{R}$, then $F(q) = F(s)\mathbf{I}$,

Functional Calculus 2

More generally, for a function $F : \sigma(Q(\mathbf{z})) \mapsto \mathbb{M}_2$, we may define a matrix (with respect to the canonical basis of $\mathbb{C}^2$) by the formula

$$F(Q(\mathbf{z}))\mathbf{w} = F(s_+(\mathbf{z}))\langle \mathbf{w}, \nu_+(\mathbf{z})\rangle \nu_+(\mathbf{z}) + F(s_-(\mathbf{z}))\langle \mathbf{w}, \nu_-(\mathbf{z})\rangle \nu_-(\mathbf{z}),$$

where $\mathbf{w} \in \mathbb{C}^2$ is arbitrary. In fact, if $U \subset \mathbb{C}$ is conjugate symmetric, the formula from above leads to a function $F : U_{\mathbb{H}} \mapsto \mathbb{M}_2$.
In particular when $q = s\mathbf{I}$, $s \in \mathbb{R}$, then $F(q) = F(s)\mathbf{I}$,

The Main Result

When $q = Q(\mathbf{z})$, so $\nu_\pm(q) = \nu_\pm(\mathbf{z})$, $s_\pm(q) = s_\pm(\mathbf{z})$, the formula from above can be written as

$$F(q))\mathbf{w} = F(s_+(q))\langle \mathbf{w}, \nu_+(q)\rangle \nu_+(q) + F(s_-(q)\langle \mathbf{w}, \nu_-(q)\rangle \nu_-(q),$$

where $\mathbf{w} \in \mathbb{C}^2$ is arbitrary.

Theorem 1

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $F : U \mapsto \mathbb{M}_2$. The matrix $F(q)$ is a quaternion for all $q \in U_{\mathbb{H}}$ if and only if F is a stem function.

Idea of Proof

We fix a point $\zeta \in U$. As $\bar{\zeta} \in U$, we may assume, with no loss of generality, that $\Im\zeta \geq 0$.
Case 1 We assume that $\Im\zeta > 0$, and choose a quaternion $q \in U_{\mathbb{H}}$ with $\sigma(q) = \{\zeta, \bar{\zeta}\}$. Writing $q = Q(\mathbf{z})$ with $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, because $\Im\zeta > 0$, we may assume $z_2 \neq 0$ Let $\nu_\pm(\mathbf{z})$ be the canonical eigenvectors of $Q(\mathbf{z})$. We have $s_+(\mathbf{z}) = \zeta,\ s_-(\mathbf{z}) = \bar{\zeta}$.
We show first that $F(Q(\mathbf{z})) \in \mathbb{H}$ if and only if

$$F(s_+(\mathbf{z}))\nu_+(\mathbf{z}) = F^{\sim}(s_-(\mathbf{z}))\nu_+(\mathbf{z}).$$

Idea of Proof cont.

Case 2 We assume that $\zeta \in U \cap \mathbb{R}$, and put $x = \zeta$. When $\mathbf{z} = (x, 0)$ with $x \in \mathbb{R}$, we have $Q(\mathbf{z})) = x\mathbf{I}$ and so $F(Q(\mathbf{z})) = F(x)\mathbf{I}$. In this case, it is obvious that $F(x)\mathbf{I}$ is a quaternion if and only if $F(x) = F^{\sim}(x)$.
Consequently, if $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$, the function F is a stem one.

Final Case To finish the proof, we have to show that if F is a stem function, we must have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$.

Idea of Proof cont.

Case 2 We assume that $\zeta \in U \cap \mathbb{R}$, and put $x = \zeta$. When $\mathbf{z} = (x, 0)$ with $x \in \mathbb{R}$, we have $Q(\mathbf{z})) = x\mathbf{I}$ and so $F(Q(\mathbf{z})) = F(x)\mathbf{I}$. In this case, it is obvious that $F(x)\mathbf{I}$ is a quaternion if and only if $F(x) = F^{\sim}(x)$.
Consequently, if $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$, the function F is a stem one.

Final Case To finish the proof, we have to show that if F is a stem function, we must have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$.

Corollary 1 Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. We have $f(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Proof. We apply Theorem 1 to the function $F = f\mathbb{I} : U \mapsto \mathbb{M}_2$. This function is a stem one if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Corollary 2 Let $U \subset \mathbb{C}$ be an open conjugate symmetric subset, and let $F : U \mapsto \mathbb{H}$. Then we have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Proof. The property $F : U \mapsto \mathbb{H}$ implies that $F^{\sim} = F$. Therefore, F is a stem function if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Consequences

Corollary 1 Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. We have $f(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Proof. We apply Theorem 1 to the function $F = f\mathbf{I} : U \mapsto \mathbb{M}_2$. This function is a stem one if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Corollary 2 Let $U \subset \mathbb{C}$ be an open conjugate symmetric subset, and let $F : U \mapsto \mathbb{H}$. Then we have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Proof. The property $F : U \mapsto \mathbb{H}$ implies that $F^{\sim} = F$. Therefore, F is a stem function if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Consequences

Corollary 1 Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. We have $f(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Proof. We apply Theorem 1 to the function $F = f\mathbf{I} : U \mapsto \mathbb{M}_2$. This function is a stem one if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Corollary 2 Let $U \subset \mathbb{C}$ be an open conjugate symmetric subset, and let $F : U \mapsto \mathbb{H}$. Then we have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Proof. The property $F : U \mapsto \mathbb{H}$ implies that $F^{\sim} = F$. Therefore, F is a stem function if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Consequences

Corollary 1 Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. We have $f(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Proof. We apply Theorem 1 to the function $F = f\mathbf{I} : U \mapsto \mathbb{M}_2$. This function is a stem one if and only if $f(\zeta) = \overline{f(\bar{\zeta})}$ for all $\zeta \in U$.

Corollary 2 Let $U \subset \mathbb{C}$ be an open conjugate symmetric subset, and let $F : U \mapsto \mathbb{H}$. Then we have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Proof. The property $F : U \mapsto \mathbb{H}$ implies that $F^{\sim} = F$. Therefore, F is a stem function if and only if $F(\zeta) = F(\bar{\zeta})$ for all $\zeta \in U$.

Extension 1

Let $U \subset \mathbb{C}$ be a conjugate symmetric set, and let $F : U \mapsto \mathbb{M}_2$ be a stem function. The formula

$$F(q)\mathbf{w} = F(s_+(q))\langle \mathbf{w}, \nu_+(q)\rangle \nu_+(q) + F(s_-(q))\langle \mathbf{w}, \nu_-(q)\rangle \nu_-(q),$$

where $q \in U_{\mathbb{H}}$ and $\mathbf{w} \in \mathbb{C}^2$ are arbitrary, is an "extension" of the function F to $U_{\mathbb{H}}$, in a sense to be specified.

Extension 2

Note that we have an embedding $U \ni \zeta \mapsto q_\zeta := Q((\zeta, 0)) \in \mathbb{H}$, which is the restriction of an $\mathbb{R}$-linear isometry. In fact, writing $\zeta = x + iy$, with $x, y \in \mathbb{R}$ unique, we have

$$q_\zeta = \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix} = x\mathbf{I} + y\mathbf{J},$$

allowing us to identify the set U with the set

$$U_\mathbf{J} := \{q_\zeta; \zeta \in U\} = \{x\mathbf{I} + y\mathbf{J}; x + iy \in U\} \subset \mathbb{H}.$$

Because F is a stem function, we must have

$$F(\zeta) = \begin{pmatrix} \frac{f_1(\zeta)}{-f_2(\bar\zeta)} & \frac{f_2(\zeta)}{f_1(\bar\zeta)} \end{pmatrix}, \quad \zeta \in U.$$

Extension 3

A direct computation shows that

$$F(q_\zeta) = \begin{pmatrix} \frac{f_1(\zeta)}{-f_2(\bar{\zeta})} & \frac{f_2(\bar{\zeta})}{f_1(\zeta)} \end{pmatrix}, \ \zeta \in U.$$

Let $\mathcal{F}_s(U, \mathbb{M}_2) = \{F : U \mapsto \mathbb{M}_2; F(\bar{\zeta}) = F^\sim(\zeta), \zeta \in U\}$, which is the $\mathbb{R}$-algebra of $\mathbb{M}_2$-valued stem functions on U. Let also $\mathcal{F}(U, \mathbb{H}) = \{G : U \mapsto \mathbb{H}\}$, which an $\mathbb{R}$-algebra of $\mathbb{H}$-valued functions on U. Setting $\kappa(\zeta) = q_\zeta$, $\zeta \in U$, we get an injective unital morphism of $\mathbb{R}$-algebras given by $\mathcal{F}_s(U, \mathbb{M}_2) \ni F \mapsto F \circ \kappa \in \mathcal{F}(U, \mathbb{H})$. Therefore, the map $U_\mathbb{H} \ni q \mapsto F(q) \in \mathbb{H}$, which extends the map $q_\zeta \mapsto F(q_\zeta)$, may be also regarded as an "extension" of $F \in \mathcal{F}_s(U, \mathbb{M}_2)$ (modulo the map κ).

Cauchy Domain

Regarding, as before, the quaternions as normal operators, we now investigate some consequences of their analytic functional calculus, in the classical sense. The frequent use of various versions of the Cauchy formula is simplified by adopting the following:

Definition Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a **Cauchy domain** (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary $\partial\Delta$ of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented.

Note that a Cauchy domain is bounded but not necessarily connected.

Cauchy Domain

Regarding, as before, the quaternions as normal operators, we now investigate some consequences of their analytic functional calculus, in the classical sense. The frequent use of various versions of the Cauchy formula is simplified by adopting the following:

Definition Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a **Cauchy domain** (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary $\partial\Delta$ of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented.

Note that a Cauchy domain is bounded but not necessarily connected.

Cauchy Domain

Regarding, as before, the quaternions as normal operators, we now investigate some consequences of their analytic functional calculus, in the classical sense. The frequent use of various versions of the Cauchy formula is simplified by adopting the following:

Definition Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a **Cauchy domain** (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary $\partial\Delta$ of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented.

Note that a Cauchy domain is bounded but not necessarily connected.

Notation

If $U \subset \mathbb{C}$ is open, the symbol $\mathcal{O}(U, \mathbb{M}_2)$ designates the algebra of all analytic $\mathbb{M}_2$-valued functions in U.
We set $\mathcal{O}(U)$ the algebra of all analytic complex-valued functions in U.
The algebra $\mathcal{O}(U, \mathbb{M}_2)$ is clearly a $\mathcal{O}(U)$-modul.

If $U \subset \mathbb{C}$ is conjugate symmetric and open, we set

$$\mathcal{O}_s(U, \mathbb{M}_2) = \{F \in \mathcal{O}(U, \mathbb{M}_2); F \text{ stem function}\},$$

which is an $\mathbb{R}$-vector space.
Let also

$$\mathcal{O}_s(U) = \{f \in \mathcal{O}(U); f(\bar{\zeta}) = \overline{f(\zeta)}\ \forall \zeta \in U\},$$

which is an $\mathbb{R}$-algebra. In addition, $\mathcal{O}_s(U, \mathbb{M}_2)$ is an $\mathcal{O}_s(U)$-module.

Notation

If $U \subset \mathbb{C}$ is open, the symbol $\mathcal{O}(U, \mathbb{M}_2)$ designates the algebra of all analytic $\mathbb{M}_2$-valued functions in U.
We set $\mathcal{O}(U)$ the algebra of all analytic complex-valued functions in U.
The algebra $\mathcal{O}(U, \mathbb{M}_2)$ is clearly a $\mathcal{O}(U)$-modul.

If $U \subset \mathbb{C}$ is conjugate symmetric and open, we set

$$\mathcal{O}_s(U, \mathbb{M}_2) = \{F \in \mathcal{O}(U, \mathbb{M}_2); F \text{ stem function}\},$$

which is an $\mathbb{R}$-vector space.
Let also

$$\mathcal{O}_s(U) = \{f \in \mathcal{O}(U); f(\bar{\zeta}) = \overline{f(\zeta)}\ \forall \zeta \in U\},$$

which is an $\mathbb{R}$-algebra. In addition, $\mathcal{O}_s(U, \mathbb{M}_2)$ is an $\mathcal{O}_s(U)$-module.

Gelfand-Dunford Formula

Lemma 3 Let $U \subset \mathbb{C}$ be a conjugate symmetric open set and let $F : \mathcal{O}(U, \mathbb{M}_2)$. For every $q \in U_{\mathbb{H}}$ we set

$$F_{\mathbb{H}}(q) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta \mathbf{I} - q)^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(q)$. Then we have $F_{\mathbb{H}}(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F \in \mathcal{O}_s(U, \mathbb{M}_2)$.

Remark 3

It follows from the proof of the previous lemma that the element $F_{\mathbb{H}}(q)$, given by formula Gelfand-Dunford formula coincides with the element $F(q)$ given by functional calculus for normal operators.
Nevertheless, we keep the notation $F_{\mathbb{H}}(q)$ whenever we want to emphasize that it is defined via the Cauchy type integral from above.

Corollary 3

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set and let $f : U \mapsto \mathbb{C}$ be an analytic function. For every $q \in U_{\mathbb{H}}$ we set

$$f_{\mathbb{H}}(q) = \frac{1}{2\pi i} \int_{\Gamma} f(\zeta)(\zeta \mathbf{I} - q)^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(q)$. Then we have $f_{\mathbb{H}}(q) \in \mathbb{H}$ if and only if $f \in \mathcal{O}_s(U)$.

Proof. The assertion is a direct consequence of Lemma 3, applied to the the function $f\mathbf{I}$.

Corollary 3

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set and let $f : U \mapsto \mathbb{C}$ be an analytic function. For every $q \in U_{\mathbb{H}}$ we set

$$f_{\mathbb{H}}(q) = \frac{1}{2\pi i} \int_{\Gamma} f(\zeta)(\zeta \mathbf{I} - q)^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(q)$. Then we have $f_{\mathbb{H}}(q) \in \mathbb{H}$ if and only if $f \in \mathcal{O}_s(U)$.

Proof. The assertion is a direct consequence of Lemma 3, applied to the the function $f\mathbf{I}$.

Notation

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$ (which is also open !).
We put

$$\mathcal{R}(\Omega) = \{f_{\mathbb{H}}; f \in \mathcal{O}_s(U)\},$$

and

$$\mathcal{R}(\Omega, \mathbb{H}) = \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U, \mathbb{M}_2)\},$$

which are $\mathbb{R}$-linear spaces. Using them, we get the following:

Analytic Functional Calculus

Theorem 2

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$. The space $\mathcal{R}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}(\Omega, \mathbb{H})$ is a right $\mathcal{R}(\Omega)$-module, and the map

$$\mathcal{O}_s(U, \mathbb{M}_2) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H})$$

is a right module isomorphism. Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Corollary 4

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$. The map

$$\mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega)$$

is a unital $\mathbb{R}$-algebra isomorphism. Moreover,
(a) for every polynomial $p(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$ with a_n real for all $n = 0, 1, \ldots, m$, we have $p_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \Omega$;
(b) if $f \in \mathcal{O}_s(U)$ has no zero in U, we have $(f_{\mathbb{H}}(q))^{-1} = f_{\mathbb{H}}^{-1}(q)$ for all $q \in \Omega$.

Corollary 5

Let $r > 0$ and let $U \supset \{\zeta \in \mathbb{C}; |\zeta| \leq r\}$ be a conjugate symmetric open set. Then for every $F \in \mathcal{O}_s(U, \mathbb{M}_2)$ one has

$$F_{\mathbb{H}}(q) = \sum_{n \geq 0} \frac{F^{(n)}(0)}{n!} q^n, \ \|q\| < r,$$

where the series is absolutely convergent.

For every function $F \in \mathcal{O}_s(U, \mathbb{M}_2)$, the derivatives $F^{(n)}$ also belong to $\mathcal{O}_s(U, \mathbb{M}_2)$, where $U \subset \mathbb{C}$ is a conjugate symmetric open set. Next, fixing $F \in \mathcal{O}_s(U, \mathbb{M}_2)$, we may define its **extended derivatives** as follows

$$F_{\mathbb{H}}^{(n)}(q) = \frac{1}{2\pi i} \int_{\Gamma} F^{(n)}(\zeta)(\zeta \mathbf{I} - q)^{-1} d\zeta,$$

for the boundary Γ of a Cauchy domain $\Delta \subset U$, $n \geq 0$ an arbitrary integer, and $\sigma(q) \subset \Delta$.

Derivatives

In particular, if $F \in \mathcal{O}_s(\mathbb{D}_r, \mathbb{M}_2)$, and so we have a representation as a convergent series $F(\zeta) = \sum_{k\geq 0} a_k \zeta^k$ with coefficients in $\mathbb{H}$, then the previous formula gives the equality $F'_{\mathbb{H}}(q) = \sum_{k\geq 1} k a_k q^{k-1}$, which looks like a (formal) derivative of the function $F_{\mathbb{H}}(q) = \sum_{k\geq 0} a_k q^k$.

In fact, the parallelism with the usual holomorphic functions goes much further. We can obtain the Cauchy derivative inequalities, properties of the zeros of such functions, and so on. We omit the details

Derivatives

In particular, if $F \in \mathcal{O}_s(\mathbb{D}_r, \mathbb{M}_2)$, and so we have a representation as a convergent series $F(\zeta) = \sum_{k\geq 0} a_k \zeta^k$ with coefficients in $\mathbb{H}$, then the previous formula gives the equality $F'_{\mathbb{H}}(q) = \sum_{k\geq 1} k a_k q^{k-1}$, which looks like a (formal) derivative of the function $F_{\mathbb{H}}(q) = \sum_{k\geq 0} a_k q^k$.

In fact, the parallelism with the usual holomorphic functions goes much further. We can obtain the Cauchy derivative inequalities, properties of the zeros of such functions, and so on. We omit the details

Remark 4

The previous results suggest a definition for $\mathbb{H}$-valued "analytic functions" as elements of the set $\mathcal{R}(\Omega, \mathbb{H})$, where Ω is a spectrally saturated open subset of $\mathbb{H}$. Because the expression "analytic function" is quite improper in this context, the elements of $\mathcal{R}(\Omega, \mathbb{H})$ will be (temporarily) called **Q-regular functions** on Ω. In fact, the functions from $\mathcal{R}(\Omega, \mathbb{H})$ may be also regarded as *Cauchy transforms* of the (stem) functions from $\mathcal{O}_s(U, \mathbb{M}_2)$, with $U = \mathfrak{S}(\Omega)$.

Slice Regularity 1

As already mentioned, there exists a large literature dedicated to quatrenionic analysis, in particular, to the concept of "slice regularity“, which is a form of holomorphy in the context of quaternions. We recall the two monographs quoted before.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Slice Regularity 1

As already mentioned, there exists a large literature dedicated to quatrenionic analysis, in particular, to the concept of "slice regularity“, which is a form of holomorphy in the context of quaternions. We recall the two monographs quoted before.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Slice Regularity 1

As already mentioned, there exists a large literature dedicated to quatrenionic analysis, in particular, to the concept of "slice regularity“, which is a form of holomorphy in the context of quaternions. We recall the two monographs quoted before.

[1] D. Alpay, F. Colombo, and I. Sabadini: **Slice Hyperholomorphic Schur Analysis**, Operator Theory: Advances and Applications Vol. 256, Birkhäuser/Springer Basel, 2016.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Slice Regularity 2

For $\mathbb{M}_2$-valued functions defined on subsets of $\mathbb{H}$, the concept of *slice regularity* is defined as follows.
Let $\mathbb{S}$ be the unit sphere of purely imaginary quaternions. Let also $\Omega \in \mathbb{H}$ be an open set, and let $F : \Omega \mapsto \mathbb{M}_2$ be a differentiable function. In the spirit of [2], we say that F is **(right) slice regular** in Ω if for all $\mathfrak{s} \in \mathbb{S}$,

$$\bar{\partial}_{\mathfrak{s}} F(x\mathbf{I} + y\mathfrak{s}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + R_{\mathfrak{s}}\frac{\partial}{\partial y}\right) F(x\mathbf{I} + y\mathfrak{s}) = 0,$$

on the set $\Omega \cap (\mathbb{R}\mathbf{I} + \mathbb{R}\mathfrak{s})$, where $R_{\mathfrak{s}}$ is the right multiplication of the elements of $\mathbb{M}_2$ by $\mathfrak{s}$.

(1) The convergent series of the form $\sum_{k\geq 0} a_k q^k$, on a set $\{q \in \mathbb{H}; \|q\| < r\}$, with $a_k \in \mathbb{H}$ for all $k \geq 0$, are $\mathbb{H}$-valued slice regular on their domain of definition. In fact, if actually $a_k \in \mathbb{M}_2$, such functions are $\mathbb{M}_2$-valued right slice regular on their domain of definition.

(2) The matrix Cauchy kernel on the open set $\Omega \subset \mathbb{H}$, defined by

$$\Omega \ni q \mapsto (\zeta \mathbf{I} - q)^{-1} \in \mathbb{M}_2,$$

is slice regular on $\Omega \subset \mathbb{H}$, whenever $\zeta \notin \mathfrak{S}(\Omega)$.

Examples

(1) The convergent series of the form $\sum_{k\geq 0} a_k q^k$, on a set $\{q \in \mathbb{H}; \|q\| < r\}$, with $a_k \in \mathbb{H}$ for all $k \geq 0$, are $\mathbb{H}$-valued slice regular on their domain of definition. In fact, if actually $a_k \in \mathbb{M}_2$, such functions are $\mathbb{M}_2$-valued right slice regular on their domain of definition.

(2) The matrix Cauchy kernel on the open set $\Omega \subset \mathbb{H}$, defined by

$$\Omega \ni q \mapsto (\zeta \mathbf{I} - q)^{-1} \in \mathbb{M}_2,$$

is slice regular on $\Omega \subset \mathbb{H}$, whenever $\zeta \notin \mathfrak{S}(\Omega)$.

Equivalence

Theorem 3

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The following conditions are equivalent:
(i) Φ is a slice regular function;
(ii) $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, that is, Φ is Q-regular.

Some remarks

(1) Slice regular functions, as defined in [2], have a Cauchy type representation, via a Cauchy type kernel, which is not slight regular.

(2) Q-regular functions, defined via the analytic functional calculus, have a Cauchy type representation via a Cauchy type kernel (due to Gelfand and Dunford), which is commutative at a fixed quaternion, and slice regular.

Equivalence

Theorem 3

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The following conditions are equivalent:
(i) Φ is a slice regular function;
(ii) $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, that is, Φ is Q-regular.

Some remarks

(1) Slice regular functions, as defined in [2], have a Cauchy type representation, via a Cauchy type kernel, which is not slight regular.

(2) Q-regular functions, defined via the analytic functional calculus, have a Cauchy type representation via a Cauchy type kernel (due to Gelfand and Dunford), which is commutative at a fixed quaternion, and slice regular.

Equivalence

Theorem 3

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The following conditions are equivalent:
(i) Φ is a slice regular function;
(ii) $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, that is, Φ is Q-regular.

Some remarks

(1) Slice regular functions, as defined in [2], have a Cauchy type representation, via a Cauchy type kernel, which is not slight regular.

(2) Q-regular functions, defined via the analytic functional calculus, have a Cauchy type representation via a Cauchy type kernel (due to Gelfand and Dunford), which is commutative at a fixed quaternion, and slice regular.

For a real or complex Banach space $\mathcal{V}$, we denote by $\mathcal{B}(\mathcal{V})$ the algebra of all bounded $\mathbb{R}$-(respectively $\mathbb{C}$-)linear operators on $\mathcal{V}$. If necessary, the identity on $\mathcal{V}$ will be denoted by $\mathbf{I}_{\mathcal{V}}$.
Using an idea which goes back to Kaplansky (see also [2]), we give the folowing:
Definition Let $\mathcal{X}$ be a real Banach space. For a given operator $T \in \mathcal{B}(\mathcal{X})$, the set $\rho_{\mathbb{H}}(T)$, given by the equality

$$\{Q(\mathbf{z}) \in \mathbb{H}; \mathbf{z} = (z_1, z_2), (T^2 - (z_1 + \bar{z}_1)T + |z_1|^2 + |z_2|^2)^{-1} \in \mathcal{B}(\mathcal{X})\}$$

is called the **quaternionic resolvent** (or simply the Q-**resolvent**) of T.
The complement $\sigma_{\mathbb{H}}(T) = \mathbb{H} \setminus \rho_{\mathbb{H}}(T)$ is called the **quaternionic spectrum** (or simply the Q-**spectrum**) of T.

Remark 5

We note that if $q = Q(\mathbf{z})$, with $\mathbf{z} = (z_1, z_2) \in \mathbb{C}$, setting $\Re q = \Re z_1$, we have

$$T^2 - (z_1 + \bar{z}_1)T + |z_1|^2 + |z_2|^2 = T^2 - 2\Re q\, T + \|q\|^2,$$

and the right hand side is precisely the expression used in [2] to define the spectrum of $\mathbb{H}$-linear operators. Note also that the definition of Q-spectrum applies to the class of $\mathbb{R}$-linear operators, which is larger that the class of $\mathbb{H}$-linear operators.

Remark 6

Looking at the definition of Q-spectrum, we observe that $Q(\mathbf{w}) \in \rho_{\mathbb{H}}(T)$ whenever for some $Q(\mathbf{z}) \in \rho_{\mathbb{H}}(T)$ we have $\sigma(Q(\mathbf{w})) = \sigma(Q(\mathbf{z}))$.

In particular, $Q(\mathbf{z}) \in \rho_{\mathbb{H}}(T)$ if and only if $Q(\mathbf{z}^*) \in \rho_{\mathbb{H}}(T)$, where $\mathbf{z}^* = (\bar{z}_1, -z_2)$ if $\mathbf{z} = (z_1, z_2)$, and $Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(T)$ implies that $Q((s_\pm(\mathbf{z}), 0)) \in \sigma_{\mathbb{H}}(T)$.

Remark 7

The *complex spectrum* of the operator $T \in \mathcal{B}(\mathcal{X})$ on the real Banach space $\mathcal{X}$ is given by

$$\sigma_{\mathbb{C}}(T) := \{\lambda \in \mathbb{C}; Q((\lambda, 0)) \in \sigma_{\mathbb{H}}(T)\}.$$

Because the Q-spectrum of T is spectrally saturated, we have $\sigma_{\mathbb{H}}(T) = \sigma_{\mathbb{C}}(T)_{\mathbb{H}}$.

We also have $\lambda \in \sigma_{\mathbb{C}}(T)$ if and only if $\bar{\lambda} \in \sigma_{\mathbb{C}}(T)$.

In addition $\lambda \in \rho_{\mathbb{C}}(T) := \mathbb{C} \setminus \sigma_{\mathbb{C}}(T)$ if and only if the operator $T^2 - 2\Re\lambda T + |\lambda|^2$ is invertible.

Complexification

Let $\mathcal{X}$ be a real Banach space, and let $T \in \mathcal{B}(\mathcal{X})$. We denote by $\mathcal{X}_{\mathbb{C}}$ the complexification of $\mathcal{X}$, written as $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \oplus i\mathcal{X}$, or simply as $\mathcal{X} + i\mathcal{X}$.

The operator T can be extended to $\mathcal{X}_{\mathbb{C}}$ via the formula $T_{\mathbb{C}}(x + iy) = Tx + iTy$ for all $x, y \in \mathcal{X}$. It is clear that $T_{\mathbb{C}}$ is a bounded $\mathbb{C}$-linear operator.

Let $T_{\mathbb{C}}^{(2)}$ be the 2×2 diagonal operator with $T_{\mathbb{C}}$ on the diagonal, acting on $\mathcal{X}_{\mathbb{C}}^2 := \mathcal{X}_{\mathbb{C}} \oplus \mathcal{X}_{\mathbb{C}}$. As for every $\mathbf{z} \in \mathbb{C}^2$ the matrix $Q(\mathbf{z})$ also acts on $\mathcal{X}_{\mathbb{C}}^2$, we may state the following:

Complexification

Let $\mathcal{X}$ be a real Banach space, and let $T \in \mathcal{B}(\mathcal{X})$. We denote by $\mathcal{X}_{\mathbb{C}}$ the complexification of $\mathcal{X}$, written as $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \oplus i\mathcal{X}$, or simply as $\mathcal{X} + i\mathcal{X}$.

The operator T can be extended to $\mathcal{X}_{\mathbb{C}}$ via the formula $T_{\mathbb{C}}(x + iy) = Tx + iTy$ for all $x, y \in \mathcal{X}$. It is clear that $T_{\mathbb{C}}$ is a bounded $\mathbb{C}$-linear operator.

Let $T_{\mathbb{C}}^{(2)}$ be the 2×2 diagonal operator with $T_{\mathbb{C}}$ on the diagonal, acting on $\mathcal{X}_{\mathbb{C}}^2 := \mathcal{X}_{\mathbb{C}} \oplus \mathcal{X}_{\mathbb{C}}$. As for every $\mathbf{z} \in \mathbb{C}^2$ the matrix $Q(\mathbf{z})$ also acts on $\mathcal{X}_{\mathbb{C}}^2$, we may state the following:

Complexification

Let $\mathcal{X}$ be a real Banach space, and let $T \in \mathcal{B}(\mathcal{X})$. We denote by $\mathcal{X}_{\mathbb{C}}$ the complexification of $\mathcal{X}$, written as $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \oplus i\mathcal{X}$, or simply as $\mathcal{X} + i\mathcal{X}$.

The operator T can be extended to $\mathcal{X}_{\mathbb{C}}$ via the formula $T_{\mathbb{C}}(x + iy) = Tx + iTy$ for all $x, y \in \mathcal{X}$. It is clear that $T_{\mathbb{C}}$ is a bounded $\mathbb{C}$-linear operator.

Let $T_{\mathbb{C}}^{(2)}$ be the 2×2 diagonal operator with $T_{\mathbb{C}}$ on the diagonal, acting on $\mathcal{X}_{\mathbb{C}}^2 := \mathcal{X}_{\mathbb{C}} \oplus \mathcal{X}_{\mathbb{C}}$. As for every $\mathbf{z} \in \mathbb{C}^2$ the matrix $Q(\mathbf{z})$ also acts on $\mathcal{X}_{\mathbb{C}}^2$, we may state the following:

Lemma 5

A quaternion $Q(\mathbf{z})$ is in the set $\rho_{\mathbb{H}}(T)$ if and only if the operators $T_{\mathbb{C}}^{(2)} - Q(\mathbf{z})$ and $T_{\mathbb{C}}^{(2)} - Q(\mathbf{z}^*)$ are invertible in $\mathcal{B}(\mathcal{X}_{\mathbb{C}}^2)$.

Example One of the simplest possible examples is to take $\mathcal{X} = \mathbb{R}$ and T the operator $Tx = \tau x$ for all $x \in \mathbb{R}$, where $\tau \in \mathbb{R}$ is fixed. We have $\mathcal{X}_{\mathbb{C}} = \mathbb{C}$, and $T_{\mathbb{C}}$ is given by the same formula, acting on $\mathbb{C}$. The Q-spectrum of T is the set

$$\{Q(\mathbf{z}); \mathbf{z} = (z_1, z_2) \in \mathbb{C}, \Re z_1 = \tau, \Im z_1 = z_2 = 0\} = \{Q((\tau, 0))\}.$$

Consequently, $\sigma_{\mathbb{H}}(T) = \{Q((\tau, 0))\}$, and $\sigma_{\mathbb{C}}(T) = \{\tau\}$.

Lemma 5

A quaternion $Q(\mathbf{z})$ is in the set $\rho_{\mathbb{H}}(T)$ if and only if the operators $T_{\mathbb{C}}^{(2)} - Q(\mathbf{z})$ and $T_{\mathbb{C}}^{(2)} - Q(\mathbf{z}^*)$ are invertible in $\mathcal{B}(\mathcal{X}_{\mathbb{C}}^2)$.

Example One of the simplest possible examples is to take $\mathcal{X} = \mathbb{R}$ and T the operator $Tx = \tau x$ for all $x \in \mathbb{R}$, where $\tau \in \mathbb{R}$ is fixed. We have $\mathcal{X}_{\mathbb{C}} = \mathbb{C}$, and $T_{\mathbb{C}}$ is given by the same formula, acting on $\mathbb{C}$. The Q-spectrum of T is the set

$$\{Q(\mathbf{z}); \mathbf{z} = (z_1, z_2) \in \mathbb{C}, \Re z_1 = \tau, \Im z_1 = z_2 = 0\} = \{Q((\tau, 0))\}.$$

Consequently, $\sigma_{\mathbb{H}}(T) = \{Q((\tau, 0))\}$, and $\sigma_{\mathbb{C}}(T) = \{\tau\}$.

Conjugation

Let $\mathcal{X}$ be a real Banach space, and let $\mathcal{X}_{\mathbb{C}}$ be its complexification. For every $u = x + iy \in \mathcal{X}_{\mathbb{C}}$ with $x, y \in \mathcal{X}$ we put $\bar{u} = x - iy$. In other words, the map $\mathcal{X}_{\mathbb{C}} \ni u \mapsto \bar{u} \in \mathcal{X}_{\mathbb{C}}$ is a **conjugation**, also denoted by C. Hence C is $\mathbb{R}$-linear and C^2 is the identity on $\mathcal{X}_{\mathbb{C}}$.

Fixing an operator $S \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$, we define the operator $S^\flat \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ to be equal to CSC. The map $\mathcal{B}(\mathcal{X}_{\mathbb{C}}) \ni S \mapsto S^\flat \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is a unital conjugate-linear automorphism, whose square is the identity on $\mathcal{B}(\mathcal{X}_{\mathbb{C}})$.

We have $S^\flat = S$ if and only if $S(\mathcal{X}) \subset \mathcal{X}$. In particular, if $T \in \mathcal{B}(\mathcal{X})$, we have $T_{\mathbb{C}}^\flat = T_{\mathbb{C}}$.

Conjugation

Let $\mathcal{X}$ be a real Banach space, and let $\mathcal{X}_{\mathbb{C}}$ be its complexification. For every $u = x + iy \in \mathcal{X}_{\mathbb{C}}$ with $x, y \in \mathcal{X}$ we put $\bar{u} = x - iy$. In other words, the map $\mathcal{X}_{\mathbb{C}} \ni u \mapsto \bar{u} \in \mathcal{X}_{\mathbb{C}}$ is a **conjugation**, also denoted by C. Hence C is $\mathbb{R}$-linear and C^2 is the identity on $\mathcal{X}_{\mathbb{C}}$.

Fixing an operator $S \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$, we define the operator $S^\flat \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ to be equal to CSC. The map $\mathcal{B}(\mathcal{X}_{\mathbb{C}}) \ni S \mapsto S^\flat \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is a unital conjugate-linear automorphism, whose square is the identity on $\mathcal{B}(\mathcal{X}_{\mathbb{C}})$.

We have $S^\flat = S$ if and only if $S(\mathcal{X}) \subset \mathcal{X}$. In particular, if $T \in \mathcal{B}(\mathcal{X})$, we have $T_{\mathbb{C}}^\flat = T_{\mathbb{C}}$.

Lemma 6

Let $\mathcal{X}$ be a real Banach space, and let $T \in \mathcal{B}(\mathcal{X})$. We have the equality $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.

If $T \in \mathcal{B}(\mathcal{X})$, X real, we have the usual analytic functional calculus for the operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$. That is, if $U \supset \sigma(T_{\mathbb{C}})$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic, we may put

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain containing $\sigma(T_{\mathbb{C}})$ in U, where we may assume that both U and Γ are conjugate symmetric.

Lemma 6

Let $\mathcal{X}$ be a real Banach space, and let $T \in \mathcal{B}(\mathcal{X})$. We have the equality $\sigma_{\mathbb{C}}(T) = \sigma(T_{\mathbb{C}})$.

If $T \in \mathcal{B}(\mathcal{X})$, X real, we have the usual analytic functional calculus for the operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{X}_{\mathbb{C}})$. That is, if $U \supset \sigma(T_{\mathbb{C}})$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic, we may put

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain containing $\sigma(T_{\mathbb{C}})$ in U, where we may assume that both U and Γ are conjugate symmetric.

Natural question: When do we have $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$, implying $F(T_{\mathbb{C}})(\mathcal{X}) \subset \mathcal{X})$, that is, the invariance of $\mathcal{X}$?

Before trying to give an answer to this question we note a standard property of the spectra defined above.

Remark If $\mathcal{X}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{X})$, both $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$ are nonempty compact subset of $\mathbb{H}$, $\mathbb{C}$, respectively.

An answer to the previous question is given by the following:

Theorem 3

If $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic and $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{X})$.

Natural question: When do we have $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$, implying $F(T_{\mathbb{C}})(\mathcal{X}) \subset \mathcal{X}$), that is, the invariance of $\mathcal{X}$?

Before trying to give an answer to this question we note a standard property of the spectra defined above.

Remark If $\mathcal{X}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{X})$, both $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$ are nonempty compact subset of $\mathbb{H}$, $\mathbb{C}$, respectively.

An answer to the previous question is given by the following:

Theorem 3

If $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic and $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{X})$.

Natural question: When do we have $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$, implying $F(T_{\mathbb{C}})(\mathcal{X}) \subset \mathcal{X})$, that is, the invariance of $\mathcal{X}$?

Before trying to give an answer to this question we note a standard property of the spectra defined above.

Remark If $\mathcal{X}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{X})$, both $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$ are nonempty compact subset of $\mathbb{H}$, $\mathbb{C}$, respectively.

An answer to the previous question is given by the following:

Theorem 3

If $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic and $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{X})$.

Invariance

Natural question: When do we have $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$, implying $F(T_{\mathbb{C}})(\mathcal{X}) \subset \mathcal{X})$, that is, the invariance of $\mathcal{X}$?

Before trying to give an answer to this question we note a standard property of the spectra defined above.

Remark If $\mathcal{X}$ is a real Banach space, and $T \in \mathcal{B}(\mathcal{X})$, both $\sigma_{\mathbb{H}}(T)$ and $\sigma_{\mathbb{C}}(T)$ are nonempty compact subset of $\mathbb{H}$, $\mathbb{C}$, respectively.

An answer to the previous question is given by the following:

Theorem 3

If $F : U \mapsto \mathcal{B}(\mathcal{X}_{\mathbb{C}})$ is analytic and $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{X})$.

Remark 8(1)

(1) Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $\mathcal{X}$ be a real Banach space. We denote by $\mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_\mathbb{C}))$ the set of all analytic maps $F : U \mapsto \mathcal{B}(\mathcal{X}_\mathbb{C})$ such that $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$. When $\mathcal{X} = \mathbb{R}$, we put $\mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_\mathbb{C})) = \mathcal{O}_c(U)$. In this case, we have $\mathcal{O}_c(U) = \mathcal{O}_s(U)$, and $\mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_\mathbb{C}))$ is a $\mathcal{O}_c(U)$-module.
Moreover, $\mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_\mathbb{C}))$ is a unital $\mathbb{R}$-algebra, containing all polynomials $P(\zeta) = \sum_{k=0}^m (A_k)_\mathbb{C} \zeta^k$, with $A_k \in \mathcal{B}(\mathcal{X})$.

Remark 8(2)

2) The injective linear map $\mathbb{M}_2 \ni a \mapsto M_a \in \mathcal{B}(\mathcal{X})$, given $M_a b = ab$, $b \in \mathbb{M}_2$ induces an injective linear map of $\mathcal{O}_s(U, \mathbb{M}_2))$ into $\mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$. Specifically, given $F \in \mathcal{O}_s(U, \mathbb{M}_2))$, that is, an analytic stem function, we have $M_F \in \mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$, where $M_F(\zeta)b = F(\zeta)b$ for all $\zeta \in U$ and $b \in \mathbb{M}_2$. This remark shows that the space $\mathcal{O}_s(U, \mathbb{M}_2))$ may be regarded as a subspace of $\mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$.

Fixing $F \in \mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_\mathbb{C}))$, we have $F(T_\mathbb{C})^\flat = F(T_\mathbb{C})$ for all $T \in \mathcal{B}(\mathcal{X})$. This allows us to define $F(T) = F(T_\mathbb{C})|\mathcal{X}$, because $\mathcal{X}$ is invariant under $F(T_\mathbb{C})$. In addition, we have the following.

Remark 8(2)

2) The injective linear map $\mathbb{M}_2 \ni a \mapsto M_a \in \mathcal{B}(\mathcal{X})$, given $M_a b = ab$, $b \in \mathbb{M}_2$ induces an injective linear map of $\mathcal{O}_s(U, \mathbb{M}_2))$ into $\mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$. Specifically, given $F \in \mathcal{O}_s(U, \mathbb{M}_2))$, that is, an analytic stem function, we have $M_F \in \mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$, where $M_F(\zeta)b = F(\zeta)b$ for all $\zeta \in U$ and $b \in \mathbb{M}_2$. This remark shows that the space $\mathcal{O}_s(U, \mathbb{M}_2))$ may be regarded as a subspace of $\mathcal{O}_c(U, \mathcal{B}(\mathbb{M}_2))$.

Fixing $F \in \mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_{\mathbb{C}}))$, we have $F(T_{\mathbb{C}})^\flat = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{X})$. This allows us to define $F(T) = F(T_{\mathbb{C}})|\mathcal{X}$, because $\mathcal{X}$ is invariant under $F(T_{\mathbb{C}})$. In addition, we have the following.

Theorem 4

For every $T \in \mathcal{B}(\mathcal{X})$, the map

$$\mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_{\mathbb{C}})) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{X})$$

is $\mathbb{R}$-linear and has the property $(Ff)(T) = F(T)f(T)$ for all $f \in \mathcal{O}_c(U)$ and $F \in \mathcal{O}_c(U, \mathcal{B}(\mathcal{X}_{\mathbb{C}}))$. Moreover, $P(T) = \sum_{k=0}^{m} A_k T^k$ for any polynomial $P(\zeta) = \sum_{k=0}^{m} A_k \zeta^k$, $\zeta \in \mathbb{C}$, with coefficients $A_k \in \mathcal{B}(\mathcal{X})$.

Definition Let $\mathcal{X}$ be a real Banach space. We say that $\mathcal{X}$ is a **(left) $\mathbb{H}$-module** if there exists a unital $\mathbb{R}$-algebra morphism of $\mathbb{H}$ into $\mathcal{B}(\mathcal{X})$. In this case, the elements of $\mathbb{H}$ in $\mathcal{B}(\mathcal{X})$ may regarded as $\mathbb{R}$-linear operators.

Corollary

If $\mathcal{X}$ is a $\mathbb{H}$-module, for every polynomial $P(\zeta) = \sum_{k=0}^{m} A_k \zeta^k$, $\zeta \in \mathbb{C}$, with coefficients $A_k \in \mathbb{H}$, we have $P(T) = \sum_{k=0}^{m} A_k T^k$.

Definition Let $\mathcal{X}$ be a real Banach space. We say that $\mathcal{X}$ is a **(left) $\mathbb{H}$-module** if there exists a unital $\mathbb{R}$-algebra morphism of $\mathbb{H}$ into $\mathcal{B}(\mathcal{X})$. In this case, the elements of $\mathbb{H}$ in $\mathcal{B}(\mathcal{X})$ may regarded as $\mathbb{R}$-linear operators.

Corollary

If $\mathcal{X}$ is a $\mathbb{H}$-module, for every polynomial $P(\zeta) = \sum_{k=0}^{m} A_k \zeta^k$, $\zeta \in \mathbb{C}$, with coefficients $A_k \in \mathbb{H}$, we have $P(T) = \sum_{k=0}^{m} A_k T^k$.

Thank you very much for your attention !