Linear Relations and Quotient Morphisms

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1 Introduction

The importance of the quotient vector spaces and associated morphisms has been emphasized in a long series of papers by L. Waelbroeck.

The Fredholm and spectral theory, developed in the framework of quotient Banach spaces also has some interesting aspects (E. Albrecht and F.-H. V.).

Giving two (real or complex) vector spaces $\mathcal{X}$ and $\mathcal{Y}$, following Arens (1961), a *linear relation* (or simply *relation*) is any subspace $Z \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$. For such a relation Z, we use the following notation. Set $D(Z) = \{x \in \mathcal{X}; \exists y \in \mathcal{Y} : (x, y) \in Z\}$, $R(Z) = \{y \in \mathcal{Y}; \exists x \in \mathcal{X} : (x, y) \in Z\}$, which are the *domain*, respectively the *range* of Z. We also use the spaces $N(Z) = \{x \in D(Z); (x, 0) \in Z\}$,
$M(Z) = \{y \in R(Z); (0, y) \in Z\}$, which are the *kernel*, respectively the *multi-valued part* of Z. The inverse $Z^{-1} \in \mathrm{Lat}(\mathcal{Y}, \mathcal{X})$ of the relation $Z \in \mathrm{Lat}(\mathcal{X}, \mathcal{Y})$ is given by $\{(y, x); (x, y) \in Z\}$.

The concept of relation is equivalent to that of *multivalued linear operator*.

Given two relations $Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$ and $Z' \in \mathrm{Lat}(\mathcal{Y} \times \mathcal{Z})$, their product (or composition) is the relation $Z' \circ Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Z})$ defined by

$$Z' \circ Z'' = \{(x, z) \in \mathcal{X} \times \mathcal{Z}; \exists y \in \mathcal{Y} : (x, y) \in Z'', (y, z) \in Z'\}.$$

2 Fredholm applications

Having in mind the case of unbounded linear operators in normed spaces, we first discuss linear transformations which are not, in general, everywhere defined.

Specifically, we consider linear maps between two vector spaces X and Y having the form

$$T : D(T) \subset X \mapsto Y,$$

where $D(T)$, which is itself a vector space, is the *domain of definition* of T. The *range* of T, the *kernel* (or the *null space*) of T and the *graph* of T will be denoted by $R(T)$, $N(T)$ and $G(T)$, respectively.

Given $T : D(T) \subset X \mapsto Y$ and $S : D(S) \subset Y \mapsto Z$, we define their *composition* $S \circ T$ in the following way. The domain $D(S \circ T) \subset D(T)$ is given by $T^{-1}(D(S))$ and $(S \circ T)(x) = S(T(x))$ for all $x \in D(S \circ T)$.

When $T : X \mapsto Y$ and $S : Y \mapsto Z$, the composition $S \circ T$ will be simply denoted by ST.

As usually, we say that a linear map $T : D(T) \subset X \mapsto Y$ is *Fredholm* if both $\dim N(T)$ and $\dim Y/R(T)$ are finite.

In that case, the (*algebraic*) *index* of T, denoted by $\mathrm{ind}(T)$, is given by

$$\mathrm{ind}(T) = \mathrm{dim} N(T) - \mathrm{dim} Y/R(T).$$

The next assertion extends the standard multiplication result for the Fredholm index.

Theorem 2.1 *Let $T : D(T) \subset X \mapsto Y$, $S : D(S) \subset Y \mapsto Z$ be Fredholm maps. Then $S \circ T$ is Fredholm and*

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(S) + \mathrm{ind}(T) + \\ \mathrm{dim} Y/(R(T) + D(S)).$$

Remark 2.2 If $T : X \mapsto Y$ and $S : Y \mapsto Z$ are Fredholm maps, the previous theorem gives the well known classical formula $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$, which is the multiplication of the Fredholm index.

3 Quotient morphisms

Let $\mathcal{X}$ be a (real or complex) vector space and let $\mathrm{Lat}(\mathcal{X})$ denote the lattice of all vector subspaces of $\mathcal{X}$. Let also $\mathcal{Q}(\mathcal{X})$ be the family of all (quotient) vector spaces of the form X/X_0, with $X_0, X \in \mathrm{Lat}(\mathcal{X})$, $X_0 \subset X$.

Remark 3.1 (1) There is a natural partial order in $\mathcal{Q}(\mathcal{X})$, defined in the following way. We write $X/X_0 \prec Y/Y_0$ if $X \subset Y$ and $X_0 \subset Y_0$. In thie case, there exists a natural map $X/X_0 \ni x + X_0 \mapsto x + Y_0 \in Y/Y_0$ called the *q-inclusion* of X/X_0 into Y/Y_0. This map is injective iff $X \cap Y_0 = X_0$, surjective iff $X + Y_0 = Y$, and therefore bijective iff $X \cap Y_0 = X_0$ and $Y = X + Y_0$.

Note that we have $X/(X \cap Y) \prec (X+Y)/Y$ but the q-inclusion is, in this case, a classical isomorphism. Nevertheless, if $X/X_0 \prec Y/Y_0$ and $Y/Y_0 \prec X/X_0$, then $X/X_0 = Y/Y_0$.

(2) In the set $\mathcal{Q}(X)$ we may define the *q-intersection* and the *q-sum* of two (or several) spaces, denoted by $\Cap$ and $\uplus$ respectively, via the formulas

$$X/X_0 \Cap Y/Y_0 = (X \cap Y)/(X_0 \cap Y_0),$$

$$X/X_0 \uplus Y/Y_0 = (X+Y)/(X_0+Y_0),$$

for any pair of spaces $X/X_0, Y/Y_0 \in \mathcal{Q}(X)$. When $X_0 = Y_0$, the q-intersection is actually intersection and the q-sum is actually the sum of the corresponding vector spaces.

Definition 3.2 A *quotient morphism* (or, simply, a *q-morphism*) from $\mathcal{X}$ into $\mathcal{Y}$ is any linear map $T : X/X_0 \mapsto Y/Y_0$, where $X/X_0 \in \mathcal{Q}(\mathcal{X})$ and $Y/Y_0 \in \mathcal{Q}(\mathcal{Y})$.

When there exists a linear map $T_0 : X \mapsto Y$ with $T_0(X_0) \subset Y_0$ such that $T(x + X_0) = T_0 x + Y_0$, $x \in X$, the q-morphism $T : X/X_0 \mapsto Y/Y_0$ is said to be *induced* (by T_0).

This concept is similar to that of *morphism* defined by Waelbroeck (who also noticed that there exist q-morphisms which are not induced by any linear map).

The family of all quotient morphisms from $\mathcal{X}$ into $\mathcal{Y}$ will be denoted by $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$. When $\mathcal{X} = \mathcal{Y}$, the family $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$ will be denoted by $\mathcal{QM}(\mathcal{X})$.

Let $T : D(T) \mapsto Y/Y_0$ be a given

q-morphism in $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$. The space $D(T)$, which is the *domain* of T, can be written as $D_0(T)/X_0$, where $D_0(T)$ is called the *lifted domain* of T.

The *range* $R(T) = T(D(T))$ of T can be represented as $R_0(T)/Y_0$, where $R_0(T) \in$Lat$(\mathcal{Y})$ is said to the *lifted range* of T.

The *graph* $G(T)$ of T in $X/X_0 \times Y/Y_0$ is isomorphic to $G_0(T)/(X_0 \times Y_0)$, where

$$G_0(T) = \{(x, y) \in X \times Y;$$

$$T(x + X_0) = y + Y_0\} \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$$

is called the *lifted graph* of T.

Definition 3.3 Let $T_j : X_j/X_{0j} \mapsto Y_j/Y_{0j}$, $j = 1, 2, \ldots, n$, be quotient morphisms from $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$. We define the *q-sum* of these morphisms, and denote it by $T_1 \uplus T_2 \uplus \cdots \uplus T_n$ or by $\uplus_{j=1}^n T_j$, the q-morphism

$\uplus_{j=1}^n T_j : \Cap_{j=1}^n X_j/X_{0j} \mapsto \uplus_{j=1}^n Y_j/Y_{0j}$

given by the formula

$$\uplus_{j=1}^n T_j = \sum_{j=1}^n B_j T_j A_j,$$

where

$$A_j : \Cap_{k=1}^n X_k/X_{0k} \mapsto X_j/X_{0j}$$

and

$$B_j : Y_j/Y_{0j} \mapsto \uplus_{k=1}^n Y_k/Y_{0k}$$

are the q-inclusions ($j = 1, 2, \ldots, n$).

Remark 3.4 Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ and $W/W_0 \in \mathcal{Q}(\mathcal{W})$. Let also $T : X/X_0 \mapsto Y/Y_0$ and $S : Z/Z_0 \mapsto W/W_0$ be q-morphisms. We define a " composition" of the maps S and T in the following way.

We first consider the subspace

$$(Y \cap Z + Y_0)/Y_0 = (Y/Y_0) \cap ((Z + Y_0)/Y_0),$$

and the map

$$T_0 : D(T_0) \mapsto (Y \cap Z + Y_0)/Y_0,$$

where $D(T_0) = T^{-1}((Y \cap Z + Y_0)/Y_0)$. Clearly, $N(T_0) = N(T)$, $R(T_0) = (R_0(T) \cap Z + Y_0)/Y_0$, and so $R_0(T_0) = R_0(T) \cap Z + Y_0$.

Secondly, we note that there exists a natural map

$$U : (Y \cap Z + Y_0)/Y_0 \mapsto Z/(Y_0 \cap Z + Z_0).$$

The map U is the composition of the isomorphism of $(Y \cap Z + Y_0)/Y_0$ onto the space $(Y \cap Z)/(Y_0 \cap Z)$ and the q-inclusion of $(Y \cap Z)/(Y_0 \cap Z)$ into $Z/(Y_0 \cap Z + Z_0)$.

Thirdly, let S° be the restriction $S|(Y_0 \cap Z + Z_0)/Z_0$. The map S induces a map

$$S_0 : Z/(Y_0 \cap Z + Z_0) \mapsto W/R_0(S^\circ),$$

via the natural isomorphism between $(Z/Z_0)/((Y_0 \cap Z + Z_0)/Z_0)$ and $Z/(Y_0 \cap Z + Z_0)$. Moreover, the space $(W/R_0(S^\circ))/R(S_0)$ is isomorphic to $W/R_0(S)$.

Clearly, the composition S_0UT_0 is well defined. The map S_0UT_0 will be designated by $S \circ_q T$.

We therefore have

$$S \circ_q T : D(T_0) \subset X/X_0 \mapsto$$
$$R_0(S_0)/R_0(S^\circ) \subset W/R_0(S^\circ).$$

The map $S \circ_q T$ will be called the *q-composition* of the maps S and T.

An important particular case of the construction from above is obtained when $Z/Z_0 \prec Y/Y_0$. In this case, $T_0 : D(T_0) \mapsto (Y_0 + Z)/Y_0$, the map U is the natural isomorphism $U : (Y_0 + Z)/Y_0 \mapsto Z/(Y_0 \cap Z)$ and $S_0 : Z/(Y_0 \cap Z) \mapsto W/R_0(S^\circ)$.

Note that for two linear maps $T : D(T) \subset X \mapsto Y$ and $S : D(S) \subset Y \mapsto Z$, we have the equality $S \circ_q T = S \circ T$.

Remark 3.5 The q-composition defined in Remark 3.4 occurs in various situations. Here is an example.

Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $R(T) = Y/Y_0$. We have a natural map from $R(T)$ into $(X/X_0)/N(T)$, identified with $X/N_0(T)$, given by the assignement $y+Y_0 \mapsto x+N_0(T)$, whenever $(x, y) \in G_0(T)$. This q-morphism will be denoted by T^{-1} and called the *q-inverse* of T. It coincides with the usual inverse when T is bijective. Moreover,

$$T^{-1} \circ T = J_{X/X_0}^{X/N_0(T)},$$

$$T \circ_q T^{-1} = I_{R(T)},$$

where $J_{X/X_0}^{X/N_0(T)}$ is the q-inclusion $X/X_0 \prec$

$X/N_0(T)$ and $I_{R(T)}$ is the identity on $R(T)$.

The q-compositons of quotient morphisms is an associative operation:

Proposition 3.6 *Let* $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$, $W/W_0, U/U_0 \in \mathcal{Q}(\mathcal{W})$ *and* $V/V_0 \in \mathcal{Q}(\mathcal{V})$. *Let also* $T : X/X_0 \mapsto Y/Y_0$, $S : Z/Z_0 \mapsto W/W_0$ *and* $P : U/U_0 \mapsto V/V_0$ *be* q-*morphisms. Then*

$$P \circ_q (S \circ_q T) = (P \circ_q S) \circ_q T.$$

Remark 3.7 (1) From now on we write the q-composition $S \circ_q T$ simply $S \circ T$.

(2) Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $R(T) = Y/Y_0$ and let $T^{-1} : R_0(T)/Y_0 \to X/N_0(T)$ (as in Remark 3.5). We can show that T^{-1} is

the only surjective q-morphism $S : R_0(T)/Y_0 \to X/N_0(T)$ satisfying $T \circ S = I_{R(T)}$.

(3) As in the previous section, we say that a q-morphism $T : X/X_0 \mapsto Y/Y_0$ is *Fredholm* if both $\dim N(T)$ and $\dim(Y/Y_0)/R(T) = \dim Y/R_0(T)$ are finite. In that case, the *index* of T, denoted by $\operatorname{ind}(T)$, is given by

$$\operatorname{ind}(T) = \dim N(T) - \dim Y/R_0(T).$$

In particular, the *q-inclusion* of X/X_0 into Y/Y_0, say Q, is Fredholm if and only if $\dim N(Q) = \dim(X \cap Y_0)/X_0$ and $\operatorname{codim} R(Q) = \dim Y/(X + Y_0)$ are both finite.

Theorem 2.1 applies to the case when $T : X/X_0 \mapsto Y/Y_0$ and $S : D(S) \subset Y/Y_0 \mapsto Z/Z_0$ are Fredholm q-morphisms. More generally, we have:

Theorem 3.8 *Let* $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ *and* $W/W_0 \in \mathcal{Q}(\mathcal{W})$. *Let also* $T : X/X_0 \mapsto Y/Y_0$ *and* $S : Z/Z_0 \mapsto W/W_0$ *be* q*-morphisms. Assume that the maps* S, T *are Fredholm and that* $\dim(Y \cap Z_0)/(Y_0 \cap Z_0) < \infty$ *and* $\dim Z/(Y \cap Z + Z_0) < \infty$ *are finite. Then* $\dim Y/(R_0(T) + Y \cap Z)) < \infty$, $\dim(Y_0 \cap N_0(S) + Z_0)/Z_0) < \infty$ *the map* $S \circ T$ *is Fredholm and we have*

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(T) + \mathrm{ind}(S) +$$

$$\dim(Y \cap Z_0)/(Y_0 \cap Z_0) + \dim Y/(R_0(T) + Y \cap Z)$$

$$- \dim(N_0(S) \cap Y_0 + Z_0)/Z_0 - \dim Z/(Y \cap Z + Z_0).$$

Theorem 3.8 has the following important consequence

Corollary 3.9 *Let* $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ *and* $W/W_0 \in \mathcal{Q}(\mathcal{W})$ *be such that* $Z/Z_0 \prec Y/Y_0$. *Let also* $T : X/X_0 \mapsto Y/Y_0$ *and* $S : Z/Z_0 \mapsto W/W_0$ *be q-morphisms.*

If the maps S, T *are Fredholm, then the dimensions*

$$\dim Y/(R_0(T)+Z) \quad \text{and} \quad \dim(Y_0 \cap N_0(S))/Z_0$$

are finite, the map $S \circ T$ *is Fredholm and we have*

$$\operatorname{ind}(S \circ T) = \operatorname{ind}(T) + \operatorname{ind}(S) +$$
$$\dim Y/(R_0(T)+Z) - \dim(N_0(S) \cap Y_0)/Z_0.$$

Remark 3.10 (1) The map $U : (Y \cap Z + Y_0)/Y_0 \mapsto Z/(Y_0 \cap Z + Z_0)$ from Remark 3.4 is an isomorphism if and only if $Y \cap Z_0 = Y_0 \cap Z_0$ and $Y \cap Z + Z_0 = Z$. For this reason, assuming T, S Fredholm and replacing the condition $Z/Z_0 \prec Y/Y_0$ by the more general condition $Y \cap Z_0 = Y_0 \cap Z_0$ and $Y \cap Z + Z_0 = Z$ in the statement of Corollary 3.9, we get the formula

$$\text{ind}(S \circ T) = \text{ind}(T) + \text{ind}(S)$$

$$+\text{dim}Y/(R_0(T) + Y \cap Z)$$

$$-\text{dim}(N_0(S) \cap Y_0 + Z_0)/Z_0,$$

via a similar argumrnt.

(2) Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $X/X_0 \prec Y/Y_0$. We may consider the iterates $T \circ T$, $T \circ T \circ T$ etc., which are unambiguously defined. In fact, defining T^0 as the q-inclusion $X/X_0 \prec Y/Y_0$, for every integer $n \geq 1$ we may define by induction $T^n = T \circ T^{n-1}$. Note that we may actually consider polynomials of T, as Arens did.

4 Linear relations as quotient morphisms

In this section we discuss linear relations as particular cases of quotient morphisms. The linear relation Z can be associated with the map $Q_Z : D(Z) \mapsto R(Z)/M(Z)$, where $Q_Z(x) = y + M(Z)$ whenever $(x, y) \in Z$.

Clearly, Q_Z is a quotient morphism of a particular form.

Proposition 4.1 *The map*

$$\mathrm{Lat}(\mathcal{X} \times \mathcal{Y}) \ni Z \mapsto Q_Z \in \mathcal{QM}(\mathcal{X}, \mathcal{Y})$$

is injective, and its range consists of all quotient morphisms $Q \in \mathcal{QM}(\mathcal{X}, \mathcal{Y})$ *of the form* $Q : X \mapsto Y/Y_0$*, which are surjective.*

The previous proposition allows us to designate the uniquely determined quotient morphism Q_Z as the *morphism of* Z, for any relation $Z \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$.

If $Z \subset X \times Y$ for some $X \in \mathrm{Lat}(\mathcal{X})$, $Y \in \mathrm{Lat}(\mathcal{Y})$, the q-morphism from $D(Z)$ into $Y/M(Z)$ induced by Q_Z will be designated by Q_Z^Y.

Proposition 4.2 *Given two relations $Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$ and $Z' \in \mathrm{Lat}(\mathcal{Y} \times \mathcal{Z})$, we have the equality*

$$Q_{Z' \circ Z''} = Q_{Z'} \circ Q_{Z''}.$$

Remark 4.3 Let $X \in \mathrm{Lat}(\mathcal{X}), Y \in \mathrm{Lat}(\mathcal{Y})$ and let $Z \subset X \times Y$ be a relation. Recall that the relation Z is said to be *Fredholm* if both $\mathrm{dim}N(Z)$ and $\mathrm{dim}Y/R(Z)$ are finite. One sets $\mathrm{ind}(Z) = \mathrm{dim}N(Z) - \mathrm{dim}Y/R(Z)$, which is the *index* of Z. Clearly, the index of Z depends strongly on the space Y.

Note that $Z \subset X \times Y$ is Fredholm if and only if the map $Q_Z^Y : D(Z) \mapsto Y/M(Z)$ induced by the morphism Q_Z of Z is Fredholm and $\mathrm{ind}(Z) = \mathrm{ind}(Q_Z^Y)$.

The next assertion is an extension of a result by R. Cross.

Theorem 4.4 *Let X, Y, W be linear spaces and let $Z'' \subset X \times Y$, $Z' \subset Y \times W$ be Fredholm relations. Then the dimensions* $\dim(Y/(D(Z') + R(Z''))$ *and* $\dim N(Z') \cap M(Z'')$ *are finite, $Z' \circ Z''$ is Fredholm and*

$$\mathrm{ind}(Z' \circ Z'') = \mathrm{ind}(Z') + \mathrm{ind}(Z'') +$$

$$\dim(Y/(D(Z') + R(Z'')) - \dim N(Z') \cap M(Z'').$$

Corollary 4.5 *Let X, Y, W be linear spaces and let $Z'' \subset X \times Y$, $Z' \subset Y \times W$ be Fredholm relations. Assume that $D(Z') = Y$. Then $Z' \circ Z''$ is Fredholm and*

$$\mathrm{ind}(Z' \circ Z'') = \mathrm{ind}(Z') + \mathrm{ind}(Z'')$$

$$-\dim N(Z') \cap M(Z'').$$

The previous corollary is stated in the monograph by Cross.

5 The graph norm

Every q-morphism (in particular each relation) in normed spaces has a graph type norm on its lifted domain of definition.

Definition 5.1 Let $\mathcal{X}, \mathcal{Y}$ be normed spaces and let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism in $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$. We set

$$\|x\|_T = \inf\{(\|x\|^2 + \|y\|^2)^{1/2};$$
$$(x, y) \in G_0(T)\}, \; x \in X.$$

Theorem 5.2 *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces and let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism in $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$. The map $\| * \|_T$ is a norm on X, with the following properties:*

(1) *If* $\mathcal{X}, \mathcal{Y}$ *are Banach spaces and* $G_0(T)$ *is closed, then* $(X, \| * \|_T)$ *is a Banach space.*

(2) *If If* $\mathcal{X}, \mathcal{Y}$ *are Hilbert spaces and* $G_0(T)$ *is closed, then* $(X, \| * \|_T)$ *is a Hilbert space.*

Corollary 5.3 *Let* $\mathcal{X}, \mathcal{Y}$ *be normed spaces and let* $Z \subset \mathcal{X} \times \mathcal{Y}$ *be a relation. The map*

$$\|x\|_Z = \inf\{(\|x\|^2 + \|y\|^2)^{1/2}; (x, y) \in Z\}$$

is a norm on $D(Z)$*, with the following properties:*

(1) *If* $\mathcal{X}, \mathcal{Y}$ *are Banach spaces and* Z *is closed, then* $(D(Z), \| * \|_Z)$ *is a Banach space.*

(2) *If If* $\mathcal{X}, \mathcal{Y}$ *are Hilbert spaces and* Z *is closed, then* $(D(Z), \| * \|_Z)$ *is a Hilbert space.*

Remark 5.4 If $\mathcal{X}, \mathcal{Y}$ are normed spaces and $T : X/X_0 \mapsto Y/Y_0$ is a q-morphism in $\mathcal{QM}(\mathcal{X}, \mathcal{Y})$, we may set

$$\|x\|_T = \inf\{\|x\| + \|y\|; (x, y) \in G_0(T)\},$$

for all $x \in X$, which is also a norm on X, somewhat simpler than that in the previous theorem. Similarly, if $Z \subset \mathcal{X} \times \mathcal{Y}$ is a relation, the quantity

$$\inf\{\|x\| + \|y\|; (x, y) \in Z\},\ x \in D(Z)$$

is also a norm on $D(Z)$. Nevertheless, these expressions are inappropriate in the Hilbert space context.

6 Analytic functional calculus for linear relations

Let X denote a (nonnull) complex Banach space and let $\mathcal{B}(X)$ be the Banach algebra of bounded linear operators defined on X. The symbol $\mathbb{C}_\infty$ will stand for the one-point compactification of the complex plane $\mathbb{C}$. The spectrum and the resolvent set associated to a relation will be generally regarded as subsets of $\mathbb{C}_\infty$. As usually, we often identify a linear operator with the relation given by its graph.

Let Z denote a fixed **closed** linear relation in $X \times X$. The relation λI_X, where λ is a complex number and I_X is the "identical relation" $\{(x, x); x \in X\}$, will be simply denoted by λ.

A point $\lambda \in \mathbb{C}$ is said to be *regular* for Z if $(\lambda - Z)^{-1} \in \mathcal{B}(X)$. The point ∞ is *regular* for Z if there exists an $r > 0$ such that for every complex λ with $|\lambda| > r$ we have $(\lambda - Z)^{-1} \in \mathcal{B}(X)$ and the set $\{(\lambda - Z)^{-1}; |\lambda| > r\}$ is bounded in $\mathcal{B}(X)$.

The *resolvent set* of Z is the set of all regular points $\lambda \in \mathbb{C}_\infty$.

The *spectrum* of Z is the set $\sigma(Z) = \mathbb{C}_\infty \setminus \rho(Z)$ (possibly empty or $\mathbb{C}_\infty$).

Remark 6.1 (1) If $Z = G(A)$ with $A \in \mathcal{B}(X)$, then $\sigma(Z) = \sigma(A)$, where $\sigma(A)$ is the usual spectrum of A.

(2) The point $\lambda \in \mathbb{C}$ is a regular point for Z if and only if $N(\lambda - Z) = \{0\}$ and $R(\lambda - Z) = X$, via the closed graph theorem.

(3) If $\lambda \in \mathbb{C}$ is a regular point for Z, the operator $(\lambda - Z)^{-1} \in \mathcal{B}(X)$ is, in general, neither injective nor surjective. For instance, if $P \in \mathcal{B}(X)$ is a (proper) projection and $Z = G(P)^{-1}$, then $Z^{-1} = P$ is neither injective nor surjective.

(4) This spectrum is different from that appearing in the monograph by Cross.

Theorem 6.2 *If $\lambda, \mu \in \rho(Z) \cap \mathbb{C}$, then*

$$(\mu - Z)^{-1} - (\lambda - Z)^{-1} =$$
$$(\lambda - \mu)(\mu - Z)^{-1}(\lambda - Z)^{-1}.$$

The resolvent set $\rho(Z)$ is an open subset of $\mathbb{C}_\infty$ and the resolvent function $\rho(Z) \cap \mathbb{C} \ni \lambda \mapsto (\lambda - Z)^{-1} \in \mathcal{B}(X)$ is analytic and has an analytic extension to $\rho(Z)$.

The next result characterizes the emptiness of the spectrum of a closed relation.

Corollary 6.3 *The spectrum $\sigma(Z)$ of the relation Z is a compact subset of $\mathbb{C}_\infty$. We have $\sigma(Z) = \emptyset$ if and only if $Z = G(S)^{-1}$, where $S \in \mathcal{B}(X)$ satisfies $S^2 = 0$.*

Furthermore, if S is densely defined, then $\sigma(Z) \neq \emptyset$.

We denote by $\mathcal{O}(Z)$ the set of all functions that are analytic in a neighborhood of $\sigma(Z)$. When $\sigma(Z) \ni \infty$, the functions from $\mathcal{O}(Z)$ are supposed to be analytic at infinity.

Let $U \supset \sigma(Z)$ be open. We can find an open set Δ with $\sigma(Z) \subset \Delta \subset \bar{\Delta} \subset U$, whose boundary, say Γ, is a finite system of rectifiable Jordan curves.

If $f \in \mathcal{O}(Z)$ and U is the domain of definition of f, we set

$$f(Z) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - Z)^{-1} d\lambda,$$

when $\infty \notin \sigma(Z)$, and

$$f(Z) = f(\infty) I_X + \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - Z)^{-1} d\lambda,$$

when $\infty \in \sigma(Z)$, where Γ is an admissible contour surrounding $\sigma(T)$.

Note that, because of the analyticity of the involved functions, the operator $f(T)$ does not depend on the particular choice of the contour Γ.

Theorem 6.4 *The map $f \mapsto f(Z)$ of $\mathcal{O}(Z)$ into $\mathcal{B}(X)$ is an algebra homomorphism.*

Remark 6.5 If X_0, X_1 are closed subspaces of X with $X_0 \cap X_1 = \{0\}$ and $X_0 + X_1$ closed, we use the symbol $X_0 \dot{+} X_1$ to express that the sum of X_0 and X_1 is direct in X. If there exist closed relations $Z_j \subset X_j \times X_j$, $j = 0, 1$, then $Z = Z_0 \dot{+} Z_1$ is a closed relation with the property $\sigma(Z) = \sigma(Z_0) \cup \sigma(Z_1)$.

In the case of unbounded operators, the boundedness of the spectrum is equivalent to the boundedness of the corresponding operator. In the case of relations, we have the following:

Theorem 6.6 *Given a closed relation $Z \subset X \times X$, the boundedness of $\sigma(Z)$ is equivalent to the existence of closed subspaces X_0, X_1 with $X_0 \dot{+} X_1 = X$, and operators $D_0 \in \mathcal{B}(X_0)$ with $D_0^2 = 0$, and $A_1 \in \mathcal{B}(X_1)$, such that*

$$Z = G(D_0)^{-1} \dot{+} G(A_1).$$

In this case, one has $\sigma(Z) = \sigma(A_1)$.

Corollary 6.7 *Given a closed relation $Z \subset X \times X$ with a bounded spectrum, there exist closed subspaces X_0, X_1 with $X_0 \dot{+} X_1 = X$, and an operator $A_1 \in \mathcal{B}(X_1)$, such that $f(Z) = 0_0 \dot{+} f(A_1)$ for every analytic function f in a neighborhood of $\sigma(T)$, where 0_0 is the null operator on X_0.*

Corollary 6.8 *Let $Z \subset X \times X$ be a densely defined closed relation. The spectrum of Z is a bounded subset of $\mathbb{C}$ if and only if Z is the graph of a bounded operator.*

Remark 6.9 For a relation $Z \subset X \times X$, one can consider its "norm" given by

$$\|Z\| = \sup_{\|x| \leq 1} \inf_{(x,y) \in Z} \|y\|.$$

If $\|Z\| < \infty$, the relation Z is said to be *continuous*. It is known (Cross) that if Z is densely defined, continuous and $\sigma(Z) \cap \mathbb{C}$ is bounded, then $\lim_{\lambda \to \infty}(\lambda - Z)^{-1} = 0$. In this case, ∞ is a regular point for Z, and so Z is the graph of a bounded operator, by virtue of the preceding corollary.

Example 6.10 Let $P \in \mathcal{B}(X)$ be a projection with $R(P) \neq X$, and let $Z = G(P)^{-1}$. We have already noticed that 0 is a regular point of Z (see Remark 6.1(3)). In fact, we can now easily compute the spectrum of Z.

Setting $X_0 = N(P)$ and $X_1 = R(P)$, we have $X = X_0 \dot{+} X_1$. Therefore, $Z = G(0_0)^{-1} \dot{+} G(I_1)$, where 0_0 is the null operator on X_0 and I_1 is the identity on X_1. Using this decomposition, we obtain $\sigma(Z) = \sigma(I_1) = \{1\}$, via Theorem 6.6.

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