Normal Extensions of Linear Relations via Quaternionic Cayley Transforms

F.-H. Vasilescu

Department of Mathematics
University of Lille 1

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Outline

Abstract

- In a previous work, the author has introduced a notion of quaternionic Cayley transform, valid for some classes of pairs of symmetric operators. In a second work, the former definition was slightly modified, leading to more direct proofs, using von Neumann's Cayley transform.
- In the present work, written in cooperation with **Adrian Sandovici**, a quaternionic Cayley transform for linear relations is introduced and some of its properties are exhibited. We emphasize the role played by the linear relations whose quaternionic Cayley transforms are unitary operators, which happen to be normal relations, and investigate the class of those linear relations which extend to such normal relations.

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- In the present work, written in cooperation with **Adrian Sandovici**, a quaternionic Cayley transform for linear relations is introduced and some of its properties are exhibited. We emphasize the role played by the linear relations whose quaternionic Cayley transforms are unitary operators, which happen to be normal relations, and investigate the class of those linear relations which extend to such normal relations.

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Introduction

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let U be a unitary operator acting in $\mathcal{H}$. If I is the identity operator on $\mathcal{H}$, set

$$S = \{\{(I - U)x, i(I + U)x\}; x \in \mathcal{H}\}, \tag{1}$$

which is a linear subspace of $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. The space S is the graph of a linear transformation in $\mathcal{H}$ if and only if the operator $I - U$ is injective. In this case, the corresponding linear transformation is known as the *inverse Cayley transform* of the unitary operator U, which is, in general, a (not necessarily bounded) self-adjoint operator.

Nevertheless, the space S given by (1) is well defined without the condition $I - U$ invertible, and it is what is called as a *linear relation* in $\mathcal{H}$ (following **Arens**). Moreover, we have

$$S^* := \{\{u, v\} \in \mathcal{H}^2 : \langle x, v\rangle = \langle y, u\rangle,\ \forall \{x, y\} \in S\} = S,$$

where S^* stands for the adjoint relation of S. In other words, S is a *self-adjoint* linear relation (formal definitions will be later given).

We shall define a Cayley transform for some linear relations. Briefly, considering a linear relation S in $\mathcal{H}^2$ (that is, a linear subspace of $\mathcal{H}^2 \times \mathcal{H}^2$), we associate it with the linear relation

$$V = \{\{\{x_1', x_2'\} + \{ix_1, -ix_2\}, \{x_1', x_2'\} + \{-ix_1, ix_2\}\} : \{\{x_1, x_1'\}, \{x_2, x_2'\}\} \in S\},$$

which is a quaternionic type Cayley transform of S . Under some natural conditions, the linear space V is the graph of a partial isometry in $\mathcal{H}^2$. Our aim is to study various properties relating the linear relation S and the partial isometry V.

An "extreme" situation is when U is an operator of the form

$$U = \begin{pmatrix} T & iA \\ iA & T^* \end{pmatrix},$$

with T normal and A self-adjoint in $\mathcal{H}$, such that $TT^* + A^2 = I$ and $AT = TA$. In this case, U is unitary, and it is a quaternionic Cayley transform of the linear relation

$$\{\{i(T-I)x - Ay, Ax - i(T^*-I)y\}, (T+I)x + iAy, iAx + (T^*+I)y\},$$

$$\{x, y\} \in \mathcal{H}^2\},$$

which is normal (in a sense to be defined).

Notation and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space. A *linear relation* (briefly, a *lr*) in $\mathcal{H}$ is a vector subspace of $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. The elements of $\mathcal{H}^2$ are represented as pairs $\{x, y\}$, with $x, y \in \mathcal{H}$.
For a *lr* T in $\mathcal{H}$, we denote by $D(T), R(T), N(T), M(T)$ its *domain of definition*, its *range*, its *kernel*, and its *multivalued part*, respectively.

We often identify a linear map $S : D(S) \subset \mathcal{H} \mapsto \mathcal{H}$ with its graph $G(S)$.

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If S, T are *lr* in $\mathcal{H}^2$, their *composition* is given by

$$ST = \{\{x, z\}; \exists y : \{x, y\} \in T, \{y, z\} \in S\}.$$

The *adjoint* T^* of T is given by

$$T^* = \{\{u, v\} : \langle u, y\rangle = \langle v, x\rangle; \forall\{x, y\} \in T\}.$$

If $T \subset T^*$, T is said to be *symmetric*.

If $T = T^*$, T is said to be *self-adjoint*.

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Normal Relations

A linear relation S in a Hilbert space $\mathcal{H}$ is said to be *formally normal* (**Coddington**) if there exists an isometry $V : S \to S^*$ of the form

$$V\{x, x'\} = \{x, x''\}, \quad \{x, x'\} \in S, \quad \{x, x''\} \in S^*. \tag{2}$$

In particular, $\|x'\| = \|x''\|$.

A formally normal linear relation S in $\mathcal{H}$ is said to be *normal* (**Coddington**) if the isometry V is surjective.

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Characterization of Normality

As in the case of (unbounded) normal operators, we have the following:

Theorem (Sandovici+V)

Let S be a linear relation in a Hilbert space $\mathcal{H}$. Equivalent are:

1. S is a normal linear relation in $\mathcal{H}$;
2. S is closed, $D(S) = D(S^*)$, and $S^*S = SS^*$.

Quaternionic Cayley Transform of Linear Relations

Let S be a symmetric linear relation in $\mathcal{H}$. We define the relation

$$V = \{\{x' + ix, x' - ix\}; \{x, x'\} \in S\},$$

which is called the *Cayley transform* of S **(Arens).**

Because S is symmetric, we have

$$\|x' \pm ix\|^2 = \|x\|^2 + \|x'\|^2, \ \{x, x'\} \in S.$$

Therefore, V is (the graph of) an isometry, defined on $D(V) = \{x' + ix, ; \{x, x'\} \in S\} \subset \mathcal{H}$, whose range is $R(V) = \{x' - ix; \{x, x'\} \in S\} \subset \mathcal{H}$.

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Version of a von Neumann's Theorem

The next result was proved by **Arens**:

Theorem Let S be a symmetric relation in the Hilbert space $\mathcal{H}$. The relation S is self-adjoint if and only if the Cayley transform of S is a unitary operator in $\mathcal{H}$.

Recall: Algebra of Quaternions

Consider the 2×2-matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Hamilton algebra of quaternions $\mathbb{H}$ can be identified with the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$ of 2×2-matrices with complex entries, generated by the matrices $\mathbf{I}$, $i\mathbf{J}$, $\mathbf{K}$ and $i\mathbf{L}$. This allows us to regard the elements of $\mathbb{H}$ as matrices and to perform some operations in $\mathbb{M}_2$ rather than in $\mathbb{H}$.

Recall: Some Elementary Operations

We have $\mathbf{J}^* = \mathbf{J}$, $\mathbf{K}^* = -\mathbf{K}$, $\mathbf{L}^* = \mathbf{L}$, $\mathbf{J}^2 = -\mathbf{K}^2 = \mathbf{L}^2 = \mathbf{I}$, $\mathbf{JK} = \mathbf{L} = -\mathbf{KJ}$, $\mathbf{KL} = \mathbf{J} = -\mathbf{LK}$, $\mathbf{JL} = \mathbf{K} = -\mathbf{LJ}$, where the adjoints are computed in the Hilbert space $\mathbb{C}^2$.
We also put $\mathbf{E} = i\mathbf{J}$, $\mathbf{F} = i\mathbf{L}$, and we we have $\mathbf{E}^* = -\mathbf{E}$, $\mathbf{E}^2 = -\mathbf{I}$, $\mathbf{F}^* = -\mathbf{F}$, $\mathbf{F}^2 = -\mathbf{I}$

A Preliminary Lemma

Let $\mathcal{H}$ be a complex Hilbert space. The matrices from $\mathbb{M}_2$ clearly act on $\mathcal{H}^2$ (whose natural norm is denoted by $\| * \|_2$).

Lemma

Let S be a linear relation in $\mathcal{H}^2$. Suppose that the linear relation $\mathbf{J}S$ is symmetric. Then we have

$$\|x' \pm \mathbf{E}x\|_2^2 = \|x'\|_2^2 + \|x\|_2^2, \quad \{x, x'\} \in S. \tag{3}$$

E–Cayley Transform

Let S be a linear relation in $\mathcal{H}^2$ such that $\mathbf{J}S$ is symmetric.

E–Cayley transform of S:

$$V := \{\{x' + \mathbf{E}x, x' - \mathbf{E}x\} : \{x, x'\} \in S\}. \tag{4}$$

As we have $\|x' + \mathbf{E}x\|_2 = \|x' - \mathbf{E}x\|_2$, it follows that V is (the graph of) a partial isometry with $D(V) = R(S + \mathbf{E})$ and $R(V) = R(S - \mathbf{E})$.

In fact, V given by

$$V : R(S+\mathbf{E}) \mapsto R(S-\mathbf{E}), \quad V(x'+\mathbf{E}x) = x'-\mathbf{E}x, \quad \{x, x'\} \in S.$$

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F–Cayley Transform

If **L**S is symmetric, a similar discussion leads to the definition of an operator W given by

$$W : R(S+\mathbf{F}) \mapsto R(S-\mathbf{F}), \quad W(x'+\mathbf{F}x) = x'-\mathbf{F}x, \quad \{x, x'\} \in S,$$

which is again a partial isometry. The operator W is called the **F**–*Cayley transform* of S.
Because the two Cayley transforms defined above are alike, in the sequel we shall mainly deal with the **E**–Cayley transform.

Properties of the **E**–Cayley Transform

Lemma

Let S be a linear relation in $\mathcal{H}^2$ such that $\mathbf{J}S$ is symmetric, and let V be the **E**–Cayley transform of S. We have the following:
(a) V is closed if and only if S is closed, and if and only if the spaces $R(S \pm \mathbf{E})$ are closed;
(b) One has $N(\mathbf{I} - V) = M(S)$. The operator $\mathbf{I} - V$ is injective if and only if S is an operator. Moreover, S is densely defined if and only if the space $R(\mathbf{I} - V)$ is dense in $\mathcal{H}^2$;
(c) If $S\mathbf{K} \subset \mathbf{K}S$, then $S\mathbf{K} = \mathbf{K}S$ and $V^{-1} = -\mathbf{K}V\mathbf{K}$;
(d) The lr $\mathbf{J}S$ is self-adjoint if and only if V is unitary in $\mathcal{H}^2$.
(e) The **E**–Cayley transform is an inclusion preserving map.

Inverse **E**–Cayley Transform

Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry, for which we define the following:

Inverse E–Cayley transform:

$$S := \{\{\mathbf{E}(V - \mathbf{I})x, (V + \mathbf{I})x\} \ : \ x \in D(V)\},$$

which is a linear relation in $\mathcal{H}^2$.

In a similar way, we can define the *inverse* **F**–*Cayley transform*. These two (quaternionic) inverse Cayley transforms have similar properties, and so we shall deal only with the inverse **E**–Cayley transform.

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Properties of the Inverse **E**–Cayley Transform

Lemma Let $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ be a partial isometry. Then the linear relation

$$S = \{\{\mathbf{E}(V - \mathbf{I})x, (V + \mathbf{I})x\} \; : \; x \in D(V)\} \, .$$

has the following properties:
(i) the linear relation $\mathbf{J}S$ is symmetric and the **E**–Cayley transform of S is V;
(ii) we have $V^{-1} = -\mathbf{K}V\mathbf{K}$ if and only if $S\mathbf{K} = \mathbf{K}S$.

Global Properties

We summarize the properties of the quaternionic Cayley transform:

Theorem

The **E**-Cayley transform is an inclusion preserving bijective map assigning to each linear relation S in $\mathcal{H}^2$ with $\mathbf{J}S$ symmetric a partial isometry V in $\mathcal{H}^2$. Moreover:
(1) the operator V is closed if and only if the linear relation S is closed;
(2) the equality $V^{-1} = -\mathbf{K}V\mathbf{K}$ holds if and only if the equality $S\mathbf{K} = \mathbf{K}S$ holds;
(3) the linear relation $\mathbf{J}S$ is self-adjoint if and only if V is unitary on $\mathcal{H}^2$.

Special Unitary Operators

In this this section we are particularly interested in those unitary operators producing normal linear relations (in particular (unbounded) normal operators), via the inverse **E**–Cayley transform.

Lemma Let U be a bounded operator on $\mathcal{H}^2$. The operator U is unitary and has the property $U^* = -\mathbf{K}U\mathbf{K}$ if and only if there are a bounded operator T and bounded self-adjoint operators A, B on $\mathcal{H}$ such that $TT^* + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$ and

$$U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix},$$

where I the identity on $\mathcal{H}$.

Some Auxilliary Results

Lemma Let U be a unitary operator on $\mathcal{H}^2$ with $U^* = -\mathbf{K}U\mathbf{K}$. If we set

$$S = \{\{\mathbf{E}(U - \mathbf{I})x, (U + \mathbf{I})x\} : x \in \mathcal{H}\}.$$

we have that S is a closed linear relation and

$$S^* = \{\{(\mathbf{I} - U)x, \mathbf{E}(\mathbf{I} + U)x\} : x \in \mathcal{H}\}.$$

Lemma Let U be an operator on $\mathcal{H}^2$ having the form

$$U = \begin{pmatrix} T & iA \\ iB & T^* \end{pmatrix},$$

with T, $A = A^*$, $B = B^*$ bounded operators on $\mathcal{H}$, such that $TT^* + A^2 = I$, $T^*T + B^2 = I$, $AT = TB$. We have the equality $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$ if and only if T is normal and $A = B$.

Intertwining Isometries

Lemma Let V be a partial isometry such that $V^{-1} = -\mathbf{K}V\mathbf{K}$. Let S be the inverse **E**–Cayley transform of V. Equivalent are:
(i) $\mathbf{J}D(S) \subset D(S)$ and there exists an isometry $H : S \to S$ of the form $H\{x, x'\} = \{\mathbf{J}x, x''\}$;
(ii) there exists an isometry $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I} - V) = (\mathbf{I} - V)G$.

Remark

(1) The isometry G given by the previous Lemma is not uniquely determined. If G_1, G_2 are two isometries as in this lemma, then the operator $G_1 - G_2$ is clearly defined on $D(V)$ and has values in the kernel of $\mathbf{I} - V$.

(2) Assume that the isometry H in the previous Lemma is surjective. Then it can be shown that the isometric operator G may be chosen to be surjective. Conversely, if G is surjective, the operator H may be also chosen to be surjective.

Proposition

Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$. Let also S be the inverse **E**–Cayley transform of U. The linear relation S is normal if and only is there exists a unitary operator G_U on $\mathcal{H}^2$ such that $\mathbf{E}(\mathbf{I} - U) = (\mathbf{I} - U)G_U$ and $(G_U)^* = -G_U$.

Main result and the Class $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$

Theorem

Let U be a unitary operator on $\mathcal{H}^2$ with the property $U^* = -\mathbf{K}U\mathbf{K}$, and let S be the inverse **E**–Cayley transform of U. The linear relation S is normal if and only if $(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)$.

Definition Let $\mathcal{U}(\mathcal{H}^2)$ be the set of all unitary operators in $\mathcal{H}^2$. We also set

$$\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) = \{U \in \mathcal{U}(\mathcal{H}^2); U^* = -\mathbf{K}U\mathbf{K},$$

$$(U + U^*)\mathbf{E} = \mathbf{E}(U + U^*)\},$$

that is, those unitary operators whose inverse **E**–Cayley transform is a normal linear relation, via the previous theorem.

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The Class $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$

Definition We introduce the following class of linear relations in $\mathcal{H}^2$:

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) = \{S \subset \mathcal{H}^2 \times \mathcal{H}^2;$$

$$S \text{ normal}, \ (\mathbf{J}S)^* = \mathbf{J}S, \ \mathbf{K}S = S\mathbf{K}\}.$$

Remark The previous results show that the map

$$\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2) \ni S \mapsto U_S \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2),$$

where U_S stands for the **E**–Cayley transform of S, is bijective.

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Subnormality

Adapting the terminology from operator theory, we may say that a linear relation S in a Hilbert space $\mathcal{H}$ is *subnormal* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal relation N in $\mathcal{K}$ such that $S \subset N$. Nevertheless, we are particularly interested to solve such a problem with the restriction $\mathcal{K} = \mathcal{H}$. In fact, to apply the quaternionic Cayley transform machinery, we consider only linear relations in the Hilbert space $\mathcal{H}^2$.

The Class $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$

We start by introducing a class of linear relations having necessary properties connected to subnormality.

Let T : be a linear relation in $\mathcal{H}^2$, with $D(T) = D_0 \oplus D_0,\ D_0 \subset \mathcal{H}$, which is equivalent to the inclusions
(i) $\mathbf{J}D(T) \subset D(T)$ and $\mathbf{K}D(T) \subset D(T)$.
Furthermore, assume that T satisfies the following conditions:
(ii) $\mathbf{J}T$ is symmetric;
(iii) $T\mathbf{K} = \mathbf{K}T$;
(iv) there exists a surjective isometry $H_T : T \to T$ of the form $H_T\{x, x'\} = \{\mathbf{J}x, x''\}$; for all $\{x, x'\} \in T$.
Denote by $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ the set of those linear relations T in $\mathcal{H}^2$ such that (i)–(iv) hold.

The Class $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$

Let $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ be the set of those partial isometries $V : D(V) \subset \mathcal{H}^2 \mapsto \mathcal{H}^2$ such that:
(a) $V^{-1} = -\mathbf{K}V\mathbf{K}$;
(b) $\mathbf{E}R(\mathbf{I} - V) = R(\mathbf{I} - V)$;
(c) there exists a surjective isometry $G : D(V) \mapsto D(V)$ such that $\mathbf{E}(\mathbf{I} - V) = (\mathbf{I} - V)G$.
It was proved that the **E**–Cayley transform is a bijective map from $\mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ onto $\mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$. Note also that $\mathcal{U}_{\mathcal{C}}(\mathcal{H}^2) \subset \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$ by some of the previous results.

Normal Extensions

A decomposition result:

Proposition Let $U \in \mathcal{U}_{\mathcal{C}}(\mathcal{H}^2)$ and let $\mathcal{D} \subset \mathcal{H}^2$ be a closed subspace with the properties $\mathbf{K}U(\mathcal{D}) \subset \mathcal{D}$ and $\mathbf{E}(\mathbf{I} - U)(\mathcal{D}) \subset (\mathbf{I} - U)(\mathcal{D})$. If $V = U|\mathcal{D}$, $\mathcal{E} = \mathcal{D}^{\perp}$ and $W = U|\mathcal{E}$, then $U = V \oplus W$ and $V, W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$.

Main extension result:

Theorem

Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$. Equivalent are:
(i) the linear relation T has an extension S in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ such that $H_T = H_S|T$;
(ii) there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with $D(W) = R(T + \mathbf{E})^{\perp}$.

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The next assertion provides an extension result for linear relations. In particular, this includes the case of not necessarily densely defined operators.

Corollary

Let $T \in \mathcal{S}_{\mathcal{IC}}(\mathcal{H}^2)$ be closed and let V be the **E**–Cayley transform of T. Then the linear relation T has an extension in $\mathcal{N}_{\mathcal{IC}}(\mathcal{H}^2)$ if and only if there exists a $W \in \mathcal{P}_{\mathcal{C}}(\mathcal{H}^2)$, with the property $D(W) = R(T + \mathbf{E})^{\perp}$.

Thank you for your attention!