A Stability Equation for Truncated Moment Problems

F.-H. Vasilescu

Department of Mathematics
University of Lille 1, France

June 29, 2010 / OT 23

Outline

Truncated Moment Problems

To solve a truncated moment problem means to characterize those finite multi-sequences of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ with $\gamma_0 > 0$ (where α's are multi-indices of a given length $n \geq 1$ and $m \geq 0$ is an integer) for which there exists a positive measure μ on $\mathbb{R}^n$ (called a *representing measure for* γ) such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \leq 2m$.

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Approaches to These Problems

- A first approach is to associate the sequence γ with the Hankel matrix $M_\gamma = (\gamma_{\alpha+\beta})_{|\alpha|,|\beta|\le m}$, which is supposed to be nonnegative when acting on a corresponding Euclidean space, and using *flat extensions* (Curto and Fialkow).
- A second approach is to use the Riesz functional, induced by the assignment $t^\alpha \mapsto \gamma_\alpha$ on the space of polynomials of total degree less or equal to $2m$, supposed to be nonnegative on the cone of sums of squares of real-valued polynomials. Riesz functionals have been used to study truncated moment problems, as well as for other purposes, by Fialkow and Nie, Laurent and Mourrain, Möller, Putinar etc.

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Function Spaces

Let $n \geq 1$ be a fixed integer. Let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on $\mathbb{R}^n$ (other joint domains of definition may be considered). We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space*.
Let also $\mathcal{S}^{(1)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(1)}$, and $\mathcal{S} = \mathcal{S}^{(1)}$ when $\mathcal{S}$ is an algebra.

Square Positive Functionals

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(1)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$.
(3) $\Lambda(1) = 1$.

Adapting some existing terminology to our context, a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.
When $\mathcal{S}$ is an algebra, conditions (2) and (3) imply condition (1). In this case, a map Λ with the property (2) is usually said to be *positive (semi)definite*.
Looking for probability measures representing such a functional, we always assume (3) in the stated form, without loss of generality.

Associated Hilbert Spaces

If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ p, q \in \mathcal{S}.$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle f + \mathcal{I}_\Lambda, g + \mathcal{I}_\Lambda\rangle = \Lambda(f\bar{g}).$$

If $\mathcal{S}$ is finite dimensional, then $\mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

Associated Hilbert Spaces (cont.)

Now, let $\mathcal{T} \subset \mathcal{S}$ be a function subspace. If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, then $\Lambda|\mathcal{T}^{(1)}$ is also a uspf, and setting $\mathcal{I}_{\Lambda,\mathcal{T}} = \{f \in \mathcal{T}; \Lambda(|f|^2) = 0\} = \mathcal{I}_\Lambda \cap \mathcal{T}$, there is a natural map

$$J_{\mathcal{T},\mathcal{S}} : \mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}} \mapsto \mathcal{S}/\mathcal{I}_\Lambda,$$

$$J_{\mathcal{T},\mathcal{S}}(f + \mathcal{I}_{\Lambda,\mathcal{T}}) = f + \mathcal{I}_\Lambda,\ f \in \mathcal{T}.$$

The equality

$$\langle f + \mathcal{I}_{\Lambda,\mathcal{T}}, f + \mathcal{I}_{\Lambda,\mathcal{T}} \rangle =$$

$$\Lambda(|f|^2) = \langle f + \mathcal{I}_\Lambda, f + \mathcal{I}_\Lambda \rangle$$

shows that the map $J_{\mathcal{T},\mathcal{S}}$ is an isometry.

Dimensional Stability

Definition We say that the uspf $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ it *stable* at $\mathcal{T}$, where $\mathcal{T} \subset \mathcal{S}$ is a function subspace, if we have the equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}) = \mathcal{S}/\mathcal{I}_{\Lambda}$.

The equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{S}}) = \mathcal{S}/\mathcal{I}_{\Lambda}$ is equivalent to the property $\mathcal{T} + \mathcal{I}_{\Lambda} = \mathcal{S}$; in other words, for every $f \in \mathcal{S}$ we can find a $g \in \mathcal{T}$ such that $f - g \in \mathcal{I}_{\Lambda}$. In particular, the spaces $\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}$ and $\mathcal{S}/\mathcal{I}_{\Lambda}$ have the same dimension.
This concept is an version of that of *flatness*, defined by Curto and Fialkow.

Notation and Comments

Let $n \geq 1$ be a fixed integer. We freely use multi-indices from $\mathbb{Z}_+^n$, and the standard notation related to them.
The symbol $\mathcal{P}$ designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients (because of the systematic use of some associated complex Hilbert spaces).
For every integer $m \geq 1$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where deg(p) is the total degree of p. Note that $\mathcal{P}_m^{(1)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(1)} = \mathcal{P}$, the latter being an algebra.
We present in the following an extension theorem within the class of unital square positive functionals on finite dimensional function subspaces $\mathcal{P}_{2m}$ of the space $\mathcal{P}$, and exhibit some of its consequences.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $0 \leq k \leq m$. As in the abstract case, we put $\mathcal{I}_k = \mathcal{I}_{\Lambda,\mathcal{P}_k} = \{p \in \mathcal{P}_k; \Lambda(|p|^2) = 0\}$, and

$$\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k,$$

which is a finite dimensional Hilbert space, with the scalar product given by

$$\langle p + \mathcal{I}_k, q + \mathcal{I}_k \rangle = \Lambda(p\bar{q}),\ p, q \in \mathcal{P}_k.$$

Recall also that the map $\mathcal{P}_k \ni p \mapsto \Lambda(|p|^2)^{1/2}$ is a semi-norm.

Now, if $l \leq m$ is another integer with $k \leq l$, since $\mathcal{I}_k \subset \mathcal{I}_l$, we have a natural map $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$ given by $J_{k,l}(p + \mathcal{I}_k) = p + \mathcal{I}_l$, $p \in \mathcal{P}_{n,k}$, which is an isometry because $\|p + \mathcal{I}_k\|^2 = \Lambda(|p|^2) = \|p + \mathcal{I}_l\|^2$, whenever $p \in \mathcal{P}_k$. In particular, $J_{k,k}$ is the identity on $\mathcal{H}_k$.

Similar constructions can be performed for a uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$

Equalities of the form $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ $(k < l)$ play an important role in this work. We note that $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ if and only if $\mathcal{P}_l = \mathcal{P}_k + \mathcal{I}_l$. In this case, $J_{k,l}$ is a unitary transformation. When $l = k + 1$, we usually write J_k instead of $J_{k,k+1}$.

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Some Extension Results

The next result has been proved by H. M. Möller.

THEOREM Let $\Lambda_m : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. Let also $\Lambda_{m+1} : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ extending Λ_m. Set

$$\mathcal{O}_{k+1} = \{p \in \mathcal{P}_{m+1}; \Lambda_{m+1}(pq) = 0 \quad \forall q \in \mathcal{P}_m\}.$$

The map Λ_{m+1} is a uspf if and only if

$$\mathrm{dim}\mathcal{O}_{k+1} = \mathrm{dim}\mathcal{I}_m + \left(\begin{array}{c} m+n \\ n-1 \end{array} \right),$$

and $\Lambda_{m+1}(|p|^2) \geq 0$ for all $p \in \mathcal{O}_{k+1}$.

The next result is essentially due to Curto and Fialkow.

THEOREM Let $\Lambda_m : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf. Assume that the isomatry $J_m : \mathcal{H}_{m-1} \mapsto \mathcal{H}_m$ is surjective. Then there exists a uniquely determined uspf $\Lambda_{m+1} : \mathcal{P}_{2m+2} \mapsto \mathbb{C}\,(m \geq 1)$ extending Λ_m. Moreover, the isometry $J_{m+1} : \mathcal{H}_m \mapsto \mathcal{H}_{m+1}$ is also surjective.

In the proof of the theorem from above, the condition "$J_m : \mathcal{H}_{m-1} \mapsto \mathcal{H}_{m-1}$ is a unitary operator" (equivalent to the *flatness* of Curto and Fialkow) is essential.

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Dimensional Stability

DEFINITION

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, let $(\mathcal{H}_l)_{0 \leq l \leq m}$ be the Hilbert spaces bilt via Λ, and let $J_l : \mathcal{H}_l \mapsto \mathcal{H}_{l+1}\,(0 \leq l \leq m-1)$ be the associated isometries. If for some $k \in \{0, \dots, m-1\}$ one has $J_k(\mathcal{H}_k) = \mathcal{H}_{k+1}$, we say that Λ is *dimensionally stable* (or simply *stable*) *at* k.

The uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ is said to be *dimensionally stable* if there exist integers m, k, with $m > k \geq 0$, such that $\Lambda_\infty|\mathcal{P}_{2m}$ is stable at k.

The number $\mathrm{sd}(\Lambda_\infty) = \dim \mathcal{H}_k$ will be called the *stable dimension* of Λ_∞.

Flatness and Dimensional Stability

REMARK Let $m \geq 1$ be an integer, let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $\{\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k,\ 0 \leq k \leq m\}$ be the Hilbert spaces built via Λ. The sesquilinear form $(p, q) \mapsto \Lambda(p\bar{q})$ implies the existence of a positive operator A_k on $\mathcal{P}_k$ such that $(A_k p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_k$, where $0 \leq k \leq m$. Note that $p \in \mathcal{I}_k$ if and only if $A_k p = 0$. This implies that $\dim \mathcal{H}_k$ equals the rank of A_k. The concept of *flatness* for the finite multi-sequence associated to Λ, introduced by Curto and Fialkow, means precisely that the rank of A_{m-1} is equal to the rank of A_m, and it is equivalent to the fact that Λ is stable at $m - 1$.

Using the previous results, as well as the Cauchy-Schwarz inequality, several results by Curto and Fialkow can be recaptured.

THEOREM

Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be a uspf.
If Λ_∞ is dimensionally stable, then Λ_∞ has a unique representing measure, which is d-atomic, where $d = \mathrm{sd}(\Lambda_\infty)$.
Conversely, if Λ_∞ has a d-atomic representing measure, then Λ_∞ is dimensionally stable and $d = \mathrm{sd}(\Lambda_\infty)$.

COROLLARY

The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ has a uniquely determined d-atomic representing measure, where $d = \dim \mathcal{H}_m$, if and only if Λ is stable at $m - 1$.

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EXAMPLE Assume that $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ has a d-atomic representing measure. If $d = 1$, then there exists a point $\xi \in \mathbb{R}^n$ such that $\Lambda_\infty(p) = p(\xi)$ for all $p \in \mathcal{P}$. Then, for all $k \geq 1$, $\mathcal{I}_k = \{p \in \mathcal{P}_k; p(\xi) = 0\}$, the space $\mathcal{H}_k$ is isomorphic to $\mathbb{C}$, and so Λ_∞ is dimensionally stable with $\mathrm{sd}(\Lambda_\infty) = 1$.

Assume now that $d \geq 2$. Let $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ be distinct points and let μ be an atomic measure concentrated on Ξ, such that $\Lambda_\infty(p) = \int p d\mu$ for all $p \in \mathcal{P}$.
Consider the polynomials

$$\chi_k(t) = \frac{\prod_{j \neq k} \|t - \xi^{(j)}\|^2}{\prod_{j \neq k} \|\xi^{(k)} - \xi^{(j)}\|^2},\ t \in \mathbb{R}^n,\ k = 1, \ldots .$$

Clearly, $\chi_k \in \mathcal{P}_{2d-2}$, $k = 1, \ldots, d$, and $\chi_k(\xi^{(l)}) = \delta_{kl}$ (the Kronecker symbol) for all $k, l = 1, \ldots, d$. In fact, the set $(\chi_k)_{1 \leq k \leq d}$ is an orthonormal basis of $L^2(\mu)$.

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Since each polynomial $p \in \mathcal{P}_l$ can be written on the set Ξ as $p(t) = \sum_{j=1}^{d} p(\xi^{(j)})\chi_j(t)$, and so

$$\int |p(t) - \sum_{j=1}^{d} p(\xi^{(j)})\chi_j(t)|^2 d\mu(t) = 0,$$

it follows that, for every $l \geq 2d - 2$, we have $\mathcal{I}_l = \{p \in \mathcal{P}_l; p|\Xi = 0\}$, and so $(\chi_k + \mathcal{I}_l)_{1 \leq k \leq d}$ is an orthonormal basis of $\mathcal{H}_l$. Therefore, all spaces $\mathcal{H}_l$, $l \geq 2d - 2$, have the same dimension equal to $\dim L^2(\mu) = d$. In particular, Λ_∞ is dimensionally stable and $\mathrm{sd}(\Lambda_\infty) = d$.

An Associated C^*-Algebra

THEOREM
Let $m \geq 1$ be an integer, and let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf.
If Λ is stable at $m - 1$, then, endowed with an equivalent norm, the space $\mathcal{H}_m$ has the structure of a unital commutative C^*-algebra.

We define a product on $\mathcal{H}_m$ in the following way. For each pair $p, q \in \mathcal{P}_m$, we set

$$(p + \mathcal{I}_m) \cdot (q + \mathcal{I}_m) = J_{m,2m}^{-1}(pq + \mathcal{I}_{2m}) = (q + \mathcal{I}_m) \cdot (p + \mathcal{I}_m),$$

because, in this case, $J_{m,2m} : \mathcal{P}_m \mapsto \mathcal{P}_{2m}$ is a unitary operator.

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The Stability Equation

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with $m \geq 1$ and let k be an integer such that $0 \leq k < m$. It is easily checked that the uspf Λ is stable at k if and only if for each multi-index δ with $|\delta| = k + 1$ the equation

$$\sum_{|\xi|,|\eta| \leq k} \gamma_{\xi+\eta} c_\xi c_\eta$$

$$-2 \sum_{|\xi| \leq k} \gamma_{\xi+\delta} c_\xi + \gamma_{2\delta} = 0$$

has a solution $(c_\xi)_{|\xi| \leq k}$ consisting of real numbers, where $\gamma = (\gamma_\xi)_{|\xi| \leq 2m}$ is the finite multi–sequence associated to Λ. To study the existence of solutions for such an equation, it is convenient to use an abstract framework.

The Abstract Stability Equation

Let $N \geq 1$ be an arbitrary integer, let $A = (a_{jk})_{j,k=1}^{N}$ be a matrix with real entries, that is positive on $\mathbb{C}^N$ (endowed with the standard scalar product denoted by $(*|*)$, and associated norm $\| * \|$), let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, and let $c \in \mathbb{R}$. We look for necessary and sufficient conditions insuring the existence of a solution $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ of the equation

$$(ASE) \qquad (Ax|x) - 2(b|x) + c = 0.$$

This is a quadric equation whose solution is given in the following. The range and the kernel of A, regarded as an operator on $\mathbb{C}^N$, will be denoted by $R(A)$, $N(A)$, respectively.

Solution to ASE

PROPOSITION

We have the following alternative:
1) If $b \notin R(A)$, equation (ASE) always has solutions.
2) If $b \in R(A)$, equation (ASE) has solutions if and only if for some (and therefore for all) $d \in A^{-1}(\{b\})$ we have $c \leq (d|b)$.
In particular, if $N(A) = \{0\}$, then A is invertible and equation (ASE) has solutions if and only if $c \leq (A^{-1}b|b)$.

Stability Equation for Moments

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ the multi-sequence associated to Λ. Then $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \leq m-1}$ is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \leq m-1\}$. In fact, by identifying the space $\mathcal{P}_{m-1}$ with $\mathbb{C}^N$, A_{m-1} is the operator with the property $(A_{m-1}p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_{m-1}$.

For each multi-index δ with $|\delta| = m$, we put $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \leq m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. With this notation, equation (ASE) becomes

$$(SE) \qquad (A_{m-1}x|x) - 2(b_\delta|x) + c_\delta = 0,$$

which may be called the *stability equation* of the uspf Λ.

THEOREM
Let $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$ ($\gamma_0 = 1, m \ge 1$) be a square positive finite multi-sequence of real numbers and let $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta|\le m-1}$, acting on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \le m-1\}$. For each multi-index δ with $|\delta| = m$, set $b_\delta = (\gamma_{\xi+\delta})_{|\xi|\le m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. The multi-sequence γ has a unique r-atomic representing measure if and anly if, whenever $b_\delta \in R(A_{m-1})$, we have $c_\delta \le (d_\delta|b_\delta)$ for some (and therefore for all) $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$, where r is the rank of the matrix A_{m-1}.

COROLLARY Assume the matrix A_{m-1} invertible. There exists a d-atomic representing measure μ on $\mathbb{R}^n$ for the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ if and only if for each δ with $|\delta| = m$ we have $c_\delta \le (A_{m-1}^{-1} b_\delta | b_\delta)$, where $d = \dim \mathcal{P}_{m-1}$.

THEOREM

Let $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$ $(\gamma_0 = 1, m \ge 1)$ be a square positive finite multi-sequence of real numbers and let $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta|\le m-1}$, acting on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \le m-1\}$. For each multi-index δ with $|\delta| = m$, set $b_\delta = (\gamma_{\xi+\delta})_{|\xi|\le m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. The multi-sequence γ has a unique r-atomic representing measure if and anly if, whenever $b_\delta \in R(A_{m-1})$, we have $c_\delta \le (d_\delta|b_\delta)$ for some (and therefore for all) $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$, where r is the rank of the matrix A_{m-1}.

COROLLARY Assume the matrix A_{m-1} invertible. There exists a d-atomic representing measure μ on $\mathbb{R}^n$ for the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ if and only if for each δ with $|\delta| = m$ we have $c_\delta \le (A_{m-1}^{-1} b_\delta|b_\delta)$, where $d = \dim \mathcal{P}_{m-1}$.

Stability Equation in a Noncommutative Context

We want to show that the stability equation can be also applied in a noncommutative context. In the following, we adapt parts of the previous discussion to such a context.

Let $\mathcal{A}$ be a complex algebra with unit 1 and involution $a \mapsto a^*$, and let $\mathcal{S} \subset \mathcal{A}$ be a vector subspace containing the unit and invariant under involution. For convenience, let us say that $\mathcal{S}$, having these properties, is a $*$-*subspace* of $\mathcal{A}$. Let also $\mathcal{S}^{(1)}$ be the vector subspace spanned by all products of the form ab with $a, b \in \mathcal{S}$, which is itself a $*$-subspace. We have $\mathcal{S} \subset \mathcal{S}^{(1)}$, and $\mathcal{S} = \mathcal{S}^{(1)}$ when $\mathcal{S}$ is a subalgebra.

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Generalized USPF

Let $\mathcal{S}$ be a $*$-subspace of $\mathcal{A}$, and let $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(a^*) = \overline{\Lambda(a)}$ for all $a \in \mathcal{S}^{(1)}$;
(2) $\Lambda(a^* a) \geq 0$ for all $f \in \mathcal{S}$.
(3) $\Lambda(1) = 1$.

As in the case of ordinary polynomials, a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional* (briefly a *uspf*).

If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(a^* b)|^2 \leq \Lambda(a^* a)\Lambda(b^* b),\ a, b \in \mathcal{S}.$$

Putting $\mathcal{I}_\Lambda = \{a \in \mathcal{S}; \Lambda(a^* a) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(a^* a)^{1/2} \in \mathbb{R}_+$ is a seminorm.

In fact, $\mathcal{I}_\Lambda = \{a \in \mathcal{S}; \Lambda(ba) = 0\, \forall b \in \mathcal{S}\}$. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle a + \mathcal{I}_\Lambda, b + \mathcal{I}_\Lambda \rangle = \Lambda(b^* a).$$

If $\mathcal{S}$ is finite dimensional, then $\mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

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Let $\mathcal{T} \subset \mathcal{S}$ be a $*$-subspace. If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, then $\Lambda|\mathcal{T}^{(1)}$ is also a uspf, and setting $\mathcal{I}_{\Lambda,\mathcal{T}} = \{a \in \mathcal{T}; \Lambda(a^*a) = 0\} = \mathcal{I}_\Lambda \cap \mathcal{T}$, there is a natural map

$$J_{\mathcal{T},\mathcal{S}} : \mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}} \mapsto \mathcal{S}/\mathcal{I}_\Lambda,\ \ J_{\mathcal{T},\mathcal{S}}(a + \mathcal{I}_{\Lambda,\mathcal{T}}) = a + \mathcal{I}_\Lambda,\ a \in \mathcal{T}.$$

The equality

$$\langle a + \mathcal{I}_{\Lambda,\mathcal{T}}, a + \mathcal{I}_{\Lambda,\mathcal{T}} \rangle = \Lambda(a^*a) = \langle a + \mathcal{I}_\Lambda, a + \mathcal{I}_\Lambda \rangle$$

shows that the map $J_{\mathcal{T},\mathcal{S}}$ is an isometry, in particular it is injective.

We say that the uspf $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ it *stable* at $\mathcal{T}$, where $\mathcal{T} \subset \mathcal{S}$ is a function subspace, if we have the equality
$J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}) = \mathcal{S}/\mathcal{I}_\Lambda$

The equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{S}}) = \mathcal{S}/\mathcal{I}_\Lambda$ is equivalent to the property $\mathcal{T} + \mathcal{I}_\Lambda = \mathcal{S}$; in other words, for every $a \in \mathcal{S}$ we can find a $b \in \mathcal{T}$ such that $a - b \in \mathcal{I}_\Lambda$. In particular, the spaces $\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}$ and $\mathcal{S}/\mathcal{I}_\Lambda$ have the same dimension.

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Polynomial Type Algebras

Let $\mathcal{A}$ be a complex involutive algebra, with unit. The algebra $\mathcal{A}$ is said to be a *polynomial type algebra* if there exists an algebraic basis $\mathcal{B} = \cup_{m=0}^{\infty}\mathcal{B}_m$ of $\mathcal{A}$ such that $\mathcal{B}_0 = \{1\}$, $1 \in \mathcal{B}_m$, $\mathcal{B}_m$ is finite and invariant under involution, and $\mathcal{B}_{m_1} \cdot \mathcal{B}_{m_2} = \mathcal{B}_{m_1+m_2}$ for all integers $m, m_1, m_2 \geq 0$.

Note that $\mathcal{B}_{m_1} \subset \mathcal{B}_{m_2}$ whenever $m_1 \leq m_2$, and that the basis $\mathcal{B}$ is closed under multiplication.

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Using the previous notation, let $\mathcal{S}_m$ be the vector space spanned by $\mathcal{B}_m$. Then the collection $(\mathcal{S}_m)_{m\geq 0}$ is an increasing family of finite dimensional $*$-subspaces of $\mathcal{A}$ such that $\mathcal{S}_0 = \mathbb{C}\cdot 1$, $\mathcal{S}_{m_1}\cdot\mathcal{S}_{m_2}\subset\mathcal{S}_{m_1+m_2}$ for all integers $m_1, m_2\geq 0$, and $\cup_{m=0}^{\infty}\mathcal{S}_m = \mathcal{A}$. Moreover, we have the equality $\mathcal{S}_m^{(1)} = \mathcal{S}_{2m}$ for all integers $m\geq 1$.

The *degree* of an arbitrary element $a\in\mathcal{A}$, which is not a multiple of 1, is the least integer $m\geq 1$ such that $a\in\mathcal{S}_m\setminus\mathcal{S}_{m-1}$. The degree of a multiple of 1 is equal to 0. The degree of $a\in\mathcal{A}$ is denoted by $\deg(a)$. With this notation, we have $\mathcal{S}_m = \{a\in\mathcal{A}; \deg(a)\leq m\}$. Note also that $\deg(a)$=$\deg(a^*)$ for all $a\in\mathcal{A}$.

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EXAMPLE The algebra $\mathcal{P}$ of all polynomials in n real variables, with complex coefficients (endowed with the natural involution $p \mapsto \bar{p}$) is, of course, a polynomial algebra.
The subset $\mathcal{M} = \{t^\alpha; \alpha \in \mathbb{Z}_+^n\} = \cup_{m \geq 0} \mathcal{M}_m$ is an algebraic basis for the algebera $\mathcal{P}$, where $\mathcal{M}_m = \{t^\alpha; |\alpha| \leq m\}$, and $m \geq 0$ is an integer. Clearly, $\mathcal{P}_m$ is spanned by $\mathcal{M}_m$.

EXAMPLE Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be a finite family of indeterminates, and let $\mathcal{F}[\mathbf{X}]$ be the complex unital algebra freely generated by $\mathbf{X}$, whose unit is designated by $\mathbf{1}$. Let $\mathcal{W}$ be the monoid generated by $\mathbf{X} \cup \{\mathbf{1}\}$. The *lenght* of an element $W \in \mathcal{W} \setminus \{\mathbf{1}\}$ is equal to the number of elements of $\mathbf{X}$ which occur in the representation of W. The length of $\mathbf{1}$ is equal to zero and the multiplication of every element $W \in \mathcal{W} \setminus \{\mathbf{1}\}$ by $\mathbf{1}$ does not change its length.

If $\mathcal{W}_m$ is the subset of those elements from $\mathcal{W}$ of lenght $\leq m$, with $m \geq 0$ an arbitrary integer, then $\mathcal{W} = \cup_{m\geq 0}\mathcal{W}_m$ is an algebraic basis of $\mathcal{F}[\mathbf{X}]$. Setting $W^* = X_{j_m}X_{j_{m-1}} \cdots X_{j_1}$ for every $W = X_{j_1} \cdots X_{j_{m-1}}X_{j_m} \in \mathcal{W} \setminus \{\mathbf{1}\}$, $\mathbf{1}^* = \mathbf{1}$, and $(cW)^* = \bar{c}W$ for all complex numbers c, we define an involution $P \mapsto P^*$ on $\mathcal{F}[\mathbf{X}]$, extending this assignment by additivity. In this way, the algebra $\mathcal{F}[\mathbf{X}]$ becomes a (noncommutative) polynomial type algebra.

Let $\mathcal{F}_m$ be the subspace spanned in $\mathcal{F}[\mathbf{X}]$ by the set $\mathcal{W}_m$, for every integer $m \geq 0$. As in the case of ordinary polynomials, if $\gamma = (\gamma_W)_{W \in \mathcal{W}_{2m}}$ is a family of complex numbers, we may define a linear map $\Lambda_\gamma : \mathcal{F}_{2m} \mapsto \mathbb{C}$, extending the assignment $W \mapsto \gamma_W$ by linearity. Moreover, assuming that $\gamma_{\mathbf{0}} = 1, \gamma_{W^*} = \overline{\gamma_W}$ for all $W \in \mathcal{W}_{2m}$, and

$$\sum_{j,k=0}^{d_m} \bar{c}_j c_k \gamma_{W_j^* W_k}$$

for all complex numbers $\{c_0, \ldots, c_{d_m}\}$, where $d_m + 1$ is the cardinal of $\mathcal{W}_m = \{W_0 = \mathbf{1}, W_1, \ldots, W_{d_m}\}$, the map Λ_γ becomes a uspf.

Truncated moment problems related to a uspf $\Lambda : \mathcal{F}_{2m} \mapsto \mathbb{C}$, when Λ is a tracial map (i.e. Λ is null on commutators) have been recently studied by S. Burgdorf and I. Klep.

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Let $\mathcal{A}$ be a polynomial type algebra with the basis $\mathcal{B} = \cup_{m=0}^{\infty}\mathcal{B}_m$. For each $a \in \mathcal{A}$ there exists an integer $m \geq 0$ such that $a \in \mathcal{S}_m$. Since $\mathcal{B}_m$ an algebraic basis of $\mathcal{S}_m$, we can write $a = \sum_{k=0}^{d_m} \alpha_k b_k$, where $d_m + 1$ is the cardinal of $\mathcal{B}_m = \{b_0 = \mathbf{1}, b_1, \ldots, b_{d_m}\}$, α_k are complex numbers and $b_k \in \mathcal{B}_m$, where $b_0 = 1$. Setting $\alpha_k = 0$ if $k > d_m$, we can write $a = \sum_{k\geq 0} \alpha_k b_k$, and this representation is unique.

On the algebra $\mathcal{A}$, we may define a scalar product given by $(a_1|a_2) = \sum_{k\geq 0} \alpha_{1k}\overline{\alpha_{2k}}$, where $a_j = \sum_{k\geq 0} \alpha_{jk} b_k$, $j = 1, 2$. With respect to this scalar product, the algebraic basis $\mathcal{B}$ is also an orthonormal family.

In particular, if $m \geq 0$ is any integer, the finite dimensional space $\mathcal{S}_m$ has a Hilbert space structure induced by the scalar product from above, such that the family of elements from $\mathcal{B}_m$ is an orthonormal basis of $\mathcal{S}_m$.

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L et $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ be a uspf such that $\Lambda|\mathcal{B}_{2m}$ has real values. Lat also $A_{m-1} = (\Lambda(b_k^* b_j))_{0 \leq j,k \leq d_{m-1}}$ which is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where $N = 1 + d_{m-1}$. By identifying the space $\mathcal{S}_{m-1}$ with $\mathbb{C}^N$, A_{m-1} is the operator with the property $(A_{m-1}f|g) = \Lambda(g^* f)$ for all $f, g \in \mathcal{S}_{m-1}$.
For each index ℓ with $d_{m-1} < \ell \leq d_m$, we put $h_\ell = (\Lambda(b_\ell^* b_k)_{0 \leq k \leq d_{m-1}} \in \mathbb{R}^N$ and $c_\ell = \Lambda(b_\ell^* b_\ell)$. With this notation, the equation (ASE) becomes

$$(A_{m-1}x|x) - 2(h_\ell|x) + c_\ell = 0,$$

which is again called the *stability equation* of the uspf Λ.

For $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ a uspf, if $0 \le k \le m$, as in the commutative case, we put $\mathcal{I}_k = \mathcal{I}_{\Lambda,\mathcal{S}_k} = \{p \in \mathcal{S}_k; \Lambda(|p|^2) = 0\}$, and $\mathcal{H}_k = \mathcal{S}_k/\mathcal{I}_k$, which are finite dimensional Hilbert spaces. The stability of Λ at $m-1$ (i.e. dim$\mathcal{H}_{m-1}$=dim$\mathcal{H}_m$) is given by the following (using the previous notation).

THEOREM The uspf $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ $(m \ge 1)$ such that $\Lambda|\mathcal{B}_{2m}$ has real values is stable at $m-1$ if and only if, whenever $h_\ell \in R(A_{m-1})$, we have $c_\ell \le (f_\ell|h_\ell)$ for some (and therefore for all) $f_\ell \in A_{m-1}^{-1}(\{h_\ell\})$, where $d_{m-1} < \ell \le d_m$.

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Summary

- The stability equation leads to a local characterization to the “dimensional stability”, whch is in turn equivalent to the “flatness” of Curto and Fialkow.
- In the noncommutative case, a “solution” to the (nonstated) moment problem for a uspf $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ might follow from the identification of the final Hilbert space $\mathcal{H}_m$ with a sub-C^*-algebra of the C^*-algebra of all linear operators on $\mathcal{H}_m$.