An Idempotent Approach to Truncated Moment Problems

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Abstract. We present a new approach to truncated and full moment problems, via idempotent elements with respect to associated square positive Riesz functionals. The existence of representing measures for such functionals is characterized via some intrinsic conditions.

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1. Introduction

The study of truncated moment problems means, roughly speaking, that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a fixed length $n \ge 1$ and $m \ge 0$ is an integer, one looks for a positive measure μ on $\mathbb{R}^n$ (usually called a *representing measure* for γ) such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \le 2m$ (see [3]–[5], [7], [8] and their references, where the subject is extensively discussed). If such a measure exists, we may always assume it to be atomic (see [1], [6], [12], [17]).

We now introduce the terminology used in the paper and recall some elementary facts, most of them well known, presented here in a slightly more general context than the usual one (see also [18]).

Let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on a topological space Ω. We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space* (on Ω). Occasionally, we use the notation $\mathcal{RS}$ to designate the "real part" of $\mathcal{S}$, that is $\{f \in \mathcal{S}; f = \bar{f}\}$.

Let also $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:

(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$;
(3) $\Lambda(1) = 1$.

Adapting some terminology from [11] to our context (see also [21]), a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.

When $\mathcal{S}$ is an algebra, conditions (2) and (3) imply condition (1). In this case, a map Λ with the property (2) is usually said to be *positive (semi)definite.*

Condition (3) may be replaced by $\Lambda(1) > 0$ but (looking for probability measures representing such a functional) we always assume (3) in the stated form, without loss of generality.

The *(abstract) moment problem* for a given uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{S}$ is a fixed function space on a topological space Ω, means to find conditions insuring the existence of a probability measure μ with support in Ω, such that $\Lambda(f) = \int f d\mu$, $f \in \mathcal{S}^{(2)}$. When such a measure μ exists, it is said to be a *representing measure* for Λ.

Note that the map $\mathcal{S}^{(2)} \ni f \mapsto \int f d\mu \in \mathbb{C}$, where μ is a probability measure with support in Ω, is a uspf, as one can easily see.

If $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality:*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ f, g \in \mathcal{S}. \tag{1.1}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle f + \mathcal{I}_\Lambda, g + \mathcal{I}_\Lambda \rangle = \Lambda(f\bar{g}). \tag{1.2}$$

In fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\ \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$. If $\mathcal{S}$ is finite dimensional, then $\mathcal{H}_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

Throughout this paper $n \geq 1$ will be a fixed integer. To present the most significant examples (from our point of view) of function spaces, we freely use multi-indices from $\mathbb{Z}_+^n$ and the standard notation related to them.

If not otherwise specified, the symbol $\mathcal{P}$ will designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients. (Although the polynomials with real coefficients seem to be more appropriate for these problems, we prefer polynomials with complex coefficients because of the systematic use of some associated complex Hilbert spaces.)

For every integer $m \geq 0$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. Note that $\mathcal{P}_m^{(2)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

We occasionally use the notation $\mathcal{P}_m^n$ instead of $\mathcal{P}_m$, when the number n should be specified.

Choosing a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$, $\gamma_0 = 1$, we associate it with a map $\Lambda_\gamma : \mathcal{P}_{2m} \mapsto \mathbb{C}$ given by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, extended to $\mathcal{P}_{2m}$ by linearity. The map Λ_γ is usually called the *Riesz functional associated to* γ. We clearly have $\Lambda_\gamma(1) = 1$ and $\Lambda_\gamma(\bar{p}) = \overline{\Lambda_\gamma(p)}$ for all $p \in \mathcal{P}_{2m}$. If, moreover, $\Lambda_\gamma(|p|^2) \ge 0$ for all $p \in \mathcal{P}_m$, then Λ_γ is a uspf. In this case, we say that γ itself is *square positive.*

Conversely, if $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf, setting $\gamma_\alpha = \Lambda(t^\alpha)$, $|\alpha| \le 2m$, we have $\Lambda = \Lambda_\gamma$, as above. The square positive multi-sequence γ is said to be the *multi-sequence associated to the uspf* Λ.

To find a representing measure for the map Λ_γ means to solve a *truncated moment problem* (see [3]–[8] for other details).

Similarly, to solve the *full (or the multidimensional Hamburger) moment problem* means to find a representing measure for the map $\Lambda_\gamma : \mathcal{P} \mapsto \mathbb{C}$, defined for a multi-sequence $\gamma = (\gamma_\alpha)_{\alpha\ge 0}$, $\gamma_0 = 1$ (see [2] for other details). Various results concerning the integral representations for truncated (and full) moment problems will be given throughout this text.

Let $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbb{R}^n$ and let $C(\Xi)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ξ, endowed with the sup-norm. If $t = (t_1, \dots, t_n)$ is the n-tuple of coordinate functions in $\mathbb{R}^n$, every element of $C(\Xi)$ is a polynomial in the restrictions $t_1|\Xi, \dots, t_n|\Xi$, via Lagrange (or other) interpolating polynomials (or by a weak form of the Weierstrass-Stone theorem).

For every integer $m \ge 0$ we have the restriction map $\mathcal{P}_m \ni p \mapsto p|\Xi \in C(\Xi)$. Let us fix an integer m for which this map is surjective (which exists again by using interpolating polynomials). Let also $\mu = \sum_{j=1}^d \lambda_j \delta_{\xi^{(j)}}$, with $\delta_{\xi^{(j)}}$ the Dirac measure at $\xi^{(j)}$, $\lambda_j > 0$ for all $j = 1, \dots, d$, and $\sum_{j=1}^d \lambda_j = 1$. We put $\Lambda(p) = \int_\Xi p d\mu$ for all $p \in \mathcal{P}_{2m}$, which is a uspf, for which μ is a representing measure.

Let now $f \in C(\Xi)$ be an idempotent. In other words, f is the caracteristic function of a subset of Ξ. Our assumption on the restriction map implies the existence of a polynomial $p \in \mathcal{P}_m$, which may be supposed to have real coefficients, such that $p|\Xi = f$. Consequently, $\Lambda(p^2) = \int_\Xi p^2 d\mu = \int_\Xi p d\mu = \Lambda(p)$. This shows that some of the solutions the equation $\Lambda(p^2) = \Lambda(p)$, which can be expressed only in terms of Λ, play an important role when trying to reconstruct the representing measure μ. This simple remark is the starting point of our approach to truncated moment problems.

In most of the papers by Curto and Fialkow (see especially [3],[4]), the approach to truncated moment problems is based on an associated moment matrix, whose positivity and flatness (see Remark 11(2)) lead to the existence (and uniqueness) of the solutions. The use of the Riesz functional to solve various moment problems and related topics appears in several works, as for instance [9], [10], [11], [13]-[15], [19]-[21] etc. Introducing a concept of idempotent element with respect to a unital square positive functional (see Definition 1), we attempt, in the following, to give a new approach to truncated moment problems, using only intrinsic conditions.

Let us briefly present the contents of this work. In the next section, idempotents associated to unital square positive functionals are introduced and some of their elementary properties are discussed. Of particular interest are those families of idempotents, mutually orthogonal with respect to a given unital square positive functional.

The third section deals with integral representations of unital square positive functionals, via orthogonal familes of associated idempotents, which are our main results. Theorem 2 and Corollary 3 characterize, in terms of idempotents, and in an intrinsic manner, the existence of representing measures having a number of atoms equal to the maximal cardinality of an orthogonal family of idempotents. The key of this characterization is our condition (3.3), which is a weighted multiplicativity of the corresponding unital square positive functional, and which is more general than the *flatness* condition but still implying the *recursiveness* property (for these notions see [3],[4]; see also Remark 11). In fact, condition (3.3) provides a finite system of second degree equations, whose solutions solve, in principle, the corresponding truncated moment problem (see Remark 8(1)). Some criteria (see Example 4, Proposition 4, Remark 10 etc.) lead to effective solutions for some truncated moment problems, as illustrated by examples. A version of the well-known Tchakaloff theorem is also obtained via our methods (see Corollary 4). Theorem 3 presents the case when the associated Hankel matrix of a uspf (see Remark 3) is invertible. Section 3 ends with a characterization of the solutions of the full moment problems in terms of families of orthogonal idempotents (see Theorem 4).

Finally, the last section contains a discussion concerning the connection between point evaluations and integral representations of unital square positive functionals. Theorem 5 characterizes the existence of representing measures of unital square positive functionals, having an arbitrary number of atoms, in terms of projections of idempotent elements.

2. Idempotents with respect to a uspf

In this section we define the concept of idempotent element with respect to a given uspf, and present some elementary properties of idempotents.

Let $\mathcal{S}$ be a finite dimensional function space on a topological space Ω. Fixing a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, let $\mathcal{I}_\Lambda = \{p \in \mathcal{S}; \Lambda(|p|^2) = 0\}$, and let $\mathcal{H}_\Lambda = \mathcal{S}/\mathcal{I}_\Lambda$, which has a Hilbert space structure induced by Λ (see the Introduction). We denote $\langle *, * \rangle$, $\| * \|$, the inner product, as in (1.2), and the norm induced on $\mathcal{H}_\Lambda$ by Λ, respectively. For every $p \in \mathcal{S}$, we put $\hat{p} = p + \mathcal{I}_\Lambda \in \mathcal{H}_\Lambda$, and the representative p will be freely chosen, once an equivalent class is given.

The symbol $\mathcal{RH}_\Lambda$ will designate the subspace $\{\hat{p} \in \mathcal{H}_\Lambda; p \in \mathcal{RS}\}$, that is, the set of "real" elements from $\mathcal{H}_\Lambda$, which is a real Hilbert space.

If $\hat{p} \in \mathcal{RH}_\Lambda$, we always suppose the representative $p \in \mathcal{RS}$.

Definition 1. An element $\hat{p} \in \mathcal{RH}_\Lambda$ is said to be Λ-*idempotent* (or simply *idempotent* if Λ is fixed) if it is a solution of the equation

$$\|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle. \tag{2.1}$$

Remark 1. (i) Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is idempotent if and only if $\Lambda(p^2) = \Lambda(p)$, via (1.2).

Set

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{2.2}$$

which is the family of nonnull idempotent elements from $\mathcal{RH}_\Lambda$. This family is nonempty because $\hat{1} \in \mathcal{ID}(\Lambda)$.

Note that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

(ii) If $\mathcal{T}$ is a another finite dimensional function space on Ω such that $\mathcal{T} \supset \mathcal{S}$, and $\Lambda_2 : \mathcal{T}^{(2)} \mapsto \mathbb{C}$ is a uspf, then obviously $\Lambda_1 = \Lambda_2|\mathcal{S}^{(2)}$ is a uspf. Moreover, $\mathcal{ID}(\Lambda_1) \subset \mathcal{ID}(\Lambda_2)$. Indeed, it is known (see [21]) and easily seen (via (1.1) and (1.2)) that $\mathcal{I}_{\Lambda_1} \subset \mathcal{I}_{\Lambda_2}$ and $\mathcal{S} \cap \mathcal{I}_{\Lambda_2} = \mathcal{I}_{\Lambda_1}$, showing that $\mathcal{H}_{\Lambda_1}$ can be isometrically embedded into $\mathcal{H}_{\Lambda_2}$. For this reason, $\mathcal{H}_{\Lambda_1}$ may and will be regarded as a subspace of $\mathcal{H}_{\Lambda_2}$, and we have the desired inclusion.

Lemma 1. (1) *If $\hat{p}, \hat{q}, \hat{p} - \hat{q} \in \mathcal{ID}(\Lambda)$, then $\hat{q}$ and $\hat{p} - \hat{q}$ are orthogonal.*

(2) *If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, and $\hat{q}$, $\hat{1} - \hat{q}$ are orthogonal.*

(3) *If $\{\hat{p}_1, \dots, \hat{p}_d\} \subset \mathcal{ID}(\Lambda)$ are mutually orthogonal, then $\sum_{j=1}^d \hat{p}_j \in \mathcal{ID}(\Lambda)$.*

Proof. (1) Indeed, by Remark 1(i),

$$\Lambda(p) = \Lambda(p^2) = \Lambda(q^2 + 2q(p-q) + (p-q)^2) =$$
$$\Lambda(q) + 2\Lambda(q(p-q)) + \Lambda(p-q) = \Lambda(p) + 2\Lambda(q(p-q)),$$

whence $\Lambda(q(p-q)) = 0$.

(2) If $\hat{q} \in \mathcal{ID}(\Lambda)$, $\hat{q} \neq \hat{1}$, then

$$\Lambda((1-q)^2) = \Lambda(1-q),$$

so $\hat{1} - \hat{q} \in \mathcal{ID}(\Lambda)$, implying $\hat{q}$, $\hat{1} - \hat{q}$ orthogonal, by (1).

(3) Setting $p = \sum_{j=1}^d p_j$, we have

$$\Lambda(p^2) = \Lambda(\sum_{j,k=1}^d p_j p_k) = \Lambda(\sum_{j=1}^d p_j^2) = \Lambda(\sum_{j=1}^d p_j) = \Lambda(p),$$

so $\hat{p} \in \mathcal{ID}(\Lambda)$. □

Lemma 2. *Let $\{\hat{b}_1, \dots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$, consisting of mutually orthogonal elements. The family $\{\hat{b}_1, \dots, \hat{b}_d\}$ is maximal with respect to inclusion if and only if $\hat{b}_1 + \dots + \hat{b}_d = \hat{1}$.*

Proof. Assume the family $\{\hat{b}_1, \ldots, \hat{b}_d\}$ to be maximal with respect to inclusion. Note that $\hat{b} = \sum_{j=1}^{d} \hat{b}_j \in \mathcal{ID}(\Lambda)$, by Lemma 1(3).

Assume now that $\hat{b}_0 = \hat{1} - \hat{b} \neq 0$. Then we have $\hat{b}_0 \in \mathcal{ID}(\Lambda)$ by Lemma 1(2). Moreover

$$\Lambda(b_0 b_k) = \Lambda(b_k - \sum_{j=1}^{d} b_j b_k) = \Lambda(b_k - b_k^2) = 0,$$

showing that the family $\{\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_d\}$ consists of mutually orthogonal elements, which contradicts the maximality of $\{\hat{b}_1, \ldots, \hat{b}_d\}$. Consequently, $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$.

Conversely, let $\{\hat{b}_1, \ldots, \hat{b}_d\}$ be such that $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$. If $\hat{b} \in \mathcal{ID}(\Lambda)$ is orthogonal to $\hat{b}_1, \ldots, \hat{b}_d$, then

$$\|\hat{b}\|^2 = \langle \hat{b}, \hat{1} \rangle = \sum_{j=1}^{d} \langle \hat{b}, \hat{b_j} \rangle = 0,$$

which is not possible. Hence the family $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is maximal with respect to inclusion. $\square$

Remark 2. (1) Obviously, the cardinal of every family consisting of mutually orthogonal elements in $\mathcal{ID}(\Lambda)$ is necessarily less or equal to $\dim\mathcal{H}_\Lambda$, which is finite. Moreover, the cardinal of a family consisting of mutually orthogonal elements in $\mathcal{ID}(\Lambda)$, which is maximal with respect to inclusion, may be strictly less than $\dim\mathcal{H}_\Lambda$. Indeed, if $\{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ is a family of mutually orthogonal elements, with $\hat{b}_1 + \cdots + \hat{b}_d = \hat{1}$ and $3 \leq d \leq \dim\mathcal{H}_\Lambda$, setting $\hat{c}_1 = \hat{b}_1$ and $\hat{c}_2 = \hat{b}_2 + \cdots + \hat{b}_d$, we get a family $\{\hat{c}_1, \hat{c}_2\} \subset \mathcal{ID}(\Lambda)$ of two orthogonal elements, which is maximal with respect to inclusion by Lemma 2, but whose cardinal is strictly less than $\dim\mathcal{H}_\Lambda$.

Let us denote by $\mathrm{mc}(\Lambda)$ the greatest cardinal of an orthogonal family in $\mathcal{ID}(\Lambda)$ (which is necessarily maximal with respect to inclusion). We shall show (see Theorem 1) that actually $\mathrm{mc}(\Lambda) = \dim\mathcal{H}_\Lambda$.

(2) Of course, the notation $\dim\mathcal{H}_\Lambda$ used above means the (complex) dimension of the complex vector space $\mathcal{H}_\Lambda$. As we have $\mathcal{H}_\Lambda = \mathcal{RH}_\Lambda + i\mathcal{RH}_\Lambda$, it follows that every orthonormal basis of $\mathcal{RH}_\Lambda$ is also an orthonormal basis of $\mathcal{H}_\Lambda$. Consequently, the dimension of the real space $\mathcal{RH}_\Lambda$ coincides with the $\dim\mathcal{H}_\Lambda$.

Definition 2. Let $\hat{p} \in \mathcal{ID}(\Lambda)$. We say that $\hat{p}$ is *decomposable* if there exists an element $\hat{q}$ such that

(1) $\hat{q}$, $\hat{p} - \hat{q} \in \mathcal{ID}(\Lambda)$;

(2) if $\hat{p}, \hat{r}$ are orthogonal for some $\hat{r} \in \mathcal{ID}(\Lambda)$, then $\hat{q}, \hat{r}$ are orthogonal.

We say that $\hat{p} \in \mathcal{ID}(\Lambda)$ is *minimal* if $\hat{p}$ is not decomposable.

Lemma 3. *Every element in $\mathcal{ID}(\Lambda)$ is either minimal or a sum of mutually orthogonal and minimal idempotents.*

Proof. Let $\hat{p} \in \mathcal{ID}(\Lambda)$. If $\hat{p}$ is minimal there is nothing to prove. Hence we may assume $\hat{p} = \hat{q}_1 + \hat{q}_2$, with $\hat{q}_1, \hat{q}_2 \in \mathcal{ID}(\Lambda)$, by Definition 2(1). Then $\hat{q}_1, \hat{q}_2$ are orthogonal, by Lemma 1(1). If both $\hat{q}_1, \hat{q}_2$ are minimal, we have the assertion. If $\hat{q}_1 = \hat{q}_{11} + \hat{q}_{12}$ with $\hat{q}_{11}, \hat{q}_{12} \in \mathcal{ID}(\Lambda)$, then $\hat{q}_{11}, \hat{q}_{12}$ are orthogonal again by Lemma 1(1). Moreover, $\hat{q}_{11}, \hat{q}_2$ and $\hat{q}_{12}, \hat{q}_2$ are orthogonal by Definition 2(2), because $\hat{q}_1, \hat{q}_2$ are orthogonal. If the elements $\hat{q}_{11}, \hat{q}_{12}, \hat{q}_2$ are minimal, we are done. If not, we decompose again those of them which are not minimal, and continue the procedure obtaining at each stage a family of mutually orthogonal elements, whose sum is $\hat{p}$. The procedure has an end, because the basic space is finite dimensional. □

From now on we investigate the existence of orthogonal families of idempotents with respect to a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$.

Let $\mathcal{B}_\Lambda = \{\hat{v} \in \mathcal{RH}_\Lambda; \|\hat{v}\| = 1\}$.

Lemma 4. *Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. We have the following properties.*

(1) $\mathcal{ID}(\Lambda) = \{\langle \hat{v}, \hat{1}\rangle \hat{v}; \hat{v} \in \mathcal{B}_\Lambda, \langle \hat{v}, \hat{1}\rangle \neq 0\} = \{\Lambda(v)\hat{v}; \hat{v} \in \mathcal{B}_\Lambda, \Lambda(v) \neq 0\}$.

(2) *The map*

$$\mathcal{B}^1_\Lambda \ni \hat{v} \mapsto \langle \hat{v}, \hat{1}\rangle \hat{v} \in \mathcal{ID}(\Lambda) \tag{2.3}$$

is bijective, where $\mathcal{B}^1_\Lambda = \{\hat{v} \in \mathcal{B}_\Lambda; \langle \hat{v}, \hat{1}\rangle \neq 0\}$.

(3) *If $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{B}_\Lambda$ is an orthogonal family satisfying the condition $\langle \hat{v}_j, \hat{1}\rangle \neq 0$, $j = 1, \ldots, d$, then $\{\langle \hat{v}_1, \hat{1}\rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1}\rangle \hat{v}_d\}$ is an orthogonal family of nonnull idempotents.*

(4) *Let $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{B}_\Lambda$ is an orthonormal basis of $\mathcal{H}_\Lambda$ with $\langle \hat{v}_j, \hat{1}\rangle \neq 0$, $j = 1, \ldots, d$. Then $\{\langle \hat{v}_1, \hat{1}\rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1}\rangle \hat{v}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. Moreover,*

$$\langle \hat{v}_1, \hat{1}\rangle \hat{v}_1 + \cdots + \langle \hat{v}_d, \hat{1}\rangle \hat{v}_d = \hat{1}.$$

Proof. (1) Indeed, if $\hat{b} \in \mathcal{ID}(\Lambda)$, then $\hat{b} \neq 0$, and $\hat{v} = \hat{b}/\|\hat{b}\| \in \mathcal{B}_\Lambda$ satisfies the equation $\langle \hat{v}, \hat{1}\rangle \hat{v} = \hat{b}$.

Conversely, if $\hat{b} = \langle \hat{v}, \hat{1}\rangle \hat{v}$ for some $\hat{v} \in \mathcal{B}_\Lambda$ with $\langle \hat{v}, \hat{1}\rangle \neq 0$, then $\|\hat{b}\|^2 = \langle \hat{b}, \hat{1}\rangle \neq 0$.

(2) and (3) follow directly from (1).

(4) Let $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{B}_\Lambda$ be an orthonormal basis of $\mathcal{H}_\Lambda$ (see Remark 2(2)) with $\langle \hat{v}_j, \hat{1}\rangle \neq 0$, $j = 1, \ldots, d$. Then $\{\langle \hat{v}_1, \hat{1}\rangle \hat{v}_1, \ldots \langle \hat{v}_d, \hat{1}\rangle \hat{v}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents, via (3).

The last equality follows by Lemma 2. □

Theorem 1. *For every uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ we have the equality* $\mathrm{mc}(\Lambda) = \dim\mathcal{H}_\Lambda$.

Proof. If $d := \dim\mathcal{H}_\Lambda = 1$, the assertion is clear. Hence we may assume $d > 1$. Using Lemma 4(4), we have to prove the existence of orthonormal basis $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{B}_\Lambda$ of $\mathcal{RH}_\Lambda$ such that $\langle \hat{v}_j, \hat{1}\rangle \neq 0$, $j = 1, \ldots, d$. Note first that we have the orthogonal decomposition $\mathcal{RH}_\Lambda = \mathbb{R}\hat{1} \oplus \mathcal{RH}^0_\Lambda$, where $\mathcal{RH}^0_\Lambda = \{\hat{p} \in \mathcal{RH}_\Lambda; \Lambda(p) = 0\}$. Then we choose an orthonormal basis $\{\hat{w}_2, \ldots, \hat{w}_d\}$ of

$\mathcal{RH}_\Lambda^0$, and put $\hat{w}_1 = \hat{1}$. Applying an appropriate rotation to the orthonormal basis $\{\hat{w}_1, \ldots, \hat{w}_d\}$, we can get an orthonormal basis $\{\hat{v}_1, \ldots, \hat{v}_d\}$ such that $\hat{v}_j \notin \mathcal{RH}_\Lambda^0$ for all $j = 1, \ldots, d$. Therefore, $\langle \hat{v}_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$. □

Corollary 1. *Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Then there are functions $b_1, \ldots, b_d \in \mathcal{RS}$ such that $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, and every $p \in \mathcal{S}$ can be uniquely represented as*

$$p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0,$$

with $p_0 \in \mathcal{I}_\Lambda$ and $d = \dim \mathcal{H}_\Lambda$.

Proof. Theorem 1 asserts that the Hilbert space $\mathcal{H}_\Lambda$ has orthogonal bases consisting of idempotent elements. If $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is such a basis, then

$$\hat{p} = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) \hat{b}_j, \ \hat{p} \in \mathcal{H}_\Lambda,$$

where $d = \dim \mathcal{H}_\Lambda$, which leads to formula $p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0$, for every $p \in \mathcal{S}$, with $p_0 \in \mathcal{I}_\Lambda$, by fixing representatives $b_1, \ldots, b_d \in \mathcal{RS}$ for $\hat{b}_1, \ldots, \hat{b}_d$, respectively.

Clearly, $b_1, \ldots, b_d \in \mathcal{RS}$ is a linearly independent family of vectors in $\mathcal{S}$. Denoting by $\mathcal{G}$ the linear span of $\{b_1, \ldots, b_d\}$ in $\mathcal{S}$, we have $\mathcal{G} \cap \mathcal{I}_\Lambda = \{0\}$. Indeed, if $q = \sum_{j=1}^{d} \theta_j b_j \in \mathcal{I}_\Lambda$, with θ_j complex scalars, then

$$\Lambda(|q|^2) = \sum_{j,k=1}^{d} \theta_j \overline{\theta_k} \Lambda(b_j b_k) = \sum_{j=1}^{d} |\theta_j|^2 \Lambda(b_j) = 0,$$

whence $q = 0$, because $\Lambda(b_j) > 0$ for all j. Therefore, the representation $p = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(p b_j) b_j + p_0$, for every $p \in \mathcal{S}$, with $p_0 \in \mathcal{I}_\Lambda$, is unique. □

Remark 3. We are especially interested by the following particular case.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ be the multi-sequence associated to Λ. Then $A = A_\Lambda = (\gamma_{\xi+\eta})_{|\xi|, |\eta| \leq m}$ is a positive matrix with real entries, acting as an operator on $\mathcal{P}_m$, whose Hilbert space structure is built by identifying this space with $\mathbb{C}^N$ via the isomorphism

$$\mathbb{C}^N \ni x = (x_\alpha)_{|\alpha| \leq m} \mapsto p_x = \sum_{|\alpha| \leq m} x_\alpha t^\alpha \in \mathcal{P}_m, \tag{2.4}$$

where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \leq m\} = \dim \mathcal{P}_m$. We therefore have $(p_x | p_y) = (x|y)$, and $|||p_x||| = |||x|||$ for all $x, y \in \mathbb{C}^N$, where $(*|*)$ (resp. $||| * |||$) is the standard scalar product (resp. norm) on $\mathbb{C}^N$. Then $A = A_\Lambda$ is the (positive) operator with the property $(Ap|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_m$. The operator A will be occasionally called the *Hankel operator* of the uspf Λ. Note that $\mathcal{I}_\Lambda$ is equal to null-space $N(A)$ of A, and $\mathcal{H}_\Lambda$ is isomorphic to range $R(A)$ of A. Note also that the elements $\hat{p}, \hat{q}$ are orthogonal in $\mathcal{H}_\Lambda$ if and only if $(Ap|q) = (Bp|Bq) = 0$, where $B = A^{1/2}$.

Let us write an equation equivalent to (2.1) in this particular context, that is, an equation of the form $\Lambda(p^2) = \Lambda(p)$, $p \in \mathcal{RP}_m$, using the isomorphism (2.4).

As we have $\Lambda(p_x^2) = (Ax|x)$ and $\Lambda(p_x) = \Lambda(p_\iota p_x) = (A\iota|x)$, where $\iota = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $p_\iota = 1$, the equation we look for has the form

$$(Ax|x) - (A\iota|x) = 0, \tag{2.5}$$

which is called *idempotent equation* of the uspf Λ.

Note that $\Lambda(p_x^2) = (Ax|x) = 0$ implies $\Lambda(p_x) = (A\iota|x) = 0$, via the Cauchy-Schwarz inequality, showing that all real elements from $\mathcal{I}_\Lambda$ are solutions of eq.(2.5). Since all these elements are equivalent to zero, we are intrested only in nonnull real solutions $x = x^{(1)} \in R(A) = R(A_1)$, where $A_1 = A|R(A)$.

Proceeding as in Lemma 4, specifically using as parameters the elements of the set $S_1 := \{v_1 \in R(A_1) \cap \mathbb{R}^N; |||v_1||| = 1\}$, the nonnull solutions of the equation (2.5) in $R(A_1)) \cap \mathbb{R}^N$ are given by

$$x^{(1)} = \frac{(\iota|A_1 v_1)}{(A_1 v_1|v_1)} v_1,\ v_1 \in S_1,\ (\iota|A_1 v_1) \neq 0.$$

Example 1. As in [7], Example 2.1, we consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

acting as an operator on $\mathbb{C}^3$. In fact, the matrix A is the Hankel operator associated to a certain uspf (see Example 3).

If $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{C}^3$ is arbitrary, we have

$$A\mathbf{x} = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + 2x_3)$$

and

$$(A\mathbf{x}|\mathbf{x}) = |x_1 + x_2 + x_3|^2 + |x_3|^2 \geq 0,$$

so the operator A is positive.

We are interested in the solutions of the idempotent equation $(A\mathbf{x}|\mathbf{x}) = (A\iota|\mathbf{x})$, where $\iota = (1, 0, 0)$. It is easily seen that

$$N(A) = \{(x, -x, 0); x \in \mathbb{C}\},\ R(A) = \{(y, y, z); y, z \in \mathbb{C}\}.$$

As we have $\mathbb{C}^3 = N(A) \oplus R(A)$, each $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{C}^3$ can be uniquely written under the form

$$(x_1, x_2, x_3) = \left(\frac{x_1 - x_2}{2}, \frac{x_2 - x_1}{2}, 0\right) \oplus \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3\right)$$

as an element of $N(A) \oplus R(A)$. Looking only for solutions $(y, y, z) \in R(A)$ of the idempotent equation, we must have

$$(A(y, y, z)|(y, y, z)) = ((1, 1, 1))|(y, y, z)),$$

because $A\iota = (1, 1, 1) \in R(A)$. This is equivalent to the equality

$$4y^2 + 4yz + 2z^2 - 2y - z = 0, \tag{2.6}$$

which represents an ellipse passing through the origin.

For instance, the vectors $\mathbf{u} = (0, 0, 1/2)$ and $\mathbf{v} = (1/2, 1/2, -1/2)$ are solutions in $R(A)$ of the idempotent equation, with $(B\mathbf{u}|B\mathbf{v}) = 0$, where $B = A^{1/2}$, as one can easily see.

Example 2. As in the Introduction, let $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbb{R}^n$ and let $C(\Xi)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ξ, endowed with the sup-norm. Let also $\mu = \sum_{j=1}^d \lambda_j \delta_{\xi^{(j)}}$, with $\delta_{\xi^{(j)}}$ the Dirac measure at $\xi^{(j)}$, $\lambda_j > 0$ for all $j = 1, \dots, d$, and $\sum_{j=1}^d \lambda_j = 1$. We put $M(p) = \int_\Xi p d\mu$ for all $p \in C(\Xi)$, which is a uspf. Endowed with the Hilbert space structure induced by the measure μ, the space $C(\Xi)$ will be denoted by $L^2(\Xi, \mu)$. Therefore, if $\mathcal{S} = \mathcal{S}^{(2)} = C(\Xi)$ is the given function space, and $M : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is the given uspf, we have $\mathcal{I}_M = \{p \in \mathcal{S}^{(2)}; M(|p|^2) = 0\} = \{0\}$ and $\mathcal{H}_M = \mathcal{S}/\mathcal{I}_M = L^2(\Xi, \mu)$.

The space $L^2(\Xi, \mu)$ has a standard family of mutually orthogonal M-idempotents, say $\{\chi_1, \dots, \chi_d\}$, where χ_j is the characteristic function of the set $\{\xi^{(j)}\}$, $j = 1, \dots, d$, which is, in fact, an orthogonal basis of $L^2(\Xi, \mu)$.

Let us fix an integer $m \geq 0$ and let $\rho : \mathcal{P}_{2m} \mapsto C(\Xi)$ be the restriction map. Then $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, given by $\Lambda(p) = M(\rho(p))$, $p \in \mathcal{P}_{2m}$, is a uspf. In addition, we have $\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; p|\Xi = 0\}$, and the map $\hat\rho : \mathcal{H}_\Lambda \mapsto L^2(\Xi, \mu)$ induced by ρ is injective. In fact, as we clearly have

$$\|\hat\rho(\hat p)\|_{L^2(\Xi,\mu)} = \|\hat p\|_{\mathcal{H}_\Lambda},\ \hat p \in \mathcal{H}_\Lambda,$$

this map is acuallly an isometry.

As noticed in the Introduction, for a sufficiently large m, the map $\hat\rho$ is also surjective. In this case, the operator $\hat\rho : \mathcal{H}_\Lambda \mapsto L^2(\Xi, \mu)$ is unitary.

Assuming the map $\hat\rho : \mathcal{H}_\Lambda \mapsto L^2(\Xi, \mu)$ unitary, and setting $\hat b_j = \hat\rho^{-1}(\chi_j)$, $j = 1, \dots, d$, we can write that

$$\Lambda(b_j b_k) = M(\rho(b_j b_k)) = M(\rho(b_j)\rho(b_k)) = M(\chi_j \chi_k),\ j, k = 1, \dots, d,$$

showing that $\{\hat b_1, \dots, \hat b_d\}$ is an orthogonal basis consisting of Λ-idempotent elements.

Let us finally note that

$$\Lambda(t^{\alpha+\beta} b_j) = \int_{\{\xi^{(j)}\}} t^{\alpha+\beta} d\mu(t) = M(\chi_j)(\xi^{(j)})^{\alpha+\beta} =$$

$$M(\chi_j)(\xi^{(j)})^\alpha (\xi^{(j)})^\beta = \Lambda(b_j)^{-1}\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j).$$

for all α, β with $|\alpha| + |\beta| \leq m$ and $j = 1, \dots, d$. This equality, which is a "weighted multiplicativity" with respect to Λ, plays an important role in the characterization of those uspf having a representing measure with $\dim \mathcal{H}_\Lambda$ atoms (see Definition 3 and Theorem 2).

3. Integral representations of uspf's

This section is dedicated to the study of various integral representations of uspf's or of their restrictions to some subspaces. As in the Introduction, if $\mathcal{S}$ is a given finite dimensional function space, we set $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, while $\mathcal{H}_\Lambda$ is the finite dimensional Hilbert space $\mathcal{S}/\mathcal{I}_\Lambda$.

Remark 4. Let $\mathcal{S}$ be a finite dimensional function space, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. According to Theorem 1, the space $\mathcal{H}_\Lambda$ has orthogonal bases consisting of idempotent elements. If $\mathcal{B}$ is such a basis, we may speak about the *C^*-algebra structure of $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$*, in a sense to be explained in the following. More generally, if $\mathcal{B} \subset \mathcal{ID}(\Lambda)$ is a collection of nonnull mutually orthogonal elements whose sum is $\hat{1}$, and if $\mathcal{H}_\mathcal{B}$ is the complex vector space spanned by $\mathcal{B}$ in $\mathcal{H}_\Lambda$, we may speak about the *C^*-algebra (structure of) $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$*. Using the basis $\mathcal{B}$ of the space $\mathcal{H}_\mathcal{B}$, we may define a multiplication, an involution, and a norm on $\mathcal{H}_\mathcal{B}$, making it a unital, commutative, finite dimensional C^*-algebra. Specifically, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ with $\hat{1} = \sum_{j=1}^d \hat{b}_j$, and if $\hat{p} = \sum_{j=1}^d \alpha_j \hat{b}_j$, $\hat{q} = \sum_{j=1}^d \beta_j \hat{b}_j$, are elements from $\mathcal{H}_\mathcal{B}$, their product is given by $\hat{p} \cdot \hat{q} = \sum_{j=1}^d \alpha_j \beta_j \hat{b}_j$. The involution is defined by $\hat{p}^* = \sum_{j=1}^d \overline{\alpha_j} \hat{b}_j$, and the norm is given by $\|\hat{p}\|_\infty = \max_{1 \le j \le d} |\alpha_j|$, for $\hat{p} = \sum_{j=1}^d \alpha_j \hat{b}_j$.

Note that if for $p, q \in \mathcal{S}$ we also have $pq \in \mathcal{S}$, the element $\hat{p} \cdot \hat{q}$ is, in general, different from $\widehat{pq}$.

It is easily seen that the space of characters of the C^*-algebra $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$, say $\Delta = \{\delta_1, \ldots, \delta_d\}$, coincides with the dual basis of $\mathcal{B}$. As $\mathcal{H}_\mathcal{B}$ is also a Hilbert space as a subspace of $\mathcal{H}_\Lambda$, we note that $\delta_j(\hat{p}) = \Lambda(b_j)^{-1} \langle \hat{p}, \hat{b}_j \rangle$, $\hat{p} \in \mathcal{H}_\mathcal{B}$, $j = 1, \ldots, d$.

Although some of the following assertions hold true in the context of finite dimensional function spaces, from now on we assume $\mathcal{S} = \mathcal{P}_m$ for some given integer $m \ge 0$, which is the most significant case for this type of problem.

Proposition 1. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be a collection of mutually orthogonal elements with $\hat{1} = \sum_{j=1}^d \hat{b}_j$, and let $\mathcal{H}_\mathcal{B}$ be the complex vector space spanned by $\mathcal{B}$ in $\mathcal{H}_\Lambda$.*

Let Δ be the space of characters of the C^*-algebra $\mathcal{H}_\mathcal{B}$, induced by $\mathcal{B}$. If $\mathcal{S}_\mathcal{B} = \{p \in \mathcal{P}_m; \hat{p} \in \mathcal{H}_\mathcal{B}\}$, there exists a linear map $\mathcal{S}_\mathcal{B} \ni p \mapsto p^\# \in C(\Delta)$, whose kernel is $\mathcal{I}_\Lambda$, such that

$$\Lambda(p) = \int_\Delta p^\#(\delta) d\mu(\delta), \ p \in \mathcal{S}_\mathcal{B},$$

where μ is a d-atomic probability measure on Δ.

Proof. For a fixed choice $b_1, \ldots, b_d$ in $\mathcal{RP}_m$ of representatives from the corresponding classes $\hat{b}_1, \ldots, \hat{b}_d$, we put $\mathcal{G}_\mathcal{B}$ to be the linear span of the set $\{b_1, \ldots, b_d\}$. Then we have $\mathcal{S}_\mathcal{B} = \mathcal{G}_\mathcal{B} + \mathcal{I}_\Lambda$, which is a direct sum, by an argument from the proof of Corollary 1. This decomposition allows us to define a linear map $\mathcal{S}_\mathcal{B} \ni p \mapsto p^\# \in C(\Delta)$ via the equality $p^\#(\delta) = \delta(\hat{p})$ for all $p \in \mathcal{S}_\mathcal{B}$ and $\delta \in \Delta$. It is obvious that the kernel of the map $\mathcal{S}_\mathcal{B} \ni p \mapsto p^\# \in C(\Delta)$ is precisely $\mathcal{I}_\Lambda$.

For an arbitrary $p \in \mathcal{S}_\mathcal{B}$, we have a representation of the form $p = \tau_1 b_1 + \cdots + \tau_d b_d + r_p$, where $\tau_j = \delta_j(\hat{p})$, $j = 1, \ldots, d$, and $r_p \in \mathcal{I}_\Lambda$. Hence,

$$\Lambda(p) = \sum_{j=1}^{d} \tau_j \Lambda(b_j) = \int_\Delta p^\#(\delta) d\mu(\delta),$$

where μ is the measure with weights $\Lambda(b_j)$ at δ_j, $j = 1, \ldots, d$, which is a d-atomic probability measure on Δ, because $\Lambda(b_j) = \Lambda(b_j^2) > 0$ for all j and $\Lambda(b_1) + \cdots + \Lambda(b_d) = \Lambda(1) = 1$. □

Proposition 2. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements. Let also $\mathcal{H}_\mathcal{C}$ be the sub-C^*-algebra generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ in $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal is $\le \dim \mathcal{H}_\Lambda$, and a linear map $\mathcal{S}_\mathcal{C} \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, such that*

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi),\ u \in \mathcal{S}_\mathcal{C},$$

where $\mathcal{S}_\mathcal{C} = \{u \in \mathcal{P}_m; \hat{u} \in \mathcal{H}_\mathcal{C}\}$, and μ is a probability measure on Ξ.

Proof. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, inducing the C^*-algebra structure of $\mathcal{H}_\Lambda$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the set of all characters of the C^*-algebra $\mathcal{H}_\Lambda$. First of all, we shall deal with the structure of the sub-C^*-algebra $\mathcal{H}_\mathcal{C}$ generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ in $\mathcal{H}_\Lambda$. Obviously, the algebra $\mathcal{H}_\mathcal{C}$ consists of arbitrary polynomials in $\hat{t}_1, \ldots, \hat{t}_n$. We can write $\hat{t}_k = \tau_{k1}\hat{b}_1 + \cdots + \tau_{kd}\hat{b}_d$, where $\tau_{kj} = \delta_j(\hat{t}_k)$, $k = 1, \ldots, n$, $j = 1, \ldots, d$. Put $\tau^{(j)} = (\tau_{1j}, \ldots, \tau_{nj}) \in \mathbb{R}^n$, $j = 1, \ldots, d$. Let us show, by recurrence, that

$$\hat{t}^\alpha = \sum_{j=1}^{d} (\tau^{(j)})^\alpha \hat{b}_j, \tag{3.1}$$

where $\hat{t} = (\hat{t}_1, \ldots, \hat{t}_n)$, so $\hat{t}^\alpha = (\hat{t}_1)^{\alpha_1} \cdots (\hat{t}_n)^{\alpha_n}$, $(\tau^{(j)})^\alpha = (\tau_{1j})^{\alpha_1} \cdots (\tau_{nj})^{\alpha_n}$, for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \le m$. Indeed, assuming that (3.1) holds if $|\alpha| < m$, we have, for some fixed $k \in \{1, \ldots, n\}$,

$$\hat{t}_k \cdot \hat{t}^\alpha = \sum_{l=1}^{n} \sum_{j=1}^{d} \tau_{kl} (\tau^{(j)})^\alpha \hat{b}_l \cdot \hat{b}_j = \sum_{j=1}^{d} \tau_{kj} (\tau^{(j)})^\alpha \hat{b}_j = \hat{t}^{\alpha_{(k)}},$$

where $\alpha_{(k)} = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1} \ldots, \alpha_n)$, showing that (3.1) holds whenever $|\alpha| \le m$. Consequently, $p(\hat{t}) = \sum_{j=1}^{d} p(\tau^{(j)})\hat{b}_j$ for every $p \in \mathcal{P}$.

Let $\Xi = \{\xi^{(1)} \ldots, \xi^{(v)}\}$ be the distinct points from the set $\{\tau^{(1)} \ldots, \tau^{(d)}\}$, where $v \le d$. Let also $I_j = \{k; \tau^{(k)} = \xi^{(j)}\}$, $j = 1, \ldots, v$.

If $p \in \mathcal{P}$ is arbitrary, then, as above,

$$p(\hat{t}) = \sum_{j=1}^{v} p(\xi^{(j)})\hat{c}_j, \tag{3.2}$$

where $\hat{c}_j = \sum_{k \in I_j} \hat{b}_k$, $j = 1, \dots, v$, which is a family of mutually orthogonal idempotents, whose sum is $\hat{1}$.

Consider now the space $\mathcal{S}'_{\mathcal{C}}$ given by

$$\mathcal{S}'_{\mathcal{C}} = \{\sum_{j=1}^{v} p(\xi^{(j)})c_j + r;\, p \in \mathcal{P}|\Xi,\, r \in \mathcal{I}_\Lambda\} = \mathcal{G}_{\mathcal{C}} + \mathcal{I}_\Lambda,$$

with $\mathcal{G}_{\mathcal{C}} = \{\sum_{j=1}^{v} p(\xi^{(j)})c_j;\, p \in \mathcal{P}|\Xi\}$, where $\mathcal{P}|\Xi$ is the space of all restrictions of arbitrary polynomials to the set Ξ.

Let us remark that the sum $\mathcal{G}_{\mathcal{C}} + \mathcal{I}_\Lambda$ is direct. If $w = \sum_{j=1}^{v} p(\xi^{(j)})c_j \in \mathcal{I}_\Lambda$, then, as in the proof of Corollary 1, namely using the identity $\Lambda(|w|^2) = 0$, we infer that $p(\xi^{(j)}) = 0$, $j = 1, \dots, v$, and thus $w = 0$. In particular, if $u = \sum_{j=1}^{v} p(\xi^{(j)})c_j + r \in \mathcal{S}'_{\mathcal{C}}$, the function $p|\Xi$ is uniquely determined.

Further, we have a linear map $\mathcal{S}'_{\mathcal{C}} \ni u \mapsto u^{\#} \in C(\Xi)$, defined in the following way. Taking an element $u = \sum_{j=1}^{v} p(\xi^{(j)})c_j + r \in \mathcal{S}'_{\mathcal{C}}$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$, we put $u^{\#}(\xi) = p(\xi)$, $\xi \in \Xi$. As the function $p|\Xi$ is uniquely determined by u, the definition of $u^{\#}$ is correct, the assignment $u \mapsto u^{\#}$ is linear, and its kernel is precisely $\mathcal{I}_\Lambda$. In addition, $\mathcal{S}'_{\mathcal{C}} \supset \{u \in \mathcal{P}_m; \hat{u} \in \mathcal{H}_{\mathcal{C}}\} = \mathcal{S}_{\mathcal{C}}$, via (3.2).

Consequently, if $u = \sum_{j=1}^{v} p(\xi^{(j)})c_j + r$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$, we have

$$\Lambda(u) = \sum_{j=1}^{v} p(\xi^{(j)})\Lambda(c_j) = \int_\Xi u^{\#}(\xi)d\mu(\xi),$$

where μ is the measure with weights $\Lambda(c_j)$ at $\xi^{(j)}$, $j = 1, \dots, v$, which concludes the proof. $\square$

Remark 5. With the notation of the previous proof, the idempotents $\hat{b}_1, \dots, \hat{b}_d$ are minimal because they form an orthogonal basis of $\mathcal{H}_\Lambda$, while the idempotents $\hat{c}_1, \dots, \hat{c}_v$ are, in general, decomposable (see Definition 2).

Proposition 3. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by an orthogonal basis consisting of idempotent elements. Also assume that the elements $\{\hat{1}, \hat{t}_1, \dots, \hat{t}_n\}$ generate the C^*-algebra $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal equals* $\dim \mathcal{H}_\Lambda$*, and a surjective linear map $\mathcal{P}_m \ni u \mapsto u^{\#} \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, with the property*

$$\Lambda(u) = \int_\Xi u^{\#}(\xi)d\mu(\xi),\ u \in \mathcal{P}_m,$$

where μ is a probability measure on Ξ.

Moreover, the map $\mathcal{P}_m \ni u \mapsto u^{\#} \in C(\Xi)$ induces a $*$-isomorphism between C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$.

If $r(\hat{t}_1,\dots,\hat{t}_n)=0$ *for all* $r\in\mathcal{I}_\Lambda$, $b_j(\xi^{(l)})=0$ *for* $j\ne l$ *and* $b_j(\xi^{(j)})=0$, $j,l=1,\dots,d$, *then* $u^\#=u|\Xi$ *for all* $u\in\mathcal{P}_m$.

Proof. We follow the lines and use the notation of the preceding proof. We must have $\mathcal{H}_\mathcal{C}=\mathcal{H}_\Lambda$, and $\mathcal{S}_\mathcal{C}=\mathcal{P}_m$. Moreover, if $\mathcal{B}=\{\hat{b}_1,\dots,\hat{b}_d\}$ is the orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements given by the hypothesis, and $\Delta=\{\delta_1,\dots,\delta_d\}$ is the set of the characters of the C^*-algebra $\mathcal{H}_\Lambda$, the points $\tau^{(j)}\in\mathbb{R}$, $j=1,\dots,d$, are distinct because the family of generators $\{\hat{t}_1,\dots,\hat{t}_n\}$ separates the points of Δ, so

$$\delta_j(\hat{t})=\tau^{(j)}=\xi^{(j)}=(\xi_1^{(j)},\dots,\xi_n^{(j)})\in\mathbb{R}^n,\ j=1,\dots,d,$$

and also $c_j=b_j$, $\xi_k^{(j)}=\delta_j(\hat{t}_k)$, $j=1,\dots,d$, $k=1,\dots,n$.

Note that the space $\mathcal{P}_m$ can be written as

$$\mathcal{P}_m=\{\sum_{j=1}^d p(\xi^{(j)})b_j+r;\, p\in\mathcal{P},\, r\in\mathcal{I}_\Lambda\}=\mathcal{G}_m+\mathcal{I}_\Lambda,$$

with $\mathcal{G}_m=\{\sum_{j=1}^d p(\xi^{(j)})b_j;\, p\in\mathcal{P}\}$, and where the sum of spaces is direct. Consequently, if $u\in\mathcal{P}_m$, we must have $u=\sum_{j=1}^d p(\xi^{(j)})b_j+r$ for some $p\in\mathcal{P}$ and $r\in\mathcal{I}_\Lambda$. Moreover, the function $p|\Xi$ is uniquely determined by u, and setting $u^\#=p|\Xi$, we have a linear map $\mathcal{P}_m\ni u\mapsto u^\#\in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$. In addition, as in Proposition 2, we also have the formula

$$\Lambda(u)=\int_\Xi u^\#(\xi)d\mu(\xi),\ u\in\mathcal{P}_m,$$

where μ is the measure with weights $\Lambda(b_j)$ at $\xi^{(j)}$, $j=1,\dots,d$.

Note that the map $\mathcal{P}_m\ni u\mapsto u^\#\in C(\Xi)$ is also surjective because, taking an arbitrary element of $C(\Xi)$ written under the form $p|\Xi$ for some $p\in\mathcal{P}$, the polynomial $u=\sum_{j=1}^d p(\xi^{(j)})b_j\in\mathcal{G}_m$ has the property $u^\#=p|\Xi$.

Since the map $\mathcal{P}_m\ni u\mapsto u^\#\in C(\Xi)$ is surjective and its kernel is precisely $\mathcal{I}_\Lambda$, the induced map $\mathcal{H}_\Lambda\ni\hat{u}\mapsto\hat{u}^\#\in C(\Xi)$ is correctly defined and bijective, where $\hat{u}^\#(\xi)=u^\#(\xi)$, $\xi\in\Xi$. This map is actually a $*$-isomprphism.

To prove this assertion, let us first choose the polynomials $p_k\in\mathcal{P}$ and $r_k\in\mathcal{I}_\Lambda$ with the property $b_k=\sum_{j=1}^d p_k(\xi^{(j)})b_j+r_k$, $k=1,\dots,d$. The uniqueness of this representation shows that $r_k=0$, $p_k(\xi^{(j)})=1$ if $k=j$, and $=0$ otherwise, for all $k,j=1,\dots,d$. In addition, $\hat{b}_k^\#=p_k|\Xi$, $k=1,\dots,d$.

Because $(\hat{b}_j\cdot\hat{b}_k)^\#(\xi)=0=p_j(\xi)p_k(\xi)$ if $j\ne k$, and $(\hat{b}_j\cdot\hat{b}_j)^\#(\xi)=\hat{b}_j^\#(\xi)=p_j(\xi)=p_j(\xi)^2$, for all $\xi\in\Xi$ and $j,k=1,\dots,d$, it follows that the map $\mathcal{H}_\Lambda\ni\hat{u}\mapsto\hat{u}^\#\in C(\Xi)$ is multiplicative. Taking into account the definitions given in Remark 4, the equalities

$$\hat{1}^\#(\xi)=\sum_{j=1}^d(\hat{b}_j)^\#(\xi)=\sum_{j=1}^d p_j(\xi)=1,$$

as well as $(\hat{u}^*)^\# = \overline{\hat{u}^\#}$, show that the map $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^\# \in C(\Xi)$ is a unital $*$-morphism. In addition, if $\hat{u} = \sum_{j=1}^d p(\xi^{(j)})\hat{b}_j \in \mathcal{H}_\Lambda$ is arbitrary,

$$\|\hat{u}\|_\infty = \max_{1\le j\le d} |p(\xi^{(j)})| = \|\hat{u}^\#\|_\infty,$$

proving that $\mathcal{H}_\Lambda \ni \hat{u} \mapsto \hat{u}^\# \in C(\Xi)$ is a $*$-isomorphism.

Finally, assume that $r(\hat{t}_1, \dots, \hat{t}_n) = 0$ for all $r \in \mathcal{I}_\Lambda$. Then

$$r(\xi^{(l)}) = r(\delta_l(\hat{t})) = \delta_l(r(\hat{t})) = 0,\ l = 1, \dots, d.$$

Consequently, if $u \in \mathcal{P}_m$ has the form $u = \sum_{j=1}^d p(\xi^{(j)})b_j + r$ for some $p \in \mathcal{P}$ and $r \in \mathcal{I}_\Lambda$, then

$$u(\xi^{(l)}) = \sum_{j=1}^d p(\xi^{(j)})b_j(\xi^{(l)}) + r(\xi^{(l)}) = \sum_{j=1}^d u^\#(\xi^{(j)})b_j(\xi^{(l)}),\ l = 1, \dots, d.$$

If, moreover, $b_j(\xi^{(l)}) = 0$ for $j \neq l$ and $b_j(\xi^{(j)}) = 0$, $j, l = 1, \dots, d$, then $u^\# = u|\Xi$ for all $u \in \mathcal{P}_m$, which completes the proof of the proposition. $\square$

Remark 6. Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and assume that the space $\mathcal{H}_\Lambda$ is endowed with the C^*-algebra structure induced by the orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$, consisting of idempotent elements. Also assume that the elements $\{\hat{1}, \hat{t}_1, \dots, \hat{t}_n\}$ generate the C^*-algebra $\mathcal{H}_\Lambda$. In particular, for each j there exists a polynomial $\pi_j \in \mathcal{P}$ such that $\hat{b}_j = \pi_j(\hat{t})$, $j = 1, \dots, d$. If Δ is the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$, for every $\delta \in \Delta$ we have $\delta(\hat{b}_j) = \pi_j(\delta(\hat{t}))$, $j = 1, \dots, d$, showing that $\{\pi_1, \dots, \pi_d\}$ is an interpolating family for the set $\{\delta(\hat{t}) \in \mathbb{R}^n; \delta \in \Delta\}$. A similar property has been already obtained in the previous proof, via a different argument.

Remark 7. (1) Assume that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ given by

$$\Lambda(p) = \sum_{j=1}^d \lambda_j p(\xi^{(j)}),\ p \in \mathcal{P}_{2m},$$

with $\lambda_j > 0$ for all $j = 1, \dots, d$, and $\sum_{j=1}^d \lambda_j = 1$, where $d = \dim\mathcal{H}_\Lambda$.

Let $r \ge m$ be an integer such that $\mathcal{P}_r$ contains interpolating polynomials for the family of points $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\}$. Setting $\Lambda_\mu(p) = \int_\Xi p d\mu$, $p \in \mathcal{P}_{2r}$, we have $\Lambda_\mu|\mathcal{P}_{2m} = \Lambda$, and $\mathcal{I}_{\Lambda_\mu} = \{p \in \mathcal{P}_r; p|\Xi = 0\}$, as one can easily see. Moreover, the space $\mathcal{H}_r := \mathcal{P}_r/\mathcal{I}_{\Lambda_\mu}$ is at least linearly isomorphic to $C(\Xi)$, via the map $\mathcal{H}_r \ni p + \mathcal{I}_{\Lambda_\mu} \mapsto p|\Xi \in C(\Xi)$. As $\mathcal{H}_\Lambda$ may be regarded as a subspace of $\mathcal{H}_r$ (see Remark 1(ii)), and $\dim\mathcal{H}_\Lambda = \dim C(\Xi)$, the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in C(\Xi)$ is a linear isomorphism. Let $\chi_k \in C(\Xi)$ be the characteristic function of the set $\{\xi^{(k)}\}$ and let $\hat{b}_k \in \mathcal{H}_\Lambda$ be the element with $b_k|\Xi = \chi_k$, $k = 1, \dots, d$. As in Example 2, we obtain that $\{\hat{b}_1, \dots, \hat{b}_d\}$ is a basis of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents. Consequently, if $\mathcal{H}_\Lambda$ is given the C^*-algebra structure induced by $\{\hat{b}_1, \dots, \hat{b}_d\}$, then $\mathcal{H}_\Lambda$ and $C(\Xi)$ are isomorphic as C^*-algebras (as in the proof of Proposition 3). In addition,

$\Lambda(b_j) = \lambda_j$ for all $j = 1, \ldots, d$, and that if $\hat{p} = \tau_1 \hat{b}_1 + \cdots + \tau_d \hat{b}_d \in \mathcal{H}_\Lambda$ is arbitrary, then $\tau_j = \lambda_j^{-1}\Lambda(pb_j) = \lambda_j^{-1}\Lambda_\mu(p\chi_j) = p(\xi^{(j)})$ for all $j = 1, \ldots, d$.

In other words, if $\Delta = \{\delta_1, \ldots, \delta_d\}$ is the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, we have $\delta_k(\hat{b}_j) = \lambda_k^{-1}\Lambda_\mu(b_j\chi_k) = b_j(\xi^{(k)})$, $j = 1, \ldots, d$.

This discussion also shows that the Hilbert spaces $\mathcal{H}_\Lambda$ and $L^2(\Xi, \mu)$ are unitarily equivalent via the unitary map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in L^2(\Xi, \mu)$.

(2) Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf such that $\mathcal{I}_\Lambda = \{0\}$; therefore, $\mathcal{P}_m = \mathcal{H}_\Lambda$. Let $\mathcal{B} = \{b_1, \ldots, b_d\}$ be an orthogonal basis of $\mathcal{P}_m$ consisting of idempotents (where $d = \dim\mathcal{P}_m$). Assume that the family $\{1, t_1, \ldots, t_n\}$ generates the C^*-algebra $\mathcal{P}_m$ induced by $\mathcal{B}$. In particular, defining the set $\Xi = \{\xi^{(1)} \ldots, \xi^{(d)}\}$ as in Proposition 3, we obtain the equality $\mathcal{P}_m = \{\sum_{j=1}^d p(\xi^{(j)})b_j;\, p \in \mathcal{P}\}$.

Assume now that $\delta_j(b_k(t)) = 0$ if $j \neq k$, and $\delta_j(b_j(t)) = 1$, where δ_j is a character and $b_k(t)$ is computed in the the C^*-algebra $\mathcal{P}_m$ $(j, k = 1, \ldots, d)$. As $\delta_j(b_k(t)) = b_k(\delta_j(t)) = b_k(\xi^{(j)})$, then $b_k(\xi^{(j)}) = 0$ for $j \neq k$ and $b_j(\xi^{(j)}) = 0$, $j, k = 1, \ldots, d$. By Proposition 3, we must have $p = \sum_{j=1}^d p(\xi^{(j)})b_j$, $p \in \mathcal{P}_m$, and so, $\Lambda|\mathcal{P}_m$ has a d-atomic representing measure μ on Ξ given by

$$\Lambda(p) = \sum_{j=1}^d \lambda_j p(\xi^{(j)}) = \int_\Xi p(t)d\mu(t)),\ p \in \mathcal{P}_m,$$

with $\lambda_j = \Lambda(b_j)$, $j = 1, \ldots, d$.

In fact, in this case the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has itself a representing measure. Indeed, fixing a multi-index θ with $|\theta| \leq 2m$, we write $\theta = \alpha + \beta$, with $|\alpha| \leq m$, $|\beta| \leq m$. As we have $t^\alpha = \sum_{j=1}^d (\xi^{(j)})^\alpha b_j$, $t^\beta = \sum_{j=1}^d (\xi^{(j)})^\beta b_j$, using the Hilbert space structure of $\mathcal{P}_m$ induced by Λ, we deduce that

$$\Lambda(t^\theta) = \langle t^\alpha, t^\beta \rangle = \sum_{j,k=1}^d (\xi^{(j)})^\alpha (\xi^{(k)})^\beta \Lambda(b_j b_k) = \sum_{j=1}^d \lambda_j (\xi^{(j)})^\theta = \int_\Xi t^\theta d\mu(t).$$

As $|\theta| \leq 2m$ is arbitrary and the result does not depend of the decomposition $\theta = \alpha + \beta$, the general case follows by linearity. In this way, we get the following.

Corollary 2. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf such that $\mathcal{I}_\Lambda = \{0\}$. Let $\mathcal{B} = \{b_1, \ldots, b_d\}$ be an orthogonal basis of $\mathcal{P}_m$ consisting of idempotents, where $d = \dim\mathcal{P}_m$. Assume that the family $\{1, t_1, \ldots, t_n\}$ generates the C^*-algebra $\mathcal{P}_m$ induced by $\mathcal{B}$, and that $\delta_j(b_k(t)) = 0$ if $k \neq j$, and $\delta_j(b_j(t)) = 1$, where δ_j is a character and $b_k(t)$ is computed in the the C^*-algebra $\mathcal{P}_m$ $(j, k = 1, \ldots, d)$. Then the uspf Λ has a representig measure consisting of d atoms.*

(3) With the notation from the proof of Proposition 2, the monomial $\hat{t}^\alpha$ is an element of the algebra $\mathcal{H}_\mathcal{C}$, not necessarily equal to $\widehat{t^\alpha} = t^\alpha + \mathcal{I}_\Lambda \in \mathcal{H}_\Lambda$ (see also Remark 4).

Theorem 2, which will be proved in the sequel, characterizes the existence of representing measures for a uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, having $d = \dim \mathcal{H}_\Lambda$

atoms, in terms of orthogonal bases of $\mathcal{H}_\Lambda$ consisting of idempotent elements. In other words, we use only intrinsic conditions. Other characterizations can be found in [3], Theorem 7.10 or in [8], Theorem 2.8, stated in terms of flat extensions, which are, in general, not intrinsic.

Before proving the theorem, we need some preparation.

Definition 3. Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* if

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j) \tag{3.3}$$

whenever $|\alpha| + |\beta| \le m$, $j = 1, \ldots, d$.

Lemma 5. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. The basis $\mathcal{B}$ is Λ-multiplicative if and only if $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \le m$ and δ is any character of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$.*

Proof. Let $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the set of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$. It follows from Remark 4 that $\delta_j(\hat{p}) = \Lambda(b_j)^{-1}\Lambda(pb_j)$, $p \in \mathcal{P}_m$, $j = 1, \ldots, d$.

Assuming $\mathcal{B}$ to be Λ-multiplicative, we have

$$\delta_j(\widehat{t^{\alpha+\beta}}) = \Lambda(b_j)^{-1}\Lambda(t^{\alpha+\beta} b_j) = \Lambda(b_j)^{-2}\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \delta_j(\widehat{t^\alpha})\delta_j(\widehat{t^\beta})$$

whenever $|\alpha| + |\beta| \le m$, $j = 1, \ldots, d$, which is equivalent to the condition $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \le m$ and δ is a character of the C^*-algebra $\mathcal{H}_\Lambda$ associated to $\mathcal{B}$

The same calculation shows that the condition $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \le m$ and δ is a character of the C^*-algebra $\mathcal{H}_\Lambda$ associated to $\mathcal{B}$ implies that the basis $\mathcal{B}$ is Λ-multiplicative. ☐

Theorem 2. *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a Λ-multiplicative basis of $\mathcal{H}_\Lambda$.*

Proof. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, and let $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the set of the characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$. First assume that $\mathcal{B}$ is Λ-multiplicative. Therefore, $\delta(\widehat{t^\alpha}) = \delta(\hat{t}^\alpha)$ whenever $|\alpha| \le m$ and $\delta \in \Delta$, by Lemma 5. Denote by $\mathcal{H}_\mathcal{C}$ the sub-C^*-algebra generated by the set $\mathcal{C} = \{\hat{1}, \hat{t}_1, \ldots, \hat{t}_n\}$ in $\mathcal{H}_\Lambda$. Our hypothesis implies the equality $\widehat{t^\alpha} = \hat{t}^\alpha$ whenever $|\alpha| \le m$, because the algebra $\mathcal{H}_\Lambda$ is semi-simple. Moreover, as the elements $\{\widehat{t^\alpha}; |\alpha| \le m\}$ span the linear space $\mathcal{H}_\Lambda$, the elements $\hat{t}_1, \ldots, \hat{t}_n$ have to generate the algebra $\mathcal{H}_\Lambda$. In particular, we must have the equality $\mathcal{H}_\mathcal{C} = \mathcal{H}_\Lambda$, and the family $\{\hat{t}_1, \ldots, \hat{t}_n\}$ separates the points of Δ. In this way, the map

$$\Delta \ni \delta \mapsto (\delta(\hat{t}_1), \ldots, \delta(\hat{t}_n)) \in \mathbb{R}^n$$

is injective. Set $\xi^{(j)} = (\delta_j(\hat{t}_1), \ldots, \delta_j(\hat{t}_n))$, $j = 1, \ldots, d$, $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$.

As in (the proof of) Proposition 2, we have $\hat{t}^\alpha = \sum_{j=1}^d (\xi^{(j)})^\alpha \hat{b}_j$. Therefore, $\widehat{t^\alpha} = \sum_{j=1}^d (\xi^{(j)})^\alpha \hat{b}_j$ whenever $|\alpha| \le m$. If $p(t) = \sum_{|\alpha|\le m} c_\alpha t^\alpha \in \mathcal{P}_m$, then

$$\Lambda(p) = \sum_{|\alpha|\le m} c_\alpha \sum_{j=1}^d (\xi^{(j)})^\alpha \Lambda(b_j) = \sum_{j=1}^d p(\xi^{(j)})\Lambda(b_j) = \int_\Xi p(\xi) d\mu(\xi)$$

where μ is the measure with weights $\Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \dots, d$.

Now, as in Remark 7(2), if θ is a multi-index with $|\theta| \le 2m$, we write $\theta = \alpha + \beta$, with $|\alpha| \le m$, $|\beta| \le m$. Then, using the Hilbert space structure of $\mathcal{H}_\Lambda$,

$$\Lambda(t^\theta) = \langle \hat{t}^\alpha, \hat{t}^\beta \rangle = \langle \sum_{j=1}^d (\xi^{(j)})^\alpha \hat{b}_j, \sum_{k=1}^d (\xi^{(k)})^\beta \hat{b}_k \rangle = \sum_{j=1}^d (\xi^{(j)})^\theta \Lambda(b_j),$$

leading to the equality

$$\Lambda(p) = \int_\Xi p(\xi) d\mu(\xi) \tag{3.4}$$

for all polynomials $p \in \mathcal{P}_{2m}$, which provides a d-atomic representation measure for Λ.

Conversely, assume that the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ with $d = \dim \mathcal{H}_\Lambda$ atoms. From the discussion in Remark 7(1), we know that the C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$ are isomorphic via the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in C(\Xi)$, which leads to the existence of an orthogonal basis $\mathcal{B}$ of the Hilbert space $\mathcal{H}_\Lambda$ consisting of idempotent elements. In addition, the maps $\delta_j(\hat{p}) = p(\xi^{(j)})$, $j = 1, \dots, d$, are the characters of $\mathcal{H}_\Lambda$. Therefore,

$$\delta_j(\widehat{t^\alpha}) = t^\alpha(\xi^{(j)}) = (\xi^{(j)})^\alpha = \delta_j(\hat{t}^\alpha),$$

whenever $|\alpha| \le m$ and $j = 1, \dots, d$, showing that $\mathcal{B}$ is a Λ-multiplicative basis, via Lemma 5. This concludes the proof of Theorem 2. □

A more explicit form of Theorem 2 is provided by the following assertion.

Corollary 3. *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ possessing $d := \dim \mathcal{H}_\Lambda$ atoms if and only if there exists a family of polynomials $\{b_1, \dots, b_d\} \subset \mathcal{RP}_m$ with the following properties:*

(i) $\Lambda(b_j^2) = \Lambda(b_j) > 0,\ j = 1, \dots, d$;

(ii) $\Lambda(b_j b_k) = 0,\ j, k = 1, \dots, d,\ j \ne k$;

(iii)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $0 \ne |\alpha| \le |\beta|$, $|\alpha| + |\beta| \le m$, $j = 1, \dots, d$.

The assertion follows directly from Theorem 2. We omit the details.

Example 3. The matrix A from Example 1 is the Hankel operator of the uspf $\Lambda : \mathcal{P}_4^1 \mapsto \mathbb{C}$, where $\mathcal{P}_4^1$ is the space of of polynomials in one real variable t, with complex coefficients, of degre ≤ 4, and Λ is the Riesz functional associated to the sequence $\gamma = (\gamma_k)_{0\le k\le 4}$, $\gamma_0 = \cdots = \gamma_3 = 1$, $\gamma_4 = 2$. This matrix has been used in [7] to show that this truncated moment problem has no representing measure in $\mathbb{R}$. We shall obtain the same conclusion, via our methods.

Note that $\mathcal{I}_\Lambda = \{p(t) = a - at; a \in \mathbb{C}\}$, and

$$\mathcal{H}_\Lambda = \{\hat{p}\,;\, p(t) = a + at + bt^2,\, a, b \in \mathbb{C}\}.$$

Setting $p_0(t) = 1/2 - t/2$, $p_1(t) = 1/2 + t/2$, we have $1 = p_0 + p_1$ and $t = p_1 - p_0$. But $p_0 \in \mathcal{I}_\Lambda$, and so $\hat{t} = \hat{1}$. Consequently, for any choice of an orthogonal basis in $\mathcal{H}_\Lambda$ consisting of idempotents, we cannot have $\widehat{t^2} = \hat{t}^2$ because $\hat{t}^2 = \hat{t} = \hat{1}$, while $\widehat{t^2} = t^2 + \mathcal{I}_\Lambda \neq \hat{1}$. This shows that Λ has no representing measure consisting of two atoms, via Theorem 2. As a matter of fact, the element $\hat{t}$ does not separate the points of the space of characters of $\mathcal{H}_\Lambda$ for any choice of an orthogonal basis $\{\hat{b}_1, \hat{b}_2\}$ consisiting of idempotent elements. Indeed, identifying the space of characters with the pair $\{\hat{b}_1/\Lambda(b_1), \hat{b}_2/\Lambda(b_2)\}$, we have

$$\langle \hat{t}, \hat{b}_j/\Lambda(b_j)\rangle = \langle \hat{1}, \hat{b}_j/\Lambda(b_j)\rangle = \Lambda(b_j/\Lambda(b_j)) = 1,\ j = 1, 2.$$

Example 4. Corollary 3 implies that all uspf $\Lambda : \mathcal{P}_2 \mapsto \mathbb{C}$ have representing measures in $\mathbb{R}^n$ with $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an arbitrary orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, then the condition (3.3)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

is automatically fulfilled when $|\alpha| + |\beta| \leq 1$, $j = 1, \ldots, d$.

In this case, we may write explicitly all representing measures of Λ. Indeed, with $b_1, \ldots, b_d$ as above, the support of the corresponding representing measure, say $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, is given by

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d,$$

while the corresponding weights are $\Lambda(b_1), \ldots, \Lambda(b_d)$, via the proof of Theorem 2. (See also [9], Section 4, for a different argument.)

The next result is a version of Tchakaloff's theorem (see also [1, 6, 12] etc.), obtained with our methods.

Corollary 4. *Let ν be a positive Borel measure on $\mathbb{R}^n$ such that $\int |||t|||^2 d\nu(t)$ is finite. Then there exist a subset $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and positive numbers $\lambda_1, \ldots, \lambda_d$, where $d \leq n + 1$, such that*

$$\int p(t) d\mu(t) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ p \in \mathcal{P}_2.$$

Moreover, the weights $\lambda_1, \ldots, \lambda_d$, and the nodes $\xi^{(1)}, \ldots, \xi^{(d)}$ as well, are given by explicit formulas.

Proof. With no loss of generality, we may assume $\nu(\mathbb{R}^n) = 1$. Then the map $\Lambda(p) = \int p d\nu$ is a uspf on $\mathcal{P}_2$. According to the previous example, each orthogonal bases of $\mathcal{H}_\Lambda$ consisting of idempotents, and whose cardinal d is less or equal to $\dim \mathcal{P}_1 = n + 1$, is automatically Λ-multiplicative. Consequently, the subset $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbb{R}^n$, and the positive numbers $\lambda_1, \dots, \lambda_d$ are given by the corresponding representing measure of Λ.

The description of the weights $\lambda_1, \dots, \lambda_d$, and that of the nodes $\xi^{(1)}, \dots, \xi^{(d)}$ as well, is also given by Example 4.

Remark 8. (1) Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. Condition (3.3) can be used, at least in principle, to get a solution of the moment problem having a number of atoms equal to $\dim\mathcal{H}_\Lambda$. Specifically, according to Corollary 3, we must find a family of polynomials $\{b_1, \dots, b_d\} \subset \mathcal{RP}_m$ with the properties $(i) - (iii)$.

Setting $b_j = \sum_\alpha x_{j\alpha} t^\alpha$, where $x_{j\alpha} = 0$ if $|\alpha| > m$, condition (i) means that

$$(i') \qquad \sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{j\beta} = \sum_\alpha \gamma_\alpha x_{j\alpha}, \ j = 1, \dots, d.$$

Condition (ii) is equivalent to

$$(ii') \qquad \sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{k\beta} = 0, \ j, k = 1, \dots, d, \ j < k.$$

Condition (iii) can be expressed as

$$(iii') \qquad \sum_{\xi,\eta} \gamma_{\alpha+\xi} \gamma_{\beta+\eta} x_{j\xi} x_{j\eta} = \sum_{\xi,\eta} \gamma_\xi \ \gamma_{\alpha+\beta+\eta} x_{j\xi} x_{j\eta},$$

$$0 \neq |\alpha| \leq |\beta|, \ |\alpha| + |\beta| \leq m, \ j = 1, \dots d.$$

Finding a solution $\{x_{j\alpha}, \ j = 1, \dots, d, \ |\alpha| \leq m\}$ of equations $(i') - (iii')$ with $b_1, \dots, b_d$ nonnull, provided it exists, means to solve the corresponding moment problem.

(2) The case $d = 1$ is easily obtained (see also [3], [4]). We approach this case from our point of view. We must have $\mathcal{H}_\Lambda = \mathbb{C}\hat{1}$, because $\dim\mathcal{H}_\Lambda = 1$ and $1 \notin \mathcal{I}_\Lambda$. For this reason, for each polynomial $p \in \mathcal{P}_m$ there exists a complex number θ_p such that $\hat{p} = \theta_p \hat{1}$. In fact, θ_p is uniquely determined, and we must have $\theta_p = \Lambda(p)$.

Clearly, $\mathcal{B} = \{\hat{1}\}$ is a basis of $\mathcal{H}_\Lambda$ consisting of one idempotent. Moreover, the basis $\mathcal{B}$ is Λ-multiplicative. Indeed, if α, β are multi-indices with $|\alpha| + |\beta| \leq m$, writing $t^\alpha = \Lambda(t^\alpha) + r_\alpha$, $t^\beta = \Lambda(t^\beta) + r_\beta$, where $r_\alpha, r_\beta \in \mathcal{I}_\Lambda$, we have

$$\Lambda(t^{\alpha+\beta}) = \Lambda(\Lambda(t^\alpha)\Lambda(t^\beta) + r_{\alpha,\beta}) = \Lambda(t^\alpha)\Lambda(t^\beta)$$

because $r_{\alpha,\beta} := \Lambda(t^\alpha) r_\beta + \Lambda(t^\beta) r_\alpha + r_\alpha r_\beta$ is in the kernel of Λ. According to Theorem 2, the uspf Λ must have a representing measure (clearly a Dirac measure) concentrated at the point $\xi := (\Lambda(t_1), \dots, \Lambda(t_n)) \in \mathbb{R}^n$, because the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto \Lambda(p) \in \mathbb{C}$ is the only character of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B} = \{\hat{1}\}$.

Example 5. We can use eq. $(i') - (iii')$ to get a solution for some moment problems. Here is an example.

Let $\Lambda : \mathcal{P}_4^1 \mapsto \mathbb{C}$ be given by the sequence $\Lambda(1) = 1$, $\Lambda(t) = -1/3$, $\Lambda(t^2) = 2/3, \Lambda(t^3) = -1/3$, $\Lambda(t^4) = 2/3$, extended by linearity. Hence, for $p(t) = x_0 + x_1 t + x_2 t^2 + x_3 t^3 + x_4 t^4$, we have

$$\Lambda(p) = x_0 - \frac{x_1}{3} + \frac{2x_2}{3} - \frac{x_3}{3} + \frac{2x_4}{3}.$$

Note that if $p(t) = x_0 + x_1 t + x_2 t^2 \in \mathcal{P}_2^1$, we have

$$\Lambda(|p|^2) = |x_0|^2 - \frac{1}{3}(x_0\bar{x}_1 + \bar{x}_0 x_1) + \frac{2}{3}(x_0\bar{x}_2 + \bar{x}_0 x_2 + |x_1|^2)$$
$$-\frac{1}{3}(x_1\bar{x}_2 + \bar{x}_1 x_2) + \frac{2}{3}|x_2|^2 =$$
$$\frac{1}{3}|x_0|^2 + \frac{1}{2}|x_0 - x_1 + x_2|^2 + \frac{1}{6}|x_0 + x_1 + x_2|^2,$$

via a direct computation, which shows that Λ is a uspf. In particular, $\Lambda(|p|^2) = 0$ if and only if $p = 0$, so $\mathcal{I}_\Lambda = \{0\}$, and $\mathcal{H}_\Lambda = \mathcal{P}_2^1$. In addition, $\dim\mathcal{H}_\Lambda = 3$.

Note also that, a polarization argument leads to the equality

$$\Lambda(pq) = \frac{1}{3}(x_0 y_0) + \frac{1}{2}(x_0 - x_1 + x_2)(y_0 - y_1 + y_2) + \frac{1}{6}(x_0 + x_1 + x_2)(y_0 + y_1 + y_2),$$

whenever $p(t) = x_0 + x_1 t + x_2 t^2 \in \mathcal{P}_2^1$ and $q(t) = y_0 + y_1 t + y_2 t^2 \in \mathcal{P}_2^1$ have real coefficients. Let us look for a Λ-multiplicative basis of the Hilbert space $\mathcal{H}_\Lambda$ given by the polynomials $b_j = x_{j0} + x_{j1} t + x_{j2} t^2$, with real coefficients, for $j = 1, 2, 3$. They should satisfy the equations

$$\frac{1}{3}(x_{j0}^2 - x_{j0}) + \frac{1}{2}((x_{j0} - x_{j1} + x_{j2})^2 - (x_{j0} - x_{j1} + x_{j2}))+$$

(j')

$$\frac{1}{6}((x_{j0} + x_{j1} + x_{j2})^2 - (x_{j0} + x_{j1} + x_{j2})),\ j = 1, 2, 3,$$

corresponding to (i') from Remark 8.

The orthogonality if the family $\{b_1, b_2, b_3\}$ is given by the equations

$$\frac{1}{3}x_{j0}x_{k0} + \frac{1}{2}(x_{j0} - x_{j1} + x_{j2})(x_{k0} - x_{k1} + x_{k2})+$$

(jj')

$$\frac{1}{6}(x_{j0} + x_{j1} + x_{j2})(x_{k0} + x_{k1} + x_{k2}) = 0,\ 1 \le j < k \le 3.$$

The Λ-multiplicativity is expressed by

$$\left(-\frac{1}{3}x_{j0} + \frac{2}{3}x_{j1} - \frac{1}{3}x_{j2}\right)^2 =$$

(jjj')

$$\left(x_{j0} - \frac{1}{3}x_{j1} + \frac{2}{3}x_{j2}\right)\left(\frac{2}{3}x_{j0} - \frac{1}{3}x_{j1} + \frac{2}{3}x_{j2}\right),\ j = 1, 2, 3,$$

derived from (iii').

We now try to find a solution of eq. $(j') - (jjj')$, taking advantage of their special form. Assuming $x_{j0} = 0$, $j = 1, 2$, we infer from (jjj') that

$$(2x_{j1} - x_{j2})^2 = (-x_{j1} + 2x_{j2})^2, \ j = 1, 2,$$

whence $x_{j1}^2 = x_{j2}^2$, $j = 1, 2$. Further, taking $x_{11} = x_{12}$ and $x_{21} = -x_{22}$, equation (jj') is satisfied if $j = 1$, $k = 2$. From equation (j'), we infer that either $x_{11} = x_{21} = 0$ or $x_{11} = x_{21} = 1/2$. Similarly, either $x_{21} = x_{22} = 0$ or $-x_{21} = x_{22} = 1/2$. As only nonnull solutions are of interest, we keep the solutions $x_{10} = 0, x_{11} = x_{21} = 1/2$ and $x_{20} = 0, -x_{21} = x_{22} = 1/2$.

It remains to find a third solution. Let us assume that $x_{31} = 0$. In this case we must have $(x_{30} + 2x_{32})^2 = x_{32}^2$, and we choose the solution $x_{30} = -x_{32}$, convenient for (jj'). Then (jj') is clearly satisfied for either $j = 1, k = 3$ or $j = 2, k = 3$. Finally, from (j') we deduce that $x_{30}^2 = x_{30}$, and so $x_{30} = 1$.

We associate the solutions found above with the polynomials

$$b_1(t) = \frac{1}{2}t + \frac{1}{2}t^2, \ b_2(t) = -\frac{1}{2}t + \frac{1}{2}t^2, \ b_3(t) = 1 - t^2.$$

Noting also that

$$\Lambda(b_1) = \frac{1}{6}, \ \Lambda(b_2) = \frac{1}{2}, \ \Lambda(b_3) = \frac{1}{3},$$

we deduce that $\{b_1, b_2, b_3\}$ form a Λ-multiplicative basis of $\mathcal{H}_\Lambda = \mathcal{P}_2^1$.

Accordingly, the characters of the C^*-algebra induced by $\{b_1, b_2, b_3\}$ are given by

$$\delta_1(p) = 6\Lambda(pb_1), \ \delta_2(p) = 2\Lambda(pb_2), \ \delta_3(p) = 3\Lambda(pb_3), \ p \in \mathcal{P}_2^1.$$

In particular,

$$\delta_1(t) = 1, \ \delta_2(t) = -1, \ \delta_3(t) = 0,$$

showing that Λ has a representing measure with weights $\{1/6, 1/2, 1/3\}$ at the points $\{1, -1, 0\} \subset \mathbb{R}$, respectively. In other words, the formula

$$\Lambda(p) = \frac{1}{2}p(-1) + \frac{1}{3}p(0) + \frac{1}{6}p(1), \ p \in \mathcal{P}_4^1$$

provides a representing measure for Λ.

Using an idea inspired by the diagonalization of symmetric matrices, we give in the following a criterion of existence of Λ-multiplicative bases by means of the representation of the quadratic form associated to Λ as a sum of squares of degree one homogeneous polynomials.

Proposition 4. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, let $d = \dim\mathcal{H}_\Lambda$, and let $g_1, \ldots, g_d$ be degree one homogeneous polynomials from $\mathcal{P}_1^{d_m}$, with $d_m = \dim\mathcal{P}_m$, such that $\Lambda(p_x^2) = \sum_{j=1}^d g_j(x)^2$, where $p_x(t) = \sum_\alpha x_\alpha t^\alpha$, $x = (x_\alpha)_\alpha \in \mathbb{R}^{d_m}$.*

Assume that for each $k = 1, \ldots, d$, the system of equations $g_k(y) = 1$ and $g_j(y) = 0$ if $j \neq k$ admits a solution, say $y^{(k)} = (y_{k,\alpha})_\alpha \in \mathbb{R}^{d_m}$, with the property $\rho_k := \sum_\alpha y_{k,\alpha}\gamma_\alpha > 0$ for all $k = 1, \ldots, d$.

Setting $b_k(t) = \sum_\alpha x_{k,\alpha} t^\alpha$, $k = 1, \dots, d$, *where* $x_{k,\alpha} = \rho_k y_{k,\alpha}$ *the family* $\{\hat{b}_1, \dots, \hat{b}_d\}$ *is an orthogonal basis of* $\mathcal{H}_\Lambda$ *consisting of idempotents, and* $\sum_{j=1}^d \lambda_j = 1$, *where* $\lambda_j := \rho_j^2$, $j = 1, \dots, d$.

In addition, writing $g_j(x) = \sum_\alpha v_{j,\alpha} x_\alpha$, $j = 1, \dots, d$, $x \in \mathbb{R}^{d_m}$, *and assuming*

$$\rho_j v_{j,\xi+\eta} = v_{j,\xi} v_{j,\eta},\ 0 \neq |\xi| \leq |\eta|,\ |\xi| + |\eta| \leq m,$$

the basis $\{\hat{b}_1, \dots, \hat{b}_d\}$ *is* Λ*-multiplicative.*

Proof. As we have $b_k(t) = p_{x^{(k)}}(t)$, with $x^{(k)} := (x_{k,\alpha})_\alpha \in \mathbb{R}^{d_m}$, and $\Lambda(b_k) = \rho_k^2 = \lambda_k$, we obtain

$$\Lambda(b_k^2) = \sum_{j=1}^d \rho_k^2 g_j(y^{(k)})^2 = \lambda_k = \Lambda(b_k),\ k = 1, \dots, d,$$

showing that $\{\hat{b}_1, \dots, \hat{b}_d\}$ are idempotents.

To prove the orthogonality of the family $\{\hat{b}_1, \dots, \hat{b}_d\}$, we use the formula

$$\Lambda(p_x p_y) = \sum_{j=1}^d g_j(x) g_j(y),\ x, y \in \mathbb{R}^{d_m}, \tag{3.5}$$

via the linearlty of the map $x \mapsto p_x$ (as in formula (2.4)), and a polarization argument. It follows from (3.5) that

$$\Lambda(b_k b_l) = \sum_{j=1}^d g_j(x^{(k)}) g_j(x^{(l)}) = \sum_{j=1}^d \rho_k \rho_l g_j(y^{(k)}) g_j(y^{(l)}) = 0,$$

whenever $k \neq l$, proving the orthogonality of the family $\{\hat{b}_1, \dots, \hat{b}_d\}$. In other words, $\{\hat{b}_1, \dots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. In addition we must have:

$$\sum_{j=1}^d \lambda_j = \sum_{j=1}^d \Lambda(b_j) = 1,$$

via Lemma 2.

Let us deal with the Λ-multiplicativity of $\{\hat{b}_1, \dots, \hat{b}_d\}$. Fixing a multi-index ξ, the monomial t^ξ corresponds to the vector $1_\xi = (1_{\xi\alpha})_\alpha \in \mathbb{R}^{d_m}$, with $1_{\xi\alpha} = 1$ if $\xi = \alpha$, and $1_{\xi\alpha} = 0$ otherwise. Then, by (3.5), we have

$$\Lambda(t^\xi b_j) = \Lambda(p_{1_\xi} p_{x^{(j)}}) = \sum_{k=1}^d g_k(p_{1_\xi}) g_k(p_{x^{(j)}}) =$$

$$\rho_j \sum_{k=1}^d v_{k,\xi} g_k(p_{y^{(j)}}) = \rho_j v_{j,\xi}.$$

Consequently,

$$\Lambda(t^\xi b_j)\Lambda(t^\eta b_j) = \rho_j^2 v_{j,\xi} v_{j,\eta} = \lambda_j \rho_j v_{j,\xi+\eta} = \Lambda(b_j)\Lambda(t^{\xi+\eta} b_j)$$

whenever $0 \neq |\xi| \leq |\eta|,\ |\xi| + |\eta| \leq m$, which completes the proof.

Remark 9. The representation of the quadratic form associated to the uspf Λ as a sum of squares of homogeneous polynomials of degree one, as in Proposition 3.5, can be obtained in the presence of an orthogonal basis of idempotents. Indeed, let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $d = \dim \mathcal{H}_\Lambda$, and let $d_m = \dim \mathcal{P}_m$. Let also $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be an orthogonal basis of $\mathcal{H}_\Lambda$. If $p_x = \sum_\alpha x_\alpha t^\alpha \in \mathcal{P}_m$ is arbitrary, wtiting $t^\alpha = \sum_{j=1}^d c_{\alpha,j} b_j + r_\alpha$, with $c_{\alpha,j} = \Lambda(b_j)^{-1}\Lambda(t^\alpha b_j)$ and $r_\alpha \in \mathcal{I}_\Lambda$, we obtain

$$p_x^2 = \sum_{\alpha,\beta} x_\alpha x_\beta \left(\sum_{j,k=1}^d c_{\alpha,j} c_{\beta,k} b_j b_k \right) + q,$$

where q is in the kernel of Λ. Therefore, $\Lambda(p_x^2) = \sum_{j=1}^d g_j(x)^2$, where $g_j(x) = \sum_\alpha \Lambda(b_j)^{-1/2}\Lambda(t^\alpha b_j) x_\alpha$.

Example 6. We can alternatively treat Example 5 using Proposition 4, whose notation is adapted to this situation. Specifically, for $x = (x_0, x_1, x_2) \in \mathbb{R}^3$, we have obtained the equation

$$\Lambda(p_x^2) = \frac{1}{3}|x_0|^2 + \frac{1}{2}|x_0 - x_1 + x_2|^2 + \frac{1}{6}|x_0 + x_1 + x_2|^2.$$

Writing

$$g_1(x) := (1/\sqrt{3})x_0,\ \ g_2(x) := (1/\sqrt{2})(x_0 - x_1 + x_2),$$
$$g_3(x) := (1/\sqrt{6})(x_0 + x_1 + x_2),$$

we obtain the representation $\Lambda(p_x^2) = g_1(x)^2 + g_2(x)^2 + g_3(x)^2$. Note that the equations $g_k(y) = 1$ and $g_j(y) = 0$ if $j \neq k$ have the solutions

$$y^{(1)} = (y_{1,0}, y_{1,1}, y_{1,2}) = (\sqrt{3}, 0, -\sqrt{3});$$
$$y^{(2)} = (y_{2,0}, y_{2,1}, y_{2,2}) = (0, -\sqrt{2}/2, \sqrt{2}/2);$$
$$y^{(3)} = (y_{3,0}, y_{3,1}, y_{3,2}) = (0, \sqrt{6}/2, \sqrt{6}/2).$$

Using these solutions, we can now compute the quantities ρ_1, ρ_2, ρ_3. A direct computation leads to

$$\rho_1 = y_{1,0}\gamma_0 + y_{1,1}\gamma_1 + y_{1,2}\gamma_2 = \sqrt{3}/3;$$
$$\rho_2 = y_{2,0}\gamma_0 + y_{2,1}\gamma_1 + y_{2,2}\gamma_2 = \sqrt{2}/2;$$
$$\rho_3 = y_{3,0}\gamma_0 + y_{3,1}\gamma_1 + y_{3,2}\gamma_2 = \sqrt{6}/6.$$

Hence

$$x^{(1)} = \rho_1 y^{(1)} = (1, 0, -1);$$
$$x^{(2)} = \rho_2 y^{(2)} = (0, -1/2, 1/2);$$
$$x^{(3)} = \rho_3 y^{(3)} = (0, 1/2, 1/2).$$

For this reason, as in Example 5, (with a slightly different notation) the polynomials

$$b_1(t) = 1 - t^2,\ \ b_2(t) = -\frac{1}{2}t + \frac{1}{2}t^2,\ b_3(t) = \frac{1}{2}t + \frac{1}{2}t^2$$

form a Λ-multiplicative basis of $\mathcal{H}_\Lambda = \mathcal{P}_2^1$. Of course, we can check the Λ-multiplicativity of the family $\{b_1, b_2, b_3\}$ using the last part of Proposition 4. Specifically, we have to check that

$$(*) \qquad \rho_j v_{j,k+l} = v_{j,k} v_{j,l},\ j = 1, 2, 3,\ 0 \neq k \leq l,\ k + l \leq 2.$$

Indeed, from the polynomials g_1, g_2, g_3, we derive that

$$v_{1,0} = 1/\sqrt{3},\ v_{1,1} = 0,\ v_{1,2} = 0, v_{2,0} = 1/\sqrt{2},\ v_{2,1} = -1/\sqrt{2},$$
$$v_{2,2} = 1/\sqrt{2},\ v_{3,0} = 1/\sqrt{6}, v_{3,1} = 1/\sqrt{6},\ v_{3,2} = 1/\sqrt{6}.$$

Note that, in $(*)$, we only must have $j = 1, 2, 3$ and $k = 1, l = 1$. Since

$$v_{1,1}^2 = 0 = \rho_1 v_{1,2};\ v_{2,1}^2 = (-1/\sqrt{2})^2 = (\sqrt{2}/2)(1/\sqrt{2}) = \rho_2 v_{2,2};$$
$$v_{3,1}^2 = (1/\sqrt{6})^2 = (\sqrt{6}/6)(1/\sqrt{6}) = \rho_3 v_{3,2},$$

eqs. $(*)$ are satisfied, providing another argument for the basis $\{b_1, b_2, b_3\}$ to be Λ-multiplicative.

Remark 10. $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with $d = \dim\mathcal{H}_\Lambda = 2$. We can sketch an algorithm to decide whether or not there exists a representing measure for Λ.

As in the proof of Theorem 1, we have the decomposition $\mathcal{H}_\Lambda = \mathbb{C}\hat{1} \oplus \mathcal{H}_\Lambda^0$, where $\mathcal{H}_\Lambda^0 = \{\hat{p} \in \mathcal{H}_\Lambda; \Lambda(p) = 0\}$. Then we can find a nonnull element $\hat{g}_0$ which spans the space $\mathcal{H}_\Lambda^0$. Moreover, we may choose g_0 such that $\Lambda(g_0^2) = 1$.

Lemma 1(2) suggests to look for (nonnull) idempotents $\hat{b}$ which are not equal to $\hat{1}$. We may assume that $b(t) = u + v g_0$, for some real numbers u, v. The necessary conditions $\Lambda(b) > 0$, $\Lambda(1 - b) > 0$, lead to the constraint $0 < u < 1$.

The idempotent equation $\Lambda(b^2 - b) = 0$ can be written as $v^2 + u^2 - u = 0$. Keeping u as a parameter, we have the solutions $v_\pm = \pm\sqrt{u - u^2}$, so $b_\pm = u \pm \sqrt{u - u^2} g_0$ are the corresponding idempotents.

Taking for instance $b_1 = b_+$, $b_2 = 1 - b_+$, then $\{\hat{b}_1, \hat{b}_2\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$, consisting of idempotents (via Lemma 1(2)). If moreover we have (3.3), then $\{\hat{b}_1, \hat{b}_2\}$ is Λ-multiplicative.

The parameter u must be the solution of a second degree equation. For instance, the equation $\Lambda(t_1 b_1)^2 = \Lambda(b_1)\Lambda(t_1^2 b_1)$ (derived from (3.3)) shows that we must have

$$(\gamma_1 u + \sqrt{u - u^2}\Lambda(t_1 g_0))^2 = u(\gamma_2 u + \sqrt{u - u^2}\Lambda(t_1^2 g_0)),$$

where $\gamma_1 := \Lambda(t_1)$, $\gamma_2 := \Lambda(t_1^2)$. Also setting $\theta_1 := \Lambda(t_1 g_0)$ and $\theta_2 := \Lambda(t_1^2 g_0)$, as $u \neq 0$, we obtain the second degree equation

$$[(\gamma_1^2 - \gamma_2 - \theta_1^2)^2 + (2\gamma_1\theta_1 - \theta_2)^2]u^2 + [2\theta_1^2(\gamma_1^2 - \gamma_2 - \theta_1^2) - (2\gamma_1\theta_1 - \theta_2)^2]u + \theta_1^4 = 0. \tag{3.6}$$

The discriminant of eq. (3.6) is given by

$$(2\gamma_1\theta_1 - \theta_2)^2[(2\gamma_1\theta_1 - \theta_2)^2 - 4\theta_1^2(\gamma_1^2 - \gamma_2 - 2\theta_1^2)].$$

When $2\gamma_1\theta_1 - \theta_2 = 0$, the only solution of equation (3.6) is $u = \theta_1^2/(\theta_1^2 - \gamma_1^2 + \gamma_2)$, provided, $\theta_1^2 - \gamma_1^2 + \gamma_2 \neq 0$. Of course, we should also have $0 < u < 1$. If $2\gamma_1\theta_1 - \theta_2 \neq 0$, the condition $(2\gamma_1\theta_1 - \theta_2)^2 - 4\theta_1^2(\gamma_1^2 - \gamma_2 - 2\theta_1^2) \geq 0$

is clearly necessary. With a convenient solution u of eq. (3.6) (that is, so that $u \in (0,1)$), we may check the remaining equations. If one of them is not satisfied, the moment problem has no solution.

When both $2\gamma_1\theta_1 - \theta_2 = 0$ and $\theta_1^2 - \gamma_1^2 + \gamma_2 = 0$, then we must have $\gamma_1^2 = \gamma_2$, and each $u \in (0,1)$ is a solution of eq. (3.6). In this case, another equation from (3.3) may be used.

This idea leads to an algorithm to decide whether or not there exists a solution of the problem, using only algebraic operations.

Example 7. Here is an example, in two variables, related to the previous remark. Consider the uspf $\Lambda : \mathcal{P}_4^2 \mapsto \mathbb{C}$ given by the multi-sequence $\gamma_{00} = 1$; $\gamma_{k0} = 2/3\,(k = 1,2,3,4)$; $\gamma_{kl} = 0\,(k,l = 0,1,2,3,4; l \neq 0)$.

Therefore, if $p(t_1,t_2) = \sum_{0\le j+k\le 2} x_{jk} t_1^j t_2^k \in \mathcal{P}_2^2$, a direct calculation leads to

$$\Lambda(|p|^2) = \frac{1}{3}|x_{00}|^2 + \frac{2}{3}|x_{00} + x_{10} + x_{20}|^2.$$

This formula shows that

$$\mathcal{I}_\Lambda = \{p \in \mathcal{P}_2^2; x_{00} = x_{10} + x_{20} = 0\},$$

and so

$$\mathcal{H}_\Lambda = \{\hat{p}; p = x_{00} + x_{20}t_1^2\}.$$

In addition,

$$\mathcal{H}_\Lambda^0 = \{\hat{p} \in \mathcal{H}_\Lambda; x_{00} + 2x_{20}/3 = 0\}.$$

Now we fix a polynomial g_0 such that $\Lambda(g_0) = 0$ and $\Lambda(g_0^2) = 1$. We may take $g_0 = (2 - 3t_1^2)/\sqrt{2}$, as one can easily see. If $p := x_{00} + x_{20}t_1^2$ is such that $x_{00} + 2x_{20}/3 = 0$, then we have

$$x_{00} + x_{20}t_1^2 = -\frac{x_{20}\sqrt{2}}{3} g_0,$$

showing that the element $\hat{g}_0$ spans $\mathcal{H}_\Lambda^0$. In particular, $\dim\mathcal{H}_\Lambda^0 = 1$.

According to the discussion from Remark 10, and choosing a parameter $u \in (0,1)$, an idempotent $\hat{b}$ may be given by the polynomial $b = u + \sqrt{u - u^2}g_0$. Set $\gamma_1 := \gamma_{10} = 2/3$, $\gamma_2 := \gamma_{20} = 2/3$, $\theta_1 = \Lambda(t_1 g_0) = -2/3\sqrt{2}$, $\theta_2 = \Lambda(t_1^2 g_0) = -2/3\sqrt{2}$. Introducing these data in eq. (3.6), we obtain the equation $9u^2 - 9u + 2 = 0$, whose roots are $1/3$ and $2/3$.

If we take $u = 1/3$, we find the polynomial $b_1(t) = 1/3 + \sqrt{2}g_0(t)/3 = 1 - t_1^2$. Setting $b_2(t) = 1 - b_1(t) = t_1^2$, we get an orthogonal basis $\{\hat{b}_1, \hat{b}_2\}$ of $\mathcal{H}_\Lambda$, consisting of idempotents. Moreover, $\Lambda(t_1 b_1)^2 = 0 = \Lambda(b_1)\Lambda(t_1^2 b_1)$. Checking also the equality $\Lambda(t_1 b_2)^2 = \Lambda(b_2)\Lambda(t_1^2 b_2)$, which is obvious, and noting that $\Lambda(t_2 b_j)^2 = \Lambda(b_j)\Lambda(t_2^2 b_j) = 0$, $j = 1,2$, we deduce that $\{\hat{b}_1, \hat{b}_2\}$ is Λ-multiplicative. In this way, Λ has a representing measure concentrated at two points. The weights of the associated atomic measure are then given by $\lambda_1 = \Lambda(b_1) = 1/3$ and $\lambda_2 = \Lambda(b_2) = 2/3$, at the points

$$\xi^{(1)} = \lambda_1^{-1}(\Lambda(t_1 b_1), \Lambda(t_2 b_1) = (0,0),$$

$$\xi^{(2)} = \lambda_2^{-1}(\Lambda(t_1 b_2), \Lambda(t_2 b_2) = (1,0),$$

respectively.

Take now $u = 2/3$. In this case the corresponding idempotent is given by $b_1'(t) = 4/3 - t_1^2$, and the orthogonal idempotent is $b_2'(t) = -1/3 + t_1^2$. Nevertheless, $\Lambda(t_1 b_1')^2 = 4/81$, while $\Lambda(b_1')\Lambda(t_1^2 b_1') = 4/27$. Consequently, the orthogonal basis $\{\hat{b'}_1, \hat{b'}_2\}$ is not Λ-multiplicative. In other words, the solution $u = 2/3$ is not admissible.

The next results illustrate the strong connection between moment problems and polynomial interpolation.

Corollary 5. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with invertible Hankel operator. The uspf Λ has a representing measure in $\mathbb{R}^n$ having $d = \dim\mathcal{P}_m$ atoms if and only if there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{H}_\Lambda = \mathcal{P}_m$ such that*

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d,\ p \in \mathcal{P}_m,$$

where

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d.$$

Proof. Assume that Λ has a representing measure in $\mathbb{R}^n$ having $d = \dim\mathcal{P}_m$ atoms and support $\Xi = \{\xi^{(1)} \ldots, \xi^{(d)}\}$. As $\mathcal{I}_\Lambda = \{0\}$, it follows from Remark 7(1) that there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{H}_\Lambda = \mathcal{P}_m$ such that

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d,\ p \in \mathcal{P}_m.$$

Moreover, $\xi_k^{(j)} = \Lambda(b_j)^{-1}\Lambda(t_k b_j)$ for all $j = 1, \ldots, d$, $k = 1, \ldots, n$, and so

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n,\ j = 1, \ldots, d.$$

Conversely, asume that there exists a family of orthogonal idempotents $\{b_1, \ldots, b_d\}$ in $\mathcal{P}_m$ such that

$$p = p(\xi^{(1)})b_1 + \cdots + p(\xi^{(d)})b_d,\ p \in \mathcal{P}_m.$$

Hence

$$\Lambda(p) = p(\xi^{(1)})\Lambda(b_1) + \cdots + p(\xi^{(d)})\Lambda(b_d) = \int_\Xi p d\mu,\ p \in \mathcal{P}_m,$$

where μ is the probability measure with weights $\Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \ldots, d$.

Proceeding as in Remark 7(2), we obtain that the equality

$$\Lambda(p) = \int_\Xi p(\xi)d\mu(\xi)$$

also holds for all polynomials $p \in \mathcal{P}_{2m}$, providing a d-atomic representation measure for Λ. □

The next result characterizes the existence of representing measures in the context of invertible Hankel matrices (via Theorem 2). It is somehow related to Question 1.2 from [9].

Theorem 3. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with invertible Hankel operator, and let $\mathcal{B} = \{b_1, \dots, b_d\} \subset \mathcal{H}_\Lambda = \mathcal{P}_m \, (d = \dim \mathcal{P}_m)$ be an orthogonal basis consisting of idempotent elements. Let also $\Delta = \{\delta_1, \dots, \delta_d\}$ be the dual basis of $\mathcal{B}$. Assume that $\mathcal{P}_m$ is endowed with the C^*-algebra structure induced by $\mathcal{B}$. The following conditions are equivalent.*

(i) $\mathcal{B}$ is Λ-multiplicative.

(ii) The polynomials $\{1, t_1, \dots, t_n\}$ generate the C^*-algebra $\mathcal{P}_m$, and $\delta_k(b_j(t)) = 0$, $k \neq j$, $\delta_j(b_j(t)) = 1$, $j, k = 1, \dots, d$.

(iii) The points

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \dots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \dots, d,$$

are distinct, and $\delta_k(b_j(t)) = 0, k \neq j, \delta_j(b_j(t)) = 1$, $j, k = 1, \dots, d$.

(Here the elements $b_j(t)$ are computed in the C^*-algebra $\mathcal{P}_m$.)

Proof. Assuming condition (i), as in the proof of Theorem 2, we deduce that the elements $t_1, \dots, t_n$ generate the unital C^*-algebra $\mathcal{H}_\Lambda = \mathcal{P}_m$. In fact, as we have a representing measure for Λ, Corollary 5 shows, in particular, that $\{b_1, \dots, b_d\}$ is an interpolating family for the set $\{\xi^{(1)}, \dots, \xi^{(d)}\}$. Therefore, $\delta_k(b_j(t)) = b_j(\xi^{(k)})$, and so $\delta_k(b_j(t)) = 0$, $k \neq j$, $\delta_j(b_j(t)) = 1$, $j, k = 1, \dots, d$, that is, $(i) \Longrightarrow (ii)$.

Assume now that the polynomials $\{1, t_1, \dots, t_n\}$ generate the C^*-algebra $\mathcal{P}_m$. Then the map

$$\Delta \ni \delta \mapsto (\delta(t_1), \dots, \delta(t_n)) \in \mathbb{R}^n \tag{3.7}$$

must be injective. As $\delta_j(p) = \Lambda(b_j)^{-1}\Lambda(p b_j)$, $j = 1, \dots, d$, $p \in \mathcal{P}_m$, the injectivity of (3.7) implies that the points

$$\xi^{(j)} := (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \dots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \dots, d$$

are distinct, and so $(ii) \Longrightarrow (iii)$.

Conversely, $(iii) \Longrightarrow (ii)$. Indeed, if the points

$$\xi^{(j)} = (\Lambda(b_j)^{-1}\Lambda(t_1 b_j), \dots, \Lambda(b_j)^{-1}\Lambda(t_n b_j)) \in \mathbb{R}^n, \; j = 1, \dots, d$$

are distinct, then the map (3.7) is injective. Hence the polynomials $1, t_1, \dots, t_n$ generate the C^*-algebra $\mathcal{P}_m$, via the finite-dimensional version of the Stone-Weierstrass theorem.

Finally, the implication $(ii) \Longrightarrow (i)$ follows from Corollary 2, via Theorem 2. □

Remark 11. (1) Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with $\mathcal{H}_\Lambda$ having a Λ-multiplicative basis $\mathcal{B}$. Then we have the property

$$p \in \mathcal{P}_{m-k} \cap \mathcal{I}_\Lambda, \; q \in \mathcal{P}_k \quad \Rightarrow \quad pq \in \mathcal{I}_\Lambda \tag{3.8}$$

whenever $0 \leq k \leq m$ is an integer. Indeed, if Δ is the space of characters of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$, then we have

$$\delta(\widehat{pq}) = \delta(\hat{p})\delta(\hat{q}), \; \delta \in \Delta, \; p \in \mathcal{P}_{m-k}, \; q \in \mathcal{P}_k,$$

showing, in particular, that (3.8) holds. In other words, property (3.3) implies that the associated Hankel matrix is *recursively generated* (see [3], especially

Lemma 4.2). In addition, for $n = 1$, Λ-multiplicativity is equivalent (via Theorem 2) to the recursiveness property

$$p \in \mathcal{P}^1_{m-1} \cap \mathcal{I}_\Lambda \quad \Rightarrow \quad tp \in \mathcal{I}_\Lambda,$$

which is a necessary and sufficient condition for the existence of a representing measures in one variable (see [3, 4] etc.).

(2) Let $M : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ be a uspf. Following [3], we say that the uspf M is *flat* if $\mathcal{P}_m + \mathcal{I}_M = \mathcal{P}_{m+1}$. Setting $\Lambda = M|\mathcal{P}_{2m}$, the flatness of M is equivalent to saying that the natural isometry

$$\mathcal{H}_\Lambda \ni p + \mathcal{I}_\Lambda \mapsto p + \mathcal{I}_M \in \mathcal{H}_M \tag{3.9}$$

is a unitary operator. In particular, $d := \dim\mathcal{H}_\Lambda = \dim\mathcal{H}_M$. In our terms, the flatness of M is equivalent to the existence of an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}_M$, consisting of idempotents, such that $b_1, \ldots, b_d \in \mathcal{P}_m$.

Many important results obtained in [3, 4], as well as in other papers by the same authors, have as a starting point the assumption of the existence of a flat extension $M : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ for a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$. This hypothesis leads to the existence and the uniqueness of a representing measure of M. Nevertheless, explicit conditions for the existence of flat extensions are known only in some particular cases. The existence of flat extension of a uspf Λ implies the existence of a Λ-multiplicative basis by Theorem 2, but the representing measure given by a Λ-multiplicative basis via Theorem 2 is not necessarily unique (see Example 4).

Let us also note that a parallel construction of representing measures for uspf's has been developed in [21], under a hypothesis equivalent to flatness, obtaining several more direct proofs.

The Λ-multiplicativity of an orthogonal basis in $\mathcal{ID}(\Lambda)$ for a given uspf Λ can be characterized in terms of the existence of a uspf extension of Λ, a priori not necessarily flat.

Proposition 5. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be an orthogonal basis of $\mathcal{H}_\Lambda$. The basis $\mathcal{B}$ is Λ-multiplicative if and only if there exists a uspf $M : \mathcal{P}_{4m} \mapsto \mathbb{C}$ extending Λ such that*

$$t^\alpha b_k - \theta_{\alpha k} b_k \in \mathcal{I}_M, \ |\alpha| \le m, \ k = 1, \ldots, d, \tag{3.10}$$

where $\theta_{\alpha k} = (\eta^{(k)})^\alpha$ for some vectors $\eta^{(1)}, \ldots, \eta^{(d)} \in \mathbb{R}^n$.

Proof. Assume first that the orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ is Λ-multiplicative. In virtue of Theorem 2, Λ has a representing measure μ, with support $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, $d = \dim\mathcal{H}_\Lambda$, and weights $\lambda_j = \Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \ldots, d$. In addition, $b_j|\Xi$ is the characteristic function of the set $\{\xi^{(j)}\}$, $j = 1, \ldots, d$, by Remark 7(1). Therefore, denoting by M the extension of Λ to $\mathcal{P}_{4m}$ via the measure μ, and setting $\theta_{\alpha k} = M(t^\alpha b_k)/\Lambda(b_k) = (\xi^{(k)})^\alpha$, we deduce that

$$M((t^\alpha b_k - \theta_{\alpha k} b_k)^2) = \int_\Xi (t^\alpha b_k - \theta_{\alpha k} b_k)^2 d\mu = \lambda_k((\xi^{(k)})^\alpha - \theta_{\alpha k})^2 = 0,$$

and so $t^\alpha b_k - \theta_{\alpha k} b_k \in \mathcal{I}_M$, and we have (3.10), with $\eta^{(k)} = \xi^{(k)}$.

Conversely, if if there exists a uspf $M : \mathcal{P}_{4m} \mapsto \mathbb{C}$ extending Λ and satisfying (3.10), we have, whenever $|\alpha| + |\beta| \le m$ and $k = 1, \dots, d$,

$$t^{\alpha+\beta} b_k - (\eta^{(k)})^{\alpha+\beta} b_k \in \mathcal{I}_M \cap \mathcal{P}_{2m} \subset \ker(\Lambda)$$

and thus

$$\Lambda(t^{\alpha+\beta} b_k) = \lambda_k (\eta^{(k)})^{\alpha+\beta}.$$

Similarly,

$$\Lambda(t^\alpha b_k)\Lambda(t^\beta b_k) = \lambda_k^2 (\eta^{(k)})^\alpha (\eta^{(k)})^\beta,$$

showing that the basis $\mathcal{B}$ is Λ-multiplicative. □

Remark 12. If $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf having a representing measure μ, as in Remark 7(1), the Hilbert spaces $\mathcal{H}_\Lambda$ and $L^2(\Xi, \mu)$ are unitarily equivalent via the unitary map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in L^2(\Xi, \mu)$. As in first part of the proof of Proposition 5 we have that $b_j|\Xi = \chi_j$, the characteristic function of $\xi^{(j)}$, $j = 1, \dots, d$. Relations (3.10) reflect the fact that the multiplication operators with independent variables in $L^2(\Xi, \mu)$, specifically $T_j f = (t_j|\Xi) f$, $f \in L^2(\Xi, \mu)$, are commuting self-adjoint operators, and χ_j are their eigenvectors ($j = 1, \dots, d$).

We end this section with a characterization of the existence of representing measures for full moment problems, in terms of idempotent elements (for other characterizations see for instance [4], Proposition 5.9, or [21], Corollary 2.15).

Theorem 4. *A uspf $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure in $\mathbb{R}^n$ if and only if there exists an increasing sequence of nonnegative integers $\{m_k\}_{k \ge 1}$ such that every Hilbert space $\mathcal{H}_{\Lambda_k}$ has a Λ_k-multiplicative basis, where $\Lambda_k = \Lambda|\mathcal{P}_{2m_k}$, $k \ge 1$ an arbitrary integer.*

Proof. First assume the existence of a sequence $\{m_k\}_{k \ge 1}$ with the stated properties. According to Theorem 2, for every $k \ge 1$ there exists an atomic probability measure μ_k such that

$$\Lambda_k(p) = \int_{\Xi_k} p(\xi) d\mu_k(\xi),\ p \in \mathcal{P}_{2m_k},$$

where Ξ_k is the support of μ_k. As we have

$$\Lambda(p) = \int_{\Xi_k} p(\xi) d\mu_k(\xi) = \int_{\Xi_{k+1}} p(\xi) d\mu_{k+1}(\xi),\ p \in \mathcal{P}_{2m_k},$$

for all $k \ge 1$, the assertion follows from [16], Theorem 4.

Conversely, if $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure, then $\Lambda|\mathcal{P}_{2k}$ has a representing measure, say ν_k for every integer $k \ge 1$. If Ξ_k is the support of ν_k, proceeding as in Remark 7(1), we can find an integer $m_k \ge k$ such that $\mathcal{H}_{\Lambda_k}$ is isomprphic, as a C^*-algebra, with $C(\Xi_k)$, where where $\Lambda_k = \Lambda|\mathcal{P}_{2m_k}$. In particular, the Hilbert space $\mathcal{H}_{\Lambda_k}$ has a basis $\mathcal{B}_k$ which is Λ_k-multiplicative by Theorem 2. Clearly, we may also assume that $m_{k+1} > m_k$ for all $k \ge 1$. □

4. Continuous point evaluations

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. For every point $\xi \in \mathbb{R}^n$, we denote by δ_ξ the point evaluation at ξ, that is, $\delta_\xi(p) = p(\xi)$, for every polynomial $p \in \mathcal{P}$. As before, we set $\mathcal{I}_\Lambda = \{f \in \mathcal{P}_m; \Lambda(|f|^2) = 0\}$, while $\mathcal{H}_\Lambda$ is the finite dimensional Hilbert space $\mathcal{P}_m/\mathcal{I}_\Lambda$.

Definition 4. The point evaluation δ_ξ is said to be Λ-*continuous* if there exists a constant $c_\xi > 0$ such that

$$|\delta_\xi(p)| \le c_\xi \Lambda(|p|^2)^{1/2}, \ p \in \mathcal{P}_m.$$

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\xi \in \mathbb{R}^n$ such that δ_ξ is Λ-continuous. For every polynomial p let us denote by $\mathcal{Z}(p)$ the set of its zeros.

Lemma 6. *We have the equality*

$$\mathcal{Z}_\Lambda = \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p).$$

Proof. If $\xi \in \mathcal{Z}_\Lambda$ and $p \in \mathcal{I}_\Lambda$ we clearly have $p(\xi) = \delta_\xi(p) = 0$. Therefore, $\mathcal{Z}_\Lambda \subset \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p)$.

Conversely, if $\xi \in \cap_{p \in \mathcal{I}_\Lambda} \mathcal{Z}(p)$, then $\delta_\xi(p) = 0$ for all $p \in \mathcal{I}_\Lambda$. Therefore, δ_ξ induces a linear functional on the Hilbert space $\mathcal{H}_\Lambda$, denoted by δ_ξ^Λ. As the seminorm $p \mapsto \Lambda(|p|^2)^{1/2}$ is actually a norm on the finite dimensional space $\mathcal{H}_\Lambda$, the linear functional δ_ξ^Λ is automatically continuous, and so δ_ξ is Λ-continuous. This shows that the equality in the statement holds. ☐

Remark 13. The previous lemma shows that the set $\mathcal{Z}_\Lambda$ coincides with the algebraic variety of the moment sequence associated to Λ (see for instance (1.6) from [4]).

The next result can be found in [3]. For the sake of completeness, we give it here, with a different proof.

Lemma 7. *Suppose that the uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ *has an atomic representing measure* μ *in* $\mathbb{R}^n$. *Then* $\mathrm{supp}(\mu) \subset \mathcal{Z}_\Lambda$.

Proof. Assume that $\mu = \sum_{j=1}^d \lambda_j \delta_{\xi^{(j)}}$ is a representing measure for Λ, with $\lambda_j > 0$ for all $j = 1, \ldots, d$, $\sum_{j=1}^d \lambda_j = 1$, and with $\xi^{(1)}, \ldots, \xi^{(d)}$ distinct points. Note that

$$|p(\xi^{(k)})|^2 \le \frac{1}{\lambda_k} \sum_{j=1}^d \lambda_j |p(\xi^{(j)}|^2 \le \frac{1}{\lambda_k} \Lambda(|p|^2),$$

for all $k = 1, \ldots, d$ and $p \in \mathcal{P}_m$, showing that the set $\{\xi^{(1)}, \ldots, \xi^{(d)}\}$ is a subset of $\mathcal{Z}_\Lambda$. ☐

Remark 14. It follows from Lemma 7 that a necessary condition for the existence of a representing measure for Λ is $\mathcal{Z}_\Lambda \neq \emptyset$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ be a uspf with the property $\mathcal{Z}_\Lambda \neq \emptyset$. As previously noted, the set $\{\delta_\xi^\Lambda; \xi \in \mathcal{Z}_\Lambda\}$ is a subset in the dual of the Hilbert space $\mathcal{H}_\Lambda$. Therefore, for every $\xi \in \mathcal{Z}_\Lambda$ there exists a vector $\hat{v}_\xi \in \mathcal{H}_\Lambda$ such that $\delta_\xi^\Lambda(\hat{p}) = \langle \hat{p}, \hat{v}_\xi \rangle = \Lambda(pv_\xi) = p(\xi)$ for all $p \in \mathcal{P}_m$. Since $m \geq 1$, the space $\mathcal{P}_m$ separates the points of the set $\mathcal{Z}_\Lambda$, and so the assignment $\xi \mapsto \hat{v}_\xi$ is injective. In addition, we may and shall always assume that a chosen representative v_ξ is in the space $\mathcal{R}\mathcal{P}_m$, so $\hat{v}_\xi \in \mathcal{R}\mathcal{H}_\Lambda$.

Set $\mathcal{V}_\Lambda = \{\hat{v}_\xi; \xi \in \mathcal{Z}_\Lambda\}$.

The next result is an approach to truncated moment problems when the number of the atoms of the representing measures is not necessarily equal to the maximal cardinal of a family of orthogonal idempotents. The basic elements are in this case projections of idempotents.

Theorem 5. *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ with $\mathcal{Z}_\Lambda$ nonempty. The uspf Λ has a representing measure in $\mathbb{R}^n$ consisting of d-atoms, where $d \geq \dim \mathcal{H}_\Lambda$, if and only if there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{R}\mathcal{H}_\Lambda$ such that*

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d, \tag{4.1}$$

$$\hat{p} = \Lambda(v_1)^{-1}\Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(pv_d)\hat{v}_d,\ p \in \mathcal{P}_m, \tag{4.2}$$

and

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \Lambda(v_j)^{-1}\Lambda(v_j v_k)\Lambda(v_j v_l),\ k, l = 1, \ldots, d. \tag{4.3}$$

Proof. We use the notation and some arguments from Remark 7(1). Assume that $\mu = \sum_{j=1}^d \lambda_j \delta_{\xi^{(j)}}$ is a representing measure for Λ, with $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^d \lambda_j = 1$. The set $\operatorname{supp}(\mu) = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$, consisting of distinct points, is a subset of $\mathcal{Z}_\Lambda$, by Lemma 7.

We proceed now as in Remark 7(1). Let $r \geq m$ be an integer such that $\mathcal{P}_r$ contains interpolating polynomials for the family of points $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Setting $\Lambda_\mu(p) = \int_\Xi p d\mu$, $p \in \mathcal{P}_{2r}$, we have that the space $\mathcal{H}_r = \mathcal{P}_r/\mathcal{I}_{\Lambda_\mu}$ is a C^*-algebra isomorphic to $C(\Xi)$, where $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be the basis of $\mathcal{H}_r$ with $b_j|\Xi$ the characteristic function of the set $\{\xi^{(j)}\}$ for all $j = 1, \ldots, d$. Of course, $\mathcal{B}$ consists of orthogonal idempotents. In addition, $p(\xi^{(j)}) = \lambda_j^{-1}\langle \hat{p}, \hat{b}_j \rangle$, $p \in \mathcal{P}_m,\ j = 1, \ldots, d$.

As $\mathcal{H}_\Lambda$ may be regarded as a vector subspace of $\mathcal{H}_r$ (see Remark 1(2)), we denote by P the orthogonal projection of $\mathcal{H}_r$ onto $\mathcal{H}_\Lambda$. In particular, $P\hat{1} = \hat{1}$.

Let $\hat{v}_j = P\hat{b}_j$, $j = 1, \ldots, d$. Then

$$\Lambda(pv_j) = \langle \hat{p}, P\hat{b}_j \rangle = \langle \hat{p}, \hat{b}_j \rangle = \lambda_j p(\xi^{(j)}),\ p \in \mathcal{P}_m,\ j = 1, \ldots, d,$$

so $\Lambda(v_j) = \Lambda(b_j) = \lambda_j > 0$, and $v_j/\lambda_j = v_{\xi^{(j)}}$, which is precisely (4.1). In addition, as $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}_r$,

$$\hat{p} = P\hat{p} = P(\Lambda(b_1)^{-1}\langle \hat{p}, \hat{b}_1 \rangle \hat{b}_1 + \cdots + \Lambda(b_d)^{-1}\langle \hat{p}, \hat{b}_d \rangle \hat{b}_d) =$$

$$\Lambda(v_1)^{-1}\Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(pv_d)\hat{v}_d,$$

for all $p \in \mathcal{P}_m$, showing that (4.2) holds. Note also that

$$\Lambda(v_k v_l) = \sum_{j=1}^{d} \lambda_j (v_k v_l)(\xi^{(j)}) = \sum_{j=1}^{d} \Lambda(v_j)^{-1} \Lambda(v_k v_j)\Lambda(v_l v_j),\ k, l = 1, \dots, d,$$

because

$$(v_k v_l)(\xi^{(j)}) = \Lambda(v_k v_{\xi^{(j)}})\Lambda(v_l v_{\xi^{(j)}}) = \lambda_j^{-2} \Lambda(v_k v_j)\Lambda(v_l v_j)$$

for all $k, l = 1, \dots, d$, proving that (4.3) also holds.

Conversely, assume that there exists a family $\{\hat{v}_1, \dots, \hat{v}_d\} \subset \mathcal{RH}_\Lambda$ such that (4.1), (4.2), (4.3) hold. We must have $v_j/\lambda_j = v_{\xi^{(j)}}$ for a uniquely determined $\xi^{(j)} \in \Xi$, with $\lambda_j = \Lambda(v_j) > 0$ for all $j = 1, \dots, d$.

Consider the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p|\Xi \in C(\Xi)$. Note that this map is correctly defined because the equality $\hat{p}_1 = \hat{p}_2$, which is equivalent to $p_1 - p_2 \in \mathcal{I}_\Lambda$, implies $p_1|\Xi = p_2|\Xi$, by Lemma 7. Moreover, the map is injective because $p(\xi^{(j)}) = \lambda_j^{-1}\Lambda(pv_j) = 0$ for all $j = 1, \dots, d$ implies $\hat{p} = 0$, via (4.2).

Since, in virtue of (4.2),

$$\Lambda(p) = \langle \hat{p}, \hat{v}_1 \rangle + \cdots + \langle \hat{p}, \hat{v}_d \rangle = \lambda_1 p(\xi^{(1)}) + \cdots + \lambda_d p(\xi^{(d)}),$$

for all $p \in \mathcal{P}_m$, the map $\Lambda|\mathcal{P}_m$ admits the extension $M(f) = \sum_{j=1}^{d} \lambda_j f(\xi^{(j)})$, $f \in C(\Xi)$, which provides an integral representation for $\Lambda|\mathcal{P}_m$.

We want to show that the map M also extends Λ. For, let $p = \sum_{j \in J} p_j q_j$, with $p_j, q_j \in \mathcal{P}_m$ for all $j \in J$, where J is a finite set of indices. Note first that

$$p(\xi^{(k)}) = \sum_{j \in J} p_j(\xi^{(k)}) q_j(\xi^{(k)}) = \lambda_k^{-2} \sum_{j \in J} \Lambda(p_j v_k)\Lambda(q_j v_k), \tag{4.4}$$

for all $k = 1, \dots, d$. Then, on one hand,

$$M(p) = \sum_{k=1}^{d} \lambda_k p(\xi^{(k)}) = \sum_{k=1}^{d} \lambda_k \sum_{j \in J} p_j(\xi^{(k)}) q_j(\xi^{(k)}),$$

so that, using (4.4),

$$M(p) = \sum_{k=1}^{d} {\lambda_k}^{-1} \sum_{j \in J} \Lambda(p_j v_k)\Lambda(q_j v_k). \tag{4.5}$$

On the other hand, writing by (4.2)

$$\hat{p}_j = \sum_{l=1}^{d} {\lambda_l}^{-1} \Lambda(p_j v_l)\hat{v}_l,\ \hat{q}_j = \sum_{s=1}^{d} {\lambda_s}^{-1} \Lambda(q_j v_s)\hat{v}_s$$

for all $j \in J$, we have

$$p - \sum_{j \in J} \sum_{l,s=1}^{d} {\lambda_l}^{-1} {\lambda_s}^{-1} \Lambda(p_j v_l)\Lambda(q_j v_s) v_l v_s \in \ker(\Lambda),$$

so

$$\Lambda(p) = \sum_{j\in J}\sum_{l,s=1}^{d} {\lambda_l}^{-1}{\lambda_s}^{-1}\Lambda(p_jv_l)\Lambda(q_jv_s)\Lambda(v_lv_s) =$$

$$\sum_{j\in J}\sum_{l,s=1}^{d} {\lambda_l}^{-1}{\lambda_s}^{-1}\Lambda(p_jv_l)\Lambda(q_jv_s)\sum_{k=1}^{d}\lambda_k^{-1}\Lambda(v_kv_l)\Lambda(v_kv_s) =$$

$$\sum_{k=1}^{d}{\lambda_k}^{-1}\sum_{j\in J}\sum_{l=1}^{d}{\lambda_l}^{-1}\Lambda(p_jv_l)\Lambda(v_lv_k)\sum_{s=1}^{d}{\lambda_s}^{-1}\Lambda(q_jv_s)\Lambda(v_sv_k) =$$

$$\sum_{k=1}^{d}{\lambda_k}^{-1}\sum_{j\in J}\Lambda(p_jv_k)\Lambda(q_jv_k),$$

via (4.3), because of the equalities

$$\Lambda(p_jv_k) = \sum_{l=1}^{d}{\lambda_l}^{-1}\Lambda(p_jv_l)\Lambda(v_lv_k),\ \Lambda(q_jv_k) = \sum_{s=1}^{d}{\lambda_s}^{-1}\Lambda(q_jv_s)\Lambda(v_sv_k),$$

derived from (4.2). This computation leads to the equality $M(p) = \Lambda(p)$, for each p of the given form. Formula (4.5) shows that, in fact, the equality $M(p) = \Lambda(p)$ does not depend on the particular representation of p as a finite sum of the form $\sum_{j\in J} p_jq_j$, with $p_j, q_j \in \mathcal{P}_m$, and so $M(p) = \Lambda(p)$ holds for all $p \in \mathcal{P}_{2m}$. $\square$

Corollary 6. *Let* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, $(m \geq 1)$, *with* $\mathcal{Z}_\Lambda$ *nonempty.*

If there exist a family $\{\hat{v}_1, \ldots, \hat{v}_d\} \subset \mathcal{H}_\Lambda$ *such that*

$$\Lambda(v_j) > 0, \quad \hat{v}_j/\Lambda(v_j) \in \mathcal{V}_\Lambda, \quad j = 1, \ldots, d,$$

and

$$\hat{p} = \Lambda(v_1)^{-1}\Lambda(pv_1)\hat{v}_1 + \cdots + \Lambda(v_d)^{-1}\Lambda(pv_d)\hat{v}_d,\ p \in \mathcal{P}_m,$$

the functional $\Lambda|\mathcal{P}_m$ *has a representing measure in* $\mathbb{R}^n$ *consisting of d-atoms, where* $d \geq \dim \mathcal{H}_\Lambda$

Proof. The statement shows that conditions (4.1) and (4.2) are fulfilled, and the assertion follows from the proof of Theorem 5. $\square$

Remark 15. Condition $d \geq \dim \mathcal{H}_\Lambda$, appearing in the two previous statements, is a necessary one, as follows from [3], Corollary 3.7.

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