Normality, Moments, Quaternions

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Outline

Unbounded Normal Operators

Let $\mathcal{H}$ be a complex Hilbert space. A natural extension of the concept of bounded normal operator to the class of densely defined linear operators is given by the following.

Definition Let $N : D(N) \subset \mathcal{H} \mapsto \mathcal{H}$ be a densely defined closed linear operator. We say that N is **normal** if $D(N^*) = D(N)$ and $\|N^*x\| = \|Nx\|$ for all $x \in D(N)$.

As in the bounded case, normal operators have many important properties, useful for both theory and applications. As the conditions from the definition are not easily verified, one seeks for conditions implying normaliy.

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Formally Normal Operators

A larger class of operators for which the conditions from the definition are more easily checked is the following.

Definition Let $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ be a densely defined closed linear operator. We say that S is **formally normal** if $D(S) \subset D(S^*)$ and $\|S^*x\| = \|Sx\|$ for all $x \in D(S)$.

Clearly, every normal operator is formally normal but the converse is not true, in general.

An important problem in this context is to describe those formally normal operators having normal extensions.

Coddington's Theorem

For an operator T, $N(T)$ denotes its null-space.

Coddington's Theorem (1960)

Let S be a formally normal operator in a Hilbert space $\mathcal{H}$. Let also $\bar{S} := S^*|D(S)$, and $\mathcal{M} := N(I + S^*\bar{S}^*)$.
There exists a normal extension N of S in $\mathcal{H}$ if and only if there exists a linear map W from $\mathcal{M}$ into itself, with the following properties:
(i) $W^2 = I|\mathcal{M}$;
(ii) $\|x\|^2 + \|\bar{S}^*x\|^2 = \|Wx\|^2 + \|\bar{S}^*Wx\|^2$, $x \in \mathcal{M}$;
(iii) $(I - W)\mathcal{M} = \bar{S}^*(I - W)\mathcal{M}$;
(iv) $\|\bar{S}^*(I - W)x\| = \|S^*(I - W)x\|$, $x \in \mathcal{M}$.
The operator N is given by $Nx = \bar{S}^*x$ for all $x \in D(N) := D(S) + (I - W)\mathcal{M}$.

A Matricial Strategy

Unlike in Coddington's work, a different approach to the extension problem, based on matrices, can be adopted.
Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. Let also T be a densely defined linear operator in $\mathcal{H}$, with the property that T and its adjoint T^* are both defined on $\mathcal{D}$. Writing $T = A + iB$, with $A = (T + T^*)/2$ and $B = (T - T^*)/2i$, and so A and B are symmetric operators on $\mathcal{D}$, we can associate the operator T with the matrix operator

$$Q_T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

It is known that T is normal in $\mathcal{H}$ if and only if if the operator Q_T is normal in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$.

Continuation

Because our techniques, based on a quaternionic Cayley transform, give conditions to insure the existence of a normal extension for a matrix operator resembling to Q_T, we can go back to the operator T, which satisfies only some verifiable conditions. In fact, we have such results actually for the case when A and B are symmetric operators defined on a not necessarily dense domain in $\mathcal{H}$, so the final extension result applies to larger class than that of formally normal operators.

Recall: Algebra of Quaternions

We start with some notation. Consider the 2×2-matrices

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hamilton algebra of quaternions $\mathbb{H}$ is identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$ of 2×2-matrices with complex entries, generated by the matrices $i\mathbf{J}$, $\mathbf{K}$, $i\mathbf{L}$, and the identity $\mathbf{I}$. Nevertheless, we regard the elements of $\mathbb{H}$ as matrices and we perform some operations in $\mathbb{M}_2$ rather than in $\mathbb{H}$.

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Continuation

We have the properties

$$\mathbf{J}^* = \mathbf{J},\ \mathbf{K}^* = -\mathbf{K},\ \mathbf{L}^* = \mathbf{L},\ \mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I},$$

$$\mathbf{JK} = \mathbf{L} = -\mathbf{KJ},\ \mathbf{KL} = \mathbf{J} = -\mathbf{LK},\ \mathbf{JL} = \mathbf{K} = -\mathbf{LJ},$$

where the adjoints are computed in the Hilbert space $\mathbb{C}^2$.

Setting $\mathbf{E} = i\mathbf{J}$, we have $\mathbf{E}^* = -\mathbf{E}$, $\mathbf{E}^2 = -\mathbf{I}$.
We also set $\mathbf{F} = i\mathbf{L}$, and we have $\mathbf{F}^* = -\mathbf{F}$, $\mathbf{F}^2 = -\mathbf{I}$.

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Some Notation

Let $\mathcal{H}$ be a complex Hilbert space, whose scalar product is denoted by $\langle *, * \rangle$, and whose norm is denoted by $\| * \|$. We especially work in the Hilbert space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$, whose scalar product, naturally induced by that from $\mathcal{H}$, is denoted by $\langle *, * \rangle_2$, and whose norm is denoted by $\| * \|_2$.
The matrices from $\mathbb{M}_2$ naturally act on $\mathcal{H}^2$ simply by replacing their entries with the corresponding multiples of the identity on $\mathcal{H}$. In particular, the matrices $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}, \mathbf{E}, \mathbf{F}$, defined in the previous section, naturally act on $\mathcal{H}^2$, and we still have the relations already mentioned.

Continuation

We fix some notation and terminology for Hilbert space (always linear) operators. For an operator T acting in $\mathcal{H}$, we denote by $D(T)$ its **domain of definition.** The **range** of T is denoted by $R(T)$, while $N(T)$ stands for the **kernel** of T.
If T is densely defined, let T^* be its **adjoint**.
If T_2 extends T_1, we write $T_1 \subset T_2$ or $T_2 \supset T_1$, which is an order relation in the space of linear operators.

An operator $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ is said to be **symmetric** if $\langle Sx, y\rangle = \langle x, Sy\rangle$ for all $x, y \in D(S)$. If, moreover, S is densely defined, then $S \subset S^*$.

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A Preliminary Result

Lemma 1 Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$. Suppose that the operator $\mathbf{J}S$ is symmetric. Then we have

$$\|(S \pm \mathbf{E})x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S).$$

If, in addition, $\mathbf{J}D(S) \subset D(S)$, we have

$$\|(S \pm \mathbf{E})\mathbf{E}x\|_2^2 = \|Sx\|_2^2 + \|x\|_2^2, \quad x \in D(S),$$

if and only if $\|S\mathbf{J}x\|_2 = \|Sx\|_2$ for all $x \in D(S)$.

E-Cayley Transform

Let $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be such that $\mathbf{J}S$ is symmetric. Lemma 1 allows us to correctly define the operator

$$V : R(S+\mathbf{E}) \mapsto R(S-\mathbf{E}), \quad V(S+\mathbf{E})x = (S-\mathbf{E})x, \quad x \in D(S),$$

which is a partial isometry. In other words, $V = (S-\mathbf{E})(S+\mathbf{E})^{-1}$, defined on $D(V) = R(S+\mathbf{E})$.

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F-Cayley Transform

Similarly, if **L**S is symmetric, the corresponding version of Lemma 1 leads to the definition of an operator

$$W : R(S+\mathbf{F}) \mapsto R(S-\mathbf{F}), \quad V(S+\mathbf{F})x = (S-\mathbf{F})x, \quad x \in D(S),$$

which is again a partial isometry, and $W = (S-\mathbf{F})(S+\mathbf{F})^{-1}$, defined on $D(W) = R(S+\mathbf{F})$.

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The operator W is called the **F–Cayley transform** of S.

Because the two Cayley transforms defined above are alike, in the sequel we shall mainly deal with the **E**–Cayley transform, also designated by the expression *quaternionic Cayley transform* (briefly, QCT).

For a symmetric operator, by *Cayley transform* we always mean the classical concept, as defined by von Neumann.

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Properties of the QCT

We summarize the properties of the QCT, stated for not necessarily densely defined operators.

Theorem 1

The **E**-Cayley transform is an order preserving bijective map assigning to each operator S with $S : D(S) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ and $\mathbf{J}S$ symmetric a partial isometry V in in $\mathcal{H}^2$ with $\mathbf{I} - V$ injective. Moreover:
(1) V is closed if and only if S is closed;
(2) the equality $V^{-1} = -\mathbf{K}V\mathbf{K}$ holds if and only if the equality $S\mathbf{K} = \mathbf{K}S$ holds;
(3) $\mathbf{J}S$ is self-adjoint if and only if V is unitary on $\mathcal{H}^2$.

Inverse QCT

Theorem 1 shows that given a partial isometry V in $\mathcal{H}^2$ with $\mathbf{I} - V$ injective, we can find a unique operator S in $\mathcal{H}^2$ with $\mathbf{J}S$ symmetric, such that the quaternionic Cayley transform of S is V. In fact, S is given by the formula

$$S = (\mathbf{I} + V)(\mathbf{I} - V)^{-1}\mathbf{E}, \ \text{defined on } D(S) = \mathbf{E}R(\mathbf{I} - V),$$

and called the *inverse* **E**–*Cayley transform* of V.

When V is unitary, the operator $\mathbf{J}S$ is actually self-adjoint. For our extension problem, it is of great interest to characterize those unitary operators U, with with $\mathbf{I} - U$ injective, such that the inverse **E**–Cayley transform S is a normal operator.

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Unitaries in the Range of QCT

Theorem 2

Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and such that $\mathbf{I} - U$ is injective. Let also S be the inverse $\mathbf{E}$–Cayley transform of U. The operator S is normal if and only if $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.

If U is a unitary operator as in the previous statement, then U has necessarily the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal and A self-adjoint in $\mathcal{H}$, $TT^* + A^2 = I$, and $AT = TA$. In this case, $S = (\mathbf{I} + U)(\mathbf{I} - U)^{-1}\mathbf{E}$.

Some Notation

Let $\mathcal{U}(\mathcal{H}^2)$ be the set of all unitary operators in $\mathcal{H}^2$. We also set

$$\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \{U \in \mathcal{U}(\mathcal{H}^2); U^* = -\mathbf{K}U\mathbf{K}, N(\mathbf{I} - U) = \{0\},$$
$$(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)\},$$

that is, those unitary operators whose inverse **E**–Cayley transform is a normal operator, via the previous theorem.

Let also

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) = \{S : D(S) \subset \mathcal{H}^2 \to \mathcal{H})^2;$$
$$S \text{ normal}, (\mathbf{J}S)^* = \mathbf{J}S, \mathbf{K}S = S\mathbf{K}\}.$$

Theorems 1 and 2 show the bijectivity of the map

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) \ni S \mapsto (S - \mathbf{E})(S + \mathbf{E})^{-1} \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2).$$

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Remark 1

In fact, we have $S \in \mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$ if and only if S is a densely defined operator in $\mathcal{H}^2$ having the form

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B are commuting self-adjoint operators. (The commutativity of A and B means that $(A - i)^{-1}$ and $(B - i)^{-1}$ commute, which is a **strong commutativity**.)

A Special Class

Let $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$. In order that T have a normal extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$, the following conditions are necessary:
(i) $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.
(ii) $\mathbf{J}T$ is symmetric;
(iii) $T\mathbf{K} = \mathbf{K}T$;
(iv) $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ for all $x \in D(T)$.

We denote by $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ the set of those operators $T : D(T) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that (i)–(iv) hold.

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Another Special Class

Let $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ be the set of those partial isometries $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that:
(a) $V^{-1} = -\mathbf{K}V\mathbf{K}$;
(b) $\mathbf{I} - V$ is injective;
(c) $\mathbf{E}R(\mathbf{I} - V) = R(\mathbf{I} - V)$ and $(\mathbf{I} - V)^{-1}\mathbf{E}(\mathbf{I} - V)$ is an isometry on $D(V)$.

It follows from our results that the $\mathbf{E}$–Cayley transform is a bijective map from $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ onto $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Note also that $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2) \cap \mathcal{U}(\mathcal{H}^2)$.

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Main Result

The interesting question concerning the existence of an extension $S \in \mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ of an operator $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ is equivalent to the description of those partial isometries in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ having extensions in the family $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$. Here is an answer to this question.

Theorem 3

Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be densely defined. The operator T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with $D(W) = R(T + \mathbf{E})^{\perp}$.

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Non-density

The hypothesis in Theorem 3 concerning the density on the domain of the given operator may be replaced by a hypothesis concerning some ranges.

Corollary

Let $T \in \mathcal{S}_{\mathcal{I}\mathcal{C}}(\mathcal{H}^2)$ be closed and let V be the **E**–Cayley transform of T. The operator T has an extension in $\mathcal{N}_{\mathcal{I}\mathcal{C}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with the properties $D(W) = R(T + \mathbf{E})^{\perp}$ and $R(\mathbf{I} - V) \cap R(\mathbf{I} - W) = \{0\}$.

Commuting Self-Adjoint Extensions

By our results, if $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$ is densely defined and the space $R(T + \mathbf{E})$ is dense in $\mathcal{H}^2$, then T has an extension in $\mathcal{N}_{\mathcal{I}C}(\mathcal{H}^2)$. This remark can be applied in the following situation. Let A, B be a pair of linear operators having a joint domain of definition $D_0 \subset \mathcal{H}$. As in the Introduction, we associate this pair with a matrix operator

$$T = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

defined on $D(T) = D_0 \oplus D_0 \in \mathcal{H}^2$. First of all, let us find equivalent conditions on A, B such that $T \in \mathcal{S}_{\mathcal{I}C}(\mathcal{H}^2)$. Clearly, $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.

Continuation

It is easily seen that T is symmetric if and only if both A, B are symmetric. The equality $\mathbf{K}T = T\mathbf{K}$ is also easily verified. Finally, the equality $\|T\mathbf{J}x\|_2 = \|Tx\|_2$ holds for all $x \in D(T)$ if and only if

$$\langle Au, Bv\rangle + \langle Bv, Au\rangle = \langle Bu, Av\rangle + \langle Av, Bu\rangle \qquad (\mathrm{C})$$

for all $u, v \in D_0$, which is a weak commutativity condition. Consequently, if A, B are symmetric and condition (C) holds, then $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$. In that case, the **E**–Cayley transform of T is in the class $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Commuting Extensions

Theorem 4

Let A, B be symmetric operators on a dense joint domain of definition $D_0 \subset \mathcal{H}$, satisfying condition (C). If the space

$$\{((A + iI)u + Bv) \oplus ((A - iI)v - Bu); u, v \in D_0\} \qquad \text{(D)}$$

is dense in $\mathcal{H}^2$, then the operators A and B have commuting self-adjoint extensions.

The density of the space from (D) is precisely the density of $R(T + \mathbf{E})$ in $\mathcal{H}^2$, implying $R(T + \mathbf{E})^\perp = \{0\}$.
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A Moment Problem with Constraints

Let (s, t, u) denote the variable in $\mathbb{R}^3$, and let $\mathcal{P}$ be the algebra of all polynomials in s, t, u, with complex coefficients.

We recall that the linear map $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ is a *square positive functional* (briefly, a *spf*) if $\Lambda(\bar{p}) = \overline{\Lambda(p)}$, and $\Lambda(|p|^2) \geq 0$ for all $p \in \mathcal{P}$. If, moreover, $\Lambda(1) = 1$, we say that Λ is *unital square positive functional* (briefly, a *uspf*).

A *representing measure* for the *uspf* $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ with *support* in the measurable subset $\Sigma \subset \mathbb{R}^3$ is a probability measure μ on Σ such that $\Lambda(p) = \int_\Sigma p d\mu$ all $p \in \mathcal{P}$.

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Consequence of Schmüdgen's Theorem

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{R}^3$, and let $\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{S}^3; 0 \leq s \leq 1\}$, which is a compact semi-algebraic set. Let also $\theta(s,t,u) = 1 - s^2 - t^2 - u^2$ and $\sigma(s) = s$. As we have

$\mathbb{S}^3_+ = \{(s,t,u) \in \mathbb{R}^3; \theta(s,t,u) = 0, \sigma(s) \geq 0, (1-\sigma)(s) \geq 0\}$, we obtain from a well-known theorem by Schmüdgen that the unital square positive functional $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ has a representing measure with support in $\mathbb{S}^3_+$ if and only if

$$\Lambda_\theta = 0, \textit{ and } \Lambda_\sigma, \Lambda_{1-\sigma}, \Lambda_{\sigma(1-\sigma)} \qquad \text{(P)}$$

are square positive functionals (where for a given $q \in \mathcal{P}$ and a map $\Lambda : \mathcal{P} \mapsto \mathbb{C}$, we put $\Lambda_q(p) = \Lambda(qp)$ for all $p \in \mathcal{P}$).

A Problem with Constraints

A more complicated situation, which can be treated with our methods, occurs when we impose some constraints, and the solution given by Schmüdgen 's theorem is not necessarily the "good" one.

Problem. *Characterize those unital square positive functionals* Λ *on* $\mathcal{P}$ *with the property* (P), *which have a representing measure with support in the set* $\mathbb{S}^3_{++} := \{(s, t, u) \in \mathbb{S}^3_+ ; 0 \leq s < 1\}$, *such that all functions* $(1 - s)^{-m}$*(*$m \geq 1$ *an integer) are integrable*.

Notation

From now on, let $\Lambda : \mathcal{P} \mapsto \mathbb{C}$ be a square positive functional with the property (P). This implies that $\Lambda(q) = 0$ for each polynomial q with $q|\mathbb{S}^3_+ = 0$. We denote by $\mathcal{P}(\mathbb{S}^3_+)$ the algebra consisting of all (classes of) functions of the form $p|\mathbb{S}^3_+$, $p \in \mathcal{P}$, modulo the ideal of those polynomials q with $q|\mathbb{S}^3_+ = 0$. The map induced by Λ on $\mathcal{P}(\mathbb{S}^3_+)$ will still be designated by Λ.

A Useful Formula

To give a solution to the Problem, we should first extend the map Λ to the algebra $\mathcal{R}(\mathbb{S}^3_{++})$ generated by the rational functions $s^j t^k u^l (1-s)^{-m}$ restricted to $\mathbb{S}^3_{++}$, where j, k, l, m are nonnegative integers.

First of all, we note the formula

$$\frac{1}{(1-s)^{m+1}} = \sum_{\alpha \geq m} \binom{\alpha}{m} s^{\alpha - m}, \tag{5.1}$$

valid for all integers $m \geq 0$, where the series is convergent at each point $s \in [0, 1)$.

A Necessary Condition

The series (5.1) suggests the following supplementary hypothesis on Λ:

Condition. *Setting*

(5.2) $$p_{m,n}(s) = \sum_{\alpha=m}^{n} \binom{\alpha}{m} s^{\alpha-m},$$

for all nonnegative integers $m, n\,(n \geq m)$ *and* $s \in [0,1)$*, we assume that*

(L) $$\lim_{n_1,n_2\to\infty} \Lambda(|p_{m,n_1} - p_{m,n_2}|^2) = 0$$

for all $m \geq 0$*.*

Condition (L) is necessary via the Lebesgue theorem of dominated convergence.

Sufficiency of (L): Step 1

We shall prove that (L) is also sufficient.
Using (L), for each polynomial $p \in \mathcal{P}(\mathbb{S}^3_+)$ and every integer $m \geq 0$, we may define

$$\tilde{\Lambda}(pr_m) = \lim_{n\to\infty} \Lambda(pp_{m,n}), \tag{5.3}$$

where $r_m(s) = (1 - s)^{-m-1}$. Note that the limit exists via the Cauchy-Schwarz inequality. Moreover,

$$\tilde{\Lambda}(pr_{m_1}) = \tilde{\Lambda}((1 - \sigma)^{m_2 - m_1} pr_{m_2}) \tag{5.4}$$

if $m_2 \geq m_1$.

Step 2

Let now $p_1, p_2 \in \mathcal{P}(\mathbb{S}^3_+)$, and let m_1, m_2 be nonnegative integers such that $r_{m_2}^{-1} p_1 - r_{m_1}^{-1} p_2 = q$, where $q|\mathbb{S}^3_+ = 0$. Assuming, with no loss of generality, that $m_2 \geq m_1$, we infer $p_2 = (1-\sigma)^{m_2 - m_1} p_1 - q r_{m_1}$. This relation also shows that $q r_{m_1}$ is a polynomial, which is null on $\mathbb{S}^3_+$. Therefore

$$\lim_{n\to\infty} \Lambda(p_2 p_{m_2,n}) = \lim_{n\to\infty} \Lambda(p_1 p_{m_1,n}).$$

Consequently,

$$\tilde{\Lambda}(p_2 r_{m_2}) = \tilde{\Lambda}(p_1 r_{m_1}). \tag{5.5}$$

Step 3

Relation (5.5) shows that $\tilde{\Lambda}$ induces a map on the algebra of fractions $\mathcal{F}(\mathbb{S}^3_{++})$ build from the algebra $\mathcal{P}(\mathbb{S}^3_{+})$, with denominators in the set $\mathcal{S} = \{(1-s)^m; m \geq 0\}$. This map is denoted again by Λ. The map $\Lambda : \mathcal{F}(\mathbb{S}^3_{++}) \mapsto \mathbb{C}$ is a unital square positive functional. Indeed, fixing $f = p/(1-\sigma)^m$, we have, via the properties of $\Lambda : \mathcal{P}(\mathbb{S}^3_{+}) \mapsto \mathbb{C}$,

$$\Lambda(\bar{f}) = \lim_{n\to\infty} \Lambda(\bar{p}p_{m,n}) = \overline{\Lambda(f)},\ \Lambda(|f|^2) = \lim_{n\to\infty} \Lambda(|f|^2 p_{2m,n}) \geq 0,$$

$$\Lambda_\sigma(|f|^2) = \lim_{n\to\infty} \Lambda(\sigma|f|^2 p_{2m,n}) \geq 0, \tag{5.6}$$

$$\Lambda_{1-\sigma}(|f|^2) = \lim_{n\to\infty} \Lambda((1-\sigma)|f|^2 p_{2m,n}) \geq 0.$$

Step 4

In particular, the map $\Lambda : \mathcal{F}(\mathbb{S}^3_{++}) \mapsto \mathbb{C}$ satisfies the Cauchy-Schwartz inequality, and so the set $\mathcal{I}_\Lambda = \{f \in \mathcal{F}(\mathbb{S}^3_{++}); \Lambda(|f|^2) = 0\}$ is an ideal in the algebra $\mathcal{F}(\mathbb{S}^3_{++})$. Moreover, the assignment $(f, g) \mapsto \Lambda(f\bar{g})$ induces an inner product on the quotient $D_0 = \mathcal{F}(\mathbb{S}^3_{++})/\mathcal{I}_\Lambda$. The completion of this quotient with respect to this inner product is a Hilbert space denoted by $\mathcal{H}$.

Step 5

We now consider in $\mathcal{H}$ the multiplication operators B_0, C_0 induced by the functions $-t/(1-s)$ and $u/(1-s)$, respectively, defined on D_0. Then B_0, C_0 leave invariant the space D_0 and commute. Moreover, for every pair $g_1, g_2 \in D_0$, the system

$$\left(\frac{-t}{1-s} + i\right) f_1 + \frac{u}{1-s} f_2 = g_1$$

$$\frac{-u}{1-s} f_1 + \left(\frac{-t}{1-s} - i\right) f_2 = g_2 \qquad (5.7)$$

has the solution $f_1 = -2^{-1}((t + i - is)g_1 + ug_2)$,
$f_2 = 2^{-1}(ug_1 - (t - i + is)g_2)$, via $s^2 + t^2 + u^2 = 1$, so $f_1, f_2 \in D_0$.

Step 6

Setting $S_0 = B_0\mathbf{I} + C_0\mathbf{K}$ on $D_0 \oplus D_0$, the system (5.7) is precisely the equation $(S_0 + \mathbf{E})(f_1 \oplus f_2) = g_1 \oplus g_2$, showing that $R(S_0 + \mathbf{E})$ is equal to $D_0 \oplus D_0$. Hence, denoting by U_0 the $\mathbf{E}$-Cayley transform of S_0, a direct computation shows that U_0 is the matrix multiplication operator

$$U_0 = \begin{pmatrix} s + it & iu \\ iu & s - it \end{pmatrix},$$

defined on $D_0 \oplus D_0$.

Step 7

We clearly have $S_0 \in \mathcal{S}_{\mathcal{I}\mathcal{C}}(\mathcal{H}^2)$. Then the closure S of S_0 also belongs to $\mathcal{S}_{\mathcal{I}\mathcal{C}}(\mathcal{H}^2)$. If U is the **E**-Cayley transform of S, then U should be closed. As U extends U_0, U must be a unitary operator on $\mathcal{H}^2$. Specifically, $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ because U is a unitary operator in $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$. In particular, $\mathbf{I} - U$ is injective

Let T, A be the bounded operators associated to U, via Theorem 2.
Note that we also have that $I - \mathrm{Re}(T)$ is injective.
In fact, the multiplication by $s + it$ on D_0 is extended by T, and the multiplicatin by u on D_0 is extended by A.

Step 8

Let E be the joint spectral measure of the pair (T, A), which is concentrated on S^3_+. Indeed, if $\mathcal{A}$ is the unital (commutative) C^*-algebra generated by T and A, the equality $T^*T + A^2 = I$ shows that the joint spectrum of the pair (T, A) may be identified with a compact subset of the sphere $\mathbb{S}^3$. In addition, as $0 \leq \mathrm{Re}(T) \leq I$, which is implied by the properties of the square positive forms Λ_σ and $\Lambda_{1-\sigma}$ given by (5.6), it follows that the measure E is concentrated in the set $\mathbb{S}^3_+$. As the operator $I - \mathrm{Re}(T)$ is injective, it follows that $E(\{(1, 0, 0)\}) = 0$. Consequently, the measure E is supported by the set $\mathbb{S}^3_{++}$.

Last Step

Since $1 + \mathcal{I}_\Lambda = (I - \mathrm{Re}(T))^m((1-\sigma)^{-m} + \mathcal{I}_\Lambda)$, it follows that $1 + \mathcal{I}_\Lambda$ is in the domain of $(I - \mathrm{Re}(T))^{-m}$ for all integers $m \geq 1$. Therefore, setting $\mu(*) = \langle E(*)(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle$, we obtain

$$\Lambda(pr_m) = \langle pr_m + \mathcal{I}_\Lambda, 1 + \mathcal{I}_\Lambda\rangle =$$

$$\langle (p(\mathrm{Re}(T), \mathrm{Im}(T), A)(I - \mathrm{Re}(T))^{-m}(1 + \mathcal{I}_\Lambda), 1 + \mathcal{I}_\Lambda\rangle = \int_{S^3_{++}} pr_m d\mu,$$

for all $f = pr_m \in \mathcal{F}(\mathbb{S}^3_{++})$, showing that μ is a representing measure for $\Lambda : \mathcal{F}(\mathbb{S}^3_{++}) \mapsto \mathbb{C}$. In addition

$$\int_{S^3_{++}} (1-s)^{-2m} d\mu = \|(I - \mathrm{Re}(T))^{-2m}(1 + \mathcal{I}_\Lambda)\|^2 < \infty,$$

for all integers $m \geq 1$, which completes our assertion.

Reference

More details concerining these results can be found in the author's paper

Quaternionic Cayley Transform Revisited
J. Math. Anal. Appl. 409 (2014) 790–807

Merci beaucoup pour votre attention !