A DIDACTIC INSIGHT INTO A MOMENT PROBLEM WITH CONSTRAINTS

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OT 25: Timişoara, June 30-July 5, 2014

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ABSTRACT

We discuss a concrete moment problem, stated in the framework of an algebra of rational functions on a hemisphere, whose specificity imposes some constraints on the existence of a representing measure. Trying to illustrate how to overtake some inherent difficulties, we exhibit the most significant arguments by inserting definitions and techniques related to a quaternionic Cayley transform.

The Moments of a Measure

In what follows, we restrict our discussion in the euclidean space $\mathbb{R}^3$.

Let Σ be a Borel measurable subset in $\mathbb{R}^3$, and let μ be a positive Borel measure on Σ. Let also (s, t, u) denote the variable in $\mathbb{R}^3$. Assuming the integrability of all monomials in (s, t, u) on Σ, the real numbers

$$\gamma_{jkl} := \int_\Sigma s^j t^k u^l d\mu(s, t, u), \quad j, k, l \in \mathbb{Z}_+,$$

are the **moments** of the measure μ.

The numbers $(\gamma_{jkl})_{j,k,l\in\mathbb{Z}_+}$ may or may not determine the measure μ.

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Formulation of the Problem

Some measurements in physics (or even in practice) may lead to a 3-sequence of real numbers $\gamma := (\gamma_{jkl})_{j,k,l\in\mathbb{Z}_+}$. The **moment problem** for such a sequence means to find a finite positive Borel measure (initially on $\mathbb{R}^3$) having these numbers as moments.

When such a measure exists, it is called a **representing measure** for γ.

If, moreover, we ask the support of a representing measure to be in a given Borel measurable subset Σ in $\mathbb{R}^3$, the moment problem is said to be a Σ-**moment problem.**

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The Riesz Functional

The general moment problem, especially in several variables, is still a difficult mathematical problem, generating numerous open questions. To fix our particular framework in a more suitable context, we shall use an equivalent formulation.

Let $\mathcal{P}^3$ be the algebra of all polynomials in s, t, u, with complex coefficients. Let also $\gamma := (\gamma_{jkl})_{j,k,l \in \mathbb{Z}_+}$ be a 3-sequence of real numbers. The assignment

$$s^j t^k u^l \mapsto \gamma_{jkl}, \quad j, k, l \in \mathbb{Z}_+,$$

extended by linearity, leads to a (linear) functional $\Lambda_\gamma : \mathcal{P}^3 \mapsto \mathbb{C}$, called the **Riesz functional** (associated to γ).

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Continuation

When the 3-sequence $\gamma := (\gamma_{jkl})_{j,k,l\in\mathbb{Z}_+}$ has a representing measure, it is easy to see that the associated Riesz functional $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$ has the properties
(1) $\Lambda_\gamma(\bar{p}) = \overline{\Lambda_\gamma(p)}$,
(2) $\Lambda_\gamma(|p|^2) \geq 0$ for all $p \in \mathcal{P}^3$, and
(3) $\Lambda_\gamma(1) > 0$.

Note that if $\Lambda_\gamma(1) = 0$, then $\gamma = 0$ because of the positivity of the representing measure, which is a trivial case to be, in general, avoided.

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Square Positive Functionals

Inspired by the properties of the Riesz functional in the presence of a representing measure, we give the following:

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Representing Measures

A **representing measure** for the spf $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$ with **support** in the measurable subset $\Sigma \subset \mathbb{R}^3$ is a positive measure μ on Σ such that

$$\Lambda(p) = \int_\Sigma p d\mu \text{ all } p \in \mathcal{P}^3.$$

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A Hemisphere as a Semi-Algebraic Set

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{R}^3$, and consider the hemisphere

$$\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{S}^3; 0 \leq s \leq 1\}.$$

As we have

$$\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{R}^3; \theta(s,t,u) = 0, \sigma(s) \geq 0, (1-\sigma)(s) \geq 0\},$$

where $\theta(s,t,u) = 1 - s^2 - t^2 - u^2$ and $\sigma(s) = s$, it follows that $\mathbb{S}^3_+$ is a compact semi-algebraic set.

A Consequence of Schmüdgen's Theorem

For a given polynomial $q \in \mathcal{P}^3$ and a map $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$, we put $\Lambda_q(p) = \Lambda(qp)$ for all $p \in \mathcal{P}^3$.

A well-known theorm by **K. Schmüdgen** implies that a unital square positive functional $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$ has a representing measure with support in $\mathbb{S}^3_+$ if and only if

$$\Lambda_\theta = 0, \textit{ and } \Lambda_\sigma, \Lambda_{1-\sigma}, \Lambda_{\sigma(1-\sigma)} \textit{ are spf's.} \qquad \text{(P)}$$

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A Moment Problem with Constraints

Let

$$\Sigma = \{(s, t, u) \in \mathbb{S}^3_+; 0 \le s < 1\},$$

which is measurable, but noncompact.

Problem. Characterize those unital square positive functionals Λ on $\mathcal{P}^3$, having a representing measure with support in the set Σ, such that all functions $(1-s)^{-m} (m \ge 1$ an integer) are integrable.

Of course, the requirement on the integrability of the functions $(1-s)^{-m} (m \ge 1)$, makes Schmüdgen's theorem invalid, in the actual form. Nevertheless, this theorem remains a useful tool, as an auxiliary result.

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Necessity of Condition (P)

Remark A solution to the previous Problem is, in particular, a solution of the $\mathbb{S}^3_+$-moment problem concerning a uspf Λ. For this reason, the condition (P) is **necessary.**

To solve the Problem, we need a condition stronger than (P). In the sequel we shall present such a condition.

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A Restriction

From now on, let $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$ be a uspf with the property (P). This implies that $\Lambda(q) = 0$ for each polynomial q with $q|\mathbb{S}^3_+ = 0$, via Schmüdgen's theorem.

We denote by $\mathcal{P}^3(\mathbb{S}^3_+)$ the algebra consisting of all (classes of) functions of the form $p|\mathbb{S}^3_+$, $p \in \mathcal{P}^3$, modulo the ideal of those polynomials q with $q|\mathbb{S}^3_+ = 0$. This allows us to define correctly the map $\Lambda_+ : \mathcal{P}^3(\mathbb{S}^3_+) \mapsto \mathbb{C}$ by the formula

$$\Lambda_+(p|\mathbb{S}^3_+) = \Lambda(p), \; p \in \mathcal{P}^3,$$

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A Useful Formula

To give a solution to the Problem, we should first extend the map Λ_+ to the algebra $\mathcal{R}(\Sigma)$ generated by the rational functions $s^j t^k u^l (1-s)^{-m}$ restricted to Σ, where j, k, l, m are nonnegative integers.

First of all, we note the formula

$$\frac{1}{(1-s)^{m+1}} = \sum_{r \geq m} \binom{r}{m} s^{r-m}, \qquad (1)$$

valid for all integers $m \geq 0$, where the series is convergent at each point $s \in [0,1)$.

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A Necessary Condition

The series (1) suggests the following supplementary hypothesis on Λ:

Condition. *Setting*

$$p_{m,n}(s) = \sum_{r=m}^{n} \binom{r}{m} s^{r-m}, \tag{2}$$

for all nonnegative integers $m, n\,(n \geq m)$ *and* $s \in [0, 1)$, *we assume that*

$$\lim_{n_1, n_2 \to \infty} \Lambda(|p_{m,n_1} - p_{m,n_2}|^2) = 0 \tag{L}$$

for all $m \geq 0$.

Remark

Condition (L), expressed only in terms of the given map Λ, is necessary via the Lebesgue theorem of dominated convergence.

Sufficiency of Condition (L): Step 1

We shall prove in the following that condition (L) is also sufficient.
Using (L), for each element $p \in \mathcal{P}(\mathbb{S}^3_+)$ and every integer $m \geq 0$, we may define

$$\tilde{\Lambda}(pr_m) = \lim_{n\to\infty} \Lambda(pp_{m,n}), \tag{3}$$

where $r_m(s) = (1-s)^{-m}$. Note that the limit exists via the Cauchy-Schwarz inequality. Moreover,

$$\tilde{\Lambda}(pr_{m_1}) = \tilde{\Lambda}((1-\sigma)^{m_2-m_1} pr_{m_2}) \tag{4}$$

if $m_2 \geq m_1$.

Step 2

Let now $p_1, p_2 \in \mathcal{P}(\mathbb{S}^3_+)$, and let m_1, m_2 be nonnegative integers such that $r_{m_2}^{-1} p_1 - r_{m_1}^{-1} p_2 = q$, where $q|\mathbb{S}^3_+ = 0$. Assuming, with no loss of generality, that $m_2 \geq m_1$, we infer
$p_2 = (1-\sigma)^{m_2 - m_1} p_1 - q r_{m_1}$. This relation also shows that $q r_{m_1}$ is a polynomial, which is null on $\mathbb{S}^3_+$. Therefore, via (4),

$$\lim_{n \to \infty} \Lambda(p_2 p_{m_2,n}) = \lim_{n \to \infty} \Lambda(p_1 p_{m_1,n}).$$

Consequently,

$$\tilde{\Lambda}(p_2 r_{m_2}) = \tilde{\Lambda}(p_1 r_{m_1}) \quad \text{if} \quad (r_{m_2}^{-1} p_1 - r_{m_1}^{-1} p_2)|\mathbb{S}^3_+ = 0. \qquad (5)$$

Step 3

Relation (5) shows that $\tilde{\Lambda}$ induces a map on the algebra of fractions $\mathcal{F}(\Sigma)$ build from the algebra $\mathcal{P}^3(\mathbb{S}^3_+)$, with denominators in the set $\mathcal{S} = \{(1-s)^m; m \geq 0\}$. This map, denoted by $\tilde{\Lambda}_+$, is given by

$$\tilde{\Lambda}_+(p(1-\sigma)^{-m}|\Sigma) = \lim_{n\to\infty} \Lambda(pr_{m,n}),\ \ p \in \mathcal{P}^3,\ \ m \geq 0,$$

which clearly extends the map Λ_+.

Step 4

The map $\tilde{\Lambda}_+ : \mathcal{F}(\Sigma) \mapsto \mathbb{C}$ is a uspf. Indeed, fixing $f = p/(1-\sigma)^m|\Sigma$, we have, via the properties of Λ,

$$\tilde{\Lambda}_+(\bar{f}) = \lim_{n\to\infty} \Lambda(\bar{p}p_{m,n}) = \overline{\Lambda(f)},\ \Lambda(|f|^2) = \lim_{n\to\infty} \Lambda(|f|^2 p_{2m,n}) \geq 0,$$

$$\tilde{\Lambda}_\sigma(|f|^2) = \lim_{n\to\infty} \Lambda(\sigma|f|^2 p_{2m,n}) \geq 0, \tag{6}$$

$$\tilde{\Lambda}_{1-\sigma}(|f|^2) = \lim_{n\to\infty} \Lambda((1-\sigma)|f|^2 p_{2m,n}) \geq 0,$$

where $\tilde{\Lambda}_\sigma(f) = \tilde{\Lambda}_+(\sigma f)$, and similar relations for $\tilde{\Lambda}_{1-\sigma}$.

Step 5

In particular, the map $\tilde{\Lambda}_+ : \mathcal{F}(\Sigma) \mapsto \mathbb{C}$ satisfies the Cauchy-Schwartz inequality, and so the set

$$\mathcal{I}_\Lambda = \{f \in \mathcal{F}(\Sigma); \Lambda(|f|^2) = 0\}$$

is an ideal in the algebra $\mathcal{F}(\Sigma)$.

Moreover, the assignment $(f, g) \mapsto, \tilde{\Lambda}_+(f\bar{g})$ induces an inner product on the quotient $D_0 = \mathcal{F}(\Sigma)/\mathcal{I}_\Lambda$.

The completion of the quotient $D_0 = \mathcal{F}(\Sigma)/\mathcal{I}_\Lambda$ with respect to the inner product $(f, g) \mapsto, \tilde{\Lambda}_+(f\bar{g})$ is a Hilbert space denoted by $\mathcal{H}$.

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Step 6

We now consider in $\mathcal{H}$ the multiplication operators B_0, C_0 induced by the functions $-t/(1-s)$ and $u/(1-s)$, respectively, defined on D_0. In other words,

$$B_0 f = \left(\frac{-t}{1-s}\right) f,$$

$$C_0 f = \frac{u}{1-s} f,$$

for all $f \in D_0$. Clearly, B_0, C_0 are densely defined, leave invariant the space D_0 and commute.

A Matricial Notation

To continue our investigation, we need some ingredients fro the theory of quaterninic Cayley transform.
We use the notation

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which act as operators on $\mathcal{H} \oplus \mathcal{H}$.
We also set $\mathbf{E} = i\mathbf{J}$, and denote by $\mathbf{I}$ the identity on $\mathcal{H} \oplus \mathcal{H}$.

E-Cayley Transform

We denote by $R(T)$ the range of a given operator T.

Definition Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric. Then we may correctly define the operator

$$V : R(S+\mathbf{E}) \mapsto R(S-\mathbf{E}), \quad V(S+\mathbf{E})x = (S-\mathbf{E})x, \quad x \in D(S),$$

which is a partial isometry.

In other words, $V = (S-\mathbf{E})(S+\mathbf{E})^{-1}$, defined on $D(V) = R(S+\mathbf{E})$.

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Properties of the **E**-Cayley Transform

We recall the properties of the **E**-Cayley Transform.

Theorem 1

The **E**-Cayley transform is an order preserving bijective map assigning to each operator S with $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ and $\mathbf{J}S$ symmetric a partial isometry V in in $\mathcal{H}^2$ with $\mathbf{I} - V$ injective.
Moreover:
(1) V is closed if and only if S is closed;
(2) the equality $V^{-1} = -\mathbf{K}V\mathbf{K}$ holds if and only if the equality $S\mathbf{K} = \mathbf{K}S$ holds;
(3) $\mathbf{J}S$ is self-adjoint if and only if V is unitary on $\mathcal{H}^2$.

Step 7

Coming back to our notation, we set

$$S_0 = B_0\mathbf{I} + C_0\mathbf{K}$$

defined on $D_0 \oplus D_0$. In fact,

$$S_0 = \frac{1}{1-s}\begin{pmatrix} -t & u \\ -u & -t \end{pmatrix}$$

Then $\mathbf{J}S_0$, givn by

$$\mathbf{J}S_0 = \frac{1}{1-s}\begin{pmatrix} -t-u & -u \\ -u & -t+u \end{pmatrix},$$

is symmetric on $D_0 \oplus D_0$.

E-Cayley Transform of some Matrices

Let $a, b \in \mathbb{R}$, and let $S = a\mathbf{I} + b\mathbf{K}$. A direct calculation shows that the **E**–Cayley transform of S is given by

$$U = (a^2 + b^2 + 1)^{-1}((a^2 + b^2 - 1)\mathbf{I} - 2ai\mathbf{J} + 2bi\mathbf{L}) =$$

$$\frac{1}{a^2 + b^2 + 1}\begin{pmatrix} a^2 + b^2 - 1 - 2ai & 2bi \\ 2bi & a^2 + b^2 - 1 + 2ai \end{pmatrix}$$

Step 8

We apply the previous formula to

$$S_0 = B_0\mathbf{I} + C_0\mathbf{K}$$

with $a = -t(1-s)^{-1}$, and $b = u(1-s)^{-1}$. Hence, denoting by U_0 the **E**-Cayley transform of S_0, a direct computation shows that U_0 is the matrix multiplication operator

$$U_0 = \begin{pmatrix} s+it & iu \\ iu & s-it \end{pmatrix},$$

defined on $R(S_0 + \mathbf{E})$.

Step 9

Note that, for every pair $g_1, g_2 \in D_0$, the system

$$\left(\frac{-t}{1-s} + i\right) f_1 + \frac{u}{1-s} f_2 = g_1 \qquad (7)$$

$$\frac{-u}{1-s} f_1 + \left(\frac{-t}{1-s} - i\right) f_2 = g_2$$

has the solution

$$f_1 = -2^{-1}((t + i - is)g_1 + ug_2),$$
$$f_2 = 2^{-1}(ug_1 - (t - i + is)g_2),$$

via the equality $s^2 + t^2 + u^2 = 1$.
Consequently $f_1, f_2 \in D_0$.

Step 10

Then the system (7) is precisely the equation

$$(S_0 + \mathbf{E})(f_1 \oplus f_2) = g_1 \oplus g_2,$$

showing that $R(S_0 + \mathbf{E})$ is equal to $D_0 \oplus D_0$.
Hence, if U_0 the **E**-Cayley transform of S_0, the previous discussion shows that the matrix multiplication operator

$$U_0 = \begin{pmatrix} s+it & iu \\ iu & s-it \end{pmatrix},$$

is defined on the space $D_0 \oplus D_0$, which is clearly invariant under U_0.

A Special Class of Operators

Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$, with $D(S) = D_0 \oplus D_0,\ D_0 \subset \mathcal{H}$. The equality $D(S) = D_0 \oplus D_0$ is equivalent to the inclusions
(i) $\mathbf{J}D(S) \subset D(S)$ and $\mathbf{K}D(S) \subset D(S)$.

In order to have a normal extension of S (with some convenient properties to be later mentioned), the following conditions are necessary:
(ii) $\mathbf{J}S$ is symmetric;
(iii) $S\mathbf{K} = \mathbf{K}S$;
(iv) $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$.

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Step 11

The operator S_0, previously defined, has the properties (i)-(iv). One can verify that the closure S of S_0 has similar properties. If U is the **E**-Cayley transform of S, then U should be closed. As U extends U_0, U must be a unitary operator on $\mathcal{H}^2$. Moreover, $\mathbf{I} - U$ is injective, as a **E**-Cayley transform, via Theorem 1.

Unitaries as Inverse **E**-Cayley Transforms

Theorem 2 Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be the inverse **E**–Cayley transform of U. The operator S is normal if and only if $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.

Any operator U as in Theorem 2 has necessarily the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal and A self-adjoint in $\mathcal{H}$, such that $TT^* + A^2 = I$ and $AT = TA$

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Step 12

Let T, A be the operators associated to U, via Theorem 2.

In fact, the operator T is an extension of the multiplication by $s + it$ on D_0, and the operator A is an extension of the multiplication by u on D_0

Since $\mathbf{I} - U$ is injective, the operator $I - \mathrm{Re}(T)$ must be also injective.

Step 13: A Joint Spectral Measure

Because the operators T, A are commuting normal operators, they must have a joint spectral measure in $\mathbb{C}^2$.

If E is the joint spectral measure of the pair (T, A), then E must be concentrated on the sphere S^3. Indeed, if $\mathcal{A}$ is the unital (commutative) C^*-algebra generated by T and A, the equality $T^*T + A^2 = I$ shows that the joint spectrum of the pair (T, A) may be identified with a compact subset of the sphere $\mathbb{S}^3$.

Step 14

We can refine the conclusion of the previous Step.
Because $0 \leq \mathrm{Re}(T) \leq I$, which is implied by the properties of the square positive forms $\tilde{\Lambda}_\sigma$ and $\tilde{\Lambda}_{1-\sigma}$ given by (6), it results that the measure E is concentrated in the set $\mathbb{S}^3_+$. As the operator $I - \mathrm{Re}(T)$ is injective, it follows that $E(\{(1,0,0)\}) = 0$. Consequently, the measure E is supported by the set Σ.

Step 15

Since $1 + \mathcal{I}_\Lambda = (I - \mathrm{Re}(T))^m((1-\sigma)^{-m} + \mathcal{I}_\Lambda)$, it follows that $1 + \mathcal{I}_\Lambda$ is in the domain of $(I - \mathrm{Re}(T))^{-m}$ for all integers $m \geq 1$. Therefore, setting $\mu(*) = \langle E(*)(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle$, we obtain

$$\Lambda(pr_m) = \langle pr_m + \mathcal{I}_\Lambda, 1 + \mathcal{I}_\Lambda\rangle =$$

$$\langle (p(\mathrm{Re}(T), \mathrm{Im}(T), A)(I - \mathrm{Re}(T))^{-m}(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle = \int_\Sigma pr_m d\mu,$$

for all $f = pr_m \in \mathcal{F}(\Sigma)$, showing that μ is a representing measure for $\Lambda : \mathcal{F}(\Sigma) \mapsto \mathbb{C}$.

Last Step

Finally

$$\int_{\Sigma}(1-s)^{-2m}d\mu = \|(I-\mathrm{Re}(T))^{-2m}(1+\mathcal{I}_{\Lambda})\|^2 < \infty,$$

for all integers $m \geq 1$, which completes our assertion.

Conclusion

Summerizing all steps of the discussion, we obtain the following statement:

Theorem 3

Let $\Lambda : \mathcal{P}^3 \mapsto \mathbb{C}$ be a unital square positive map, and let

$$\Sigma = \{(s, t, u) \in \mathbb{S}^3; 0 \leq s < 1\}.$$

There exists a uniquely determined positive measure on Σ such that all functions $(1 - s)^{-m} (m \geq 1$ an integer) are integrable if and only if conditions (P) and (L) are fulfilled.

Reference

More details concerining these results can be found in the author's paper

Quaternionic Cayley Transform Revisited
J. Math. Anal. Appl. 409 (2014) 790–807

Extensions of the results in the previous paper to the case of linear relations can be found in the work

Normal extensions of subnormal linear relations via quaternionic Cayley transforms (with A. Sandovici), *Monatsh. Math.* 170 (2013), no. 3-4, 437–463.

Thank you very much for your attention !