Spectral Approach to Quaternionic-Valued Functions

(Abordare Spectrală a Funcţiilor cu Valori Cuaternioni)

Florian-Horia Vasilescu

Department of Mathematics
University of Lille, France

The Ninth Congress of Romanian Mathematicians
June 30, 2019, Galaţi, Romania

Abstract

Denoting by $\mathbb{M}$ the complexification of the quaternionic algebra $\mathbb{H}$, we characterize the family of those $\mathbb{M}$-valued functions, defined on subsets of $\mathbb{H}$, whose values are actually quaternions, using an intrinsic approach based on elementary spectral theory. In particular, we show that the so-called slice quaternionic regularity is equivalent to an analytic functional calculus with $\mathbb{M}$-valued stem functions, via a Cauchy type transform.

Rezumat

Notând cu $\mathbb{M}$ complexificarea algebrei cuaternionice $\mathbb{H}$, caracterizăm familia acelor funcţii definite pe submulţimi ale lui $\mathbb{H}$, a priori cu valori in $\mathbb{M}$, care iau valori chiar în $\mathbb{H}$, folosind o metodă intrinsecă de teorie spectrală. În particular, arătăm că $\mathbb{C}$ regularitatea direcţionala a funcţiilor cuaternionice este echivalentă cu un calcul funcţional analitic, printr-o transformare de tip Cauchy.

Aducem un omagiu matematicienilor români **Grigore Moisil** şi **Nicolae Theodorescu**, pentru pionieratul lor în dezvoltarea teorei funcţiilor cu valori cuaternioni.

Alte contribuţii româneşti legate de acest domeniu sunt datorate lui **Viorel Iftimie, Dan Pascali, Marcel Roşculeţ** şi alţii.

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Outline

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[1] F. Colombo, J. Gantner, D. P. Kimsey: **Spectral Theory on the S-Spectrum for Quaternionic Operators**, Birkhäuser, 2018.

[2] F. Colombo, I. Sabadini and D. C. Struppa: **Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions**: Progress in Mathematics, Vol. 28 Birkhäuser/Springer Basel AG, Basel, 2011.

Very Short History

Introduced by W. R. Hamilton in 1843, the quaternions form a unital non commutative division algebra over the real field (by the celebrated Theorem of Frobenius: 1877), with numerous applications in mathematics and physics. In mathematics, one of the most important investigation in the quaternionic context has been to find a convenient manner to express the "analyticity" of functions depending on quaternions. Among the pioneer contributions in this direction one should mention the works by Moisil and Theodorescu (1931), and by Fueter (1939/40) as well. Many recent contributions can be found in the monographs [1] and [2].

Hamilton's Algebra (1)

Abstract Hamilton's algebra $\mathbb{H}$ is the four-dimensional $\mathbb{R}$-algebra with unit 1, generated by the "imaginary units" $\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$, which satisfy

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{l},\ \mathbf{kl} = -\mathbf{lk} = \mathbf{j},\ \mathbf{lj} = -\mathbf{jl} = \mathbf{k},\ \mathbf{jj} = \mathbf{kk} = \mathbf{ll} = -1.$$

We may assume that $\mathbb{H} \supset \mathbb{R}$ identifying every number $x \in \mathbb{R}$ with the element $x1 \in \mathbb{H}$.

The algebra $\mathbb{H}$ has a natural multiplicative norm given by

$$\|\mathbf{x}\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2},\ \ \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l},\ \ x_0, x_1, x_2, x_3 \in \mathbb{R}$$

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Hamilton's Algebra (2)

Every element $\mathbf{x} \in \mathbb{H} \setminus \{0\}$ is invertible, and $\mathbf{x}^{-1} = \|\mathbf{x}\|^{-2}\mathbf{x}^*$, where the map
$\mathbb{H} \ni \mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto \mathbf{x}^* = x_0 - x_1\mathbf{j} - x_2\mathbf{k} - x_3\mathbf{l} \in \mathbb{H}$
is a natural involution.

For an arbitrary quaternion
$\mathbf{x} = x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l}$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, we set
$\Re\mathbf{x} = x_0 = (\mathbf{x} + \mathbf{x}^*)/2$, and $\Im\mathbf{x} = x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} = (\mathbf{x} - \mathbf{x}^*)/2$,
that is, the *real* and the *imaginary part* of $\mathbf{x}$, respectively.

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Real C^*-Algebras

Using one of the equivalent definitions, a real C^*-algebra is a real Banach $*$-algebra A satisfying the C^*-identity $\|a^*a\| = \|a\|^2$ for all $a \in A$, also having the property $\|a^*a\| \leq \|a^*a + b^*b\|$ for all $a, b \in A$.

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Complexification

The *complexification* of the $\mathbb{R}$-vector space $\mathbb{H}$ is the tensor product $\mathbb{M} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$, which is a $\mathbb{C}$-algebra with unit, containing the $\mathbb{R}$-algebra $\mathbb{H}$ via the injective morphism $\mathbb{H} \ni x \mapsto 1 \otimes x \in \mathbb{M}$. It is more convenient to write $\mathbb{M}$ as $\mathbb{H} + i\mathbb{H}$, where the sum is direct, via the $\mathbb{R}$-linear bijection

$$\mathbb{H} + i\mathbb{H} \ni x + iy \mapsto 1 \otimes x + i \otimes y \in \mathbb{M}$$

which implies the natural multiplicative structure given by

$$(\mathbf{x}_1 + i\mathbf{x}_2)(\mathbf{y}_1 + i\mathbf{y}_2) = (\mathbf{x}_1\mathbf{y}_1 - \mathbf{x}_2\mathbf{y}_2) + i(\mathbf{x}_1\mathbf{y}_2 + \mathbf{x}_2\mathbf{y}_1),\ \ \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{H}.$$

Thus $\mathbb{M}$ is an associative complex algebra, with unit 1 and involution $(\mathbf{x}_1 + i\mathbf{x}_2)^* = \mathbf{x}_1^* - i\mathbf{x}_2^*$. Moreover, in the algebra $\mathbb{M}$, *the elements of* $\mathbb{H}$ *commute with all complex numbers.*

Embedding

We may embed the algebra $\mathbb{H}$, regarded as a real C^*-algebra into the complex algebra $\mathbb{M}$, endowed with a unique C^*-algebra structure, via the natural embedding. Specifically, we have the following.

Theorem 1

The complex algebra $\mathbb{M}$ has a unique C^*-algebra structure such that $\mathbb{H}$ is a real C^*-subalgebra of $\mathbb{M}$.

This is a particular case of a more general known result (appearing in works by I. Vidav, T.V. Palmer etc.) concerning the embedding of a real C^*-subalgebra into a complex C^*-subalgebra.

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Matrix Quaternions

Let $\mathbb{M}_2$ denote the complex C^*-algebra of all 2×2-matrices. Setting

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{L} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

with $i^2 = -1$. Note that we have

$$\mathbf{J}^2 = \mathbf{K}^2 = \mathbf{L}^2 = -\mathbf{I}, \tag{0}$$

$$\mathbf{JK} = \mathbf{L} = -\mathbf{KJ},\ \mathbf{KL} = \mathbf{J} = -\mathbf{LK},\ \mathbf{LJ} = \mathbf{K} = -\mathbf{JL}.$$

The Q-Map

The assignment

$$\mathbb{H} \ni x_0 + x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l} \mapsto x_0\mathbf{I} + x_1\mathbf{J} + x_2\mathbf{K} + x_3\mathbf{L} \in \mathbb{M}_2, \quad (1)$$

with $x_0, x_1, x_2, x_3 \in \mathbb{R}$, is a unital $\mathbb{R}$-algebra morphism, which is also a $*$-isometry, denoted by Q. The range of Q, say $\mathbb{H}_2$, is the $\mathbb{R}$-subalgebra, generated by the matrices **I**, **J**, **K** and **L** of the $\mathbb{C}$-algebra $\mathbb{M}_2$, whose elements are called *matrix quaternions*).

The R-Map

We have a $\mathbb{C}$-linear map $R : \mathbb{M} \mapsto \mathbb{M}_2$ given by

$$R(\mathbf{a} + i\mathbf{b}) = Q(\mathbf{a}) + iQ(\mathbf{b}) \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{H},$$

which is an algebraic $*$-isomorphism. Defining

$$\|\mathbf{a} + i\mathbf{b}\| = \|R(\mathbf{a} + i\mathbf{b})\|,$$

where the right hand side norm is that of $\mathbb{M}_2$, turns the algebra $\mathbb{M}$ into a complex C^*-algebra.

The norm on $\mathbb{M}$ as defined above does not depend on particular representation of $\mathbb{M}$ onto $\mathbb{M}_2$, because of the uniqueness of a C^*-norm. In addition, its restriction to $\mathbb{H}$ equals the original norm of $\mathbb{H}$. In fact, this construction provides a direct proof of **Theorem 1**.

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Conjugation

Remark 1 In the algebra $\mathbb{M}_2$ we have a *conjugation* given by $\bar{A} = B - iC$ whenever $A = B + iC$ with $B, C \in \mathbb{H}_2$.

Similarly, in the algebra $\mathbb{M}$ there also exists a *conjugation* given by $\bar{\mathbf{a}} = \mathbf{b} - i\mathbf{c}$, where $\mathbf{a} = \mathbf{b} + i\mathbf{c}$ is arbitrary in $\mathbb{M}$, with $\mathbf{b}, \mathbf{c} \in \mathbb{H}$. Setting $A = R(\mathbf{a})$, we clearly have $\bar{A} = \overline{R(\mathbf{a})}$ for all $\mathbf{a} \in M$. Moreover, $\bar{\mathbf{a}} = \mathbf{a}$ if and only if $\mathbf{a} \in \mathbb{H}$, $\overline{\mathbf{a} + \mathbf{b}} = \bar{\mathbf{a}} + \bar{\mathbf{b}}$, and $\overline{\mathbf{ab}} = \bar{\mathbf{a}}\bar{\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{M}$.

Of course, the $\mathbb{R}$-linear map $\mathbb{M} \ni \mathbf{a} \mapsto \bar{\mathbf{a}} \in \mathbb{M}$ is an $\mathbb{R}$-linear homeomorphism of $\mathbb{M}$.

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The Imaginary Unit Sphere

The concept of "slice regularity“ (introduced by Gentili and Struppa) means a form of holomorphy in the context of quaternions (see [2]). In the following, it will be introduced for $\mathbb{M}$-valued functions, defined on subsets of $\mathbb{H}$.

Let $\mathbb{S} = \{\mathfrak{s} = x_1\mathbf{j} + x_2\mathbf{k} + x_3\mathbf{l}; x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$, that is, the unit sphere of purely imaginary quaternions. Clearly, $\mathfrak{s}^* = -\mathfrak{s}$, $\mathfrak{s}^2 = -1$, $\mathfrak{s}^{-1} = -\mathfrak{s}$, and $\|\mathfrak{s}\| = 1$ for all $\mathfrak{s} \in \mathbb{S}$.

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Slice Regularity

Let $\Omega \subset \mathbb{H}$ be an open set, and let $F : \Omega \mapsto \mathbb{M}$ be a differentiable function. We say that F is *right slice regular* on Ω if we have

$$\bar{\partial}_{\mathfrak{s}} F(x + y\mathfrak{s}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + R_{\mathfrak{s}} \frac{\partial}{\partial y}\right) F(x + y\mathfrak{s}) = 0,$$

for all $\mathfrak{s} \in \mathbb{S}$, where $R_{\mathfrak{s}}$ is the right multiplication of the elements of $\mathbb{M}$ by $\mathfrak{s}$, and $x + y\mathfrak{s} \in \Omega \cap (\mathbb{R} + \mathbb{R}\mathfrak{s})$.

An Example

The convergent series of the form $\sum_{k\ge0} a_k\mathbf{q}^k$, on quaternionic balls $\{\mathbf{q}\in\mathbb{H}; \|\mathbf{q}\|<r\}$, with $r>0$ and $a_k\in\mathbb{M}$ for all $k\ge0$, are $\mathbb{M}$-valued slice regular on their domain of definition. In fact, if $a_k\in\mathbb{H}$, such functions are $\mathbb{H}$-valued slice regular on their domain of definition.

A Spectral Equation

We shall use the natural concept of spectrum in the complex algebra $\mathbb{M}$, which can be applied to quaternions.
We have the identities

$$(\lambda - \mathbf{x}^*)(\lambda - \mathbf{x}) = (\lambda - \mathbf{x})(\lambda - \mathbf{x}^*) = \lambda^2 - \lambda(\mathbf{x} + \mathbf{x}^*) + \|\mathbf{x}\|^2 \in \mathbb{C},$$

for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{H}$. Therefore, $\lambda - \mathbf{x} \in \mathbb{M}$ is invertible iff the complex number $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ is nonnull, and in that case

$$(\lambda - \mathbf{x})^{-1} = (\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2)^{-1}(\lambda - \mathbf{x}^*).$$

The Spectrum of a Quaternion

As the roots of the polynomial $\lambda^2 - 2\lambda\Re\mathbf{x} + \|\mathbf{x}\|^2$ are $\Re\mathbf{x} \pm i\|\Im\mathbf{x}\|$, we can give the following:

Definition 1 The **spectrum** of a quaternion $\mathbf{x} \in \mathbb{H}$ is given by the equality $\sigma(\mathbf{x}) = \{s_\pm(\mathbf{x})\}$, where $s_\pm(\mathbf{x}) = \Re\mathbf{x} \pm i\|\Im\mathbf{x}\|$.

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Some Remarks

(1) As usually, the *resolvent set* $\rho(\mathbf{x})$ of a quaternion $\mathbf{x} \in \mathbb{H}$ is the set $\mathbb{C} \setminus \sigma(\mathbf{x})$, while the function

$$\rho(\mathbf{x}) \ni \lambda \mapsto (\lambda - \mathbf{x})^{-1} \in \mathbb{M}$$

is the *resolvent* (*function*) of $\mathbf{x}$, which is a $\mathbb{M}$-valued analytic function on $\rho(\mathbf{x})$.

(2) Two quaternions $\mathbf{x}, \mathbf{y} \in \mathbb{H}$ have the same spectrum if and only if $\Re\mathbf{x} = \Re\mathbf{y}$ and $\|\Im\mathbf{x}\| = \|\Im\mathbf{y}\|$.

(3) If $\mathbf{q} = x + y\mathfrak{s}$, where $x, y \in \mathbb{R}$ and $\mathfrak{s} \in \mathbb{S}$, we always have $\sigma(\mathbf{q}) = \{x \pm iy\}$, so the spectrum of $\mathbf{q}$ does not depend on $\mathfrak{s}$.

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An $\mathbb{M}$-valued Cauchy Kernel

Definition 2 The $\mathbb{M}$-*valued Cauchy kernel* on the open set $\Omega \subset \mathbb{H}$ is given by

$$(\zeta, \mathbf{q}) \mapsto (\zeta - \mathbf{q})^{-1}, \quad \mathbf{q} \in \Omega, \quad \zeta \in \rho(\mathbf{q}). \tag{2}$$

Example 1 The $\mathbb{M}$-*valued Cauchy kernel* on the open set $\Omega \subset \mathbb{H}$ is slice regular. Specifically, choosing an arbitrary relatively open set $V \subset \Omega \cap (\mathbb{R} + \mathbb{R}\mathfrak{s})$, and fixing $\zeta \in \cap_{\mathbf{q} \in V} \rho(\mathbf{q})$, we can write for $\mathbf{q} = x + y\mathfrak{s} \in V$

$$\bar{\partial}_{\mathfrak{s}}((\zeta - q)^{-1}) = \bar{\partial}_{\mathfrak{s}}((\zeta - x - y\mathfrak{s})^{-1}) = 0,$$

via a direct computation, implying the assertion.

General Eigenvalues and Eigenvectors

Remark 1 The discussion about the spectrum of a quaternion can be enlarged, regarding any $\mathbf{q} \in \mathbb{H}$ as a left multiplication operator on the C^*-algebra $\mathbb{M}$, denoted by $L_{\mathbf{q}}$, and given by $L_{\mathbf{q}}\mathbf{a} = \mathbf{q}\mathbf{a}$ for all $\mathbf{a} \in \mathbb{M}$. We have $\sigma(L_{\mathbf{q}}) = \sigma(\mathbf{q})$. To find the eigenvectors of $L_{\mathbf{q}}$, we should look for solutions of the equation $\mathbf{q}\nu = s\nu$ in the algebra $\mathbb{M}$, with $s \in \sigma(\mathbf{q})$. We find the solutions

$$\nu_{\pm}(\mathbf{q}) = \left(1 \mp i\frac{\Im\mathbf{q}}{\|\Im\mathbf{q}\|}\right)\mathbf{x}$$

of the equation $\mathbf{q}\nu_{\pm} = s_{\pm}\nu_{\pm}$, where $\mathbf{x} \in \mathbb{H}$ is arbitrary, provided $\Im\mathbf{q} \neq 0$.

When $\Im\mathbf{q} = 0$, the solutions are given by $\nu = q_0\mathbf{a}$, with $q_0 = \Re\mathbf{q} = \mathbf{q}$, and $\mathbf{a} \in \mathbb{M}$ arbitrary.

Associated Idempotents

Every quaternion $\mathfrak{s} \in \mathbb{S}$ may be associated with two elements $\iota_{\pm}(\mathfrak{s}) = (1 \mp i\mathfrak{s})/2$ in $\mathbb{M}$, which are commuting idempotents such that $\iota_{+}(\mathfrak{s}) + \iota_{-}(\mathfrak{s}) = 1$ and $\iota_{+}(\mathfrak{s})\iota_{-}(\mathfrak{s}) = 0$.
In particular, if $\mathbf{q} \in \mathbb{H}$ and $\Im\mathbf{q} \neq 0$, setting $\mathfrak{s}_{\tilde{\mathbf{q}}} = \tilde{\mathbf{q}}\|\tilde{\mathbf{q}}\|^{-1} \in \mathbb{S}$, where $\tilde{\mathbf{q}} = \Im\mathbf{q}$, the elements

$$\iota_{\pm}(\mathfrak{s}_{\tilde{\mathbf{q}}}) = \frac{1}{2}\left(1 \mp i\frac{\tilde{\mathbf{q}}}{\|\tilde{\mathbf{q}}\|}\right)$$

are idempotents, as above.

Spectral Projections

The next result provides explicit formulas of the spectral projections associated to the operator $L_{\mathbf{q}}$, $\mathbf{q} \in \mathbb{M}$. This is not trivial only if $\mathbf{q} \in \mathbb{H} \setminus \mathbb{R}$. Otherwise, the only spectral projection is the identity.

Lemma 1 Let $\mathbf{q} \in \mathbb{H} \setminus \mathbb{R}$ be fixed. The spectral projections associated to $s_{\pm}(\mathbf{q})$ are given by

$$P_{\pm}(\mathbf{q})\mathbf{a} = \iota_{\pm}(\mathfrak{s}_{\tilde{\mathbf{q}}})(\mathbf{u} \pm \mathfrak{s}_{\tilde{\mathbf{q}}}\mathbf{v}),$$

respectively, for every $\mathbf{a} = \mathbf{u} + i\mathbf{v} \in \mathbb{M}$ with $\mathbf{u}, \mathbf{v} \in \mathbb{H}$.

In particular, for a fixed $\mathbf{q}$ with $\Im\mathbf{q} \neq 0$, we have the formulas

$$P_{\pm}(\mathbf{q})\mathbf{h} = \frac{1}{2}(1 \mp i\mathfrak{s}_{\tilde{\mathbf{q}}})\mathbf{h}, , \ \mathbf{h} \in \mathbb{H}.$$

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Some Definitions

Definition 3 (1) A subset $S \subset \mathbb{C}$ is said to be *conjugate symmetric* if $\zeta \in S$ if and only if $\bar{\zeta} \in S$.

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if whenever $\sigma(\mathbf{r}) = \sigma(\mathbf{q})$ for some $\mathbf{r} \in \mathbb{H}$ and $\mathbf{q} \in A$, we also have $\mathbf{r} \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put $\mathfrak{S}(A) = \cup_{\mathbf{q} \in A} \sigma(\mathbf{q})$. We also put $S_{\mathbb{H}} = \{\mathbf{q} \in \mathbb{H}; \sigma(\mathbf{q}) \subset S\}$ for an arbitrary subset $S \subset \mathbb{C}$.

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Some Properties

(1) If $A \subset \mathbb{H}$ is spectrally saturated, then $S = \mathfrak{S}(A)$ is conjugate symmetric, and conversely, if $S \subset \mathbb{C}$ is conjugate symmetric, then $S_{\mathbb{H}}$ is spectrally saturated. Moreover, the assignment $S \mapsto S_{\mathbb{H}}$ is injective.
Similarly, the assignment $A \mapsto \mathfrak{S}(A)$ is injective and $A = S_{\mathbb{H}}$ if and only if $S = \mathfrak{S}(A)$.
(2) If $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, then $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open. Conversely, if $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open.

Particular case: if $U = \{\zeta \in \mathbb{C}; |\zeta| < r\}$, for some $r > 0$, then $U_{\mathbb{H}} = \{\mathbf{q} \in \mathbb{H}; \|\mathbf{q}\| < r\}$.

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Stem Function

Definition 4 Let $U \subset \mathbb{C}$ be conjugate symmetric, and let $F : U \mapsto \mathbb{M}$. We say that F is a *stem function* if $F(\bar{\lambda}) = \overline{F(\lambda)}$ for all $\lambda \in U$.
For an arbitrary conjugate symmetric subset $U \subset \mathbb{C}$, we put

$$\mathcal{S}(U, \mathbb{M}) = \{F : U \mapsto \mathbb{M}; F(\bar{\zeta}) = \overline{F(\zeta)}, \zeta \in U\}, \tag{3}$$

that is, the $\mathbb{R}$-vector space of all $\mathbb{M}$-valued stem functions on U.
Replacing $\mathbb{M}$ by $\mathbb{C}$, we denote by $\mathcal{S}(U)$ the real algebra of all $\mathbb{C}$-valued stem functions, which is an $\mathbb{R}$-subalgebra in $\mathcal{S}(U, \mathbb{M})$.
In addition, the space $\mathcal{S}(U, \mathbb{M})$ is a $\mathcal{S}(U)$-bimodule.

Functional Calculus: Definition

Definition 5 Let $U \subset \mathbb{C}$ be conjugate symmetric. For every $F : U \mapsto \mathbb{M}$ and all $\mathbf{q} \in U_{\mathbb{H}}$ we define a function $F_{\mathbb{H}} : U_{\mathbb{H}} \mapsto \mathbb{M}$, via the assignment

$$U_{\mathbb{H}} \ni \mathbf{q} \mapsto F_{\mathbb{H}}(\mathbf{q}) = F(s_+(\mathbf{q}))\iota_+(\mathfrak{s}_{\tilde{\mathbf{q}}}) + F(s_-(\mathbf{q}))\iota_-(\mathfrak{s}_{\tilde{\mathbf{q}}}) \in \mathbb{M}, \quad (4)$$

where $\tilde{\mathbf{q}} = \Im \mathbf{q}$, $\mathfrak{s}_{\tilde{\mathbf{q}}} = \tilde{\mathbf{q}}\|\tilde{\mathbf{q}}\|^{-1}$, and $\iota_\pm(\mathfrak{s}_{\tilde{\mathbf{q}}}) = 2^{-1}(1 \mp i\mathfrak{s}_{\tilde{\mathbf{q}}})$.

Main Characterization

One of the main results is the following.

Theorem 2

Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $F : U \mapsto \mathbb{M}$. The element $F_{\mathbb{H}}(\mathbf{q})$ is a quaternion for all $\mathbf{q} \in U_{\mathbb{H}}$ if and only if F is a stem function.

Zeros

The following is a consequence of Theorem 2.

Remark Let $U \subset \mathbb{C}$ be a conjugate symmetric set and let $F \in \mathcal{S}(U, \mathbb{M})$ be arbitrary. We can easily describe the zeros of $F_{\mathbb{H}}$. Setting $\mathcal{Z}(F) := \{\lambda \in U; F(\lambda) = 0\}$, and $\mathcal{Z}(F_{\mathbb{H}}) := \{\mathbf{q} \in U_{\mathbb{H}}; F_{\mathbb{H}}(\mathbf{q}) = 0\}$, we must have

$$\mathcal{Z}(F_{\mathbb{H}}) = \{\mathbf{q} \in U_{\mathbb{H}}; \sigma(\mathbf{q}) \subset \mathcal{Z}(F)\}.$$

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A General Functional Calculus

The next result offers an $\mathbb{H}$-*valued general functional calculus* with arbitrary stem functions.

Theorem 3

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated set, and let $U = \mathfrak{S}(\Omega)$. The map

$$\mathcal{S}(U, \mathbb{M}) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{F}(\Omega, \mathbb{H})$$

is $\mathbb{R}$-linear, injective, and has the property $(Ff)_{\mathbb{H}} = F_{\mathbb{H}} f_{\mathbb{H}}$ for all $F \in \mathcal{S}(U, \mathbb{M})$ and $f \in \mathcal{S}(U)$. Moreover, the restricted map

$$\mathcal{S}(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{F}(\Omega, \mathbb{H})$$

is unital and multiplicative, where $\mathcal{F}(\Omega, \mathbb{H}) := \{\Phi : \Omega \mapsto \mathbb{H}\}$.

Definition and Notation

Using the $\mathbb{M}$-valued Cauchy kernel, we shall define a concept of quaternionic Cauchy transform. Some preparations:

Definition 5 Let $U \subset \mathbb{C}$ be open. Recall that an open subset $\Delta \subset U$ will be called a *Cauchy domain* (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary $\partial\Delta$ of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented.

Notation Let $U \subset \mathbb{C}$ be open, and $\mathcal{O}(U, \mathbb{M})$ the algebra of all $\mathbb{M}$-valued analytic functions on U. If $U \subset \mathbb{C}$ is also conjugate symmetric, let $\mathcal{O}_s(U, \mathbb{M})$ be the real subalgebra of $\mathcal{O}(U, \mathbb{M})$ consisting of all stem functions from $\mathcal{O}(U, \mathbb{M})$.

Replacing $\mathbb{M}$ by $\mathbb{C}$, we put $\mathcal{O}(U) = \mathcal{O}(U, \mathbb{C})) \subset \mathcal{O}(U, \mathbb{M})$, and $\mathcal{O}_s(U) = \mathcal{O}_s(U, \mathbb{C}) \subset \mathcal{O}_s(U, \mathbb{M})$.

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Quaternionic Cauchy Transform

Definition 6 Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, and let $F \in \mathcal{O}(U, \mathbb{M})$. For every $\mathbf{q} \in U_{\mathbb{H}}$ we set

$$C[F](\mathbf{q}) = \frac{1}{2\pi i} \int_\Gamma F(\zeta)(\zeta - \mathbf{q})^{-1} d\zeta, \tag{5}$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(\mathbf{q})$. The function $C[F] : U_{\mathbb{H}} \mapsto \mathbb{M}$ is called the *(quaternionic) Cauchy transform* of the function $F \in \mathcal{O}(U, \mathbb{M})$.

The function $C[F]$ is well defined because the function $U \setminus \sigma(\mathbf{q}) \ni \zeta \mapsto F(\zeta)(\zeta - \mathbf{q})^{-1} \in \mathbb{M}$ is analytic.

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Slice Regularity

Let $\mathcal{R}(U_{\mathbb{H}}, \mathbb{M}) = \{C[F]; F \in \mathcal{O}(U, \mathbb{M})\}$.

Proposition 1 Let $U \subset \mathbb{C}$ be open and conjugate symmetric, and let $F \in \mathcal{O}(U, \mathbb{M})$. Then function $C[F] \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{M})$ is slice regular on $U_{\mathbb{H}}$.

Remark 2 Because the function F does not necessarily commute with the left multiplication by $\mathfrak{s}$, the choice of the right multiplication in the slice regularity is necessary to get the stated property of $C[F]$.

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Series Development

It is not unexpected to have the following.

Corollary 1 Let $r > 0$ and let $U \supset \{\zeta \in \mathbb{C}; |\zeta| \leq r\}$ be a conjugate symmetric open set. Then for every $F \in \mathcal{O}(U, \mathbb{M})$ one has

$$C[F](q) = \sum_{n \geq 0} \frac{F^{(n)}(0)}{n!} q^n, \ \|q\| < r,$$

where the series is absolutely convergent.

$\mathbb{H}$-Valued Cauchy Transforms

Theorem 4

Let $U \subset \mathbb{C}$ be a conjugate symmetric open set and let $F \in \mathcal{O}(U, \mathbb{M})$. The Cauchy transform $C[F]$ is $\mathbb{H}$-valued if and only if $F \in \mathcal{O}_s(U, \mathbb{M})$.

Remark 3 It follows from the proof of Theorem 4 that the element $C[F](\mathbf{q})$, given by formula (5) for $F \in \mathcal{O}_s(U, \mathbb{M})$, coincides with the element $F_{\mathbb{H}}(\mathbf{q})$ given by Theorem 2. To unify the notation, from now on this element will be denoted by $F_{\mathbb{H}}(q)$, whenever F is a stem function, analytic or not.

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Notation

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega)$. We put

$$\mathcal{R}(\Omega, \mathbb{H}) = \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U, \mathbb{M})\},$$

which is a space of slice regular $\mathbb{H}$-valued functions, by Proposition 1 and Theorem 1.

Replacing $\mathbb{M}$ by $\mathbb{C}$, we put $\mathcal{R}(\Omega) = \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U)\}$.

Analytic Functional calculus

Theorem 5

Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega)$. The space $\mathcal{R}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}(\Omega, \mathbb{H})$ is a right $\mathcal{R}(\Omega)$-module, the map

$$\mathcal{O}_s(U, \mathbb{M}) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H})$$

is a right module isomorphism, and its restriction

$$\mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega)$$

is an $\mathbb{R}$-algebra isomorphism. Moreover, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$, $\zeta \in \mathbb{C}$, with $a_n \in \mathbb{H}$ for all $n = 0, 1, \dots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Q-Regular Functions

Theorem 5 suggests a definition for $\mathbb{H}$-valued "analytic functions" as elements of the set $\mathcal{R}(\Omega, \mathbb{H})$, where Ω is a spectrally saturated open subset of $\mathbb{H}$. Because the expression "analytic function" is quite improper in this context, the elements of $\mathcal{R}(\Omega, \mathbb{H})$ will be temporarily called *Q-regular functions* on Ω. In fact, the functions from $\mathcal{R}(\Omega, \mathbb{H})$ are Cauchy transforms of the stem functions from $\mathcal{O}_s(U, \mathbb{M})$, with $U = \mathfrak{S}(\Omega)$.

Extended Derivatives

Remark 4 For every function $F \in \mathcal{O}_s(U, \mathbb{M})$, the derivatives $F^{(n)}$ also belong to $\mathcal{O}_s(U, \mathbb{M})$, where $U \subset \mathbb{C}$ is a conjugate symmetric open set.
Fixing $F \in \mathcal{O}_s(U, \mathbb{M})$, we may define its *extended derivatives*:

$$F_{\mathbb{H}}^{(n)}(\mathbf{q}) = \frac{1}{2\pi i} \int_{\Gamma} F^{(n)}(\zeta)(\zeta - \mathbf{q})^{-1} d\zeta, \qquad (6)$$

where $\Delta \subset U$ is a Cauchy domain, $\Gamma = \partial\Delta$ and $n \geq 0$ is an integer.

In particular, if Δ is a disk centered at zero and $F \in \mathcal{O}_s(\Delta, \mathbb{M})$, so $F(\zeta) = \sum_{k\geq 0} a_k \zeta^k$ with $a_h \in \mathbb{H}$, then (6) gives the equality $F'_{\mathbb{H}}(\mathbf{q}) = \sum_{k\geq 1} k a_k \mathbf{q}^{k-1}$.

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An Inverse Formula

Proposition 2 Let $U \subset \mathbb{C}$ be conjugate symmetric, let $x, y \in \mathbb{R}$ with $x \pm iy \in U$, let $\mathfrak{s} \in \mathbb{S}$, and let $F \in \mathcal{O}_s(U, \mathbb{M})$. Then we have the formulas

$$F(x \pm iy) = F_{\mathbb{H}}(x \pm y\mathfrak{s}) \left(\frac{1 \mp i\mathfrak{s}}{2} \right) + F_{\mathbb{H}}(x \mp y\mathfrak{s}) \left(\frac{1 \pm i\mathfrak{s}}{2} \right) . \quad (7)$$

Equivalence

Lemma 2 Let $U \subset \mathbb{H}$ be a conjugate symmetric open set, let $\mathfrak{s} \in \mathbb{S}$ be fixed, and let $\Psi : U_{\mathfrak{s}} \mapsto \mathbb{H}$ be such that $\bar{\partial}_{\pm\mathfrak{s}}\Psi = 0$. Then there exists a function $\Phi \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{H})$ with $\Psi = \Phi|U_{\mathfrak{s}}$, where $U_{\mathfrak{s}} = \{x + y\mathfrak{s}; x + iy \in U\}$.

Theorem 6 Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The following conditions are equivalent:
(i) Φ is a slice regular function;
(ii) $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, that is, Φ is Q-regular.

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Conclusion

For a given spectrally saturated open set $\Omega \subset \mathbb{H}$, the class of functions from $\mathcal{R}(\Omega, \mathbb{H})$, having many properties similar to those of the (real) analytic functions might be called **(right) regular functions**. Of course, we may also define a concept of **(left) regular functions**, having analogous properties.

Author's references

This text is based on the author's recent work entitled **Quaternionic Regularity via Analytic Functional Calculus** (see http://arxiv.org/abs/1905.13051v1).

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Final Theorem

Mulţumesc mult pentru atenţia d-voastră!
(Thank you very much for your attention!)