A Stability Equation for Truncated Moment Problems

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Outline

An Abstract "Stability" Equation

Let $N \geq 1$ be an arbitrary integer, let $A = (a_{jk})_{j,k=1}^N$ be a matrix with real entries, that is positive on $\mathbb{C}^N$ (endowed with the standard scalar product denoted by $(*|*)$), let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, and let $c \in \mathbb{R}$. We look for necessary and sufficient conditions insuring the existence of a solution $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ of the equation

$$(ASE) \qquad (Ax|x) - 2(b|x) + c = 0.$$

This is a quadric equation whose solution is given in the following. The range and the kernel of A, regarded as an operator on $\mathbb{C}^N$, will be denoted by $R(A)$, $N(A)$, respectively.

Solution to ASE

PROPOSITION

We have the following alternative:
1) If $b \notin R(A)$, equation (ASE) always has solutions.
2) If $b \in R(A)$, equation (ASE) has solutions if and only if for some (and therefore for all) $d \in A^{-1}(\{b\})$ we have $c \leq (d|b)$.
In particular, if $N(A) = \{0\}$, then A is invertible and equation (ASE) has solutions if and only if $c \leq (A^{-1}b|b)$.

The previous quadratic equation has a geometric interpretation related to a certain dimensional stability, to be later discussed. In particular, it can be used to characterize locally the "flatness" of the truncated moment problems, both in commutative and non-commutative cases.

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$*$-Subspaces in Involutive Algebras

Let $\mathcal{A}$ be a complex algebra with unit 1 and involution $a \mapsto a^*$, and let $\mathcal{S} \subset \mathcal{A}$ be a vector subspace containing the unit and invariant under involution. For convenience, let us say that $\mathcal{S}$, having these properties, is a $*$-*subspace* of $\mathcal{A}$. Let also $\mathcal{S}^{(1)}$ be the vector subspace spanned by all products of the form ab with $a, b \in \mathcal{S}$, which is itself a $*$-subspace. We have $\mathcal{S} \subset \mathcal{S}^{(1)}$, and $\mathcal{S} = \mathcal{S}^{(1)}$ when $\mathcal{S}$ is a subalgebra.

Square Positive Functionals

Let $\mathcal{S}$ be a $*$-subspace of $\mathcal{A}$, and let $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ be a linear map with the following properties:
(1) $\Lambda(a^*) = \overline{\Lambda(a)}$ for all $a \in \mathcal{S}^{(1)}$;
(2) $\Lambda(a^* a) \geq 0$ for all $a \in \mathcal{S}$.
(3) $\Lambda(1) = 1$.

A linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional* (briefly a *uspf*).

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Elementary Properties

If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(a^*b)|^2 \leq \Lambda(a^*a)\Lambda(b^*b),\ a, b \in \mathcal{S}.$$

Putting $\mathcal{I}_\Lambda = \{a \in \mathcal{S}; \Lambda(a^*a) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni a \mapsto \Lambda(a^*a)^{1/2} \in \mathbb{R}_+$ is a seminorm.

In fact, $\mathcal{I}_\Lambda = \{a \in \mathcal{S}; \Lambda(ba) = 0\, \forall b \in \mathcal{S}\}$. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle a + \mathcal{I}_\Lambda, b + \mathcal{I}_\Lambda \rangle = \Lambda(b^*a).$$

If $\mathcal{S}$ is finite dimensional, then $\mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

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Let $\mathcal{T} \subset \mathcal{S}$ be a $*$-subspace. If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, then $\Lambda|\mathcal{T}^{(1)}$ is also a uspf, and setting $\mathcal{I}_{\Lambda,\mathcal{T}} = \{a \in \mathcal{T}; \Lambda(a^*a) = 0\} = \mathcal{I}_\Lambda \cap \mathcal{T}$, there is a natural map

$$J_{\mathcal{T},\mathcal{S}} : \mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}} \mapsto \mathcal{S}/\mathcal{I}_\Lambda, \ J_{\mathcal{T},\mathcal{S}}(a + \mathcal{I}_{\Lambda,\mathcal{T}}) = a + \mathcal{I}_\Lambda, \ a \in \mathcal{T}.$$

The equality

$$\langle a + \mathcal{I}_{\Lambda,\mathcal{T}}, a + \mathcal{I}_{\Lambda,\mathcal{T}} \rangle = \Lambda(a^*a) = \langle a + \mathcal{I}_\Lambda, a + \mathcal{I}_\Lambda \rangle$$

shows that the map $J_{\mathcal{T},\mathcal{S}}$ is an isometry, in particular it is injective.

Dimensional Stability

We say that the uspf $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ it *stable* at $\mathcal{T}$, where $\mathcal{T} \subset \mathcal{S}$ is a $*$-subspace, if we have the equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}) = \mathcal{S}/\mathcal{I}_\Lambda$

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Polynomial Type Algebras

Let $\mathcal{A}$ be a complex involutive algebra, with unit. The algebra $\mathcal{A}$ is said to be a *polynomial type algebra* if there exists an algebraic basis $\mathcal{B} = \cup_{m=0}^{\infty}\mathcal{D}_m$ of $\mathcal{A}$ such that $\mathcal{D}_0 = \{1\}$, $\mathcal{D}_m$ is finite and invariant under involution for all integers $m \geq 0$, $\mathcal{D}_{m_1} \cap \mathcal{D}_{m_2} = \emptyset$ for all integers $m_1, m_2 \geq 0, m_1 \neq m_2$, and $\mathcal{D}_{m_1} \cdot \mathcal{D}_{m_2} = \mathcal{D}_{m_1+m_2}$ for all integers $m_1, m_2 \geq 0$.
A polynomial type algebra as above will be briefly denoted by $(\mathcal{A}, (\mathcal{D}_m)_{m\geq 0})$.

Note also that if $(\mathcal{A}', (\mathcal{D}'_m)_{m\geq 0}), (\mathcal{A}'', (\mathcal{D}''_m)_{m\geq 0})$ are polynomial type algebras, then $(\mathcal{A}' \oplus \mathcal{A}'', (\mathcal{D}_m)_{m\geq 0})$ (resp. $(\mathcal{A}' \otimes \mathcal{A}'', (\mathcal{D}_m)_{m\geq 0})$) is a polynomial type algebra, where $\mathcal{D}_m = \mathcal{D}'_m \oplus \mathcal{D}''_m$ (resp. $\mathcal{D}_m = \cup_{m'+m''=m}\mathcal{D}'_{m'} \otimes \mathcal{D}''_{m''}$).

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Using the previous notation, let $\mathcal{B}_m = \cup_{k=0}^m \mathcal{D}_k$, so $\mathcal{B} = \cup_{m\geq 0}\mathcal{B}_m$, and let $\mathcal{S}_m$ be the vector space spanned by $\mathcal{B}_m$. Then the collection $(\mathcal{S}_m)_{m\geq 0}$ is an increasing family of finite dimensional $*$-subspaces of $\mathcal{A}$ such that $\mathcal{S}_0 = \mathbb{C}\cdot 1$, $\mathcal{S}_{m_1}\cdot\mathcal{S}_{m_2} \subset \mathcal{S}_{m_1+m_2}$ for all integers $m_1, m_2 \geq 0$, and $\cup_{m=0}^\infty \mathcal{S}_m = \mathcal{A}$. Moreover, we have the equality $\mathcal{S}_m^{(1)} = \mathcal{S}_{2m}$ for all integers $m \geq 1$.

The *degree* of an arbitrary element $a \in \mathcal{A}$, which is not a multiple of 1, is the least integer $m \geq 1$ such that $a \in \mathcal{S}_m \setminus \mathcal{S}_{m-1}$. The degree of a multiple of 1 is equal to 0. The degree of $a \in \mathcal{A}$ is denoted by $\deg(a)$. With this notation, we have $\mathcal{S}_m = \{a \in \mathcal{A};\ \deg(a) \leq m\}$. Note also that $\deg(a)=\deg(a^*)$ and that, for a $b \in \mathcal{B}$, we have $\deg(b) = m$ iff $b \in \mathcal{D}_m$.

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EXAMPLE The algebra $\mathcal{P}$ of all polynomials in n real variables, with complex coefficients (endowed with the natural involution $p \mapsto \bar{p}$) is, of course, a polynomial type algebra.
The subset $\mathcal{B} = \{t^\alpha; \alpha \in \mathbb{Z}_+^n\} = \cup_{m\geq 0}\mathcal{D}_m$ is an algebraic basis for the algebera $\mathcal{P}$, where $\mathcal{D}_m = \{t^\alpha; |\alpha| = m\}$, and $m \geq 0$ is an integer. The span of $\mathcal{B}_m = \cup_{k=0}^m \mathcal{D}_k$ is denoted by $\mathcal{P}_m$, which clearly consists of all polynomials of total degree $\leq m$.

EXAMPLE Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be a finite family of indeterminates, and let $\mathcal{F}[\mathbf{X}]$ be the complex unital algebra freely generated by $\mathbf{X}$, whose unit is designated by $\mathbf{1}$. Let $\mathcal{W}$ be the monoid generated by $\mathbf{X} \cup \{\mathbf{1}\}$. The *lenght* of an element $W \in \mathcal{W} \setminus \{\mathbf{1}\}$ is equal to the number of elements of $\mathbf{X}$ which occur in the representation of W. The length of $\mathbf{1}$ is equal to zero.

If $\mathcal{V}_m$ is the subset of those elements from $\mathcal{W}$ of lenght m, with $m \geq 0$ an arbitrary integer, and $\mathcal{W}_m = \cup_{k=0}^m \mathcal{V}_k$, then $\mathcal{W} = \cup_{m \geq 0} \mathcal{W}_m$ is an algebraic basis of $\mathcal{F}[\mathbf{X}]$. Setting $(cW)^* = \bar{c} W^*$ for all complex numbers c, where $W^* = X_{j_m} X_{j_{m-1}} \cdots X_{j_1}$ for every $W = X_{j_1} \cdots X_{j_{m-1}} X_{j_m} \in \mathcal{W} \setminus \{\mathbf{1}\}$, and $\mathbf{1}^* = \mathbf{1}$, we define an involution $P \mapsto P^*$ on $\mathcal{F}[\mathbf{X}]$, extending this assignment by additivity. In this way, the algebra $(\mathcal{F}[\mathbf{X}], (\mathcal{V}_m)_{m \geq 0})$ becomes a (noncommutative) polynomial type algebra.

Let $\mathcal{F}_m$ be the subspace spanned in $\mathcal{F}[\mathbf{X}]$ by the set $\mathcal{W}_m$, for every integer $m \geq 0$. As in the case of ordinary polynomials, if $\gamma = (\gamma_W)_{W \in \mathcal{W}_{2m}}$ is a family of complex numbers, we may define a linear map $\Lambda_\gamma : \mathcal{F}_{2m} \mapsto \mathbb{C}$, extending the assignment $W \mapsto \gamma_W$ by linearity. Moreover, assuming that $\gamma_{\mathbf{1}} = 1, \gamma_{W^*} = \overline{\gamma_W}$ for all $W \in \mathcal{W}_{2m}$, and

$$\sum_{j,k=1}^{d_m} \bar{c}_j c_k \gamma_{W_j^* W_k}$$

for all complex numbers $\{c_1, \ldots, c_{d_m}\}$, where d_m is the cardinal of $\mathcal{W}_m = \{W_1 = \mathbf{1}, W_2, \ldots, W_{d_m}\}$, the map Λ_γ becomes a uspf.

Truncated moment problems related to a uspf $\Lambda : \mathcal{F}_{2m} \mapsto \mathbb{C}$ (that is, looking for special representations of such a map), when Λ is a tracial map (i.e. Λ is null on commutators) have been recently studied by S. Burgdorf and I. Klep.

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Burgdorf and Klep considered sequences of real numbers $\gamma = (\gamma_W)_{W \in \mathcal{W}_{2m}}$, satisfying some natural conditions, and looked for their representations of the form

$$\gamma_W = \sum_{j=1}^{N} \lambda_j \mathrm{Tr}(W(\mathbf{A}^{(j)}),$$

where $\lambda_1, \ldots, \lambda_N$ are positive numbers with $\sum_{j=1}^{N} \lambda_j = 1, \mathbf{A}^{(j)} = (A_1^{(j)}, \ldots, A_n^{(j)})$ are symmetric matrices, and "Tr" is the normalized trace. They solved the problem when the associated uspf was stable at the last step, mainly working in the equivalent framework of flatness, as Curto and Fialkow did.

The Stability Equation in Polynomial Type Algebras

Let $\mathcal{A}$ be a polynomial type algebra with the basis $\mathcal{B} = \cup_{m=0}^{\infty} \mathcal{B}_m$. For each $a \in \mathcal{A}$ there exists an integer $m \geq 0$ such that $a \in \mathcal{S}_m$. Since $\mathcal{B}_m = \{b_1 = \mathbf{1}, b_2, \ldots, b_{d_m}\}$ is an algebraic basis of $\mathcal{S}_m$, we can write $a = \sum_{k=1}^{d_m} \alpha_k b_k$, with α_k complex numbers. Setting $\alpha_k = 0$ if $k > d_m$, we can write $a = \sum_{k \geq 1} \alpha_k b_k$, and this representation is unique.

On the algebra $\mathcal{A}$, we may define a scalar product given by $(a_1|a_2) = \sum_{k\geq 1} \alpha_{1k}\overline{\alpha_{2k}}$, where $a_j = \sum_{k\geq 0} \alpha_{jk} b_k$, $j = 1, 2$. With respect to this scalar product, the basis $\mathcal{B}$ is an orthonormal set. In particular, if $m \geq 0$ is any integer, the finite dimensional space $\mathcal{S}_m$ has a Hilbert space structure induced by the scalar product from above, such that the family of elements from $\mathcal{B}_m$ is an orthonormal basis of $\mathcal{S}_m$.

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Let $\mathcal{A}$ be a polynomial type algebra with the basis $\mathcal{B} = \cup_{m=0}^{\infty}\mathcal{B}_m$.
We say that the basis $\mathcal{B}$ is *real* if for every $b \in \mathcal{B}$ we have $b^* = b$.
Replacing the basis $\mathcal{B}$ by the family

$$\{b \in \mathcal{B}; b = b^*\} \cup \{\frac{b + b^*}{2}; b \in \mathcal{B}, b \neq b^*\} \cup \{\frac{b - b^*}{2\mathrm{i}}; b \in \mathcal{B}, b \neq b^*\},$$

which is again a basis, we may assume, with no loss of generality, that a given polynomial type algebra has a real basis.

Let $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf. Adapting a terminology used by Burgdorf and Klep, we say that Λ is *tracial* if $\Lambda(a_1 a_2) = \Lambda(a_2 a_1)$ for all $a_1, a_2 \in \mathcal{S}_{2m}$ with $a_1 a_2 \in \mathcal{S}_{2m}$.

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Let $\mathcal{A}$ be a polynomial type algebra with a real basis $\mathcal{B} = \cup_{m=0}^{\infty}\mathcal{B}_m$. Lat also $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a tracial uspf. We can consider the matrix $A_{m-1} = (\Lambda(b_k b_j))_{1\leq j,k\leq d_{m-1}}$, which is positive matrix, with real entries, acting as an operator on $\mathbb{C}^N$, where $N = d_{m-1}$. By identifying the space $\mathcal{S}_{m-1}$ with $\mathbb{C}^N$, A_{m-1} is the operator with the property $(A_{m-1}f|g) = \Lambda(g^*f)$ for all $f, g \in \mathcal{S}_{m-1}$.
For each index ℓ with $d_{m-1} < \ell \leq d_m$, we put $h_\ell = (\Lambda(b_\ell b_k)_{1\leq k\leq d_{m-1}} \in \mathbb{R}^N$ and $c_\ell = \Lambda(b_\ell^2)$. With this notation, the equation (ASE) becomes

$$(A_{m-1}x|x) - 2(h_\ell|x) + c_\ell = 0,$$

which is called the *stability equation* of the uspf Λ.

For $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ a uspf, if $1 \leq k \leq m$, as in the Introduction, we put $\mathcal{I}_k = \mathcal{I}_{\Lambda,\mathcal{S}_k} = \{p \in \mathcal{S}_k; \Lambda(p^*p) = 0\}$, and $\mathcal{H}_k = \mathcal{S}_k/\mathcal{I}_k$, which are finite dimensional Hilbert spaces. The stability of Λ at $m-1$ (i.e. dim$\mathcal{H}_{m-1}$=dim$\mathcal{H}_m$) is given by the following (using the previous notation).

THEOREM Let $\mathcal{A}$ be a polynomial type algebra with a real basis, and let $\Lambda : \mathcal{S}_{2m} \mapsto \mathbb{C}$ ($m \geq 1$) be a tracial uspf. The uspf Λ is stable at $m-1$ if and only if, whenever $h_\ell \in R(A_{m-1})$, we have $c_\ell \leq (f_\ell|h_\ell)$ for some (and therefore for all) $f_\ell \in A_{m-1}^{-1}(\{h_\ell\})$, where $d_{m-1} < \ell \leq d_m$.

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Classical Truncated Moment Problems

To solve a (classical) truncated moment problem means to characterize those finite multi-sequences of real numbers $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ with $\gamma_0 = 1$ (where α's are multi-indices of a given length $n \geq 1$ and $m \geq 0$ is an integer) for which there exists a probability measure μ on $\mathbb{R}^n$ (called a *representing measure for* γ) such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \leq 2m$.

According to a result firstly proved by Tchakaloff and generalized and improved by several authors, the measure μ may always be supposed to be atomic.

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Approaches to These Problems

- A first approach is to associate the sequence γ with the Hankel matrix $M_\gamma = (\gamma_{\alpha+\beta})_{|\alpha|,|\beta|\le m}$, which is supposed to be nonnegative when acting on a corresponding Euclidean space, and using *flat extensions* (Curto and Fialkow or Burgdorf and Klep, in a non-commutative context).
- A second approach is to use the Riesz functional, induced by the assignment $t^\alpha \mapsto \gamma_\alpha$ on the space of polynomials of total degree less or equal to $2m$, supposed to be nonnegative on the cone of sums of squares of real-valued polynomials. Riesz functionals have been used to study truncated moment problems, as well as for other purposes, by Fialkow and Nie, Laurent and Mourrain, Möller, Putinar etc. We describe such an approach in the following.

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Function Spaces

Let $n \geq 1$ be a fixed integer. Let $\mathcal{A}$ be a unital algebra consisting of complex-valued Borel functions, defined on $\mathbb{R}^n$, invariant under complex conjugation (other joint domains of definition may be considered). Let also $\mathcal{S} \subset \mathcal{A}$ be a vector subspace such that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space*. Note that a function space is a $*$-subspece of $\mathcal{A}$, where the involution is the complex conjugation
As before, $\mathcal{S}^{(1)}$ is the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. In particular a function space $\mathcal{S}$ we may define uspf's $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$, which linear map with the properties: (1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(1)}$; (2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$; (3) $\Lambda(1) = 1$.

Notation and Comments

Let $n \geq 1$ be a fixed integer. We freely use multi-indices from $\mathbb{Z}_+^n$, and the standard notation related to them.
The symbol $\mathcal{P}$ designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients (because of the systematic use of some associated complex Hilbert spaces).
For every integer $m \geq 1$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where deg(p) is the total degree of p. Note that $\mathcal{P}_m^{(1)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(1)} = \mathcal{P}$, the latter being an algebra.
We present in the following an extension theorem within the class of unital square positive functionals on finite dimensional function subspaces $\mathcal{P}_{2m}$ of the space $\mathcal{P}$, and exhibit some of its consequences.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $0 \le k \le m$. As in the abstract case, we put $\mathcal{I}_k = \mathcal{I}_{\Lambda,\mathcal{P}_k} = \{p \in \mathcal{P}_k; \Lambda(|p|^2) = 0\}$, and $\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k$, which is a finite dimensional Hilbert space, with the scalar product given by

$$\langle p + \mathcal{I}_k, q + \mathcal{I}_k \rangle = \Lambda(p\bar{q}),\ p, q \in \mathcal{P}_k.$$

Now, if $l \le m$ is another integer with $k \le l$, since $\mathcal{I}_k \subset \mathcal{I}_l$, we have a natural map $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$ given by $J_{k,l}(p + \mathcal{I}_k) = p + \mathcal{I}_l,\ p \in \mathcal{P}_{n,k}$, which is an isometry. In particular, $J_{k,k}$ is the identity on $\mathcal{H}_k$.
Similar constructions can be performed for a uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$

Equalities of the form $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ $(k < l)$ play an important role in this work. In this case, $J_{k,l}$ is a unitary transformation. When $l = k + 1$, we usually write J_k instead of $J_{k,k+1}$.

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LEMMA

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf. If Λ is stable at $m-1$, then $(\sum_{j=1}^{m} t_j \mathcal{I}_m) \cap \mathcal{P}_m \subset \mathcal{I}_m$. In particular, $t_j \mathcal{I}_{m-1} \subset \mathcal{I}_m$ for all $j = 1, \ldots, m$.

REMARK Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, stable at $m-1$. The previous Lemma allows us to define correctly the map $M_j : \mathcal{H}_{m-1} \mapsto \mathcal{H}_m$ by the equality $M_j(p + \mathcal{I}_{m-1}) = t_j p + \mathcal{I}_m$ for all $j = 1, \ldots, m$. Setting $J = J_{m-1}$, we may consider on the Hilbert space $\mathcal{H}_m$ the linear operators $A_j = M_j J^{-1}$ for all $j = 1, \ldots, m$. With this notation, we have the following.

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PROPOSITION

The linear maps A_j, $j = 1, \ldots, m$, are self-adjoint operators, and $A = (A_1, \ldots, A_n)$ is a commuting n-tuple on $\mathcal{H}_m$.

The next assertion is a version of a theorem by Curto and Fialkow. We obtain it with different techniques, and in a quicker way.

THEOREM Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ $(m \geq 1)$ be a uspf, stable at $m - 1$. Then there exists a unique extension $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ of Λ, which is a uspf.

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Dimensional Stability

DEFINITION

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, let $(\mathcal{H}_l)_{0 \leq l \leq m}$ be the Hilbert spaces bilt via Λ, and let $J_l : \mathcal{H}_l \mapsto \mathcal{H}_{l+1}\,(0 \leq l \leq m-1)$ be the associated isometries. If for some $k \in \{0, \ldots, m-1\}$ one has $J_k(\mathcal{H}_k) = \mathcal{H}_{k+1}$, we say that Λ is *dimensionally stable* (or simply *stable*) *at* k.

The uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ is said to be *dimensionally stable* if there exist integers m, k, with $m > k \geq 0$, such that $\Lambda_\infty | \mathcal{P}_{2m}$ is stable at k.

The number $\mathrm{sd}(\Lambda_\infty) = \dim \mathcal{H}_k$, unambiguously defined, will be called the *stable dimension* of Λ_∞.

Flatness and Dimensional Stability

REMARK Let $m \geq 1$ be an integer, let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $\{\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k,\ 0 \leq k \leq m\}$ be the Hilbert spaces built via Λ. The sesquilinear form $(p, q) \mapsto \Lambda(p\bar{q})$ insures the existence of a positive operator A_k on $\mathcal{P}_k$ such that $(A_k p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_k$, where $0 \leq k \leq m$. Note that $p \in \mathcal{I}_k$ if and only if $A_k p = 0$. This implies that $\dim \mathcal{H}_k$ equals the rank of A_k. The concept of *flatness* for the finite multi-sequence associated to Λ, introduced by Curto and Fialkow, means precisely that the rank of A_{m-1} is equal to the rank of A_m, and it is equivalent to the fact that Λ is stable at $m - 1$.

Using the previous results, as well as the Cauchy-Schwarz inequality, several results by Curto and Fialkow can be recaptured, usually with shorter proofs.

THEOREM

Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be a uspf.
If Λ_∞ is dimensionally stable, then Λ_∞ has a unique representing measure, which is d-atomic, where $d = \mathrm{sd}(\Lambda_\infty)$.
Conversely, if Λ_∞ has a d-atomic representing measure, then Λ_∞ is dimensionally stable and $d = \mathrm{sd}(\Lambda_\infty)$.

COROLLARY

The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ has a uniquely determined d-atomic representing measure, where $d = \dim \mathcal{H}_m$, if and only if Λ is stable at $m - 1$.

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EXAMPLE Assume that $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ has a *d*-atomic representing measure. If $d = 1$, then there exists a point $\xi \in \mathbb{R}^n$ such that $\Lambda_\infty(p) = p(\xi)$ for all $p \in \mathcal{P}$. Then, for all $k \geq 1$, $\mathcal{I}_k = \{p \in \mathcal{P}_k ; p(\xi) = 0\}$, the space $\mathcal{H}_k$ is isomorphic to $\mathbb{C}$, and so Λ_∞ is dimensionally stable with $\mathrm{sd}(\Lambda_\infty) = 1$.

Assume now that $d \geq 2$. Let $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\} \subset \mathbb{R}^n$ be distinct points and let μ be an atomic probability measure concentrated on Ξ, such that $\Lambda_\infty(p) = \int p d\mu$ for all $p \in \mathcal{P}$. Consider the polynomials

$$\chi_k(t) = \frac{\prod_{j \neq k} \|t - \xi^{(j)}\|^2}{\prod_{j \neq k} \|\xi^{(k)} - \xi^{(j)}\|^2}, \; t \in \mathbb{R}^n, \; k = 1, \ldots .$$

Clearly, $\chi_k \in \mathcal{P}_{2d-2}$, $k = 1, \ldots, d$, and $\chi_k(\xi^{(l)}) = \delta_{kl}$ (the Kronecker symbol) for all $k, l = 1, \ldots, d$. In fact, the set $(\chi_k)_{1 \leq k \leq d}$ is an orthonormal basis of $L^2(\mu)$.

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Since each polynomial $p \in \mathcal{P}_l$ can be written on the set Ξ as $p(t) = \sum_{j=1}^{d} p(\xi^{(j)})\chi_j(t)$, and so

$$\int |p(t) - \sum_{j=1}^{d} p(\xi^{(j)})\chi_j(t)|^2 d\mu(t) = 0,$$

it follows that, for every $l \geq 2d - 2$, we have $\mathcal{I}_l = \{p \in \mathcal{P}_l; p|\Xi = 0\}$, and so $(\chi_k + \mathcal{I}_l)_{1 \leq k \leq d}$ is an orthonormal basis of $\mathcal{H}_l$. Therefore, all spaces $\mathcal{H}_l$, $l \geq 2d - 2$, have the same dimension equal to $\dim L^2(\mu) = d$. In particular, Λ_∞ is dimensionally stable and $\mathrm{sd}(\Lambda_\infty) = d$.

An Associated C^*-Algebra

THEOREM
Let $m \geq 1$ be an integer, and let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf.
If Λ is stable at $m - 1$, then, endowed with an equivalent norm, the space $\mathcal{H}_m$ has the structure of a unital commutative C^*-algebra.

Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be the sp-extension of Λ.
We identify the space $\mathcal{H}_m$ with a commutative sub-C^*-algebra $\mathcal{A}$ of the C^*-algebra $B(\mathcal{H}_m)$ of all linear operators on $\mathcal{H}_m$ in the following way.

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We identify the space $\mathcal{H}_m$ with a commutative sub-C^*-algebra $\mathcal{A}$ of the C^*-algebra $B(\mathcal{H}_m)$ of all linear operators on $\mathcal{H}_m$ in the following way.

We define a map $\pi : \mathcal{H}_m \mapsto B(\mathcal{H}_m)$ by the equation $\hat{p} \mapsto p(A)$, where $\hat{p} = p + \mathcal{I}_m$, $p \in \mathcal{P}_m$, which is correctly defined and injective. Moreover, if $\mathcal{A} = \{p(A); p \in \mathcal{P}\}$, which is a commutative sub-C^*-algebra $\mathcal{A}$ of $B(\mathcal{H}_m)$, then the map $\pi : \mathcal{H}_m \mapsto \mathcal{A}$ is a linear isomorphism. Identifying the algebra $\mathcal{A}$ with the space $\mathcal{H}_m$, we obtain the desired structure of the latter.

The Stability Equation for Polynomial Spaces

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with $m \geq 1$ and let k be an integer such that $0 \leq k < m$. It is easily checked that the uspf Λ is stable at k if and only if for each multi-index δ with $|\delta| = k + 1$ the equation

$$\sum_{|\xi|,|\eta| \leq k} \gamma_{\xi+\eta} c_\xi c_\eta$$

$$-2 \sum_{|\xi| \leq k} \gamma_{\xi+\delta} c_\xi + \gamma_{2\delta} = 0$$

has a solution $(c_\xi)_{|\xi| \leq k}$ consisting of real numbers, where $\gamma = (\gamma_\xi)_{|\xi| \leq 2m}$ is the finite multi–sequence associated to Λ.

Stability Equation and Moments

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ the multi-sequence associated to Λ. Then $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \leq m-1}$ is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \leq m-1\}$. In fact, by identifying the space $\mathcal{P}_{m-1}$ with $\mathbb{C}^N$, A_{m-1} is the operator with the property $(A_{m-1}p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_{m-1}$.

For each multi-index δ with $|\delta| = m$, we put $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \leq m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. With this notation, equation (ASE) becomes

$$(SE) \qquad (A_{m-1}x|x) - 2(b_\delta|x) + c_\delta = 0,$$

which may be called the *stability equation* of the uspf Λ.

THEOREM
Let $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m}\,(\gamma_0 = 1, m \ge 1)$ be a square positive finite multi-sequence of real numbers and let $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \le m-1}$, acting on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \le m-1\}$. For each multi-index δ with $|\delta| = m$, set $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \le m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. The multi-sequence γ has a unique r-atomic representing measure if and anly if, whenever $b_\delta \in R(A_{m-1})$, we have $c_\delta \le (d_\delta | b_\delta)$ for some (and therefore for all) $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$, where r is the rank of the matrix A_{m-1}.

COROLLARY Assume the matrix A_{m-1} invertible. There exists a d-atomic representing measure μ on $\mathbb{R}^n$ for the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ if and only if for each δ with $|\delta| = m$ we have $c_\delta \le (A_{m-1}^{-1} b_\delta | b_\delta)$, where $d = \dim \mathcal{P}_{m-1}$.

THEOREM
Let $\gamma = (\gamma_\alpha)_{|\alpha| \le 2m} (\gamma_0 = 1, m \ge 1)$ be a square positive finite multi-sequence of real numbers and let $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \le m-1}$, acting on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \le m-1\}$. For each multi-index δ with $|\delta| = m$, set $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \le m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. The multi-sequence γ has a unique r-atomic representing measure if and anly if, whenever $b_\delta \in R(A_{m-1})$, we have $c_\delta \le (d_\delta | b_\delta)$ for some (and therefore for all) $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$, where r is the rank of the matrix A_{m-1}.

COROLLARY Assume the matrix A_{m-1} invertible. There exists a d-atomic representing measure μ on $\mathbb{R}^n$ for the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ if and only if for each δ with $|\delta| = m$ we have $c_\delta \le (A_{m-1}^{-1} b_\delta | b_\delta)$, where $d = \dim \mathcal{P}_{m-1}$.

Summary

Let $\mathcal{A}$ be a polynomial type algebra, with real basis $\mathcal{B} = \cup_{k\geq 0}\mathcal{B}_k$. To state a truncated moment problem in this context, for a fixed integer $m \geq 1$ we consider a finite sequence of real numbers $\gamma = (\gamma_b)_{b\in\mathcal{B}_{2m}}, \gamma_{\mathbf{1}} = 1$, and define the map $\Lambda_\gamma : \mathcal{S}_{2m} \mapsto \mathbb{C}$, extending the assignment $b \mapsto \gamma_b$, $b \in \mathcal{B}_{2m}$ by linearity. Natural conditions on the sequence γ insures that Λ_γ is a tracial uspf. We expect that the stability of Λ_γ at $m-1$ implies a special structure of this map, as in the case studied by Burgdorf and Klep. Indeed, partial results obtained so far seem to lead to the use of Wedderburn type decompositions for algebras of matrices, as exploited by Burgdorf and Klep.

Thank you !