Quantum chaos: beyond the Schnirelman theorem

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Texas A & M University – October 2004

INTRODUCTION

QUANTUM CHAOS : the semi-classical analysis of quantum systems having a chaotic Hamiltonian system as their classical limit.

EXAMPLES:

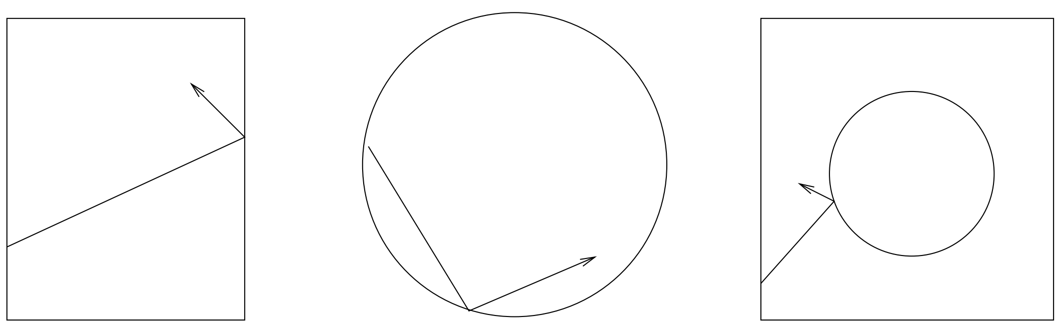
1. *Quantum* : Δ on $L^2(M, \mathrm{dvol}(g))$; $\Delta\psi_n = \lambda_n\psi_n$;
   Chaotic system : the geodesic flow on (M, g), a compact negatively curved manifold. This is a Hamiltonian flow on T^*M!
   Semi-classical : $\lambda_n \to +\infty$.

2. *Quantum* : the Dirichlet laplacian Δ on a domain $\Omega \subset \mathbb{R}^2$; $\Delta\psi_n = \lambda_n\psi_n$
   Chaotic system : the billiard flow on Ω (Bunimovich, Sinaï). This is a Hamiltonian flow on $\Omega \times \mathbb{R}^n$.
   Semi-classical : $\lambda_n \to +\infty$.

3. *Quantum* : quantum maps $=$ unitary maps on N dimensional spaces.
   Chaotic system : symplectic Anosov maps on the torus.
   Semi-classical : $N \to +\infty$.

THE BASIC QUESTION: relate the asymptotic behaviour of the eigenvalues and eigenfunctions to the statistical properties of the underlying classical dynamical system. In particular, what is the signature of chaos on the eigenfunctions and on the eigenvalues?

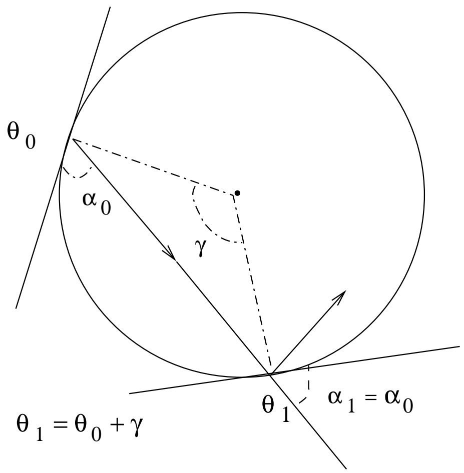
IN THIS TALK: I will talk mostly about the eigenfunctions . . . First task: illustrate the issues in the case of billiards.



A rectangular, circular and Sinaï billiard

REGULAR VERSUS CHAOTIC MOTION: the example of billiards

- Motion in the circular billiard: no chaos here!



Geometric description: initial data θ_0, α_0.

Then $\gamma = 2\alpha_0,\ r_- = \cos\alpha_0,\ \theta_n = \theta_0 + n\gamma$

Hamiltonian description: initial data $(\vec{q}, \vec{p}) \in D \times \mathbb{R}^2$.

Then $E = \frac{\vec{p}^2}{2}, L = (\vec{q} \wedge \vec{p})_z$ are constants of the motion.

E and L determine α_0 and $r_- = \frac{|L|}{\sqrt{2E}}$ and vice versa.

The motion is now regular and stable: $\gamma' = \gamma + \epsilon \Rightarrow \theta'_n = \theta_n + n\epsilon$.

The set of all orbits with fixed E and L fill an annulus $r_- \leq \| \vec{q} \| \leq 1$. If $\alpha_0/2\pi$ is not rational, each orbit fills this annulus uniformly in the sens that

$$\frac{1}{T}\int_0^T f(\vec{q}(t))dt \overset{T\to+\infty}{\longrightarrow} \int_0^{2\pi}\int_{r_-}^1 f(\vec{q})\frac{dq}{\pi(1-r_-^2)}.$$

• The corresponding spectral (= quantum) problem:

$$\begin{aligned} \Delta\psi_{\ell,m} &= \lambda_{\ell m}^2 \psi_{\ell,m}, \quad \ell \in \mathbb{N} \\ L\psi_{\ell,m} &\equiv \frac{1}{i}\partial_\theta \psi_{\ell,m} = m\psi_{\ell,m}, \quad m \in \mathbb{Z}. \end{aligned}$$

Note that $[\Delta, L] = 0$. Here

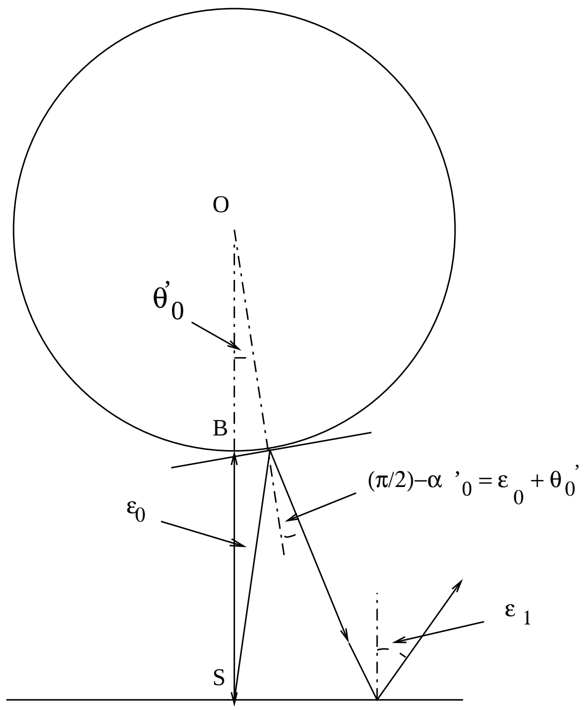
$$\psi_{\ell,m}(r,\theta) = N_{\ell,m} J_{|m|}(\lambda_{\ell,m} r)\mathrm{e}^{im\theta}.$$

One can prove that

$$\lim_{\ell,m\to+\infty;\frac{m}{\lambda_{\ell,m}}=r_-} \int_D f(r,\theta) \mid \psi_{\ell,m}(r,\theta) \mid^2 r dr d\theta = \int_0^{2\pi} \int_{r_-}^1 f(r,\theta) \frac{r dr d\theta}{\pi(1-r_-^2)}$$

LESSON : the behaviour of the classical motion seems to be reflected in the eigenfunctions at large energies (= physics speak for high eigenvalues).

- Motion in the Sinaï billiard: chaos!



Highly unstable flow: $\epsilon_n \geq 3^n \epsilon_0$.

There is only one constant of the motion: $E = \frac{p^2}{2}$.
A typical orbit explores the full energy surface (ergodicity) and goes therefore everywhere in the billiard.

Moreover:

$$\frac{1}{T}\int_0^T f(\vec{q}(t))dt \overset{T\to+\infty}{\longrightarrow} \int_\Omega f(\vec{q})\frac{dq}{\mid \Omega \mid}.$$

• The corresponding spectral (= quantum) problem: $\Delta\psi_n = \lambda\psi_n,\ n \in \mathbb{N}$.

THEOREM (Gérard-Leichtnam (93), Zelditch-Zworski (96)) Let Ω be an open set in $\mathbb{R}^2$ with a piecewise smooth boundary. Suppose the billiard flow on Ω is ergodic. Let ψ_n be the eigenfunctions of the Dirichlet Laplacian on Ω, with eigenvalues $\lambda_n \to +\infty$. Then there exists a density one subsequence ψ_{n_k} so that

$$\lim_{k\to\infty} \int_{B\subset\Omega} \mid \psi_{n_k}(x) \mid^2 dx = \frac{\mid B \mid}{\mid \Omega \mid}.$$

LESSON : the behaviour of the classical motion seems indeed to be reflected in the eigenfunctions at large energies.

The “Schnirelman theorem” (Schnirelman (74), Zelditch (87,95), Colin de Verdière (85), Helffer-Martinez-Robert (87)), ... is a robust result. It always works if the classical system is ergodic.

QUESTIONS BEYOND SCHNIRELMAN

QUESTION 1 Do there exist exceptional sequences of eigenfunctions that do NOT converge semi-classically to Lebesgue measure? In other words, is the extraction of a subsequence a necessity in the statement of this result or an artifact of the proof?

If you think the answer is NO, you are a believer in something that has been baptized “unique quantum ergodicity”.

If you think the answer is YES, you have to confront the following question:

QUESTION 2 Can you characterize all possible limit measures?

It is easy to see that any such limit measure has to be invariant under the dynamics. Particularly simple, easily understood candidate limit measures are measures supported on a periodic orbit of the dynamics. So one may try to look for subsequences $\psi_{n'_\ell}$ for which

$$\lim_{\ell \to \infty} \int_\Omega f(x) \mid \psi_{n'_\ell}(x) \mid^2 dx = \frac{1}{T_\gamma} \int_0^{T_\gamma} f(\gamma(t)) dt,$$

where $t \in [0, T_\gamma] \mapsto \gamma(t) \in \mathbb{R}$ is a periodic orbit of the flow.

ANSWERS to these questions are not available in general systems. Only in some restricted classes of models have partial answers been obtained:

- For the (Hecke) eigenfunctions of the Laplace-Beltrami operator of a (class) of constant negative curvature surfaces the ANSWER TO QUESTION 1 has been proven to be NO (Lindenstrauss 2003).

- For certain quantum maps, the ANSWER TO QUESTION 1 has been proven to be YES (Faure, Nonnenmacher, De Bièvre 2003)! A partial but reasonably complete answer is then also known to question 2.

Quantum maps: a case study in quantum chaos

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Texas A & M University – October 2004

INTRODUCTION

"Quantum maps" are a particularly simple class of systems on which to test various conjectures, ideas, techniques in quantum chaos. A quantum map is a quantum system obtained through the quantization of a discrete classical Hamiltonian system, *i.e.* a symplectic map Φ on a compact symplectic manifold. The latter is usually either the two-sphere, or the $2d$-dimensional torus, viewed as a symplectic manifold.

Stripped to its mathematical bone, a quantum map is a family $(U_\hbar, \mathcal{H}_\hbar)_{\hbar>0}$ of unitaries $U_\hbar$ on finite (!) dimensional Hilbert spaces $\mathcal{H}_\hbar$. One has

$$\dim \mathcal{H}_\hbar \stackrel{\hbar \to 0}{\longrightarrow} +\infty$$

and one is interested in studying the eigenfunctions and eigenvalues of the $U_\hbar$ as $\hbar \to 0$.

The goal is to understand the asymptotic behaviour of the eigenfunctions and eigenvalues in terms of the dynamical properties of Φ, especially when the latter is chaotic.

Would an example help?

ANOSOV MAPS ON THE TORUS

Hyperbolic automorphisms (CAT maps) :

$A \in \mathrm{SL}(2,\mathbb{Z}),\ |\ \mathrm{Tr}A\ |> 2\ \Rightarrow\ Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm.$

A acts as a symplectomorphism on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and is

- Hyperbolic : a.e. $x, x' \in \mathbb{T}^2,\ t \in \mathbb{N}$ (not too large),

$$\mathrm{d}(x,x') \sim \epsilon \Rightarrow \mathrm{d}(A^t x, A^t x') \sim \epsilon \mathrm{e}^{\gamma_0 t}.$$

- Exponentially mixing : $\forall f, g \in C^\infty(\mathbb{T}^2)$,

$$\left| \int_{\mathbb{T}^2} (f \circ A^t)(x) g(x) dx - \int_{\mathbb{T}^2} f(x) dx \int_{\mathbb{T}^2} g(x) dx \right| \leq C_{A,f} \parallel \nabla g \parallel_1\ \mathrm{e}^{-\gamma_0 t}.$$

Perturbed hyperbolic automorphisms : For $g \in C^\infty(\mathbb{T}^2)$, $\phi_s, s \in \mathbb{R}$ is the Hamiltonian flow of g. Define $\Phi_\epsilon = \phi_\epsilon \circ A$.

For small ϵ, this is still hyperbolic and exponentially mixing, with exponent γ_ϵ, $\lim_{\epsilon\to 0} \gamma_\epsilon = \gamma_0$ (Blank, Keller, Liverani 2002).

THE CORRESPONDING QUANTUM MAP

The Hilbert spaces With $U(a)\psi(y) = \mathrm{e}^{-\frac{i}{2\hbar}a_1 a_2}\mathrm{e}^{\frac{i}{\hbar}a_2 y}\psi(y-a_1) = \mathrm{e}^{-\frac{i}{\hbar}(a_1 P - a_2 Q)}\psi(y)$,

define

$$\mathcal{H}_\hbar = \{\psi \in \mathcal{S}'(\mathbb{R}) \mid U(1,0)\psi = \psi = U(0,1)\psi\}, \quad 2\pi\hbar N = 1 \Rightarrow \mathrm{d}im\mathcal{H}_\hbar = N.$$

Then

$$\psi \in \mathcal{H}_\hbar \Rightarrow \psi(y) = \sum_{\ell \in \mathbb{Z}} c_\ell \delta(y - \frac{\ell}{N}); \quad c_{\ell+N} = c_\ell.$$

Weyl quantization For $f \in C^\infty(\mathbb{T}^2)$, $x = (q,p) \in \mathbb{T}^2$, write

$$f(x) = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 p - n_2 q)}$$

and define

$$\mathrm{Op}^{\mathrm{W}} f = \hat{f} = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 P - n_2 Q)} = \sum_{n \in \mathbb{Z}^2} f_n U(\frac{n}{N}) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

The quantum dynamics Take $A \in \mathrm{SL}(2,\mathbb{Z}),\ \mid \mathrm{Tr} A \mid > 2$ and construct

$$M(A)\psi(y) = \left(\frac{i}{2\pi\hbar a_{12}}\right)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{i}{2\hbar a_{12}}(a_{22}y^2 - yy' + a_{11}y'^2)} \psi(y')dy'.$$

Then, for all $t \in \mathbb{Z}$,

$$M(A)\mathcal{H}_\hbar = \mathcal{H}_\hbar \quad \text{and} \quad M(A)^{-t}\,\mathrm{Op}^{\mathrm{W}} f\, M(A)^t - \mathrm{Op}^{\mathrm{W}}(f \circ A^t) = 0.$$

Now, for $\epsilon > 0$ define the unitary operator

$$U_\epsilon = \mathrm{e}^{-\frac{i}{\hbar}\epsilon \mathrm{Op}^{\mathrm{W}} g} M(A) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

This is the quantum map we wish to study. It is naturally related to the discrete Hamiltonian dynamics on $\mathbb{T}^2$ obtained by iterating $\Phi_\epsilon = \phi_\epsilon \circ A$. It acts on the N dimensional spaces $\mathcal{H}_\hbar$ and we are interested in the behaviour of its eigenfunctions and eigenvalues in the $N \to \infty$ limit:

$$U_\epsilon \psi_j^{(N)} = \mathrm{e}^{i\theta_j^{(N)}} \psi_j^{(N)}, \quad j = 1 \ldots N.$$

WHAT IS KNOWN?

1. The basic result : the Schnirelman theorem

THEOREM 1 (Bouzouina-DB 96) Let $\epsilon \geq 0$ and small.

Then, for "almost all" sequences $\psi_N \in \mathcal{H}_\hbar$, so that $U_\epsilon \psi_N = \mathrm{e}^{i\theta_N} \psi_N$,

$$\langle \psi_N, \mathrm{Op}^{\mathrm{W}} f \psi_N \rangle \overset{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x) dx, \quad \forall f \in C^\infty(\mathbb{T}^2) \tag{1}$$

COMMENTS: (i) This is proven by adapting known arguments, which is why it holds also for $\epsilon \neq 0$ and on higher dimensional tori.

(ii) The result can be adapted for maps that are not continuous such as the Baker and sawtooth maps (De Bièvre, Degli Esposti 1997) and to systems with a mixed phase space (Marklof, O'Keefe 2004).

2. Questions beyond Schnirelman

QUESTION 1 Do there exist exceptional sequences of eigenfunctions that do not converge semi-classically to Lebesgue measure (for quantized automorphisms this means (1) does not hold).

QUESTION 2 Can you characterize all possible limit measures?

ANSWERS to these questions are not available in general systems. Only in some restricted classes of models have partial answers been obtained:

- For the (Hecke) eigenfunctions of the Laplace-Beltrami operator of a (class) of constant negative curvature surfaces the ANSWER TO QUESTION 1 has been proven to be NO (Lindenstrauss 2003).

- For quantized hyperbolic toral automorphisms, the ANSWER TO QUESTION 1 has been proven to be YES:

THEOREM 2 (Faure-Nonnenmacher-DB 03) Let $\epsilon = 0$. Let $0 \leq \alpha \leq \frac{1}{2}$, then there exists $N_k \longrightarrow \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that

$$\langle \psi_{N_k}, \mathrm{Op}^{\mathrm{W}} f \psi_{N_k} \rangle \stackrel{N_k \to +\infty}{\longrightarrow} \alpha f(0) + (1-\alpha) \int_{\mathbb{T}^2} f(x) dx, \quad \forall f \in C^\infty(\mathbb{T}^2).$$

• And what about QUESTION 2 for those systems?

Let μ be an A-invariant probability measure on $\mathbb{T}^2$ that is singular with respect to Lebesgue measure. Then THEOREM 2 easily implies (diagonalization trick) that any probability measure on $\mathbb{T}^2$ of the form

$$\alpha\mu + (1-\alpha)\mathrm{dx}$$

is a limit measure provided $\alpha \leq 1/2$. This is more or less optimal, since:

THEOREM 3 (Bonechi-DB 03, Faure-Nonnenmacher 04) Let $\epsilon = 0$. Let μ be an A-invariant PURE POINT measure. Let $\alpha > \frac{1}{2}$. Then there does not exist $N_k \to \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that

$$\langle \psi_{N_k}, \mathrm{Op}^{\mathrm{W}} f \psi_{N_k} \rangle \stackrel{N_k \to +\infty}{\to} \alpha\mu(f) + (1-\alpha)\int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2).$$

OPEN QUESTION: What happens to this result if μ is purely singular continuous?

● THEOREM 2 plays on the existence of a sequence of integers that is in fact very special, and depends on A. It is also very sparse. By using number theoretic properties of the $M(A)$ very different results are obtained:

THEOREM 4 (Kurlberg-Rudnick 00) If $A \in \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic and $A \equiv \mathbb{I}_2$ mod 4, then there exists for each N a basis $\{\psi_1^{(N)}, \psi_2^{(N)}, \ldots \psi_N^{(N)}\}$ of eigenfunctions of $M(A)$ so that

$$\langle \psi_{j_N}^{(N)}, \mathrm{Op}^{\mathrm{W}} f \psi_{j_N}^{(N)} \rangle \stackrel{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x) dx, \quad \forall f \in C^\infty(\mathbb{T}^2) \qquad (2)$$

holds for any sequence ψ_{j_N}.

This obviously constitutes a strengthening of the Schnirelman theorem for the particular class of A considered. The basis for which the result holds is explicitly described. Indeed, the eigenvalues of $M(A)$ may be degenerate so that it is possible that exceptional sequences of eigenfunctions not belonging to the above basis have a different semi-classical limit.

THEOREM 5 (Kurlberg-Rudnick 01) If $A \in \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic and $a_{11}a_{12} \equiv 0 \equiv a_{21}a_{22}$ mod 2, then there exists a density one sequence of integers $(\tilde{N}_\ell)_{\ell \in \mathbb{N}}$ along which (1) holds.

THE UPSHOT OF ALL THIS:

The quantized hyperbolic toral automorphisms are ”uniquely quantum ergodic”for typical values of $\hbar$, but not for special ones.

A FURTHER QUESTION : CAT maps are rather special because they are linear and have many number theoretic properties that are not generic. A natural question is therefore :

Which of Theorems 2,3,4,5 has a chance to generalize to $\epsilon \neq 0$?

AN ANSWER?

THEOREM 1 is already stated for $\epsilon \neq 0$.

THEOREM 2 uses one very special arithmetic property of the problem, and I don't think it should survive perturbation ($\epsilon \neq 0$).

THEOREM 4,5,6, . . . rely on the arithmetic properties of the system . . .

CLAIM : **THEOREM 3** ought to hold for $\epsilon \neq 0$ as well.

Theorem 3 : the main ingredient

THEOREM (Bonechi-DB 03)"Congregate, and thou shalt be spread"

Let $\epsilon = 0$. Let $a_0 \in \mathbb{T}^2$. If $\varphi_N \in \mathcal{H}_\hbar$ is some sequence (not necessarily eigenfunctions!) with the property that, for all $f \in C^\infty(\mathbb{T}^2)$

$$\langle \varphi_N, \mathrm{Op}^{\mathrm{W}} f \varphi_N \rangle \overset{N\to+\infty}{\to} f(a_0)$$

then there exists a sequence of times $t_N \to \infty$ so that

$$\langle \varphi_N, U_0^{-t_N} \mathrm{Op}^{\mathrm{W}} f\, U_0^{t_N} \varphi_N \rangle \overset{N\to+\infty}{\to} \int_{\mathbb{T}^2} f(x) dx.$$

Remark: The statement involves the simultaneous limit

"$\hbar$ to zero, time to infinity".

The time scales t_N involved are logarithmic in $\hbar$. So if you want to prove something like this for $\epsilon \neq 0$ (perturbed automorphisms), you need to be able to control $U_\epsilon^{t_N}$ at such time scales. For the Schnirelman theorem, that's not needed since there, you take $\hbar \to 0$ first, and then $t \to \infty$.

A VERSION OF THEOREM 3 FOR PERTURBED AUTOMORPHISMS

DEFINITION A sequence $\varphi_N \in \mathcal{H}_\hbar$ is said to localize at $a \in \mathbb{T}^2$ if

$$\langle \varphi_N, \mathrm{Op}^{\mathrm{W}} f \, \varphi_N \rangle \stackrel{N \to +\infty}{\longrightarrow} f(a), \qquad \forall f \in C^\infty(\mathbb{T}^2).$$

This implies there exists $r_\hbar \to 0$ so that

$$\int_{|x-a| \geq r_\hbar} |\langle \varphi_N, \tilde{\varphi}_x \rangle|^2 \frac{dx}{2\pi\hbar} \stackrel{N \to +\infty}{\longrightarrow} 0.$$

THEOREM (J. M. Bouclet, SDB, 2004) Let U_ϵ be as before. There exists $\delta_0 > 0$ with the following property. Let ψ_N be a sequence of eigenfunctions of U_ϵ that localizes on a periodic orbit of Φ_ϵ. Then $r_\hbar \geq \hbar^{\frac{1}{2} - \delta_0}$.

This gives some weak control on the way eigenfunctions may concentrate. If they do so, they must do it slowly!

We have some sub-optimal control on the exponent δ. (Faure and Nonnenmacher announced a (still sub-optimal) improvement on this.)

THE EGOROV THEOREM 1

Recall: We are given $A \in \mathrm{SL}(2, \mathbb{Z})$, $g \in C^\infty(\mathbb{T}^2)$ and $\epsilon \geq 0$

The Lyapounov exponent γ_0 is defined by $Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm$.

The classical dynamics is $\Phi_\epsilon = \phi_\epsilon \circ A$.

The quantum dynamics is $U_\epsilon = \mathrm{e}^{-\frac{i\epsilon}{\hbar}\mathrm{Op}^{\mathrm{W}} g} M(A)$.

"THEOREM" Let $f \in C^\infty(\mathbb{T}^2)$. There exists

$$\Gamma_\epsilon \geq \gamma_0, \quad \lim_{\epsilon \to 0} \Gamma_\epsilon = \gamma_0,$$

such that, for all $0 < \nu$, for all $t \in \mathbb{Z}$ such that

$$0 \leq |t| \leq (\frac{2}{3\Gamma_\epsilon} - \nu)|\ln \hbar|$$

$$U_\epsilon^{-t}\, \mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^t - \mathrm{Op}^{\mathrm{W}}(f \circ \Phi_\epsilon^t) \stackrel{\hbar \to 0}{\longrightarrow} 0.$$

THE EGOROV THEOREM 2

Recall: Given $A \in \mathrm{SL}(2,\mathbb{Z})$, $g \in C^\infty(\mathbb{T}^2)$ and $\epsilon \geq 0$. $Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm$

Dynamics : $\Phi_\epsilon = \phi_\epsilon \circ A$ and $U_\epsilon = \mathrm{e}^{-\frac{i\epsilon}{\hbar}\mathrm{Op}^{\mathrm{W}} g} M(A)$.

THEOREM Let $f, g \in C^\infty(\mathbb{T}^2)$ be bounded and analytic in a strip of width δ_0. There exists

$$\Gamma_\epsilon \geq \gamma_0, \quad \lim_{\epsilon \to 0} \Gamma_\epsilon = \gamma_0,$$

such that, for all $0 < \nu$, there exists $J_0 > 0$ so that for all $J \geq J_0$ and for all $t \in \mathbb{Z}$,

$$U_\epsilon^{-t}\,\mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^t - \mathrm{Op}^{\mathrm{W}}(f \circ \Phi_\epsilon^t) = \hbar \sum_{1 \leq j < J} \hbar^{j-1} \mathrm{Op}^{\mathrm{W}}(\mathcal{L}_j^t f) + \hbar^J \rho_J(f, \epsilon, \hbar, t)$$

where,

$$\| \partial^\beta \mathcal{L}_j^t f \|_\infty \leq \| f \|_{\infty,\delta_0} C_{j,\beta} \mathrm{e}^{t\Gamma_\epsilon(|\beta| + \frac{3}{2}j)}, \quad \forall t \in \mathbb{Z}.$$

and, for $0 \leq |t| \leq (\frac{2}{3\Gamma_\epsilon} - \nu)|\ln \hbar|$

$$\| \hbar^J \rho_J(f, \epsilon, \hbar, t) \|_{\mathcal{H}_\hbar \to \mathcal{H}_\hbar} \to 0.$$

COHERENT STATE PROPAGATION

Constructing coherent states: the recipe

Ingredients : $\varphi \in \mathcal{S}(\mathbb{R})$, $0 < \sigma < 1$, $a \in \mathbb{R}^2$.

Construction: For $y \in \mathbb{R}$,

$$\varphi_\hbar(y) = \frac{1}{\hbar^{\sigma/2}}\varphi(\frac{y}{\hbar^\sigma}), \quad \varphi_{a,\hbar}(y) = U(a)\varphi_\hbar(y) \in \mathcal{S}(\mathbb{R}).$$

With P defined as $P = \sum_{n\in\mathbb{Z}^2}(-1)^{n_1 n_2}U(n)$, it is a fact that

$$P\mathcal{S}(\mathbb{R}) = \mathcal{H}_\hbar.$$

Define finally

$$\tilde{\varphi}_{a,\hbar} = S\varphi_{a,\hbar} \in \mathcal{H}_\hbar.$$

Done.

PROPOSITION :

$$\langle \tilde{\varphi}_{a,\hbar}, \mathrm{Op}^{\mathrm{W}} f \ \tilde{\varphi}_{a,\hbar}\rangle_{\mathcal{H}_\hbar} \overset{N\to+\infty}{\longrightarrow} f(a)$$

Evolving coherent states:

THEOREM

$$\langle \tilde{\varphi}_{a,\hbar}, U_\epsilon^{-t}\, \mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^{t}\, \tilde{\varphi}_{a,\hbar} \rangle_{\mathcal{H}_\hbar} \overset{N \to +\infty}{\longrightarrow} \int_{\mathbb{T}^2} f(x)dx.$$

provided

$$(\min(\sigma, 1-\sigma) + \nu)\, \frac{1}{\gamma_\epsilon} |\ln \hbar| \leq |t| \leq (\frac{2}{3\Gamma_\epsilon} - \nu)|\ln \hbar|.$$

Recall that

$$\gamma_\epsilon \leq \gamma_0 \leq \Gamma_\epsilon$$

and that

$$\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma_0 = \lim_{\epsilon \to 0} \Gamma_\epsilon$$

so that the time window is non-trivial.

The basic ingredient again:

DEFINITION A sequence $\varphi_N \in \mathcal{H}_\hbar$ is said to localize at $a \in \mathbb{T}^2$ if

$$\langle \varphi_N, \mathrm{Op}^{\mathrm{W}} f \, \varphi_N \rangle \overset{N \to +\infty}{\longrightarrow} f(a), \qquad \forall f \in C^\infty(\mathbb{T}^2).$$

This implies there exists $r_\hbar \to 0$ so that

$$\int_{|x-a| \geq r_\hbar} |\langle \varphi_N, \tilde{\varphi}_x \rangle|^2 \frac{dx}{2\pi\hbar} \overset{N \to +\infty}{\longrightarrow} 0.$$

THEOREM Assume that $\varphi_N \in \mathcal{H}_\hbar$ concentrates on some point $a \in \mathbb{T}^2$ with speed $r_\hbar$ such that

$$r_\hbar \leq \hbar^{1/2-\sigma},$$

with $\sigma > 0$. Then, if σ is small enough, there exists $t_\hbar \to \infty$ and $\epsilon(\sigma) > 0$ such that for all $|\epsilon| \leq \epsilon(\sigma)$

$$\left\langle \varphi_N, U_\epsilon^{-t} Op^W(f) U_\epsilon^t \varphi_N \right\rangle \to \int_{\mathbb{T}^2} f(x)\, dx, \qquad \hbar \downarrow 0. \tag{3}$$

BACK TO THEOREM 3

DEFINITION A sequence $\varphi_N \in \mathcal{H}_\hbar$ is said to localize at $a \in \mathbb{T}^2$ if

$$\langle \varphi_n, \mathrm{Op}^{\mathrm{W}} f \, \varphi_N \rangle \overset{N \to +\infty}{\longrightarrow} f(a), \qquad \forall f \in C^\infty(\mathbb{T}^2).$$

This implies there exists $r_\hbar \to 0$ so that

$$\int_{|x-a| \geq r_\hbar} |\langle \varphi_N, \tilde{\varphi}_x \rangle|^2 \frac{dx}{2\pi\hbar} \overset{N \to +\infty}{\longrightarrow} 0.$$

THEOREM Let U_ϵ be as before. There exists $\delta_0 > 0$ with the following property. Let ψ_N be a sequence of eigenfunctions of U_ϵ that localizes at some point $a \in \mathbb{T}^2$. Then $r_\hbar \geq \hbar^{\frac{1}{2} - \delta_0}$.

This gives some weak control on the way eigenfunctions may concentrate. If they do so, they must do it slowly! We expect to be able to improve the exponent $\frac{1}{2} - \delta_0$ (Faure and Nonnenmacher announced a result in this direction).