Spectrum and Analytic Functional Calculus in Real and Quaternionic Frameworks

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Outline

ABSTRACT

We investigate the spectrum and the analytic functional calculus for quaternionic linear operators having as a model some elements from the spectral theory of real linear operators. We show that the construction of the analytic functional calculus for real linear operators can be refined to get a similar construction for quaternionic linear ones, in a classical manner, using a Riesz-Dunford-Gelfand type kernel. A quaternionic joint spectrum for pairs of operators is also discussed, and an analytic functional calculus is constructed, via a Martinelli type kernel in two variables.

References

In connection with the subject of this text, the following works will be sometimes quoted.

[1] A. G. Baskakov and A. S. Zagorskii, **Spectral Theory of Linear Relations on Real Banach Spaces**, Mathematical Notes, 2007, Vol. 81, No. 1, pp. 15–27.

[2] F. Colombo, J. Gantner, D. P. Kimsey, **Spectral Theory on the S-Spectrum for Quaternionic Operators**, Birkhäuser, 2018.

[3] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Short Motivation

While the spectrum of a linear operator is naturally defined for complex linear operators, it is sometimes useful to have it also for real linear operators. Such a definition goes back to Kaplansky, and can be stated as follows.

If T is a real linear operator on the real vector space $\mathcal{V}$, a point $u + iv$ ($u, v \in \mathbb{R}$) is in the **spectrum** of T if the operator $(uI - T)^2 + v^2 I$ is **not** invertible on $\mathcal{V}$, where I the identity on $\mathcal{V}$.

This definition involves only operators acting in $\mathcal{V}$, but the spectrum is a subset of the complex plane.

In fact, a stronger motivation of this choice can be given via the complexification of the space $\mathcal{V}$.

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Spectrum in Real Algebras

The spectrum may be defined actually in real algebras.

Let $\mathcal{A}$ be a unital real Banach algebra, not necessarily commutative. We identify the real field $\mathbb{R}$ with the subalgebra $\mathbb{R}\mathbf{1}$, where $\mathbf{1}$ is the unit of $\mathcal{A}$.

As mentioned above, the **(complex) spectrum** of an element $a \in \mathcal{A}$ may be defined by the equality

$$\sigma_{\mathbb{C}}(a) = \{u + iv; (u-a)^2 + v^2 \text{ is not invertible}, u, v \in \mathbb{R}\}.$$

This set is, in particular, **conjugate symmetric**, that is, $u + iv \in \sigma_{\mathbb{C}}(a)$ if and only if $u - iv \in \sigma_{\mathbb{C}}(a)$.

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Complexification

Fixing a unital real Banach algebra $\mathcal{A}$, we denote by $\mathcal{A}_{\mathbb{C}}$ the **complexification** of $\mathcal{A}$, which is given by $\mathcal{A}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, written simply as $\mathcal{A} + i\mathcal{A}$, where the sum is direct, identifying the element $1 \otimes a + i \otimes b$ with $a + ib$, for all $a, b \in \mathcal{A}$.

Then $\mathcal{A}_{\mathbb{C}}$ is a unital complex algebra with the product $(a + ib)(c + id) = ac - bd + i(ad + bc)$ for all $a, b, c, d \in \mathcal{A}$.

The algebra $\mathcal{A}_{\mathbb{C}}$ can be organized as a Banach algebra, with a (not necessarily unique) convenient norm, for instance with the norm $\|a + ib\| = \|a\| + \|b\|$, where $\| * \|$ is the norm of $\mathcal{A}$.

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Conjugation

In the algebra $\mathcal{A}_{\mathbb{C}}$, the complex numbers commute with all elements of $\mathcal{A}$. Moreover, we have a **conjugation** given by

$$\mathcal{A}_{\mathbb{C}} \ni a + ib \mapsto a - ib \in \mathcal{A}_{\mathbb{C}},\ a, b \in \mathcal{A},$$

which is a unital conjugate-linear automorphism, whose square is the identity. In particular, an arbitrary element $a + ib$ is invertible if and only if $a - ib$ is invertible.

Equality $\sigma_{\mathbb{C}}(a) = \sigma(a)$

The usual spectrum, defined for each element $c \in \mathcal{A}_{\mathbb{C}}$, will be denoted by $\sigma(c)$. Regarding the algebra $\mathcal{A}$ as a real subalgebra of $\mathcal{A}_{\mathbb{C}}$, that is, identifying it with the subalgebra $\{1 \otimes a; a \in \mathcal{A}\}$ we have the following.

Lemma 1

For every $a \in \mathcal{A}$ we have the equality $\sigma_{\mathbb{C}}(a) = \sigma(a)$.

The proof is related to the obvious identity

$$(u - a)^2 + v^2 = (u + iv - a)(u - iv - a).$$

Remark 1

The spectrum $\sigma(a)$ with $a \in \mathcal{A}$ is a conjugate symmetric set.

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Algebras of Operators

We are particularly interested to apply the discussion from above to the context of linear operators. The spectral theory for real linear operators is well known, and it is developed actually in the context of linear relations (see [1]).

We restrict ourselves to the case of linear operator but, unlike in [1], we construct a functional calculus using a larger class of analytic functions.

For a real or complex Banach space $\mathcal{V}$, we denote by $\mathcal{B}(\mathcal{V})$ the algebra of all bounded $\mathbb{R}$-(respectively $\mathbb{C}$-)linear operators on the space $\mathcal{V}$.
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Complex Extension of Real Operators

Let $\mathcal{V}$ be a real Banach space, and let $\mathcal{V}_{\mathbb{C}} = \mathcal{V} + i\mathcal{V}$ be its complexification.

Each operator $T \in \mathcal{B}(\mathcal{V})$ has a natural extension to an operator $T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, given by $T_{\mathbb{C}}(x + iy) = Tx + iTy$, $x, y \in \mathcal{V}$.

The map $\mathcal{B}(\mathcal{V}) \ni T \mapsto T_{\mathbb{C}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is $\mathbb{R}$-linear and multiplicative.

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Conjugation on $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$

Fixing an operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we define the operator $S^{\flat} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ to be equal to CSC, where $C : \mathcal{V}_{\mathbb{C}} \mapsto \mathcal{V}_{\mathbb{C}}$ is the conjugation of $\mathcal{V}$.

Some properties:
(1) The map $\mathcal{B}(\mathcal{V}_{\mathbb{C}}) \ni S \mapsto S^{\flat} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is a unital conjugate-linear automorphism, whose square is the identity.
(2) We have $S^{\flat} = S$ if and only if $S(\mathcal{V}) \subset \mathcal{V}$. In particular, we have $T_{\mathbb{C}}^{\flat} = T_{\mathbb{C}}$.
(3) Because $(S + S^{\flat})(\mathcal{V}) \subset \mathcal{V}, i(S - S^{\flat})(\mathcal{V}) \subset \mathcal{V}$, we have the equality $\mathcal{B}(\mathcal{V}_{\mathbb{C}}) = \mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In fact, if $S = U + iV$, with $U, V \in \mathcal{B}(\mathcal{V})$, we have $S^{\flat} = U - iV$, so the map $S \mapsto S^{\flat}$ the conjugation of the complex algebra $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$, induced by the conjugation C of $\mathcal{V}_{\mathbb{C}}$.

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(3) Because $(S + S^\flat)(\mathcal{V}) \subset \mathcal{V}, i(S - S^\flat)(\mathcal{V}) \subset \mathcal{V}$, we have the equality $\mathcal{B}(\mathcal{V}_{\mathbb{C}}) = \mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In fact, if $S = U + iV$, with $U, V \in \mathcal{B}(\mathcal{V})$, we have $S^\flat = U - iV$, so the map $S \mapsto S^\flat$ the conjugation of the complex algebra $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$, induced by the conjugation C of $\mathcal{V}_{\mathbb{C}}$.

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A Spectral Equality

For every operator $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we denote, as before, by $\sigma(S)$ its usual spectrum. As $\mathcal{B}(\mathcal{V})$ is a real algebra, the (complex) spectrum of an operator $T \in \mathcal{B}(\mathcal{V})$ is defined as before:

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A Comment

Having a concept of spectrum for real operators, an important step for further development is the construction of an analytic functional calculus. Such a construction has been done actually in the context of real linear relations in [1]. In what follows we shall present a similar construction for real linear operators. Although the case of linear relations is in principle, more general, unlike in [1], we perform our construction using a class of operator valued analytic functions insted of scalar valued analytic functions. Moreover, our arguments look simpler, and the construction is a model for a similar one, to get an analytic functional calculus for quaternionic linear operators.

Cauchy Domain

The use of vector versions of the Cauchy formula is simplified by adopting the following definition. Let $U \subset \mathbb{C}$ be open. An open subset $\Delta \subset U$ will be called a *Cauchy domain* (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented.

Note that a Cauchy domain is bounded but not necessarily connected.

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A Question

If $\mathcal{V}$ is a real Banach space, and so each operator $T \in \mathcal{B}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$, which is compact and nonempty, one can use the classical Riesz-Dunford functional calculus, applied to $T_{\mathbb{C}}$, in a slightly generalized form. Nevertheless, the values of this functional calculus are, in general, operators from $\mathcal{B}(\mathcal{V}_{\mathbb{C}})$.

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A Vector-Valued Cauchy's Formula

To give an answer to the previous question, let us have a look at the general case, recalling a fundamental formula.

Definition 1

If $U \supset \sigma(T_{\mathbb{C}}) = \sigma_{\mathbb{C}}(T)$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic, we put

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain containing $\sigma(T_{\mathbb{C}})$ in U.

In fact, because $\sigma(T_{\mathbb{C}})$ is conjugate symmetric, we may and shall assume that both U and Γ are conjugate symmetric.

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If $U \supset \sigma(T_{\mathbb{C}}) = \sigma_{\mathbb{C}}(T)$ is an open set in $\mathbb{C}$ and $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic, we put

$$F(T_{\mathbb{C}}) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T_{\mathbb{C}})^{-1} d\zeta,$$

where Γ is the boundary of a Cauchy domain containing $\sigma(T_{\mathbb{C}})$ in U.

In fact, because $\sigma(T_{\mathbb{C}})$ is conjugate symmetric, we may and shall assume that both U and Γ are conjugate symmetric.

Invariance of $\mathcal{V}$

The previous **question** will be answered once we have $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$, implying the invariance of $\mathcal{V}$ under $F(T_{\mathbb{C}})$. With the conditions of the remark from above we have the following.

Theorem 1

If $F : U \mapsto \mathcal{B}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^{\flat} = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}(\mathcal{V})$.

Algebraic Stem Functions

If $\mathcal{A}$ is a unital real Banach algebra, $\mathcal{A}_{\mathbb{C}}$ its complexification, and $U \subset \mathbb{C}$ is open, we denote by $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ the algebra of all analytic $\mathcal{A}_{\mathbb{C}}$-valued functions. If U is conjugate symmetric, and $C : \mathcal{A}_{\mathbb{C}} \mapsto \mathcal{A}_{\mathbb{C}}$ is its natural conjugation, we denote by $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ the real subalgebra of $\mathcal{O}(U, \mathcal{A}_{\mathbb{C}})$ consisting of those functions F with the property $F(\bar{\zeta}) = CF(\zeta)$ for all $\zeta \in U$.

Adapting a well known terminology, the functions from $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ will be called $\mathcal{A}_{\mathbb{C}}$-**valued (analytic) stem functions.**

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Scalar and Operator (Analytic) Stem Functions

When $\mathcal{V} = \mathbb{R}$, so $\mathcal{B}(\mathcal{V})_{\mathbb{C}} = \mathbb{C}$, the corresponding space of analytic functions will be denoted by $\mathcal{O}_s(U)$, which is a real algebra. Note that $\mathcal{O}_s(U, \mathcal{A}_{\mathbb{C}})$ is also a bilateral $\mathcal{O}_s(U)$-module.

When $\mathcal{A} = \mathcal{B}(\mathcal{V})$, the conjugation on $\mathcal{A}_{\mathbb{C}} = \mathcal{B}(\mathcal{V})_{\mathbb{C}}$ is given by the map $S \mapsto S^\flat$. Consequently, an analytic function $F : U \mapsto \mathcal{B}(\mathcal{V})_{\mathbb{C}}$ is a $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$-stem function if $F(\zeta)^\flat = F(\bar{\zeta})$ for all $\zeta \in U$.

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The Map $T \mapsto F(T)$

In the next result, we identify the algebra $\mathcal{B}(\mathcal{V})$ with a subalgebra of $\mathcal{B}(\mathcal{V})_{\mathbb{C}}$. In ths case, when $F \in \mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, we may write

$$F(T) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)(\zeta - T)^{-1} d\zeta,$$

because the right hand side of this formula belongs to $\mathcal{B}(\mathcal{V})$, via the previous Theorem.

Analytic Functional Calculus in $\mathcal{B}(\mathcal{V})$

The properties of the map $F \mapsto F(T)$, which can be called the *analytic functional calculus of T*, are summarized by the following.

Theorem 2

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}(\mathcal{V})$, with $\sigma_{\mathbb{C}}(T) \subset U$. Then the map

$$\mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{V})$$

is a real right module morphism. Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} A_n \zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^{m} A_n T^n \in \mathcal{B}(\mathcal{V})$.

Idea of Proof

The arguments are more or less standard. The $\mathbb{R}$-linearity of the maps
$\mathcal{O}_s(U, \mathcal{B}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}(\mathcal{V})$, $\mathcal{O}_s(U) \ni f \mapsto f(T) \in \mathcal{B}(\mathcal{V})$
is clear.

We also have
$(Ff)(T) = \frac{1}{2\pi i} \int_{\Gamma_0} F(\zeta) f(\zeta)(\zeta - T)^{-1} d\zeta =$
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$F(T)f(T)$,
for all $F \in \mathcal{B}(\mathcal{V})_{\mathbb{C}})$, $f \in \mathcal{O}_s(U)$, which means that the map is a real right module morphism.
Here Γ, Γ_0 are the boundaries of two Cauchy domains Δ, Δ_0 respectively, such that $\Delta \supset \bar{\Delta}_0$, and Δ_0 contains $\sigma_{\mathbb{C}}(T)$.

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Hamilton's Algebra

We start by recalling some well known facts.

Let $\mathbb{H}$ be the abstract algebra of quaternions, which is the four-dimensional $\mathbb{R}$-algebra with unit 1 (and so $\mathbb{H} \supset \mathbb{R}$) generated by the "imaginary units" $\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$, which satisfy $\mathbf{jk} = -\mathbf{kj} = \mathbf{l}$, $\mathbf{kl} = -\mathbf{lk} = \mathbf{j}$, $\mathbf{lj} = -\mathbf{jl} = \mathbf{k}$, $\mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1$.

The algebra $\mathbb{H}$ has a natural multiplicative norm given by $\|\mathbf{x}\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_0^2}$, $\mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l}$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and a natural involution
$\mathbb{H} \ni \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto \mathbf{x}^* = x_0 - x_1\mathbf{j} - x_2\mathbf{k} - x_3\mathbf{l} \in \mathbb{H}$.

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Other Information about $\mathbb{H}$

Every element $\mathbf{x} \in \mathbb{H} \setminus \{0\}$ is invertible, and

$$\mathbf{x}^{-1} = \|\mathbf{x}\|^{-2}\mathbf{x}^*.$$

For an arbitrary quaternion $\mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l},\ \ x_0, x_1, x_2, x_3 \in \mathbb{R}$, we set $\Re\mathbf{x} = x_0 = (\mathbf{x} + \mathbf{x}^*)/2$, and $\Im\mathbf{x} = x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} = (\mathbf{x} - \mathbf{x}^*)/2$, that is, the *real* and the *imaginary part* of $\mathbf{x}$, respectively.

The "imaginary units" $\mathbf{j}, \mathbf{k}, \mathbf{l}$ of the algebra $\mathbb{H}$ are considered independent of the imaginary unit i of the complex plane $\mathbb{C}$.

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The Algebra $\mathbb{M}$

We construct the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ of the $\mathbb{R}$-algebra $\mathbb{H}$, which is identified with the direct sum $\mathbb{H} + i\mathbb{H}$, denoted by $\mathbb{M}$, which is a unital complex algebra containing $\mathbb{C}$, such that the elements of $\mathbb{H}$ commute with all complex numbers.

As before, in the algebra $\mathbb{M}$ there exists a natural conjugation given by $\bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{a} = \mathbf{b} + i\mathbf{c}$ is arbitrary in $\mathbb{M}$, with $\mathbf{b}, \mathbf{c} \in \mathbb{H}$, which is a conjugate linear map. Obviously, $\bar{\mathbf{a}} = \mathbf{a}$ if and only if $\mathbf{a} \in \mathbb{H}$, which is a useful characterization of the elements of $\mathbb{H}$ among those of $\mathbb{M}$.

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Spectrum of a Quaternion

In the algebra $\mathbb{M}$, we have the identities $(\lambda - \mathbf{x}^*)(\lambda - \mathbf{x}) = (\lambda - \mathbf{x})(\lambda - \mathbf{x}^*) = \lambda^2 - \lambda(\mathbf{x} + \mathbf{x}^*) + \|\mathbf{x}\|^2 \in \mathbb{C}$, for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{H}$. The element $\lambda - \mathbf{x} \in \mathbb{M}$ is invertible if and only if the complex number $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is nonnull.

Because $\lambda - \mathbf{x} \in \mathbb{M}$ is not invertible if and only if

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the **spectrum** of a quaternion $\mathbf{x} \in \mathbb{H}$ in the algebra $\mathbb{M}$ is given by the equality $\sigma(\mathbf{x}) = \{s_\pm(\mathbf{x})\}$, where

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$\mathbb{H}$-Vector Spaces

A **right $\mathbb{H}$-vector space** $\mathcal{V}$ is a real vector space having a right multiplication with the elements of $\mathbb{H}$, such that
$(x+y)\mathbf{q} = x\mathbf{q} + y\mathbf{q}$, $x(\mathbf{q}+\mathbf{s}) = x\mathbf{q} + x\mathbf{s}$, $x(\mathbf{qs}) = (x\mathbf{q})\mathbf{s}$
for all $x, y \in \mathcal{V}$ and $\mathbf{q}, \mathbf{s} \in \mathbb{H}$.

In a similar way, one defines the concept of a **left $\mathbb{H}$-vector space.**

A real vector space $\mathcal{V}$ will be said to be an **$\mathbb{H}$-vector space** if it is simultaneously a right $\mathbb{H}$- and a left $\mathbb{H}$-vector space.

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$\mathbb{H}$-Linearity

Let $\mathcal{V}$ be an $\mathbb{H}$-vector space, which is also a Banach space. If the maps $L_{\mathbf{q}} : \mathcal{V} \mapsto \mathcal{V}$, $L_{\mathbf{q}}x = \mathbf{q}x$, $x \in \mathcal{V}$, and $R_{\mathbf{q}} : \mathcal{V} \mapsto \mathcal{V}$, $R_{\mathbf{q}}x = x\mathbf{q}$, $x \in \mathcal{V}$ are continuous for all $\mathbf{q} \in \mathbb{H}$, the space $\mathcal{V}$ will be called a **Banach $\mathbb{H}$- space.**

If $\mathcal{V}$ is a Banach $\mathbb{H}$-space the operator $T \in \mathcal{B}(\mathcal{V})$ is **right $\mathbb{H}$-linear** if $T(x\mathbf{q}) = T(x)\mathbf{q}$ for all $x \in \mathcal{V}$ and $\mathbf{q} \in \mathbb{H}$. The set of right $\mathbb{H}$ linear operators will be denoted by $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, which is, in particular, a unital real algebra. Moreover, $TR_{\mathbf{q}} = R_{\mathbf{q}}T$ for all $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ and $\mathbf{q} \in \mathbb{H}$.

One can also define the concept of **left $\mathbb{H}$-linear operator** but we shall mainly work with right $\mathbb{H}$-linear operators.

The operators $L_{\mathbf{q}} : \mathcal{V} \mapsto \mathcal{V}$, $L_{\mathbf{q}}x = \mathbf{q}x$, $x \in \mathcal{V}$, are right $\mathbb{H}$-linear operators.

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Extending $\mathbb{H}$-Right Linear Operators

To adapt the previus discussion regarding the real algebras, we first consider the complexification $\mathcal{V}_{\mathbb{C}}$ of a Banach $\mathbb{H}$-space. $\mathcal{V}$. Because $\mathcal{V}$ is an $\mathbb{H}$-bimodule, the space $\mathcal{V}_{\mathbb{C}}$ is actually an $\mathbb{M}$-bimodule, via the multiplications

$$(\mathbf{q} + i\mathbf{s})(x + iy) = \mathbf{q}x - \mathbf{s}y + i(\mathbf{q}y + \mathbf{s}x),$$

$$(x + iy)(\mathbf{q} + i\mathbf{s}) = x\mathbf{q} - y\mathbf{s} + i(y\mathbf{q} + x\mathbf{s}),$$

for all $\mathbf{q} + i\mathbf{s} \in \mathbb{M}$, $x + iy \in \mathcal{V}_{\mathbb{C}}$.

Let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$. Then the operator $T_{\mathbb{C}}$ is right $\mathbb{M}$-linear, that is $T_{\mathbb{C}}((x + iy)(\mathbf{q} + i\mathbf{s})) = T_{\mathbb{C}}(x + iy)(\mathbf{q} + i\mathbf{s})$ for all $\mathbf{q} + i\mathbf{s} \in \mathbb{M}$, $x + iy \in \mathcal{V}_{\mathbb{C}}$, by a direct computation.

Extending $\mathbb{H}$-Right Linear Operators

To adapt the previus discussion regarding the real algebras, we first consider the complexification $\mathcal{V}_{\mathbb{C}}$ of a Banach $\mathbb{H}$-space. $\mathcal{V}$. Because $\mathcal{V}$ is an $\mathbb{H}$-bimodule, the space $\mathcal{V}_{\mathbb{C}}$ is actually an $\mathbb{M}$-bimodule, via the multiplications

$$(\mathbf{q} + i\mathbf{s})(x + iy) = \mathbf{q}x - \mathbf{s}y + i(\mathbf{q}y + \mathbf{s}x),$$

$$(x + iy)(\mathbf{q} + i\mathbf{s}) = x\mathbf{q} - y\mathbf{s} + i(y\mathbf{q} + x\mathbf{s}),$$

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Let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$. Then the operator $T_{\mathbb{C}}$ is right $\mathbb{M}$-linear, that is $T_{\mathbb{C}}((x + iy)(\mathbf{q} + i\mathbf{s})) = T_{\mathbb{C}}(x + iy)(\mathbf{q} + i\mathbf{s})$ for all $\mathbf{q} + i\mathbf{s} \in \mathbb{M}$, $x + iy \in \mathcal{V}_{\mathbb{C}}$, by a direct computation.

The Algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$

Let $\mathcal{V}$ be a Banach $\mathbb{H}$-space and let C be the conjugation on $\mathcal{V}_{\mathbb{C}}$. As before, for every $S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$, we put $S^\flat = CSC$. The right multiplication $R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V})$ with the quaternion $\mathbf{q}$ on $\mathcal{V}_{\mathbb{C}}$ will be also denoted by $R_{\mathbf{q}} \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})$. We set

$$\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}) = \{S \in \mathcal{B}(\mathcal{V}_{\mathbb{C}}); SR_{\mathbf{q}} = R_{\mathbf{q}}S\}.$$

Because $CR_{\mathbf{q}} = R_{\mathbf{q}}C$, if $S \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$, then $S^\flat \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$. In fact, as we have $(S + S^\flat)(\mathcal{V}) \subset \mathcal{V}$ and $i(S - S^\flat)(\mathcal{V}) \subset \mathcal{V}$, it folows, as in the real case, that $\mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}}) = \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$.

Quaternionic Spectrum

The quaternionic spectrum, largely used in the literature (see [2] or [3]) is the following.

Definition

For a given operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, its **quaternionic spectrum** (or simply the *Q*-**spectrum**) is given by the equality

$$\sigma_{\mathbb{H}}(T) := \{\mathbf{q} \in \mathbb{H};\ T^2 - 2(Re\mathbf{q})T + \|\mathbf{q}\|^2) \text{ non invertible}\}.$$

The complement $\rho_{\mathbb{H}}(T) = \mathbb{H} \setminus \sigma_{\mathbb{H}}(T)$ is called the **quaternionic resolvent** (or simply the *Q*-**resolvent**) of T.

Note that, if $\mathbf{q} \in \sigma_{\mathbb{H}}(T))$ then

$$\{\mathbf{s} \in \mathbb{H};\ \Re\mathbf{s} = \Re\mathbf{q},\ \|\Im\mathbf{s}\| = \|\Im\mathbf{q}\|\} \subset \sigma_{\mathbb{H}}(T).$$

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Quaternionic Spectrum via Complex Spectrum

If $\mathcal{V}$ is a Banach $\mathbb{H}$-space, then the complexification $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} = \mathcal{B}^{\mathrm{r}}(\mathcal{V}) + i\mathcal{B}^{\mathrm{r}}(\mathcal{V}) = \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ is a unital complex Banach algebra, where the complex numbers commute with the elements of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$. For this reason, for each $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the complex spectrum

$$\sigma_{\mathbb{C}}(T) = \{\lambda \in \mathbb{C}; \lambda - T_{\mathbb{C}} \text{ not invertible}\},$$

and the associated complex resolvent set $\rho_{\mathbb{C}}(T) = \mathbb{C} \setminus \sigma_{\mathbb{C}}(T)$.

Lemma 2

For every $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ we have the equality

$$\sigma_{\mathbb{H}}(T) = \{\mathbf{q} \in \mathbb{H}; \sigma_{\mathbb{C}}(T) \cap \sigma(\mathbf{q}) \neq \emptyset\}. \quad (1)$$

Analytic Functions of a Right $\mathbb{H}$-Linear Operator

If $\mathcal{V}$ is a Banach $\mathbb{H}$-space, so each operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ has a complex spectrum $\sigma_{\mathbb{C}}(T)$, as in the case of real operators we may construct an analytic functional calculus using the classical Riesz-Dunford functional calculus, in a generalized form. In this case, our basic algebra will be $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$, endowed with the continuous conjugation $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}} \ni S \mapsto S^{\flat} \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$

Theorem 3

If $F : U \mapsto \mathcal{B}^{\mathrm{r}}(\mathcal{V}_{\mathbb{C}})$ is analytic and $F(\zeta)^{\flat} = F(\bar{\zeta})$ for all $\zeta \in U$, then $F(T_{\mathbb{C}})^{\flat} = F(T_{\mathbb{C}})$ for all $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

The proof of this theorem is similar to Theorem 1, valid for real linear operators.

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A Formula for $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$

As in the real case, we identify the algebra $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with a subalgebra of $\mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$. Let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$. In ths case, when $F \in \mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}})$, we may write

$$F(T) = \frac{1}{2\pi i} \int_{\Gamma} F(\zeta)(\zeta - T)^{-1} d\zeta,$$

that is, replacing $T_{\mathbb{C}}$ simply by T, because the right hand side of this formula belongs to $\mathcal{B}^{\mathrm{r}}(\mathcal{V})$, and so $F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, via the previous theorem.

The next result is a version of Theorem 2, valid for some quaternionic operators.

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Quaternionic Analytic Functional Calculus

Theorem 4

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with $\sigma_{\mathbb{C}}(T) \subset U$. Then the map

$$\mathcal{O}_s(U, \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}) \ni F \mapsto F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$$

is a real right module morphism. Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} A_n \zeta^n$, $\zeta \in \mathbb{C}$, with $A_n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ for all $n = 0, 1, \ldots, m$, we have $P(T) = \sum_{n=0}^{m} A_n T^n \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$.

The proof of this result is similar to that of Theorem 2. We only note that $F(T) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ rather than $F(T) \in \mathcal{B}(\mathcal{V})$ because $F(\zeta) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})_{\mathbb{C}}$ for all $\zeta \in U$.

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Spectrally Saturared Sets in $\mathbb{H}$

For further use, we recall that a subset $\Omega \subset \mathbb{H}$ is said to be **spectrally saturated** if whenever $\sigma(\mathbf{h}) = \sigma(\mathbf{q})$ for some $\mathbf{h} \in \mathbb{H}$ and $\mathbf{q} \in \Omega$, we also have $\mathbf{h} \in \Omega$. This concept coincides with that of axially symmetric set, appearing in [2], [3] etc...

Note that, for every $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$, the Q-spectrum $\sigma_{\mathbb{H}}(T)$ is spectrally saturated.

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Special Spaces of $\mathbb{H}$-Valued Functions

We may obtain a particular case of Theorem 4 regarding the algebra $\mathbb{H}$ as a Banach $\mathbb{H}$-space.

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let

$$U = \mathfrak{S}(\Omega) := \{\lambda \in \mathbb{C}, \exists \mathbf{q} \in \Omega, \lambda \in \sigma(\mathbf{q})\},$$

which is an open set.

Denoting by $f_{\mathbb{H}}$ the function $\Omega \ni \mathbf{q} \mapsto f(\mathbf{q}), \mathbf{q} \in \Omega$ for every $f \in \mathcal{O}_s(U)$, we set

$$\mathcal{R}(\Omega) = \{f_{\mathbb{H}}; f \in \mathcal{O}_s(U)\},$$

which is an $\mathbb{R}$-algebra. Defining the $F_{\mathbb{H}}$ in a similar way for each $F \in \mathcal{O}_s(U, \mathbb{M})$, we set

$$\mathcal{R}(\Omega, \mathbb{H}) = \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U, \mathbb{M})\}.$$

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Some Isomorphisms

The next result is a consequence of Theorem 4.

Theorem 5

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega)$. The space $\mathcal{R}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}(\Omega, \mathbb{H})$ is a right $\mathcal{R}(\Omega)$-module, the map

$$\mathcal{O}_s(U, \mathbb{M}) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H})$$

is a right module isomorphism, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega)$$

is a real algebra isomorphism.
Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Two Analytic Functional Calculi

The space $\mathcal{R}(\Omega, \mathbb{H})$ is independently introduced in [3], consisting of so-called slice hyperholomorphic functions, and being used to construct a quaternionic functional calculus (see also [2]). Specifically, fixing a spectrally saturated open set $\Omega \subset \mathbb{H}$ and an operator $T \in \mathcal{B}^{\mathrm{r}}(\mathcal{V})$ with $\sigma_{\mathbb{H}}(T) \subset \Omega$, one defines a map

$$\mathcal{R}(\Omega, \mathbb{H}) \ni F \mapsto \Phi_T(F) \in \mathcal{B}^{\mathrm{r}}(\mathcal{V}),$$

also said to be an analytic functional calculus for T, having properties similar to those of the functional calculus given by Theorem 4, and constructed via a non-commutative Cauchy type kernel. However, this calculus is NOT more general than that from Theorem 4 because the spaces $\mathcal{O}_s(U, \mathbb{M})$ and $\mathcal{R}(\Omega, \mathbb{H})$ are isomorphic, as the same Theorem 4 shows.

Quaternionic Joint Spectrum of Pairs

A spectral theory for commuting pairs of real operators can be also related to the quaternionic framework.

If $\mathcal{V}$ is an arbitrary vector space, we denote by $\mathcal{V}^2$ the Cartesian product $\mathcal{V} \times \mathcal{V}$.

Let $\mathcal{V}$ be a real Banach space, and let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators. The extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ also consists of commuting operators. Inspired by what follows, we set

$$Q(\mathbf{T}_{\mathbb{C}}) := \begin{pmatrix} T_{1\mathbb{C}}, & T_{2\mathbb{C}}, \\ -T_{2\mathbb{C}}, & T_{1\mathbb{C}}, \end{pmatrix}$$

which acts on the complex Banach space $\mathcal{V}_{\mathbb{C}}^2$.

A Representation of $\mathbb{H}$

In many applications, it is more convenient to work with matrix quaternions rather than with abstract quaternions. Specifically, one considers the injective unital algebra morphism

$$\mathbb{H} \ni x_1 + y_1\mathbf{j} + x_2\mathbf{k} + y_2\mathbf{l} \mapsto \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & x_1 - iy_1 \end{pmatrix} \in \mathbb{M}_2,$$

with $x_1, y_1, x_2, y_2 \in \mathbb{R}$, where $\mathbb{M}_2$ is the complex algebra of 2×2-matrix, whose image, denoted by $\mathbb{H}_2$ is the real algebra of matrix quaternions. The elements of $\mathbb{H}_2$ can be also written as matrices of the form

$$Q(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2.$$

Q-Joint Spectrum

Definition

Let $\mathcal{V}$ be a real Banach space. For a given pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ of commuting operators, the set of those matrix quaternions $Q(\mathbf{z}) \in \mathbb{H}_2$, $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, such that the operator

$$T_1^2 + T_1^2 - 2\Re z_1 T_1 - 2\Re z_2 T_2 + |z_1|^2 + |z_2|^2$$

is invertible in $\mathcal{B}(\mathcal{V})$ is said to be the **quaternionic joint resolvent** (or simply the Q-**joint resolvent**) of $\mathbf{T}$, and is denoted by $\rho_{\mathbb{H}}(\mathbf{T})$. The complement $\sigma_{\mathbb{H}}(\mathbf{T}) = \mathbb{H}_2 \setminus \rho_{\mathbb{H}}(\mathbf{T})$ is called the **quaternionic joint spectrum** (or simply the Q-**joint spectrum**) of $\mathbf{T}$.

Q-Joint Spectrum via Invertibility

For every pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ we put $\mathbf{T}_{\mathbb{C}}^c = (T_{1\mathbb{C}}, -T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$, and for every pair $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ we put $\mathbf{z}^c = (\bar{z}_1, -z_2) \in \mathbb{C}^2$

Lemma 3

A matrix quaternion $Q(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^2$) is in the set $\rho_{\mathbb{H}}(\mathbf{T})$ if and only if the operators $Q(\mathbf{T}_{\mathbb{C}}) - Q(\mathbf{z})$, $Q(\mathbf{T}_{\mathbb{C}}^c) - Q(\mathbf{z}^c)$ are invertible in $\mathcal{B}(\mathcal{V}_{\mathbb{C}}^2)$.

This Lemma shows that the set $\sigma_{\mathbb{H}}(\mathbf{T})$ has the property $Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T})$ if and only if $Q(z^c) \in \sigma_{\mathbb{H}}(\mathbf{T}^c)$. Putting $\sigma_{\mathbb{C}^2}(\mathbf{T}) := \{\mathbf{z} \in \mathbb{C}^2; Q(\mathbf{z}) \in \sigma_{\mathbb{H}}(\mathbf{T}\}$, the set $\sigma_{\mathbb{C}^2}(\mathbf{T})$ has a similar property.

Connexion with Taylor's Spectrum

For the extended pair $\mathbf{T}_{\mathbb{C}} = (T_{1\mathbb{C}}, T_{2\mathbb{C}}) \in \mathcal{B}(\mathcal{V}_{\mathbb{C}})^2$ of the commuting pair $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})$ there is an interesting connexion with the *joint spectral theory* of J. L. Taylor. Specifically, if the operator

$$T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2$$

is invertible, then the point $\mathbf{z} = (z_1, z_2)$ belongs to the joint resolvent of $\mathbf{T}_{\mathbb{C}}$. Therefore, if $\sigma(\mathbf{T}_{\mathbb{C}})$ designates the Taylor spectrum of $T_{\mathbb{C}}$, we have the inclusion $\sigma(\mathbf{T}_{\mathbb{C}}) \subset \sigma_{\mathbb{C}^2}(\mathbf{T})$. In particular, for every complex-valued function f analytic in a neighborhood of $\sigma_{\mathbb{C}^2}(\mathbf{T})$, the operator $f(\mathbf{T}_{\mathbb{C}})$ can be computed via Taylor's analytic functional calculus. In fact, we have a Martinelli type formula for the analytic functional calculus.

An Analytic Functional Calculus for Pairs

Theorem 6

Let $\mathcal{V}$ be a real Banach space, let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$ be a pair of commuting operators, let $U \subset \mathbb{C}^2$ be an open set, let $D \subset U$ be a bounded domain containing $\sigma_{\mathbb{C}^2}(\mathbf{T})$, with piecewise-smooth boundary Σ, and let $f \in \mathcal{O}(U)$. Then $f(\mathbf{T}_{\mathbb{C}}) = \frac{1}{(2\pi i)^2} \int_\Sigma f(\mathbf{z}) L(\mathbf{z}, T_{\mathbb{C}})^{-2}[(\bar{z}_1 - T_{1\mathbb{C}})d\bar{z}_2 - (\bar{z}_2 - T_{2\mathbb{C}})d\bar{z}_1]d\mathbf{z}$, where $d\mathbf{z} = dz_1 dz_2$, and

$$L(\mathbf{z}, \mathbf{T}_{\mathbb{C}}) = T_{1\mathbb{C}}^2 + T_{2\mathbb{C}}^2 - 2\Re z_1 T_{1\mathbb{C}} - 2\Re z_2 T_{2\mathbb{C}} + |z_1|^2 + |z_2|^2.$$

An Open Problem

Find conditions implying the equality $f(\mathbf{T}_{\mathbb{C}})^{\flat} = f(\mathbf{T}_{\mathbb{C}})$, leading to an analytic functional calculus for commuting pairs $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(\mathcal{V})^2$.

Conclusion

Our investigations show that there exists a parallel approach to the spectral theory of quaternionic linear operators, starting directly from spaces of analytic functions instead of spaces of regular quaternionic functions, leading, practically, to the same results, via more transparent techniques.

The present work is a continuation of the author's work entitled **Quaternionic Regularity via Analytic Functional Calculus,** which can be found in the collection

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Thank you very much for your attention !