An Idempotent Approach to Truncated Moment Problems

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Outline

Truncated Moment Problems

The study of truncated moment problems means, roughly speaking, that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a given length $n \geq 1$ and $m \geq 0$ is an integer, one looks for a positive measure μ on $\mathbb{R}^n$ such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \leq 2m$. As Tchakaloff firstly proved, if such a measure exists, we may always assume it to be atomic.

Framework

Let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on a topological space Ω. We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space* (on Ω). Occasionally, we use the notation $\mathcal{RS}$ to designate the "real part" of $\mathcal{S}$, that is $\{f \in \mathcal{S}; f = \bar{f}\}$.

Let also $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

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Unital Square Positive Functionals

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$.
(3) $\Lambda(1) = 1$.

A linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.
When $\mathcal{S}$ is an algebra, conditions (2) and (3) imply condition (1). In this case, a map Λ with the property (2) is usually said to be *positive (semi)definite*.
Condition (3) may be replaced by $\Lambda(1) > 1$ but (looking for probability measures representing such a functional) we always assume (3) in the stated form, without loss of generality.

Elementary Properties

If $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ p, q \in \mathcal{S}. \tag{1}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle f + \mathcal{I}_\Lambda, g + \mathcal{I}_\Lambda \rangle = \Lambda(f\bar{g}). \tag{2}$$

Note that, in fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\, \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$.
If $\mathcal{S}$ is finite dimensional, then $\mathcal{H}_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

Framework Again

Let $n \geq 1$ will be a fixed integer. We freely use multi-indices from $\mathbb{Z}_+^n$ and the standard notation related to them.
The symbol $\mathcal{P}$ will designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients.
For every integer $m \geq 1$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where deg(p) is the total degree of p. Note that $\mathcal{P}_m^{(2)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

Giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$, $\gamma_0 = 1$, we associate it with a map $\Lambda_\gamma : \mathcal{P}_{2m} \mapsto \mathbb{C}$ given by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, extended to $\mathcal{P}_{2m}$ by linearity. The map Λ_γ is called the *Riesz functional associated to* γ.

We clearly have $\Lambda_\gamma(1) = 1$ and $\Lambda_\gamma(\bar{p}) = \overline{\Lambda_\gamma(p)}$ for all $p \in \mathcal{P}_{2m}$. If, moreover, $\Lambda_\gamma(|p|^2) \ge 0$ for all $p \in \mathcal{P}_m$, then Λ_γ is a uspf. In this case, we say that γ itself is *square positive*.

Conversely, if $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf, setting $\gamma_\alpha = \Lambda(t^\alpha)$, $|\alpha| \le 2m$, we have $\Lambda = \Lambda_\gamma$, as above. The multi-sequence γ is said to be the *multi-sequence associated to the uspf* Λ.

Introducing idempotents

Let $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbb{R}^n$ and let $\ell^\infty(\Xi)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ξ, endowed with the sup-norm. For every integer $m \geq 0$ we have the restriction map $\mathcal{P}_m \ni p \mapsto p|\Xi \in \ell^\infty(\Xi)$. Let us fix an integer m for which this map is surjective. (Such an m always exists via the Lagrange or other interpolation polynomials.) Let also $\mu = \sum_{j=1}^{d} \lambda_j \delta_{\xi^{(j)}}$, with $\lambda_j > 0$ for all $j = 1, \dots, d$ and $\sum_{j=1}^{d} \lambda_j = 1$. We put $\Lambda(p) = \int_\Xi p d\mu$ for all $p \in \mathcal{P}_{2m}$, which is a uspf, for which μ is a representing measure.

Let now $f \in \ell^\infty(\Xi)$ be an idempotent, that is, the caracteristic function of a subset of Ξ. Then there exists a polynomial $p \in \mathcal{P}_m$, supposed to have real coefficients, such that $p|\Xi = f$. Consequently, $\Lambda(p^2) = \int_\Xi p^2 d\mu = \int_\Xi p d\mu = \Lambda(p)$. This shows that the solutions the equation $\Lambda(p^2) = \Lambda(p)$, which can be expressed only in terms of Λ, play an important role when trying to reconstruct the representing measure μ.

This remark is the starting point of our approach to truncated moment problems.
Idempotents (with respect to a given uspf Λ) will be objects related to the solutions of the equation $\Lambda(p^2) = \Lambda(p)$, where where p is a polynomial with real coefficients.
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General Integral Representations

For a complex vector space $\mathcal{V}$, we denote by $\mathcal{V}^*$ its (algebraic) dual. First of all, we extend the concept of representing measure to arbitrary functionals from $\mathcal{V}^*$. In fact, this is a sort of demystification of the concept of representing measure.

Definition 1 We say that $\phi \in \mathcal{V}^*$ has an *integral representation* on a subset $\Delta \subset \mathcal{V}^*$ if there exists a probability measure μ on Δ such that

$$\phi(x) = \int_\Delta \delta(x) d\mu(\delta), \ x \in \mathcal{V}.$$

The measure μ is said to be a *representing measure* for the functional ϕ. The measure μ is said to be *d-atomic* if the support of μ consists of d distinct points in Δ.

Such integral representations can be easily obtained for functionals on finite dimensional vector spaces.

Theorem 1 If $\mathcal{V}$ is a finite dimensional complex vector space, then every nonnull functional from $\mathcal{V}^*$ has a d-atomic integral representation, where d is the dimension of $\mathcal{V}$.

Sketch of proof Let $\phi \in \mathcal{V}^*$, and let $\iota \in \mathcal{V}$ be such that $\phi(\iota) = 1$. There exists a basis $\{b_1, \ldots, b_d\}$ of $\mathcal{V}$ such that $\phi(b_j) > 0$ for all $j = 1, \ldots, d$, and $\iota = b_1 + \cdots + b_d$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\} \subset \mathcal{V}^*$ be the dual basis. We may carry the C^*-algebra structure of $\ell^\infty(\Delta)$ onto $\mathcal{V}$ and get the formula

$$\phi(x) = \sum_{j=1}^{d} \lambda_j \delta_j(x) = \int_\Delta \delta(x) d\mu(\delta)), \; x \in \mathcal{V},$$

where $\lambda_j = \phi(b_j) > 0$ for all $j = 1, \ldots, d$ and $\phi(\iota) = 1 = \lambda_1 + \cdots + \lambda_d$. Therefore, μ is a d-atomic probability measure on Δ, with weights λ_j at δ_j, $j = 1, \ldots, d$.

An Integral Representation Theorem

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Theorem 1 shows that every linear functional on a finite dimensional space has an integral representation via a probability measure, for some C^*-algebra structure of the ambient space, depending upon the given functional. We can refine the previous construction, relating it to a preexistent multiplicative structure.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\}$, and let $\mathcal{H}_\Lambda = \mathcal{P}_m/\mathcal{I}_\Lambda$, which has a Hilbert space structure induced by Λ. We denote $\langle *, * \rangle$, $\| * \|$, the inner product and the norm induced on $\mathcal{H}_\Lambda$ by Λ, respectively. For every $p \in \mathcal{P}_m$, we put $\hat{p} = p + \mathcal{I}_\Lambda \in \mathcal{H}_\Lambda$. When $\hat{p} \in \mathcal{H}_\Lambda$, we freely choose a fixed representative p.

The symbol $\mathcal{RH}_\Lambda$ will designate the set $\{\hat{p} \in \mathcal{RH}_\Lambda; p - \bar{p} \in \mathcal{I}_\Lambda\}$, that is, the set of “real” elements from $\mathcal{RH}_\Lambda$. If $\hat{p} \in \mathcal{RH}_\Lambda$, we always choose $p \in \mathcal{RP}_m$.

Definition of Idempotents

Definition 2 An element $\hat{p} \in \mathcal{RH}_\Lambda$ is said to be *idempotent* if it is a solution of the equation $\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle$.
Remark (i) Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is idempotent if and only if $\Lambda(p^2) = \Lambda(p)$, via relation (2). Set

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{3}$$

which the family of nonnull idempotent elements from $\mathcal{RH}_\Lambda$. This family is nonempty because $\hat{1} \in \mathcal{ID}(\Lambda)$.
Note that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.
(ii) If $m_1 \leq m_2$ and $\Lambda_2 : \mathcal{P}_{2m_2} \mapsto \mathbb{C}$ is a uspf, then $\Lambda_1 = \Lambda_2 | \mathcal{P}_{2m_2}$, which is obviously a uspf, has the property $\mathcal{ID}(\Lambda_1) \subset \mathcal{ID}(\Lambda_2)$. Indeed, since $\mathcal{I}_{\Lambda_1} \subset \mathcal{I}_{\Lambda_2}$ and $\mathcal{P}_{m_1} \cap \mathcal{I}_{\Lambda_2} = \mathcal{I}_{\Lambda_1}$, $\mathcal{H}_{\Lambda_1}$ can be isometrically embedded into $\mathcal{H}_{\Lambda_2}$. Thus $\mathcal{H}_{\Lambda_1}$ may be regarded as a subspace of $\mathcal{H}_{\Lambda_2}$.

Some Lemmas

Lemma 2 (1) If $\hat{p}, \hat{q}, \hat{p} - \hat{q} \in \mathcal{ID}(\Lambda)$, then $\hat{q}$ and $\hat{p} - \hat{q}$ are orthogonal.
(2) If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, and $\hat{q}$, $\hat{1} - \hat{q}$ are orthogonal.
(3) If $\hat{p}, \hat{q} \in \mathcal{ID}(\Lambda)$ are orthogonal, then $\hat{p} + \hat{q} \in \mathcal{ID}(\Lambda)$.

Lemma 3 Let $\{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$, consistig of mutually orthogonal elements. If the family $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is maximal with respect to the inclusion, then $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$.

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Abstract Idempotent Equation

We are interested inthe existence of the orthogonal families of idempotents with respect to a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$.
It is easily checked that $p \in \mathcal{RP}_m$, $p = \sum_{|\xi| \leq m} c_\xi t^\xi$, is a solution of the equation $\Lambda(p^2) = \Lambda(p)$ if and only if

$$\sum_{|\xi|,|\eta| \leq m} \gamma_{\xi+\eta} c_\xi c_\eta - \sum_{|\xi| \leq m} \gamma_\xi c_\xi = 0,$$

where $\gamma = (\gamma_\xi)_{|\xi| \leq 2m}$ is the finite multi- sequence associated to Λ.
To study the existence of solutions for such an equation, it is convenient to use at the beginning an abstract framework.

Let $N \geq 1$ be an arbitrary integer, let $A = (a_{jk})_{j,k=1}^{N}$ be a matrix with real entries, that is positive on $\mathbb{C}^N$ (endowed with the standard scalar product denoted by $(*|*)$, and associated norm $\| * \|$), and let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$. We look for necessary and sufficient conditions insuring the existence of a solution $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ of the equation

$$(Ax|x) - 2(b|x) = 0. \tag{4}$$

The particular case which interests us will be dealt with in the following.
The range and the kernel of A, regarded as an operator on $\mathbb{C}^N$, will be denoted by $R(A)$, $N(A)$, respectively. Note also that $R(A) = R(B)$, and $N(A) = N(B)$, where $B = A^{1/2}$

We are interested by the following particular case. Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}$ the multi-sequence associated to Λ. Then $A_\Lambda = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \le m}$ is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n ; |\xi| \le m\}$. In fact, by identifying the space $\mathcal{P}_m$ with $\mathbb{C}^N$ via the isomorphism

$$\mathcal{P}_m \ni p_x = \sum_{|\alpha| \le m} x_\alpha t^\alpha \mapsto x = (x_\alpha)_{|\alpha| \le m} \in \mathbb{C}^N, \tag{5}$$

then $A = A_\Lambda$ is the operator with the property $(Ax|y) = \Lambda(p_x \bar{p}_y)$ for all $x, y \in \mathbb{C}^N$. The operator A will be occasionally called the *Hankel operator* of the uspf Λ. Note that $\mathcal{I}_\Lambda$ is isomorphic to $N(A)$, and $\mathcal{H}_\Lambda$ is isomorphic to $R(A)$, via the isomprphism (5). Note also that the elements $\hat{p}_x, \hat{p}_y$ are orthogonal in $\mathcal{H}_\Lambda$ if and only if $(Ax|y) = (Bx|By) = 0$.

Let us deal with equation (4) in this particular context. Set $2b = (\gamma_\xi)_{|\xi| \le m} \in \mathbb{R}^N$. With this notation, equation (4) will be called the *idempotent equation* of the uspf Λ.
Because $\Lambda(p_x^2) = (Ax|x) = 0$ implies $\Lambda(p_x) = 2(b|x) = 0$, we are intrested only in solutions $x = x^{(1)} \in R(A) = R(A_1)$, where $A_1 = A|R(A)$. Note also that $b = b^{(1)} \in R(A)$, because $2b = A\iota$, where $\iota = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $p_\iota = 1$. Therefore, $(A\iota|\iota) - 2(b|\iota) = 0$, and so the vector ι is always a nonnull solution of the idempotent equation.

Proposition Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $A : \mathbb{C}^N \mapsto \mathbb{C}^N$ be the associated Hankel operator.
The nonnull solutions of the idempotent equation of Λ in $R(A) \cap \mathbb{R}^n$ are given by

$$x^{(1)} = B_1^{-1}(y^{(1)} + B_1^{-1}b),\ y^{(1)} \in R(A) \cap \mathbb{R}^N,\ \|y^{(1)}\| = \|B_1^{-1}b\|, \tag{6}$$

except for $y^{(1)} = -B_1^{-1}b$. In addition, tha assignment $y^{(1)} \mapsto x^{(1)}$ is one-to one.
The idempotent equation of Λ has only one nonnull solution in $R(A) \cap \mathbb{R}^n$ if and only if $\dim R(A) \cap \mathbb{R}^n = 1$.
If $d := \dim R(A) \cap \mathbb{R}^n > 1$, there exists a family $\{x_1^{(1)}, \ldots, x_d^{(1)}\}$ of solutions in $R(A) \cap \mathbb{R}^n$ of the idempotent equation of Λ such that the vectors $\{B_1 x_1^{(1)}, \ldots, B_1 x_d^{(1)}\}$ are mutually orthogonal in $R(A)$.

Corollary Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf such that the associated Hankel operator A is invertible. The nontrivial solutions of the idempotent equation of Λ are given by

$$x = B^{-1}y + \frac{1}{2}\iota, \ y \in \mathbb{R}^N, \ \|y\| = \frac{1}{2}\|B\iota\|,$$

except for $y = -\frac{1}{2}B\iota$.
The idempotent equation of Λ has only one nonnull solution if and only if $m = 0$.
If $d := \dim \mathcal{P}_m > 0$, there exists a family $\{x_1, \ldots, x_d\}$ of solutions of the idempotent equation of Λ such that the vectors $\{Bx_1, \ldots, Bx_d\}$ are mutually orthogonal.

Remark Using (5), we deduce the existence of an ortogonal basis $\{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}_\Lambda$, consisting of idempotent elements. Specifically, if $\{x_1^{(1)}, \ldots, x_d^{(1)}\}$ is a family of solutions in $R(A) \cap \mathbb{R}^n$ of the idempotent equation of Λ with $\{B_1 x_1^{(1)}, \ldots, B_1 x_d^{(1)}\}$ mutually orthogonal, and if $\{\hat{b}_1, \ldots, \hat{b}_d\}$ are the corresponding vectors from $\mathcal{H}_\Lambda$ obtained via (5), then $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is a basis of the space $\mathcal{H}_\Lambda$, which is isomprphic to $R(A)$. In addition, as we have $(Ax_j^{(1)}|x_k^{(1)}) = 0$ for all $j \neq k$, $j, k = 1, \ldots, d$, the elements $\{\hat{b}_1, \ldots, \hat{b}_d\}$ are mutually orthogonal in $\mathcal{H}_\Lambda$.

Theorem 2 For every $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ uspf there exist orthogonal bases of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements.

Corollary Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m > 0)$ be a uspf such that the associated Hankel operator A is invertible. Then there exists a basis $\{b_1, \ldots, b_d\}$ of $\mathcal{P}_m$, consisting of polynomials with real coefficients, such that $\Lambda(b_j b_k) = 0$ for all $j \neq k$, $j, k = 1, \ldots, d$, where $d = \dim \mathcal{P}_m$.

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Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

acting as an operator on $\mathbb{C}^3$, which is positive.
We are interested in the solutions of the idempotent equation $(A\mathbf{x}|\mathbf{x}) = (\iota|\mathbf{x})$, where $\iota = (1,0,0)$. It is easily seen that $N(A) = \{(x,-x,0); x \in \mathbb{C}\}$, $R(A) = \{(y,y,y+z); y,z \in \mathbb{C}\}$.
Looking only for solutions $(y,y,y+z) \in R(A)$, the idempotent equation is given by

$$10y^2 + 8yz + 2z^2 - 3y - z = 0,$$

which represents an ellipse passing through the origin.

Remark According to Theorem 2, the space $\mathcal{H}_\Lambda$ has orthogonal bases consisting of idempotent elements. If $\mathcal{B}$ is such a basis, we may speak about the C^*-algebra structure of $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, in the spirit of Theorem 1. More generally, if $\mathcal{B} \subset \mathcal{ID}(\Lambda)$ is a collection of mutually orthogonal elements whose sum is $\hat{1}$, and if $\mathcal{H}_\mathcal{B}$ is the complex vector space generated by $\mathcal{B}$ in $\mathcal{H}_\Lambda$, we may speak about the C^*-algebra structure of $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$. Using the basis $\mathcal{B}$ of the space $\mathcal{H}_\mathcal{B}$, we may construct a multiplication, an involution, and a norm on $\mathcal{H}_\mathcal{B}$, making it a unital, commutative, finite dimensional C^*-algebra. For instance, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ with $\hat{1} = \sum_{j=1}^d \hat{b}_j$, and if $\hat{p} = \sum_{j=1}^d \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^d \beta_j \hat{b}_j$, are elements from $\mathcal{H}_\mathcal{B}$, their product is given by $\hat{p} \cdot \hat{q} = \sum_{j=1}^d \alpha_j \beta_j \hat{b}_j$,.

Proposition 2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be a collection of mutually orthogonal elements with $\hat{1} = \sum_{j=1}^{d} \hat{b}_j$, and let $\mathcal{H}_{\mathcal{B}}$ be the complex vector space generated by $\mathcal{B}$ in $\mathcal{H}_\Lambda$.
Let Δ be the space of characters of the C^*-algebra $\mathcal{H}_{\mathcal{B}}$, induced by $\mathcal{B}$. If $\mathcal{S}_{\mathcal{B}} = \{p \in \mathcal{P}_m; \hat{p} \in \mathcal{H}_{\mathcal{B}}\}$, there exists a linear map $\mathcal{S}_{\mathcal{B}} \ni p \mapsto p^\# \in \ell^\infty(\Delta)$ such that

$$\Lambda(u) = \int_\Delta p^\#(\delta) d\mu(\delta), \; p \in \mathcal{S}_{\mathcal{B}},$$

where μ is a d-atomic probability measure on Δ.

Proposition 3 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements. Let also $\mathcal{H}_\mathcal{C}$ be the sub-C^*-algebra generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ in $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal is $\leq \dim \mathcal{H}_\Lambda$, and a linear map $\mathcal{S}_\mathcal{C} \ni u \mapsto u^\# \in \ell^\infty(\Xi)$, such that

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi),\ \ u \in \mathcal{S}_\mathcal{C},$$

where $\mathcal{S}_\mathcal{C} = \{p \in \mathcal{P}_m; \hat{p} \in \mathcal{H}_\mathcal{C}\}$ and μ is a probability measure on Ξ.

Remark Assume that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ given by

$$\Lambda(p) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ \ p \in \mathcal{P}_{2m},$$

with $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^{d} \lambda_j = 1$, where $d = \mathrm{dim}\mathcal{H}_\Lambda$.
Let $r \geq m$ be an integer such that $\mathcal{P}_r$ contains interpolating polynomials for the family of points $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Setting $\Lambda_\mu(p) = \int_\Xi p d\mu$, $p \in \mathcal{P}_{2r}$, we have $\Lambda_\mu | \mathcal{P}_{2m} = \Lambda$, and $\mathcal{I}_{\Lambda_\mu} = \{p \in \mathcal{P}_r; p|\Xi = 0\}$, as one can easily see. Moreover, the space $\mathcal{H}_r := \mathcal{P}_r / \mathcal{I}_{\Lambda_\mu}$ is at least linearly isomorphic to $\ell^\infty(\Xi)$, where $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, via the map $\mathcal{H}_r \ni p + \mathcal{I}_{\Lambda_\mu} \mapsto p|\Xi \in \ell^\infty(\Xi)$.

As $\mathcal{H}_\Lambda$ may be regarded as a subspace of $\mathcal{H}_r$, and $\mathrm{dim}\mathcal{H}_\Lambda = \mathrm{dim}\ell^\infty(\Xi)$, the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in \ell^\infty(\Xi)$ is a linear isomorphism. Let $\chi_k \in \ell^\infty(\Xi)$ be the characteristic function of the set $\{\xi^{(k)}\}$ and let $\hat{b}_k \in \mathcal{H}_\Lambda$ be the element with $b_k|\Xi = \chi_k$, $k = 1, \ldots, d$. Note that

$$\Lambda(b_k^2) = \lambda_k(b_k^2)(\xi^{(k)}) = \lambda_k(b_k)(\xi^{(k)}) = \Lambda(b_k), \ , k = 1, \ldots .d$$

Similarly, $\Lambda(b_k b_l) = 0$ for all $k, l = 1, \ldots, d$, $k \neq l$. This shows that $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is a basis of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents. Consequently, if $\mathcal{H}_\Lambda$ is given the C^*-albebra structure induced by $\{\hat{b}_1, \ldots, \hat{b}_d\}$, then $\mathcal{H}_\Lambda$ and $\ell^\infty(\Xi)$ are isomorphic as C^*-algebras. Note also that $\Lambda(b_j) = \lambda_j$ for all $j = 1, \ldots, d$, and that if $\hat{p} = \alpha_1\hat{b}_1 + \cdots + \alpha_d\hat{b}_d \in \mathcal{H}_\Lambda$ is arbitrary, then $\alpha_j = \Lambda(pb_j) = \lambda_j p(\xi^{(j)})$ for all $j = 1, \ldots, d$.

Theorem 3 The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ having $d = \dim \mathcal{H}_\Lambda$ atoms if and only if there exists orthogonal basis $\mathcal{B}$ of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements such that $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \leq m$ and δ is a character of the C^*-algebra $\mathcal{H}_\Lambda$ associated to $\mathcal{B}$, where $\hat{t} = (\hat{t}_1, \ldots, \hat{t}_n)$.

Corollary The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ having $d = \dim \mathcal{H}_\Lambda$ atoms if and only if there exists orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements such that

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Example The matrix A from the previous Example is the Hankel matrix associated to the uspf $\Lambda : \mathcal{P}_4^1$, where $\mathcal{P}_4^1$ is the space of of polynomials in one real variable t, with complex coefficients, of degre ≤ 4, and Λ is the Riesz functional associated to the sequence $\gamma = (\gamma_k)_{0 \leq k \leq 4},\ \gamma_0 = \cdots = \gamma_3 = 1,\ \gamma_4 = 2$. Note that $\mathcal{I}_\Lambda = \{p(t) = a - at; a \in \mathbb{C}\}$, and $\mathcal{H}_\Lambda = \{\hat{p}\,;\ p(t) = a + at + (a+b)t^2,\ a, b \in \mathbb{C}\}$. Setting $p_0(t) = 0.5 - 0.5t,\ p_1(t) = 0.5 + 0.5t$, we have $1 = p_0 + p_1$ and $t = p_1 - p_0$. But $p_0 \in \mathcal{I}_\Lambda$, and so $\hat{t} = \hat{1}$. Consequently, for any choice of an othogonal basis $\mathcal{H}_\Lambda$ consisting of idempotents, we cannot have $\widehat{t^2} = \hat{t}^2$ because $\hat{t}^2 = \hat{t} = \hat{1}$, while $\widehat{t^2} = t^2 + \mathcal{I}_\Lambda \neq \hat{1}$. This shows that Λ has no representing measure consisting of two atoms. As a matter of fact, the element $\hat{t}$ does not separate the points of the space of characters of $\mathcal{H}_\Lambda$ for any choice of an orthogonal basis $\{\hat{b}_1, \hat{b}_2\}$ consisiting of idempotent elements.

Example A previous Corollary implies that all uspf $\Lambda : \mathcal{P}_2 \mapsto \mathbb{C}$ have representing measures in $\mathbb{R}^n$ having $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an arbitrary orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, then the condition

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

is automatically fulfilled when $|\alpha| + |\beta| \leq 1$, $j = 1, \ldots, d$

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. For every point $\xi \in \mathbb{R}^n$, we denote by δ_ξ the point evaluation at ξ, that is, $\delta_\xi(p) = p(\xi)$, for every polynomial $p \in \mathcal{P}$. As in the Introduction, we set $\mathcal{I}_\Lambda = \{f \in \mathcal{P}_m; \Lambda(|f|^2) = 0\}$, while $\mathcal{H}_\Lambda$ is the finite dimensional Hilbert space $\mathcal{P}_m/\mathcal{I}_\Lambda$.

Definition The point evaluation δ_ξ is said to be Λ-*continuous* if there exists a constant $c_\xi > 0$ such that

$$|\delta_\xi(p)| \leq c_\xi \Lambda(|p|^2)^{1/2}, \ p \in \mathcal{P}_m.$$

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\xi \in \mathbb{R}^n$ such that δ_ξ is Λ-continuous. For every polynomial p let us denote by $\mathcal{Z}(p)$ the set of its zeros.

Lemma We have the equality $\mathcal{Z}_\Lambda = \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p)$.

RemarkThe previous lemma shows that the set $\mathcal{Z}_\Lambda$ coincides with the algebraic variety of the moment sequence associated to Λ (introduced by Curto & Fialkow).

Lemma (Curto & Fialkow) Suppose that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has an atomic representing measure μ. Then $\mathrm{supp}(\mu) \subset \mathcal{Z}_\Lambda$.

Remark It follows from previous Lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

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Remark It follows from previous Lemma that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with the property $\mathcal{Z}_\Lambda \neq \emptyset$. As previously noted, the set $\{\delta_\xi^\Lambda; \xi \in \mathcal{Z}_\Lambda\}$ is a subset in the dual of the Hilbert space $\mathcal{H}_\Lambda$. Therefore, for every $\xi \in \mathcal{Z}_\Lambda$ there exists a unique vector $\hat{v}_\xi \in \mathcal{H}_\Lambda$ such that $\delta_\xi^\Lambda(\hat{p}) = \langle \hat{p}, \hat{v}_\xi \rangle = \Lambda(pv_\xi) = p(\xi)$ for all $p \in \mathcal{P}_m$. Let $\mathcal{V}_\Lambda = \{\hat{v}_\xi; \xi \in \mathcal{Z}_\Lambda\}$. We may and shall always assume that a chosen representative v_ξ from the equivalence class $\hat{v}_\xi$ is a polynomial with real coefficients.

Theorem 4 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with $\mathcal{Z}_\Lambda$ nonempty. The uspf Λ has a representing measure having d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{H}_\Lambda$ such that

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d, \tag{7}$$

$$\hat{p} = \Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(pv_d)\hat{v}_d, \; p \in \mathcal{P}_m, \tag{8}$$

and

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \Lambda(v_j)^{-1} \Lambda(v_j v_k)\Lambda(v_j v_l), \; k, l = 1, \ldots, d. \tag{9}$$

Corollary Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ with $\mathcal{Z}_\Lambda$ nonempty. The functional $\Lambda|\mathcal{P}_m$ has a representing measure having d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{H}_m$ such that

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d,$$

and

$$\hat{p} = \Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(pv_d)\hat{v}_d, \ p \in \mathcal{P}_m.$$

Summary

Thank you !