Square Positive Functionals in an Abstract Setting

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Outline

Summary

In the framework of spaces of functions on measurable spaces, and using techniques from the theory of finite-dimensional commutative Banach algebras, as well as Hilbert space methods, we discuss integral representations of square positive functionals, extending and completing some older results concerning the positive Riesz functionals in finite-dimensional spaces of polynomials.

An Abstract Setting

Let $(\Omega, \mathcal{A})$ be a *measurable space*, that is, Ω is an arbitrary (nonempty) set and $\mathcal{A}$ is σ-algabra of subsets of Ω. Let also $\mathcal{M}(\Omega)$ be the algebra of all complex-valued $\mathcal{A}$-measurable functions on Ω.

For convenience, a vector subspace $\mathcal{S} \subset \mathcal{M}(\Omega)$ such that $1 \in \mathcal{S}$ and $f \in \mathcal{S}$ implies $\bar{f} \in \mathcal{S}$ is said to be a **function space** on $(\Omega, \mathcal{A})$) (or simply on Ω).

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Fixing a function space $\mathcal{S}$ on Ω, let $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

The symbol $\mathcal{RS}$ will designate the "real part" of $\mathcal{S}$, that is $\{f \in \mathcal{S}; f = \bar{f}\}$.

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Some Examples

Important examples of function spaces are associated with the space $\mathcal{P}$ of all polynomials in $n \geq 1$ real variables, with complex coefficients. For every integer $m \geq 0$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where deg(p) is the total degree of p. Note that $\mathcal{P}_m^{(2)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

We occasionally use the notation $\mathcal{P}_m^n$ instead of $\mathcal{P}_m$ and $\mathcal{P}^n$ instead of $\mathcal{P}$ when the number n should be specified.

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Square Positive Functionals

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$;
(3) $\Lambda(1) = 1$.

Adapting some terminology used by Möller to our context, a linear map Λ with the properties (1)-(3) is said to be a **unital square positive functional**, briefly a **uspf**.

An Example

Let $(\Omega, \mathcal{A})$ be a measurable space, and let μ be a probability measure on $\mathcal{A}$. Let also $\mathcal{S} \subset \mathcal{M}(\Omega)$ be a function space on $(\Omega, \mathcal{A}))$ such that $\mathcal{S} \subset L^2(\Omega, \mu)$. Then the map

$$\Lambda(f) = \int_\Omega f(\omega) d\mu(\omega), \; f \in \mathcal{S}^{(2)}, \tag{1}$$

is a unital square positive functional.

Some General Properties

If $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality:*

$$|\Lambda(fg)|^2 \leq \Lambda(|f|^2)\Lambda(|g|^2),\ f, g \in \mathcal{S}. \tag{2}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle \hat{f}, \hat{g} \rangle = \Lambda(f\bar{g}), \tag{3}$$

where $\hat{f} = f + \mathcal{I}_\Lambda$ is the equivalence class of $f \in \mathcal{S}$ modulo $\mathcal{I}_\Lambda$.

In fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\ \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$.

If $\mathcal{S}$ is finite dimensional, then $\mathcal{H}_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

When the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is given, we shall use the notation $\mathcal{I}_\Lambda$, $\mathcal{H}_\Lambda$, $\hat{f}$, with the meaning from above, if not otherwise specified.

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Abstract Moment Problem

The **(abstract) moment problem** for a given uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{S}$ is a fixed function space on $(\Omega, \mathcal{A})$, means to find conditions insuring the existence of a probability measure μ, defined on $\mathcal{A}$, such that $\Lambda(f) = \int f d\mu,\ f \in \mathcal{S}^{(2)}$. When such a measure μ exists, it is said to be a **representing measure** for (the uspf) Λ.

Briefly, we are looking for those uspf Λ having the form (1), that is $\Lambda(f) = \int f d\mu,\ f \in \mathcal{S}^{(2)}$, where μ is a probability measure.

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Atomic Representing Measures

In some special cases, a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ may have an **atomic representing measure in** Ω, that is, there exists a finite subset $\Omega_\Lambda = \{\omega_1, \ldots, \omega_d\} \subset \Omega$ and positive numbers $\lambda_1, \ldots, \lambda_d$, with $\lambda_1 + \cdots + \lambda_d = 1$, such that $\Lambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$ for all $f \in \mathcal{S}^{(2)}$. In this case we must have $\{\omega\} \in \mathcal{A}$ for all $\omega \in \Omega$.

When $\mathcal{S}$ is finite dimensional and the uspf Λ on $\mathcal{S}^{(2)}$ has an arbitrary representing measure, then one expects that this measure may be replaced by an atomic one. Such a property, going back to Tchakaloff, will be also discussed in the following. First of all, we state a version of Tchakaloff's Theorem, due to C. Bayer and J. Teichmann, to be later used.

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A Version of Tchakaloff's Theorem

Theorem TBT *Let $(\Omega, \mathcal{A})$ be a measurable space, let μ be a positive measure on $(\Omega, \mathcal{A})$ with support in $\Sigma \in \mathcal{A}$, and let $\phi : \Omega \mapsto \mathbb{R}^n$ be a Borel measurable map. Assume that*

$$\int_{\mathbb{R}^n} \|x\| d(\phi * \mu)(x) < \infty.$$

Then there exist a positive integer $k \leq n$, points $\omega_1, \ldots, \omega_k \in \Sigma$, and positive numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\int_\Omega (p \circ \phi)(\omega) d\mu(\omega) = \sum_{j=1}^{k} \lambda_j (p \circ \phi)(\omega_j), \; p \in \mathcal{P}_1^n,$$

where $\phi * \mu$ is the measure given by $(\phi * \mu)(A) = \mu(\phi^{-1}(A))$ for any Borel set $A \subset \mathbb{R}^n$.

Example Introducing Idempotents

A concept of idempotent with respect to a uspf plays an important role in what follows.

Example 1 Let $\Omega = \{\omega_1, \ldots, \omega_d\}$ be an arbitrary (finite) set and let $C(\Omega)$ be the (finite dimensional) C^*-algebra of all complex-valued functions defined on Ω, endowed with the sup-norm. Assume that $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple in $C(\Omega)$ generating this algebra. In particular, the set $\{\theta^\alpha; \alpha \in \mathbb{Z}_+^n\}$ must contain a subset $\{\theta^\alpha; \alpha \in \mathbb{Z}_+^n, |\alpha| \le m\}$, for some integer $m \ge 1$, which spans the vector space $C(\Omega)$.
Fixing an integer $m \ge 1$ as above, the map linear map $\mathcal{P}_m \ni p \mapsto p \circ \theta \in C(\Omega)$ is surjective.

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Consider the measure $\nu = \sum_{j=1}^{d} \lambda_j \delta_j$, with δ_j the Dirac measure at ω_j, $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^{d} \lambda_j = 1$.
Setting $\xi^{(j)} := \theta(\omega_j)$, $\mu(\{\xi^{(j)}\}) = \nu(\{\omega_j\}) = \lambda_j$, $j = 1, \ldots, d$, and $\Xi := \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$, we put

$$\Lambda(p) = \int_{\Xi} p d\mu = \int_{\Omega} p \circ \theta d\nu, \ \ p \in \mathcal{P}_{2m},$$

which is a uspf, for which μ is a representing measure.
Let now $f \in C(\Omega)$ be an idempotent, that is, the caracteristic function of a subset of Ω.Then there exists a polynomial $p \in \mathcal{RP}_m$, such that $p \circ \theta = f$. Consequently, $\Lambda(p^2) = \int_{\Omega} p^2 \circ \theta d\nu = \int_{\Omega} p \circ \theta d\nu = \Lambda(p)$. This shows that some of the solutions the equation $\Lambda(p^2) = \Lambda(p)$, play an important role when trying to reconstruct the representing measure μ.

Relative Idempotents

Let $\mathcal{S}$ be a finite dimensional function space on a Hausdorff space Ω. Fixing a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, $\mathcal{H}_\Lambda = \mathcal{S}/\mathcal{I}_\Lambda$ be the associated Hilbert space, with the inner product given by (3).

The symbol $\mathcal{RH}_\Lambda$ will designate the real Hilbert space subspace $\{\hat{p} \in \mathcal{H}_\Lambda; p \in \mathcal{RS}\}$. If $\hat{p} \in \mathcal{RH}_\Lambda$, we may always suppose that $p \in \mathcal{RS}$.

Definition 1 An element $\hat{p} \in \mathcal{RH}_\Lambda$ is said to be a Λ-**idempotent** (or simply an **idempotent** if Λ is fixed) if it is a solution of the equation

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Remark 1 Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is un idempotent if and only if $\Lambda(p^2) = \Lambda(p)$, via (3). Set

$$\mathcal{ID}(\Lambda) = \{\hat{p} \in \mathcal{RH}_\Lambda; \|\hat{p}\|^2 = \langle \hat{p}, \hat{1} \rangle \neq 0\}, \tag{5}$$

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Bases Consisting of Idempotents

Note first that two elements $\hat{p}, \hat{q} \in \mathcal{H}_\Lambda$ are orthogonal if and only if $\Lambda(p\bar{q}) = 0$.

The existence of orthogonal bases consisting of idempotents with respect to a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ follows from the following result.

Lemma 1 *Let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. If the set $\{\hat{v}_1, \dots, \hat{v}_d\} \subset \mathcal{RH}_\Lambda$ is an orthonormal basis of $\mathcal{H}_\Lambda$ with $\langle \hat{v}_j, \hat{1} \rangle \neq 0$, $j = 1, \dots, d$, the set $\{\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1, \dots \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. Moreover,*

$$\langle \hat{v}_1, \hat{1} \rangle \hat{v}_1 + \cdots + \langle \hat{v}_d, \hat{1} \rangle \hat{v}_d = \hat{1}.$$

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From the Lemma above, it follows readily the next result.

Theorem 1 *For every uspf* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, *the Hilbert space* $\mathcal{H}_\Lambda$ *has infinitely many orthogonal bases consisting of idempotent elements.*

Corollary 1 *Let* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ *be a uspf. Then there are functions* $b_1, \ldots, b_d \in \mathcal{RS}$ *such that* $\Lambda(b_j^2) = \Lambda(b_j) > 0$, $\Lambda(b_j b_k) = 0$ *for all* $j, k = 1, \ldots, d$, $j \neq k$, *and every* $f \in \mathcal{S}$ *can be uniquely represented as*

$$f = \sum_{j=1}^{d} \Lambda(b_j)^{-1} \Lambda(f b_j) b_j + f_0,$$

with $f_0 \in \mathcal{I}_\Lambda$ *and* $d = \mathrm{dim}\mathcal{H}_\Lambda$.

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Multiplicative Structures

The bases consisting of idempotents can be associated with multiplicative structures in the following way.

Let $\mathcal{S}$ be a finite dimensional function space on Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\} \subset \mathcal{ID}(\Lambda)$ be a collection of nonnull mutually orthogonal elements whose sum is $\hat{1}$ (in particular an orthogonal basis), and let $\mathcal{H}_\mathcal{B}$ be the complex vector space spanned by $\mathcal{B}$ in $\mathcal{H}_\Lambda$. Using $\mathcal{B}$, we may define a multiplication, an involution, and a norm on $\mathcal{H}_\mathcal{B}$, making it a unital, commutative, finite dimensional C^*-algebra.

Specifically, if $\hat{f} = \sum_{j=1}^{d} \alpha_j \hat{b}_j$, $\hat{g} = \sum_{j=1}^{d} \beta_j \hat{b}_j$, are elements from $\mathcal{H}_{\mathcal{B}}$, their product is given by $\hat{f} \cdot \hat{g} = \sum_{j=1}^{d} \alpha_j \beta_j \hat{b}_j$. The involution is defined by $\hat{f}^* = \sum_{j=1}^{d} \overline{\alpha_j} \hat{b}_j$, and the norm is given by $\|\hat{f}\|_\infty = \max_{1 \leq j \leq d} |\alpha_j|$.

Having in mind the construction from above, we may speak about the C^*-**algebra (structure of)** $\mathcal{H}_{\mathcal{B}}$ **induced by** $\mathcal{B}$. When $\mathcal{B}$ is actually a basis, we clearly have $\mathcal{H}_{\mathcal{B}} = \mathcal{H}_{\Lambda}$.

Space of Characters

One can see that the space of characters of the C^*-algebra $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$, say $\Delta = \{\delta_1, \ldots, \delta_d\}$, coincides with the dual basis of $\mathcal{B}$. As $\mathcal{H}_\mathcal{B}$ is also a Hilbert space as a subspace of $\mathcal{H}_\Lambda$, we note that

$$\delta_j(\hat{f}) = \Lambda(b_j)^{-1}\langle \hat{f}, \hat{b}_j\rangle,\ \hat{f} \in \mathcal{H}_\mathcal{B},\ j = 1, \ldots, d.$$

This also shows that C^*-algebra $\mathcal{H}_\mathcal{B}$ induced by $\mathcal{B}$ is semi-simple.

Note that if $\theta = (\theta_1, \ldots, \theta_n)$ is an n-tuple of elements of $\mathcal{S}$, then $\hat{\theta}^\alpha = \widehat{\theta_1}^{\alpha_1} \cdots \widehat{\theta_n}^{\alpha_n}$ is, in general, different from $\widehat{\theta^\alpha}$, when $\theta^\alpha \in \mathcal{S}$ for some multi-index $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}_+^n$, where, as usually, we put $\theta^\alpha := \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}$.

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Special Finitely Generated Function Spaces

Let $\mathcal{S}$ be a function space on a Hausdorff space Ω. We assume that there exist an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RS}$, and an integer $m \geq 1$, such that such that the family $\Theta_m := \{\theta^\alpha; |\alpha| \leq m\}$ spans the space $\mathcal{S}$. In other words, $\mathcal{S} = \{p \circ \theta; p \in \mathcal{P}_m^n\}$.

When such a pair (θ, m) exists, we shortly say that the function space $\mathcal{S}$ is m-**generated by** θ.

Obviously, in this case $\mathcal{S}$ is of finite dimension, and the family Θ_{2m} spans the space $\mathcal{S}^{(2)}$.

If $\mathcal{S}$ is a finite dimensional function space, it is at least 1-generated by a basis $\theta = (\theta_1, \ldots, \theta_n)$. Nevertheless, the case of an $\mathcal{S}$ m-generated by a tuple θ with $m > 1$ is also of some interest.

In particular, if $t_1, \ldots, t_n$ are the independent variables of $\mathbb{R}^n$, and if $\theta_j = t_j$, $j = 1, \ldots, n$, then $\mathcal{P}_m^n$, is a function space m-generated by $t := (t_1, \ldots, t_n)$.

An Integral Representation Formula

Theorem 2 *Let $\mathcal{S}$ be a finite dimensional function space on Ω, let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf, and let $\mathcal{B} \subset \mathcal{ID}(\Lambda)$ be an orthogonal basis inducing a C^*-algebra structure on $\mathcal{H}_\Lambda$. Let $\theta = (\theta_1, \ldots, \theta_n)$ a given n-tuple of $\mathcal{S}$ such that $\hat{1}, \hat{\theta}_1, \ldots, \hat{\theta}_n$ generate the C^*-algebra $\mathcal{H}_\Lambda$. Then there exist a finite subset Ξ of $\mathbb{R}^n$, whose cardinal equals $\dim \mathcal{H}_\Lambda$, and a surjective linear map $\mathcal{S} \ni u \mapsto u^\# \in C(\Xi)$, whose kernel is $\mathcal{I}_\Lambda$, with the property*

$$\Lambda(u) = \int_\Xi u^\#(\xi) d\mu(\xi), \ u \in \mathcal{S},$$

where μ is a probability measure on Ξ.
Moreover, the map $\mathcal{S} \ni u \mapsto u^\# \in C(\Xi)$ induces a $*$-isomorphism between C^*-algebras $\mathcal{H}_\Lambda$ and $C(\Xi)$.

Some Consequences

Remark 2 Let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is basis of $\mathcal{H}_\Lambda$ as in the statement. Some consequences:
1) There exists a nonnegative integer m such that the set $\hat{\Theta}_m := \{\hat{\theta}^\alpha; |\alpha| \le m\}$ generates the C^*-algebra $\mathcal{H}_\Lambda$.
2) We have a direct sum decomposition

$$\mathcal{S} = \{\sum_{j=1}^{d} p(\xi^{(j)})b_j + r;\, p \in \mathcal{P}_m,\, r \in \mathcal{I}_\Lambda\} = \mathcal{G} + \mathcal{I}_\Lambda,$$

with $\mathcal{G} = \{\sum_{j=1}^{d} p(\xi^{(j)})b_j;\, p \in \mathcal{P}_m\}$.
3) The family $\{p_1, \ldots, p_d\} \subset \mathcal{P}_m$ interpolates the set Ξ, where $p_j = \hat{b}_j^\#$ for all j.

An Example

Theorem 2 offers an integral representation even for some usps not having representing measures. Here is an example.

We consider the uspf $\Lambda : \mathcal{P}_4^1 \mapsto \mathbb{C}$, given by $\Lambda(t^k) = 1$, $k = 0, 1, 2, 3$, and $\Lambda(t^4) = 2$, extended to $\mathcal{P}_4^1$ by linearity. It is known (Curto & Fialkow) that this uspf has no representing measure. Nevertheless, via Theorem 2, the restriction $\Lambda|\mathcal{P}_2^1$ has a certain integral representations, for any fixed orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. We have

$$\mathcal{I}_\Lambda = \{p(t) = u - ut; u \in \mathbb{C}\},$$

and

$$\mathcal{H}_\Lambda = \{\hat{p}; p(t) = u + ut + vt^2, u, v \in \mathbb{C}\}.$$

In particular, $\hat{1} = \hat{t}$, and so $\mathcal{H}_\Lambda = \{u\hat{1} + v\widehat{t^2}, u, v \in \mathbb{C}\}$.

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In particular, $\hat{1} = \hat{t}$, and so $\mathcal{H}_\Lambda = \{u\hat{1} + v\widehat{t^2},\ u, v \in \mathbb{C}\}$.

We fix the elements $b_1 = t^2/2$ and $b_2 = 1/2 + t/2 - t^2/2$. We have $\dim \mathcal{H}_\Lambda = 2$, and $\{\hat{b}_1, \hat{b}_2\}$ is an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents, with $\hat{b}_1 = \widehat{t^2}/2$, $\hat{b}_2 = \hat{1} - \widehat{t^2}/2$.

Put $\theta_1 = t$, $\theta_2 = t^2$. The set $\{\hat{\theta}_1, \hat{\theta}_2\}$ generates the C^*-algebra $\mathcal{H}_\Lambda$. Using the dual basis $\Delta = \{\delta_1, \delta_2\}$, we infer that $\xi^{(1)} = (1, 2)$, $\xi^{(2)} = (1, 0)$, so $\Xi = \{(1, 2), (1, 0)\} \subset \mathbb{R}^2$.

If

$$p = u + wt + vt^2 \in \mathcal{P}_2^1$$

is arbitrary, then

$$\hat{p} = u\hat{1} + v\widehat{t^2} = (u + 2v)\hat{b}_1 + u\hat{b}_2 \in \mathcal{H}_\Lambda, \ u, v \in \mathbb{C},$$

and we have

$$\Lambda(p) = p^\#(\xi^{(1)})\Lambda(b_1) + p^\#(\xi^{(2)})\Lambda(b_2) = u + v,$$

where

$$p^\#(x) = ux_1 + vx_2, \ x = (x_1, x_2) \in \mathbb{R}^2.$$

In other words,

$$\Lambda(p) = \int_{\Xi} p^{\#}(\xi) d\nu(\xi), \; p \in \mathcal{P}_2^1,$$

where ν is the atomic measure with weights $\Lambda(b_1), \Lambda(b_2)$ at $\xi^{(1)}, \xi^{(2)}$, respectively. In addition, the map $\mathcal{H}_\Lambda \ni \hat{p} \mapsto p^{\#}|\Xi \in C(\Xi)$ is a $*$-isomprphism.

Finally, a similar procedure may be applied to any pair of idempotents $\{\hat{b}_1, \hat{b}_2\}$, which is an orthogonal basis of $\mathcal{H}_\Lambda$.

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Continuous Point Evaluations

Let $\mathcal{S}$ be a finite dimensional function space on a measurable space Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. For every point $\omega \in \Omega$, we denote by δ_ω the point evaluation at ω, that is, $\delta_\omega(f) = f(\omega)$, for every function $f \in \mathcal{S}$.

Definition 2 The point evaluation δ_ω is said to be Λ-**continuous** if there exists a constant $c_\omega > 0$ such that

$$|\delta_\omega(f)| \leq c_\omega \Lambda(|f|^2)^{1/2}, \; f \in \mathcal{S}.$$

Let $\mathcal{Z}_\Lambda$ be the subset of those points $\omega \in \Omega$ such that δ_ω is Λ-continuous.

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For every function f let us denote by $\mathcal{Z}(f)$ the set of its zeros.

Lemma 2 *We have the equality*

$$\mathcal{Z}_\Lambda = \cap_{f \in \mathcal{I}_\Lambda} \mathcal{Z}(f).$$

Note that $\mathcal{I}_\Lambda = \{0\}$ implies $\mathcal{Z}_\Lambda = \Omega$.

Remark 3 Lemma 2 shows that the set $\mathcal{Z}_\Lambda$ extends the concept of algebraic variety of the moment sequence associated to Λ (Curto & Fialkow).

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Contractive USPF's

Let $\mathcal{S}$ be a finite dimensional function space on a measurable space Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. We say that Λ is **contractive** if there exists a finite set $F \subset \mathcal{Z}_\Lambda$ such that $|\Lambda(f)| \le \|f\|_F,\ f \in \mathcal{S}^{(2)}$, where $\|f\|_F = \max_{\omega \in F} |f(\omega)|$.

Proposition 1 *The uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has an atomic representing measure if and only if it is contractive.*

A Uniqueness Criterion

Assume that the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is contractive, so we have $|\Lambda(f)| \le \|f\|_F$, $f \in \mathcal{S}^{(2)}$ for a certain finite subset F in Ω. In particular, Λ has representing measure, that is $\Lambda(f) = \sum_{j=1}^{d} \lambda_j f(\omega_j),\ \ f \in \mathcal{S}^{(2)}$, by Proposition 1. Let us define the quantities

$$\sigma(f) := \sup_{g \in \mathcal{R}\mathcal{S}^{(2)}} [-\Lambda(g) - \|f+g\|_F],$$

$$\tau(f) = \inf_{g \in \mathcal{R}\mathcal{S}^{(2)}} [\|f+g\|_F - \Lambda(g)],$$

where $f \in \mathcal{R}C(F)$ is arbitrary. It follows from the standard proof of the Hahn-Banach Theorem that the equality $\sigma(f) = \tau(f)$ for all $f \in \mathcal{R}C(F)$ implies the uniqueness of the extension, that is, the uniqueness of the representing measure.

An Interpolation Approach

The existence of a representing measure can be also characterized in terms of an interpolation property.

Proposition 2 *Let* $\mathcal{S}$ *be a finite dimensional function space on* Ω. *A uspf* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ *has a representing measure in* Ω *with* $d := \dim \mathcal{H}_\Lambda$ *atoms if and only if there exist an orthogonal basis of* $\mathcal{H}_\Lambda$ *consisting of idempotents* $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$, *and a set* $\Omega_\Lambda = \{\omega_1, \ldots, \omega_d\} \subset \mathcal{Z}_\Lambda$ *such that* $b_j(\omega_j) = 1$ *and* $b_k(\omega_j) = 0$ *for all* $j, k = 1, \ldots, d$, $j \neq k$.

Multiplicativity with Respect to a USPF

Definition 3 Let $\mathcal{S}$ be a function space m-generaterd by the n-tuple θ, let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* (*with respect to* θ) if

$$\Lambda(\theta^\alpha b_j)\Lambda(\theta^\beta b_j) = \Lambda(b_j)\Lambda(\theta^{\alpha+\beta} b_j) \tag{6}$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Lemma 3 *Let $\mathcal{S}$ be a function space m-generaterd by the n-tuple θ, let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents. The basis $\mathcal{B}$ is Λ-multiplicative with respect to θ if and only if $\delta(\widehat{\theta^\alpha}) = \delta(\hat{\theta}^\alpha)$ whenever $|\alpha| \leq m$ and δ is any character of the C^*-algebra $\mathcal{H}_\Lambda$ induced by $\mathcal{B}$.*

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Main Result

Theorem 3 *Let $\mathcal{S}$ be a function space on Ω which is m-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$. A uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has a representing measure in Ω with $d = \dim \mathcal{H}_\Lambda$ atoms if and only if there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of idempotent elements which is Λ-multiplicative, and $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$.*

Corollary 2 *Let $\mathcal{S}$ be a function space on Ω which is m-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Let also $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$, consisting of idempotent elements, which induces on $\mathcal{H}_\Lambda$ a C^*-algebra structure. Assume that the basis $\mathcal{B}$ is Λ-multiplicative with respect to θ. Then there exists a finite set $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ and a probability measure μ on Ξ such that*

$$\Lambda(h(\theta)) = \int_\Xi h(\xi) d\mu(\xi), \;\; h \in \mathcal{P}^n_{2m}.$$

Corollary 3 *The uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ *has a representing measure in* $\mathbb{R}^n$ *possessing* $d := \dim \mathcal{H}_\Lambda$ *atoms if and only if there exists a* Λ*-multiplicative basis of* $\mathcal{H}_\Lambda$*.*

Corollary 4 *The uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ *has a representing measure in* $\mathbb{R}^n$ *possessing* $d := \dim \mathcal{H}_\Lambda$ *atoms if and only if there exists a family of polynomials* $\{b_1, \ldots, b_d\} \subset \mathcal{RP}_m$ *with the following properties:*
$(i)\ \Lambda(b_j^2) = \Lambda(b_j) > 0,\ j = 1, \ldots, d;$
$(ii)\ \Lambda(b_j b_k) = 0,\ j, k = 1, \ldots, d,\ j \neq k;$
(iii)

$$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$$

whenever $0 \neq |\alpha| \leq |\beta|,\ |\alpha| + |\beta| \leq m,\ j = 1, \ldots, d.$

Automatic Λ-Multiplicativity

The previous Theorem is particularly interesting when applied to a function space which is 1-generated by a given tuple. This happens because, in this case, condition (6) is automatically fulfilled.

Corollary 5 *Let $\mathcal{S}$ be a function space on Ω, which is 1-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let also $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. If either*
(1) *there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of idempotent elements such that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, or*
(2) $\theta(\Omega) = \mathbb{R}^n$,
the uspf Λ has a representing measure in Ω with $d = \dim \mathcal{H}_\Lambda$ atoms.

An Example

Corollary 5 provides an atomic representing measure for a large class of 1-generated function spaces. Here is an example. Let $r > n$, r, n be positive integers, let $\Omega' \subset \mathbb{R}^{r-n}$ be a Borel set, and let $\Omega = \mathbb{R}^n \times \Omega'$. We consider on Ω the functions $\theta_j(t, t') = t_j + \psi_j(t')$, $j = 1, \ldots, n$, where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $t' = (t'_1, \ldots, t'_{r-n}) \in \Omega'$, and $\psi_1, \ldots \psi_n$ are Borel functions. Let $\mathcal{S}$ be the function space on Ω spanned by $\theta := (\theta_1, \ldots, \theta_n)$ and $\theta_0 = 1$. Then every uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ has an atomic representing measure in Ω. This follows from Corollary 5, because we clearly have $\theta(\Omega) = \mathbb{R}^n$.

Explicit Formulas

Remark 4 We can give explicit formulas related to Corollary 5. Let $\mathcal{S}$ be a function space on Ω, which is 1-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let also $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. We fix a basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents, which is automatically Λ-multiplicative with respect to θ, where $d = \dim(\mathcal{H}_\Lambda)$. Let also $\Delta = \{\delta_1, \ldots, \delta_d\}$ be the dual basis. We set $\xi^{(j)} = \delta_j(\hat{\theta}) \in \mathbb{R}^n$, $j = 1, \ldots, d$, and $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$. Then we have

$$\Lambda(p(\theta)) = \int_\Xi p(\xi) d\mu(\xi),\ p \in \mathcal{P}_2^n,$$

where μ is probability measure with weights $\lambda_j := \Lambda(b_j)$ at $\xi^{(j)}$, $j = 1, \ldots, d$.

If $\delta_j(\hat{\theta}) = \theta(\omega_j)$, $j = 1, \ldots, d$, we actually have

$$\Lambda(h) = \sum_{j=1}^{d} \lambda_j h(\omega_j) \text{ for every } h \in \mathcal{S}^{(2)}.$$

In fact,

$$\theta(\omega_j) = (\Lambda(b_j)^{-1}\Lambda(\theta_1 b_j), \ldots, \Lambda(b_j)^{-1}\Lambda(\theta_n b_j) \in \mathbb{R}^n, \ j = 1, \ldots, d.$$

Finally, when $\theta(\Omega) = \mathbb{R}^n$, the existence of the points ω_j, $j = 1, \ldots, d$ is insured for any basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ consisting of orthogonal idempotents.

The next result is an application of Theorem TBT, stated at the beginning of this presentation.

Theorem 4 *Let $\mathcal{S}$ be a function space on Ω, m-generaterd by the n-tuple $\theta = (\theta_1, \ldots, \theta_n)$, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. If Λ has a representing measure then it has an atomic representing measure.*

Remark Let $\mathcal{S}$ be a finite dimensional function space on Ω, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf having a representing measure which may be supposed to be atomic, via the previous Theorem. Let also $\mathcal{Q} := \{q_1, \ldots, q_s\} \subset \mathcal{RS}$, and let

$$\Omega_{\mathcal{Q}} = \{\omega \in \Omega; q_j(\omega) \geq 0, j = 1, \ldots, s\}.$$

We may choose an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$of $\mathcal{H}_\Lambda$ consisting of idempotents, naturally associated with this basis. The measure μ has support in $\Omega_{\mathcal{Q}}$ if and only if

$$\Lambda(q_j b_k) \geq 0 \text{ for all } j = 1, \ldots, s;\ k = 1, \ldots, d.$$

This remark may be applied, in particular, to spaces of functions consistng of polynomials, restricted to semi-algebraic sets.

Conclusion

The moment problem for a unital square positive functional defined on a finite dimensional function space, in particular the truncated moment problem, is an ill posed one because it may have no solution, one solution or infinitely many solutions. Abstract solutions can be easily described while explicit solutions to such a problem are still an open question. The main difficulty comes from the fact that the given data are related to Hilbert space or matrix theory methods while the measure theory is related rather to spaces of functions with sup-norm. To compute the norm of a given unital square positive functional is, in general, a difficult problem, and no explicit method is available.

Thank you very much for your attention !