POSITIVE EXTENSIONS AND INTEGRAL REPRESENTATIONS VIA SPACES OF FRACTIONS

F.-H. Vasilescu
University of Lille I, France

1. Introduction

Let $\mathbf{Z}_+^n$ be the set of all multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$, let $\mathcal{P}_n$ be the algebra of all polynomial functions in $t = (t_1, \ldots, t_n) \in \mathbf{R}^n$ with complex coefficients and let $\mathcal{P}_{n,\alpha}$ be the vector space generated by the monomials $t^\beta = t_1^{\beta_1} \cdots t_n^{\beta_n}$, with $\beta_j \leq 2\alpha_j, \forall j, \alpha \in \mathbf{Z}_+^n$.

Set $(\mathbf{R}_\infty)^n = (\mathbf{R} \cup \{\infty\})^n$. Consider the family $\mathcal{Q}_n$ consisting of all rational functions of the form

$$q_\alpha(t) = (1+t_1^2)^{-\alpha_1} \cdots, (1+t_n^2)^{-\alpha_n},$$

$t \in \mathbf{R}^n$, where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}_+^n$ is arbitrary. Set also $p_\alpha(t) = q_\alpha(t)^{-1}$, $t \in \mathbf{R}^n$, $\alpha \in \mathbf{Z}_+^n$. The function q_α can be continuously extended to $(\mathbf{R}_\infty)^n \setminus \mathbf{R}^n$ for all $\alpha \in \mathbf{Z}_+^n$. The function p/p_α can be continuously extended to $(\mathbf{R}_\infty)^n \setminus \mathbf{R}^n$ for every $p \in \mathcal{P}_{n,\alpha}$, and so it can be regarded as an element of $C_{\mathbf{R}}((\mathbf{R}_\infty)^n)$. Therefore, $\mathcal{P}_{n,\alpha}$ is a subspace of $C_{\mathbf{R}}((\mathbf{R}_\infty)^n)/q_\alpha = p_\alpha C_{\mathbf{R}}((\mathbf{R}_\infty)^n)$ for all $\alpha \in \mathbf{Z}_+^n$, and so

$$\mathcal{P}_n \subset C_{\mathbf{R}}((\mathbf{R}_\infty)^n)/\mathcal{Q}_n,$$

which is an algebra of fractions.

Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ be an n-sequence of real numbers and let

$L_\gamma : \mathcal{P}_n \to \mathbf{C}$

be the *associated linear functional* given by $L_\gamma(t^\alpha) = \gamma_\alpha, \alpha \in \mathbf{Z}_+^n$, extended by linearity. Recall that the n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ of real numbers is said to be a *moment sequence* if there exists a positive measure μ on $\mathbf{R}^n$ such that $t^\alpha \in L^1(\mu)$ and $\gamma_\alpha = \int t^\alpha \, d\mu(t)$, $\alpha \in \mathbf{Z}_+^n$. The measure μ is said to be a *representing measure* for γ. Let us state a characterization of the *moment sequences.*

Theorem 1.1. *An n-sequence $\gamma = (\gamma_\alpha)_{\alpha \in \mathbf{Z}_+^n}$ $(\gamma_0 > 0)$ of real numbers is a moment sequence on $\mathbf{R}^n$ if and only if the associated linear functional L_γ has the properties $L_\gamma(p_\alpha) > 0$ and $|L_\gamma(p)| \leq L_\gamma(p_\alpha) \sup_{t \in \mathbf{R}^n} |q_\alpha(t)p(t)|$, $p \in \mathcal{P}_{n,\alpha}$, $\alpha \in \mathbf{Z}_+^n$.*

As we have $\mathcal{P}_n \subset C_{\mathbf{R}}((\mathbf{R}_\infty)^n)/\mathcal{Q}_n$, the linear map $L_\gamma : \mathcal{P}_n \mapsto \mathbb{C}$ associated to an n-sequence γ can be viewed as a linear map on a subspace of an algebra of fractions. In particular, the proof of Theorem 1.1 can be derived from general results in the framework of algebras of functions.

2. Spaces of fractions of continuous functions

Let Ω be a compact space and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, endowed with the sup norm $\|*\|_\infty$. We denote by $M(\Omega)$ the space of all complex-valued Borel measures on Ω. For every function $h \in C(\Omega)$, we set $Z(h) = \{\omega \in \Omega; h(\omega) = 0\}$. If $\mu \in M(\Omega)$, we denote by $|\mu| \in M(\Omega)$ the variation of μ.

Let $\mathcal{Q}$ be a family of nonnegative elements of $C(\Omega)$. The set $\mathcal{Q}$ is said to be a *set of denominators* if (i) $1 \in \mathcal{Q}$, (ii) $q', q'' \in \mathcal{Q}$ implies $q'q'' \in \mathcal{Q}$, and (iii) if $qh = 0$ for some $q \in \mathcal{Q}$ and $h \in C(\Omega)$, then $h = 0$. Using a set of denominators $\mathcal{Q}$, we can form the algebra of fractions $C(\Omega)/\mathcal{Q}$. If $C(\Omega)/q = \{f \in C(\Omega)/\mathcal{Q}; qf \in C(\Omega)\}$, we have $C(\Omega)/\mathcal{Q} = \cup_{q\in\mathcal{Q}} C(\Omega)/q$.

Setting $\|f\|_{\infty,q} = \|qf\|_{\infty}$ for each $f \in C(\Omega)/q$, the pair $(C(\Omega)/q, \|*\|_{\infty,q})$ becomes a Banach space. Hence, $C(\Omega)/\mathcal{Q}$ is an inductive limit of Banach spaces

Set $(C(\Omega)/q)_+ = \{f \in C(\Omega)/q; qf \geq 0\}$, which is a positive cone for each q.

Let $\mathcal{Q}_0 \subset \mathcal{Q}$, let $\mathcal{F} = \Sigma_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to \mathbb{C}$ be linear. The map ψ is continuous if the restriction $\psi|C(\Omega)/q$ is continuous for all $q \in \mathcal{Q}_0$.

Let us also remark that the linear functional $\psi : \mathcal{F} \to \mathbb{C}$ is said to be positive if $\psi|(C(\Omega)/q)_+ \geq 0$ for all $q \in \mathcal{Q}_0$.

The next result, which is an extension of the Riesz representation theorem, describes the dual of a space of fractions, defined as above.

Theorem 2.1. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$, let $\mathcal{F} = \sum_{q\in\mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{F} \to \mathbb{C}$ be linear. The functional ψ is continuous if and only if there exists a uniquely determined measure $\mu_\psi \in M(\Omega)$ such that $|\mu_\psi|(Z_q) = 0$, $1/q$ is $|\mu_\psi|$-integrable for all $q \in \mathcal{Q}_0$ and $\psi(f) = \int_\Omega f d\mu_\psi$ for all $f \in \mathcal{F}$.*

The functional $\psi : \mathcal{F} \to \mathbb{C}$ *is positive, if and only if it is continuous and the measure* μ_ψ *is positive.*

Corollary 2.2. *Let* $\mathcal{Q}_0 \subset \mathcal{Q}$ *be nonempty, let* $\mathcal{F} = \Sigma_{q \in \mathcal{Q}_0} C(\Omega)/q$*, and let* $\psi : \mathcal{F} \to \mathbb{C}$ *be linear.*

The functional ϕ *is positive if and only if* $\|\psi_q\| = \psi(1/q),\ q \in \mathcal{Q}_0$*, where* $\psi_q = \psi|C(\Omega)/q$*.*

In the family $\mathcal{Q}$ we write $q'|q''$ for $q', q'' \in \mathcal{Q}$, meaning q' *divides* q'' if there exists a $q \in \mathcal{Q}$ such that $q'' = q'q$. A subset $\mathcal{Q}_0 \subset \mathcal{Q}$ is *cofinal* in $\mathcal{Q}$ if for every $q \in \mathcal{Q}$ we can find a $q_0 \in \mathcal{Q}_0$ such that $q|q_0$.

The next assertion is an extension result of linear functionals to positive ones.

Theorem 2.3. *Let* $\mathcal{Q}_0 \ni 1$ *be a cofinal subset of* $\mathcal{Q}$. *Let* $\mathcal{F} = \Sigma_{q\in\mathcal{Q}_0}\mathcal{F}_q$, *where* $\mathcal{F}_q$ *is a vector subspace of* $C(\Omega)/q$ *such that* $1/q \in \mathcal{F}_q$ *and* $\mathcal{F}_q \subset \mathcal{F}_r$ *for all* $q, r \in \mathcal{Q}_0$, *with* $q|r$. *Let also* $\phi : \mathcal{F} \to \mathbb{C}$ *be linear with* $\phi(1) > 0$, *and set* $\phi_q = \phi|\mathcal{F}_q$, $q \in \mathcal{Q}_0$.

The linear functional ϕ *extends to a positive linear functional* ψ *on* $C(\Omega)/\mathcal{Q}$ *such that* $\|\psi_q\| = \|\phi_q\|$, *where* $\psi_q = \psi|C(\Omega)/q$, *if and only if* $\|\phi_q\| = \phi(1/q) > 0$, $q \in \mathcal{Q}_0$.

We put $Z(\mathcal{Q}_0) = \cup_{q\in\mathcal{Q}_0} Z(q)$ for each subset $\mathcal{Q}_0 \subset \mathcal{Q}$.

Corollary 2.4. *With the conditions of the previous Theorem, there exists a positive measure μ on Ω such that*

$$\phi(f) = \int_\Omega f\, d\mu,\ \ f \in \mathcal{F}.$$

For every such measure μ and every $q \in \mathcal{Q}$, we have $\mu(Z(q)) = 0$. Hence, if $\mathcal{Q}$ contains a countable subset $\mathcal{Q}_1$ with $Z(\mathcal{Q}_1) = Z(\mathcal{Q})$, then $\mu(Z(\mathcal{Q})) = 0$.

Exemple 2.5. Let $\mathcal{S}_1$ be the algebra of polynomials in $z, \bar{z}$, $z \in \mathbb{C}$. This algebra, used to characterize the moment sequences in the complex plane, can be identified with a subalgebra of an algebra of fractions of continuous functions.

Let $\mathcal{R}_1$ be the set of functions

$\{(1+|z|^2)^{-k}; z \in \mathbb{C}, k \in \mathbb{Z}_+\}$, which can be continously extended to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Identifying $\mathcal{R}_1$ with the set of their extensions in $C(\mathbb{C}_\infty)$, the family $\mathcal{R}_1$ becomes a set of denominators in $C(\mathbb{C}_\infty)$. This will allows us to identify the algebra $\mathcal{S}_1$ with a subalgebra of the algebra of fractions $C(\mathbb{C}_\infty)/\mathcal{R}_1$.

Let $\mathcal{S}_{1,k}$, $k \geq 1$ a fixed integer, be the space generated by the monomials $z^j \bar{z}^l$, $0 \leq j+l < 2k$, and the monomial $|z|^{2k}$, which may be viewed as a subspace of $C(\mathbb{C}_\infty)/r_k$, where $r_k(z) = (1+|z|^2)^{-k}$ for all $k \geq 0$.

We clearly have $\mathcal{S}_1 = \Sigma_{k\geq 0}\mathcal{S}_{1,k}$, and so the space $\mathcal{S}_1$ can be viewed as a subalgebra of the algebra $C(\mathbb{C}_\infty)/\mathcal{R}_1$. Note also that $r_k^{-1} \in \mathcal{S}_{1,k}$ for all $k \geq 1$ and $\mathcal{S}_{1,k} \subset \mathcal{S}_{1,l}$ whenever $k \leq l$.

According to Theorem 1.4, a linear map $\phi : \mathcal{S}_1 \mapsto \mathbb{C}$ has a positive extension $\psi : C(\mathbb{C}_\infty)/\mathcal{R}_1 \mapsto \mathbb{C}$ with $\|\phi_k\| = \|\psi_k\|$ if and only if $\|\phi_k\| = \phi(r_k^{-1})$, where $\phi_k = \phi|\mathcal{S}_{1,k}$ and $\psi_k = \psi|C(\mathbb{C}_\infty)/r_k$, for all $k \geq 0$. This result can be used to characterize the Hamburger moment problem in the complex plane. Specifically, given a sequence of complex numbers $\gamma = (\gamma_{j,l})_{j\geq 0, l\geq 0}$ with $\gamma_{0,0} = 1$, $\gamma_{k,k} \geq 0$ if $k \geq 1$ and

$\gamma_{j,l} = \bar{\gamma}_{l,j}$ for all $j \geq 0, l \geq 0$, the Hamburger moment problem means to find a probability measure on $\mathbb{C}$ such that $\gamma_{j,l} = \int z^j \bar{z}^l d\mu(z)$, $j \geq 0, l \geq 0$.

Defining $L_\gamma : \mathcal{S}_1 \mapsto \mathbb{C}$ by setting $L_\gamma(z^j \bar{z}^l) = \gamma_{j,l}$ for all $j \geq 0, l \geq 0$ (extended by linearity), if L_γ has the properties of the functional ϕ above insuring the existence of a positive extension to $C(\mathbb{C}_\infty)/\mathcal{R}_1$, then the measure μ is provided by Corollary 1.5.

For a fixed integer $m \geq 1$, we can state and characterize the existence of solutions for a truncated moment problem (for an extensive study of such problems we refer to the works by Curto and Fialkow). Specifically, given a finite sequence of complex numbers $\gamma = (\gamma_{j,l})_{j,l}$ with $\gamma_{0,0} = 1$, $\gamma_{j,j} \geq 0$ if $1 \leq j \leq m$ and $\gamma_{j,l} = \bar{\gamma}_{l,j}$ for all $j \geq 0, l \geq 0$, $j \neq l$, $j + l < 2m$, find a probability measure on $\mathbb{C}$ such that $\gamma_{j,l} = \int z^j \bar{z}^l d\mu(z)$ for all indices j, l. As in the previous case, a necessary and sufficient condition is that the corresponding map $L_\gamma : \mathcal{S}_{1,m} \mapsto \mathbb{C}$ have the property $\|L_\gamma\| = L_\gamma(1/r_m)$. Note also that the actual truncated moment problem is slightly different from the usual one.

3. Operator-valued moment problems

Let $\mathcal{D}$ be a complex inner product space whose completion is denoted by $\mathcal{H}$, let $SF(\mathcal{D})$ be the space of oll sesquilinear forms on $\mathcal{D}$, and let $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ be a linear map.We look for a positive measure F on the Borel subsets of $\mathbb{R}^n$, with values in $B(\mathcal{H})$, such that $\phi(p)(x, y) = \int p\, dF_{x,y}$ for all $p \in \mathcal{P}_n$ and $x, y \in \mathcal{D}$, which is an operator moment problem. When such a positive measure F exists, we say that $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ is a *moment form* and the measure F is said to be a *representing measure* for ϕ. The next result is due to Albrecht and V.

Theorem 3.1. *Let $\mathcal{D}$ be a complex inner product space and let $\phi : \mathcal{P}_n \to SF(\mathcal{D})$ be a unital, linear map. The map ϕ is a moment form if and only if*

(i) $\phi(p_\alpha)(x, x) > 0$ for all $x \in \mathcal{D} \setminus \{0\}$ and $\alpha \in \mathbb{Z}_+^n$.

(ii) For all $\alpha \in \mathbb{Z}_+^n$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with

$$\sum_{j=1}^{m} \phi(p_\alpha)(x_j, x_j) \leq 1, \sum_{j=1}^{m} \phi(p_\alpha)(y_j, y_j) \leq 1,$$

and for all $f = (f_{j,k}) \in M_m(\mathcal{P}_{n,\alpha})$ with $\sup_t \|q_\alpha(t) f(t)\|_m \leq 1$, we have

$$\left| \sum_{j,k=1}^{m} \phi(f_{j,k})(x_k, y_j) \right| \leq 1.$$

4. Completely contractive

extensions

In this section we present a version of result by Albrecht and V, concerning the existence of normal extensions. We discuss it here for infinitely many operators.

Nevertheless, we first present the case of a single operator.

Fix a Hilbert space $\mathcal{H}$ and a dense subspace $\mathcal{D}$ of $\mathcal{H}$, let, as before, $SF(\mathcal{D})$ be the space of all sesquilinear forms on $\mathcal{D}$.

We recall that $\mathcal{S}_1$, is the set of all polynomials in z and $\bar{z}$, $z \in \mathbb{C}$.

Considering an operator S, we may define a unital linear map

$\phi_S : \mathcal{S}_1 \to SF(\mathcal{D})$ by

$$\phi_S(z^j \bar{z}^k)(x, y) = \langle S^j x, S^k y \rangle,$$

$$x, y \in \mathcal{D},\ j \in \mathbb{Z}_+,$$

extended by linearity to the subspace $\mathcal{S}_1$.

Theorem 4.1. *Let $S : \mathcal{D}(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be a densely defined linear operator such that $S\mathcal{D}(S) \subset \mathcal{D}(S)$. The operator S admits a normal extension if and only if for all $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in \mathcal{D}(S)$ with*

$$\sum_{j=1}^{n} \sum_{k=0}^{m} \binom{m}{k} \langle S^k x_j, S^k x_j \rangle \le 1,$$

$$\sum_{j=1}^{n} \sum_{k=0}^{m} \binom{m}{k} \langle S^k y_j, S^k y_j \rangle \le 1,$$

and for all $p = (p_{j,k}) \in M_n(\mathcal{S}_1)$, with $\sup_{z\in\mathbb{C}} \|(1+|z|^2)^{-m} p(z)\|_n \le$

1, *we have*

$$\left|\sum_{j,k=1}^{n} \langle \phi_S(p_{j,k}) x_k, y_j \rangle\right| \le 1.$$

Theorem 4.1 is a direct consequence of a more general assertion, to be stated in the sequel. A version of the theorem above has been obtained by Stochel and Szafraniec, via a completely different approach.

Let $\mathcal{Q} \subset C(\Omega)$ be a set of positive denominators. Fix a $q \in \mathcal{Q}$. A linear map $\psi : C(\Omega)/q \to SF(\mathcal{D})$ is called *unital* if $\psi(1)(x, y) = \langle x, y \rangle$, $x, y \in \mathcal{D}$.

We say that ψ is *positive* if $\psi(f)$ is positive semidefinite for all $f \in (C(\Omega)/q)_+$.

More generally, let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty. Let $\mathcal{C} = \Sigma_{q \in \mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{C} \to SF(\mathcal{D})$ be linear. The map ψ is said to be *unital* (resp. *positive*) if $\psi | C(\Omega)/q$ is unital (resp. positive) for all $q \in \mathcal{Q}_0$.

We start with a part of a theorem by Albrecht and V.

Theorem A. *Let $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty, let $\mathcal{C} = \Sigma_{q \in \mathcal{Q}_0} C(\Omega)/q$, and let $\psi : \mathcal{C} \to SF(\mathcal{D})$ be linear and unital. The map ψ is positive if and only if*

$$\sup\{|\psi(hq^{-1})(x,x)|;\ h \in C(\Omega),\ \|h\|_\infty \le 1\}$$
$$= \psi(q^{-1})(x,x),\ q \in \mathcal{Q}_0,\ x \in \mathcal{D}.$$

Let again $\mathcal{Q}_0 \subset \mathcal{Q}$ be nonempty and let $\mathcal{F} = \Sigma_{q \in \mathcal{Q}_0} \mathcal{F}_q$, where $1/q \in \mathcal{F}_q$ and $\mathcal{F}_q$ is a vector subspace of $C(\Omega)/q$ for all $q \in \mathcal{Q}_0$. Let $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ be linear. Suppose that $\phi(q^{-1})(x,x) > 0$ for all $x \in \mathcal{D} \setminus \{0\}$ and $q \in \mathcal{Q}_0$. Then $\phi(1/q)$ induces an inner product on $\mathcal{D}$, and let $\mathcal{D}_q$ be the space $\mathcal{D}$, endowed with the norm given by $\| * \|_q^2 = \phi(1/q)(*,*)$.

Let $M_n(\mathcal{F}_q)$ (resp. $M_n(\mathcal{F})$) denote the space of $n \times n$-matrices with entries in $\mathcal{F}_q$ (resp. in $\mathcal{F}$) .

Note that $M_n(\mathcal{F}) = \Sigma_{q\in\mathcal{Q}_0} M_n(\mathcal{F}_q)$ may be identified with a subspace of the algebra of fractions $C(\Omega, M_n)/\mathcal{Q}$, where M_n is the C^*-algebra of $n\times n$-matrices with entries in $\mathbb{C}$. Moreover, the map ϕ has a natural extension $\phi^n : M_n(\mathcal{F}) \mapsto SF(\mathcal{D}^n)$, given by

$$\phi^n(\mathbf{f})(\mathbf{x}, \mathbf{y}) = \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k, y_j),$$

for all $\mathbf{f} = (f_{j,k}) \in M_n(\mathcal{F})$ and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{D}^n$.

Let $\phi_q^n = \phi^n \mid M_n(\mathcal{F}_q)$. Endowing the Cartesian product $\mathcal{D}^n$ with the norm $\|\mathbf{x}\|_q^2 = \Sigma_{j=1}^n \phi(1/q)(x_j, x_j)$ if $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{D}^n$,
and denoting it by $\mathcal{D}_q^n$, we say that

the map ϕ^n is contractive if $\|\phi_q^n\| \leq 1$ for all $q \in \mathcal{Q}_0$. Using the standard norm $\| * \|_n$ in the space of M_n, the space $M_n(\mathcal{F}_q)$ is endoved with the norm

$$\|(qf_{j,k})\|_{n,\infty} = \sup_{\omega \in \Omega} \|(q(\omega)f_{j,k}(\omega))\|_n,$$

for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$.

Following Arveson and Powers, we shall say that the map $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ is *completely contractive* if the map $\phi^n : M_n(\mathcal{F}) \mapsto SF(\mathcal{D}^n)$ is contractive for all integers $n \geq 1$.

Note that a linear map $\phi : \mathcal{F} \mapsto SF(\mathcal{D})$ with the property $\phi(1/q)(x, x) > 0$ for all $x \in \mathcal{D} \setminus \{0\}$ and $q \in \mathcal{Q}_0$ is completely contractive if and only if for all $q \in \mathcal{Q}_0$, $n \in \mathbb{N}$, $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{D}$ with

$$\sum_{j=1}^{n} \phi(q^{-1})(x_j, x_j) \le 1,$$

$$\sum_{j=1}^{n} \phi(q^{-1})(y_j, y_j) \le 1,$$

and for all $(f_{j,k}) \in M_n(\mathcal{F}_q)$ with $\|(qf_{j,k})\|_{n,\infty} \le 1$, we have

$$\left| \sum_{j,k=1}^{n} \phi(f_{j,k})(x_k, y_j) \right| \le 1.$$

Let us recall another result by Albrecht and V., given here in a shorter form.

Theorem B. *Let Ω be a compact space and let $\mathcal{Q} \subset C(\Omega)$ be a set of positive denominators. Let also $\mathcal{Q}_0$ be a cofinal subset of $\mathcal{Q}$, with $1 \in \mathcal{Q}_0$.*

Let $\mathcal{F} = \Sigma_{q\in\mathcal{Q}_0}\mathcal{F}_q$, *where* $\mathcal{F}_q$ *is a vector subspace of* $C(\Omega)/q$ *such that* $1/r \in \mathcal{F}_r \subset \mathcal{F}_q$ *for all* $r \in \mathcal{Q}_0$ *and* $q \in \mathcal{Q}_0$, *with* $r|q$. *Let also* $\phi : \mathcal{F} \to SF(\mathcal{D})$ *be linear and unital, and set* $\phi_q = \phi|\mathcal{F}_q$, $\phi_{q,x}(*) = \phi_q(*)(x,x)$ *for all* $q \in \mathcal{Q}_0$ *and* $x \in \mathcal{D}$.

The following conditions are equivalent:

(a) *The map* ϕ *extends to a unital, positive, linear map* ψ *on* $C(\Omega)/\mathcal{Q}$ *such that, for all* $x \in \mathcal{D}$ *and* $q \in \mathcal{Q}_0$*, we have:* $\|\psi_{q,x}\| = \|\phi_{q,x}\|$*, where* $\psi_q = \psi|C(\Omega)/q$*,* $\psi_{q,x}(*) = \psi_q(*)(x,x)$*.*

(b) (i) $\phi(q^{-1})(x,x) > 0$ *for all* $x \in \mathcal{D} \setminus \{0\}$ *and* $q \in \mathcal{Q}_0$*.*

(ii) *The map* ϕ *is completely contractive.*

Remark. A "minimal" subspace of $C(\Omega)/\mathcal{Q}$ to apply Theorem C is obtained as follows. If $\mathcal{Q}_0$ is a cofinal subset of $\mathcal{Q}$ with $1 \in \mathcal{Q}_0$, we define $\mathcal{F}_q$ for some $q \in \mathcal{Q}_0$ to be the vector space generated by all fractions of the form r/q, where $r \in \mathcal{Q}_0$ and $r|q$.

It is clear that the subspace $\mathcal{F} = \Sigma_{q\in\mathcal{Q}_0} \mathcal{F}_q$ has the properties required to apply Theorem B.

Corollary C. *Suppose that condition (b) in Theorem B is satisfied. Then there exists a positive $B(\mathcal{H})$–valued measure F on the Borel subsets of Ω such that*

$$\phi(f)(x, y) = \int_\Omega f\, dF_{x,y},$$

for all $f \in \mathcal{F}, x, y \in \mathcal{D}$. For every such measure F and every $q \in \mathcal{Q}_0$, we have $F(Z(q)) = 0$.

Example 4.2. We extend to infinitely many variables the Example 2.5. Let $\mathcal{I}$ be a (nonempty) family of indices.

Denote by $z = (z_\iota)_{\iota\in\mathcal{I}}$ the independent variable in $\mathbb{C}^{\mathcal{I}}$. Let also $\bar{z} = (\bar{z}_\iota)_{\iota\in\mathcal{I}}$. Let $\mathbb{Z}_+^{(\mathcal{I})}$ be the set of all collections $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}}$ of non-negative integers, with finite support. Setting $z^0 = 1$ for $0 = (0)_{\iota\in\mathcal{I}}$ and $z^\alpha = \prod_{\alpha_\iota\neq 0} z_\iota^{\alpha_\iota}$ for $z = (z_\iota)_{\iota\in\mathcal{I}} \in \mathbb{C}^{\mathcal{I}}$, $\alpha = (\alpha_\iota)_{\iota\in\mathcal{I}} \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we may consider the algebra of those complex-valued functions $\mathcal{S}_{\mathcal{I}}$ on $\mathbb{C}^{\mathcal{I}}$ consisting of expressions of the form $\sum_{\alpha,\beta\in\mathcal{J}} c_{\alpha,\beta} z^\alpha \bar{z}^\beta$, with $c_{\alpha,\beta}$ complex numbers for all $\alpha, \beta \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{Z}_+^{(\mathcal{I})}$ is finite.

We can embed the space $\mathcal{S}_{\mathcal{I}}$ into the algebra of fractions derived from the basic algebra $C((\mathbb{C}_\infty)^{\mathcal{I}})$,

using a suitable set of denominators. Specifically, we consider the family $\mathcal{R}_{\mathcal{I}}$ consisting of all rational functions of the form $r_\alpha(t) = \Pi_{\alpha_\iota \neq 0}(1+|z_\iota|^2)^{-\alpha_\iota}$, $z = (z_\iota)_{\iota \in \mathcal{I}} \in \mathbb{C}^{\mathcal{I}}$, where $\alpha = (\alpha_\iota) \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, is arbitrary. Of course, we set $r_0 = 1$. The function r_α can be continuously extended to

$(\mathbb{C}_\infty)^{\mathcal{I}} \setminus \mathbb{C}^{\mathcal{I}}$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$. In fact, actually the function $f_{\beta,\gamma}(z) = z^\beta \bar{z}^\gamma r_\alpha(z)$ can be continuously extended to $(\mathbb{C}_\infty)^{\mathcal{I}} \setminus \mathbb{C}^{\mathcal{I}}$ whenever $\beta_\iota + \gamma_\iota < 2\alpha_\iota$, and $\beta_\iota = \gamma_\iota = 0$ if $\alpha_\iota = 0$, for all $\iota \in \mathcal{I}$ and $\alpha, \beta, \gamma \in \mathbb{Z}_+^{(\mathcal{I})}$. Moreover, the family $\mathcal{R}_{\mathcal{I}}$ becomes a set of denominators in $C((\mathbb{C}_\infty)^{\mathcal{I}})$. This shows that the space $\mathcal{S}_{\mathcal{I}}$ can be embedded into the algebra of fractions $C((\mathbb{C}_\infty)^{\mathcal{I}})/\mathcal{R}_{\mathcal{I}}$.

To be more specific, for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, we denote by $\mathcal{S}_{\mathcal{I},\alpha}^{(1)}$ the linear spaces generated by the monomials $z^\beta \bar{z}^\gamma$, with $\beta_\iota + \gamma_\iota < 2\alpha_\iota$ whenever $\alpha_\iota > 0$, and $\beta_\iota = \gamma_\iota = 0$ if $\alpha_\iota = 0$. Put $\mathcal{S}_{\mathcal{I},0}^{(1)} = \mathbb{C}$.

We also define $\mathcal{S}_{\mathcal{I},\alpha}^{(2)}$, for $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $\alpha \neq 0$, to be the linear space generated by the monomials $|z|^{2\beta} = \Pi_{\beta_\iota \neq 0}(z_\iota \bar{z}_\iota)^{\beta_\iota}$, $0 \neq \beta$, $\beta_\iota \leq \alpha_\iota$ for all $\iota \in \mathcal{I}$ and $|z| = (|z_\iota|)_{\iota \in \mathcal{I}}$. We define $\mathcal{S}_{\mathcal{I},0}^{(2)} = \{0\}$.

Set $\mathcal{S}_{\mathcal{I},\alpha} = \mathcal{S}_{\mathcal{I},\alpha}^{(1)} + \mathcal{S}_{\mathcal{I},\alpha}^{(2)}$ for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$. Note that, if $f \in \mathcal{S}_{\mathcal{I},\alpha}$, the function $r_\alpha f$ extends continuously to $(\mathbb{C}_\infty)^{\mathcal{I}}$ and that $\mathcal{S}_{\mathcal{I},\alpha} \subset \mathcal{S}_{\mathcal{I},\beta}$ if $\alpha_\iota \leq \beta_\iota$ for all $\iota \in \mathcal{I}$.

It is now clear that the algebra $\mathcal{S}_{\mathcal{I}} = \sum_{\alpha \in \mathbb{Z}_+^{(\mathcal{I})}} \mathcal{S}_{\mathcal{I},\alpha}$ can be identified with a subalgebra of $C((\mathbb{C}_\infty)^{\mathcal{I}})/\mathcal{R}_{\mathcal{I}}$. This algebra has the properties of the space $\mathcal{F}$ appearing in the statement of Theorem B.

Let now $T = (T_\iota)_{\iota \in \mathcal{I}}$ be a family of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that $T_\iota(\mathcal{D}) \subset \mathcal{D}$ and $T_\iota T_\kappa x = T_\kappa T_\iota x$ for all $\iota, \kappa \in \mathcal{I}$, $x \in \mathcal{D}$.

Setting T^α as in the case of complex monomials, which is possible because of the commutativity of the family T on $\mathcal{D}$, we may define a unital linear map $\phi_T : \mathcal{S}_{\mathcal{I}} \to SF(\mathcal{D})$ by

$$\phi_T(z^\alpha \bar{z}^\beta)(x, y) = \langle T^\alpha x, T^\beta y \rangle,$$

for all $x, y \in \mathcal{D}, \alpha, \beta \in \mathbb{Z}_+^{(\mathcal{I})}$, which extends by linearity to the subspace $\mathcal{S}_{\mathcal{I}}$ generated by these monomials. For all α, β in $\mathbb{Z}_+^{(\mathcal{I})}$ with $\beta - \alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, and $x \in \mathcal{D} \setminus \{0\}$, we have

$$0 < \langle x, x \rangle \leq \phi_T(r_\alpha^{-1})(x, x) \leq \phi_T(r_\beta^{-1})(x, x) .$$

The polynomial $1/r_\alpha$ will be denoted by s_α for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$.

The family $T = (T_\iota)_{\iota \in \mathcal{I}}$ is said to have a *normal extension* if there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a family $N = (N_\iota)_{\iota \in \mathcal{I}}$ consisting of commuting normal operators in $\mathcal{K}$ such that $\mathcal{D} \subset \mathcal{D}(N_\iota)$ and $N_\iota x = T_\iota x$ for all $x \in \mathcal{D}$ and $\iota \in \mathcal{I}$.

A family $T = (T_\iota)_{\iota \in \mathcal{I}}$ having a normal extension is also called a *subnormal family*.

The following result is a version of theorem by Albrecht and V, valid for an arbitrary family of operators We mention that, the basic space has been modified.

Theorem 4.3. *Let* $T = (T_\iota)_{\iota \in \mathcal{I}}$ *be a family of linear operators defined on a dense subspace* $\mathcal{D}$ *of a Hilbert space* $\mathcal{H}$.

Assume that $\mathcal{D}$ is invariant under T_ι for all $\iota \in \mathcal{I}$ and that T is a commuting family on $\mathcal{D}$.

The family T admits a normal extension if and only if the map $\phi_T : \mathcal{S}_\mathcal{I} \mapsto SF(\mathcal{D})$ has the property that for all $\alpha \in \mathbb{Z}_+^{(\mathcal{I})}$, $m \in \mathbb{N}$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{D}$ with $\Sigma_{j=1}^m \phi_T(s_\alpha)(x_j, x_j) \leq 1$, $\Sigma_{j=1}^m \phi_T(s_\alpha)(y_j, y_j) \leq 1$, and for all $p = (p_{j,k}) \in M_m(\mathcal{S}_{\mathcal{I},\alpha})$ with $\sup_z \|r_\alpha(z)p(z)\|_m \leq 1$, we have

$$\left| \sum_{j,k=1}^{m} \phi_T(p_{j,k})(x_k, y_j) \right| \leq 1.$$

Remark. Let $S : \mathcal{D}(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be an arbitrary linear operator. If $B : \mathcal{D}(B) \subset \mathcal{K} \mapsto \mathcal{K}$ is a normal operator such that $\mathcal{H} \subset \mathcal{K}$, $\mathcal{D}(S) \subset \mathcal{D}(B)$, $Sx = PBx$ and $\|Sx\| = \|Bx\|$ for all $x \in \mathcal{D}(S)$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$, then we have $Sx = Bx$ for all $x \in \mathcal{D}(S)$. Indeed, $\langle Sx, Sx\rangle = \langle Sx, Bx\rangle$ and $\langle Bx, Sx\rangle = \langle PBx, Sx\rangle = \langle Sx, Sx\rangle = \langle Bx, Bx\rangle$. Hence, we have $\|Sx - Bx\| = 0$ for all $x \in \mathcal{D}(S)$.

Remark 4.4. Let $T = (T_\iota)_{\iota\in\mathcal{I}}$ be a family of linear operators defined on a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{D}$ is invariant under T_ι and that T is a commuting family on $\mathcal{D}$. If the map $\phi_T : \mathcal{S}_\mathcal{I} \mapsto SF(\mathcal{D}))$ is as in Theorem 2.3, the family has a proper quasi-invariant subspace. In other words, there exists a proper Hilbert subspace $\mathcal{L}$ of the Hilbert space $\mathcal{H}$ such that the subspace $\{x \in \mathcal{D}(T_\iota) \cap \mathcal{L}; Tx \in \mathcal{L}\}$ is dense in in $\mathcal{L}$ for each $\iota \in \mathcal{I}$.

For the proof of Theorem 4.3, we need the following version of the spectral theorem.

Theorem 4.5. *Let* $(N_\iota)_{\iota \in \mathcal{I}}$ *be a commuting family of normal operators in* $\mathcal{H}$*. Then there exists a unique spectral measure* G *on the Borel subsets of* $(\mathbb{C}_\infty)^{\mathcal{I}}$ *such that each coordinate function*
$(\mathbb{C}_\infty)^{\mathcal{I}} \ni z \to z_\iota \in \mathbb{C}_\infty$ *is* G*-almost everywhere finite. In ad-*

dition,

$$\langle N_\iota x, y\rangle = \int_{(\mathbb{C}_\infty)^{\mathcal{I}}} z_\iota dE_{x,y}(z),$$

for all $x \in \mathcal{D}(N_\iota)$, $y \in \mathcal{H}$, *where*

$$\mathcal{D}(N_\iota) = \{x \in \mathcal{H}; \int_{(\mathbb{C}_\infty)^{\mathcal{I}}} |z_\iota|^2 dE_{x,x}(z) < \infty\},$$

for all $\iota \in \mathcal{I}$.

If the set $\mathcal{I}$ *is at most countable, then the measure* G *has support in* $\mathbb{C}^{\mathcal{I}}$.