Chaotic dynamics of a free particle interacting with a harmonic oscillator

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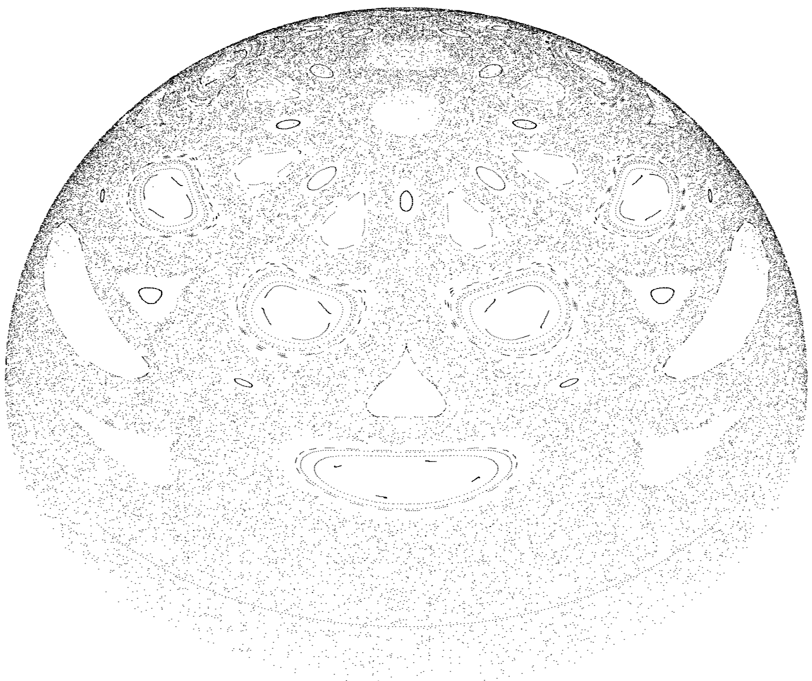
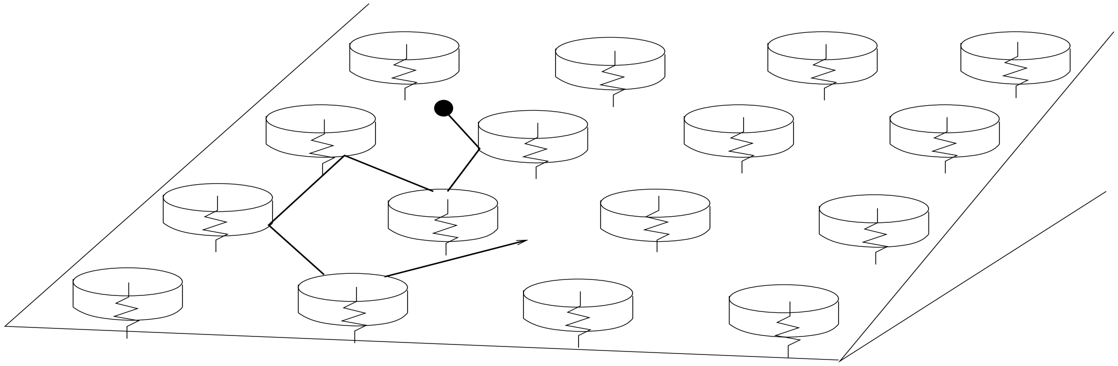


Ohm's law and the pinball machine

- Ohm's law: $V = RI$ or $\vec{E} = \rho \vec{j}$ or $\vec{v} = \frac{\mathrm{q}\tau}{m}\vec{E}$.

$$m\frac{d\vec{v}}{dt} = \mathrm{q}\vec{E} - \frac{m}{\tau}\vec{v}, \qquad \vec{v}(t) \sim \frac{\mathrm{q}\tau}{m}\vec{E} \quad (t \to \infty).$$

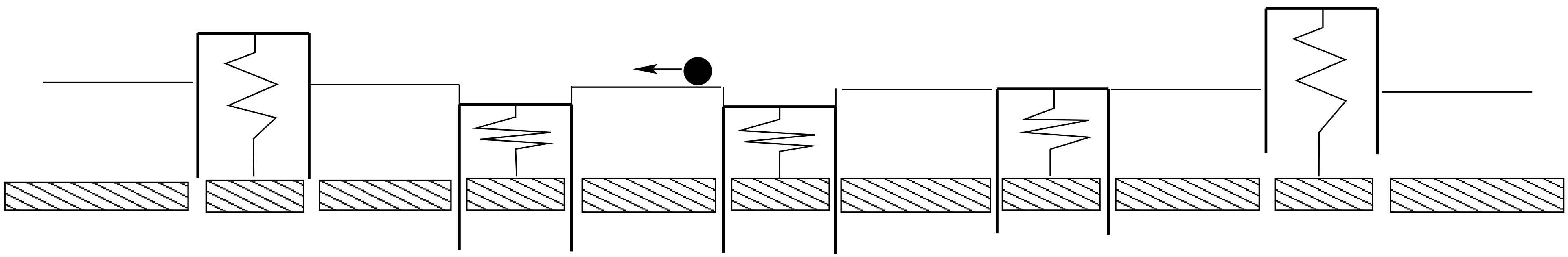
- The pinball machine (or the inelastic Lorentz gas)



Towards a Hamiltonian model for Ohm's law?

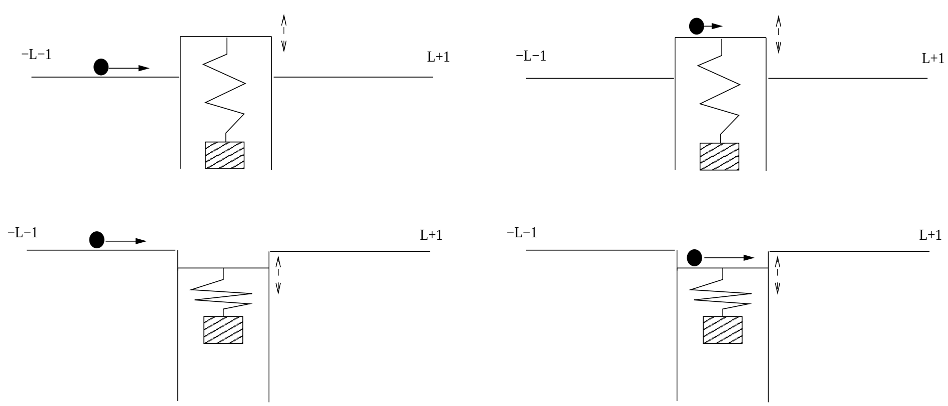
(see, e.g., L. Bruneau and S.D.B., A Hamiltonian model for linear friction in a homogeneous medium, CMP **229**, 511-542, 2002)

A 1-d inelastic Lorentz gas



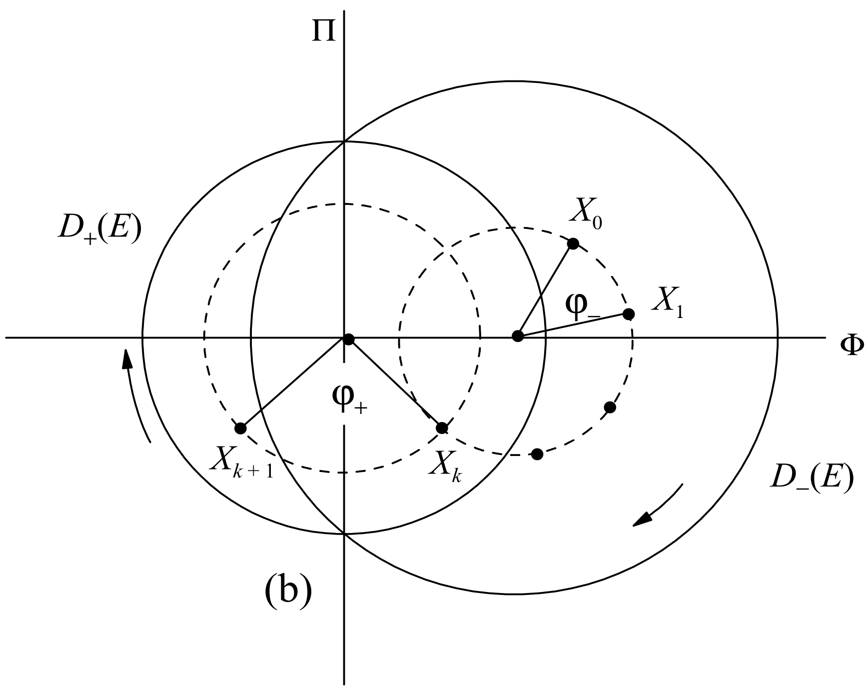
One can imagine putting the system on a slope, or that the particle is charged and an electric field is applied.

Two examples of what may happen in one cell:



The dynamics in one cell

- Phase space : $q \in]-1-L/2, 1+L/2[,\ p, \Phi, \Pi \in \mathbb{R}$
- Hamiltonian: $H_\alpha(q,p,\Phi,\Pi) = \frac{1}{2}\left(p^2+\Pi^2+\Phi^2\right) - \alpha\Phi\chi(q)$, where χ is the characteristic function of $[-1,1]$. Three parameters: α, L and E.
- The dynamics from the oscillator's point of view, at fixed $E > 0$:

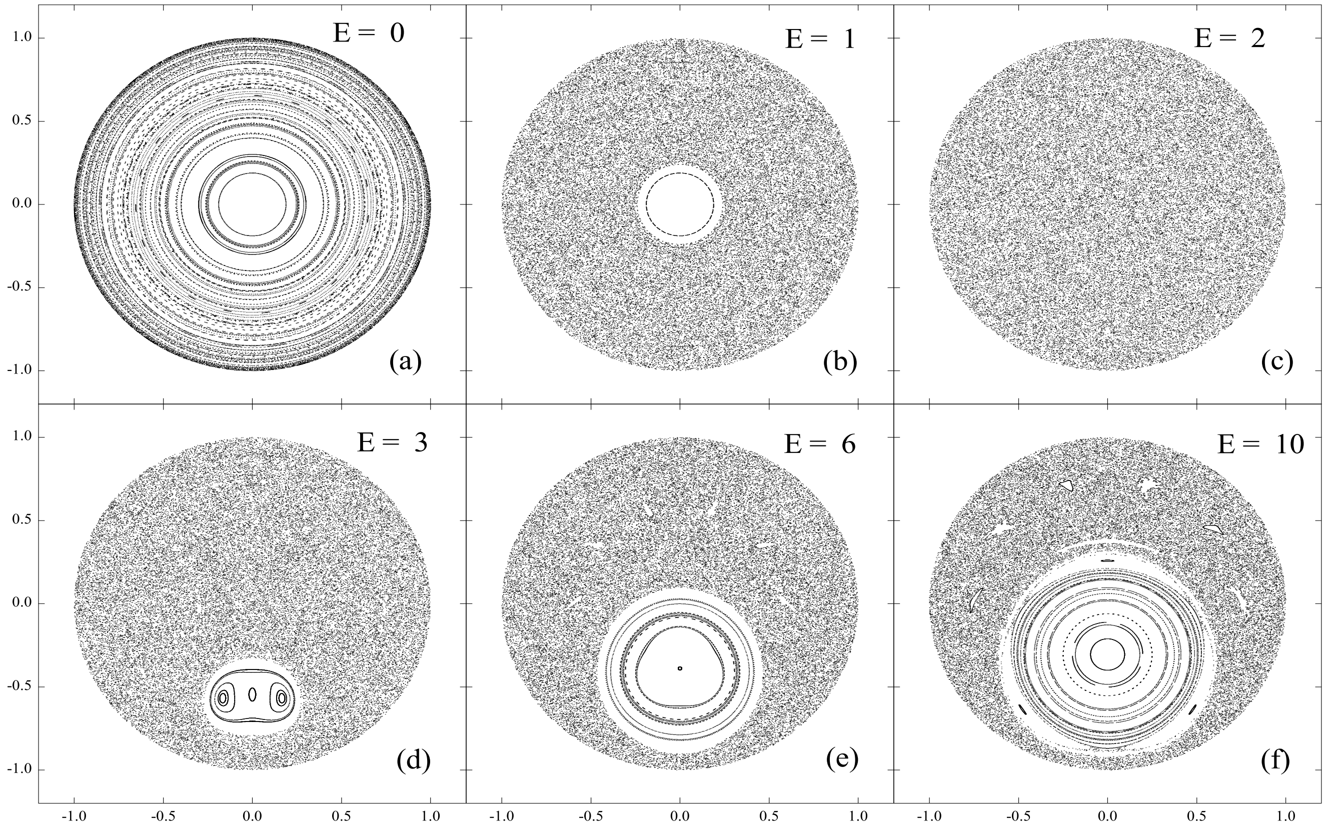


$D_-(E)$: disc at $\Phi = \alpha, \Pi = 0$, of radius $\sqrt{2E+\alpha^2}$.

$D_+(E)$: disc at origin, of radius $\sqrt{2E}$

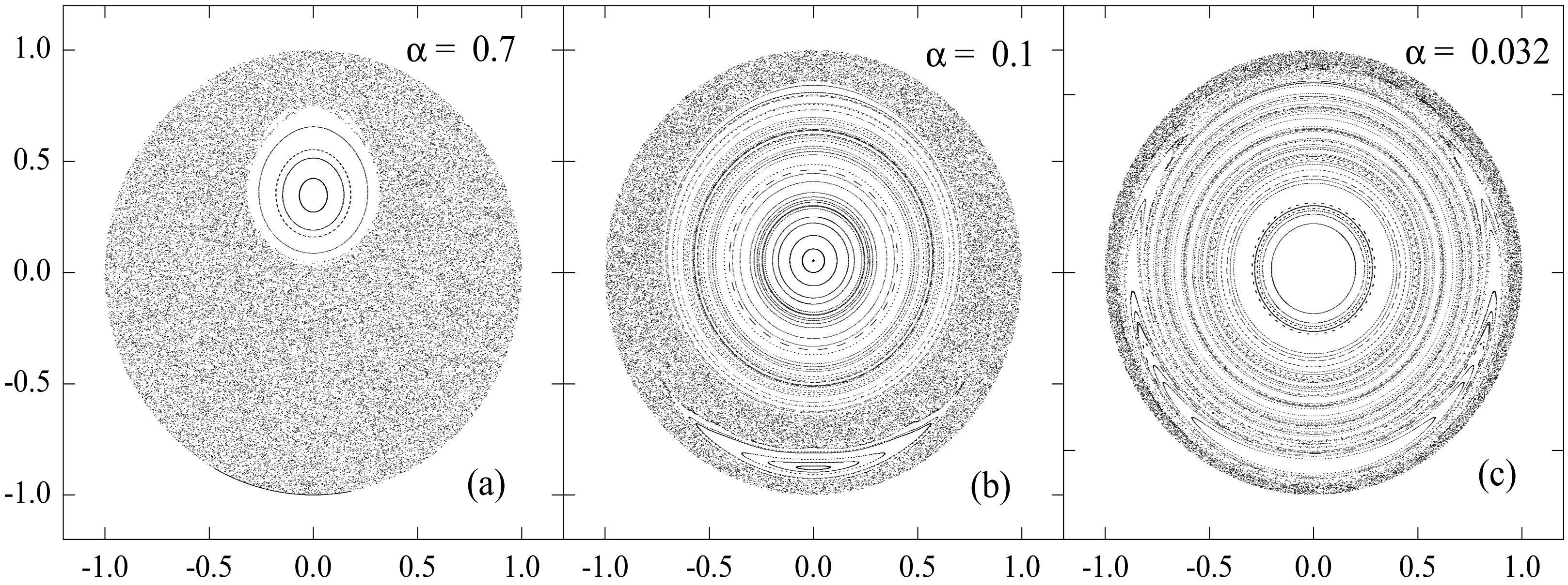
Numerical observations 1: varying E

Oscillator phase diagrams for a system with $\alpha = 2, L = 2.37$. Shown are 100 randomly chosen orbits, each visiting 500 times the Poincaré section at $q = 0$.



Numerical observations 2: varying α

Oscillator phase diagrams for a system with $E = 0.85, L = 1$ and $\alpha = 0.7, 0.1, 0.032$. Shown are 100 randomly chosen orbits, each visiting 500 times the Poincaré section at $q = 0$.



Main features of the dynamics

Depending on the values of the three system parameters α, L, E one observes the following types of behaviour:

- The phase space has one ergodic, chaotic component.
- Coexistence of a chaotic with a totally integrable component, Void I. The boundary between the two is “clean”, without KAM type structures.
- One large, prominent elliptic island along the Π-axis: Void II.
- One-parameter families of parabolic periodic orbits, similar to bouncing ball modes in billards.
- Other structures still!

The origin of the chaos: it's a linked twist map

CLAIM: The dynamics is a succession of non-aligned shears, which is a known source of hyperbolicity, hence of chaos.

INDEED:

- The simplest example: the composition of two linear shears (i.e. parabolic linear maps) on the torus ($x, y \in [0,1]\,, a, b \in \mathbb{Z}$)

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ay \\ y \end{pmatrix} \text{ and } T_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y + bx \end{pmatrix}$$

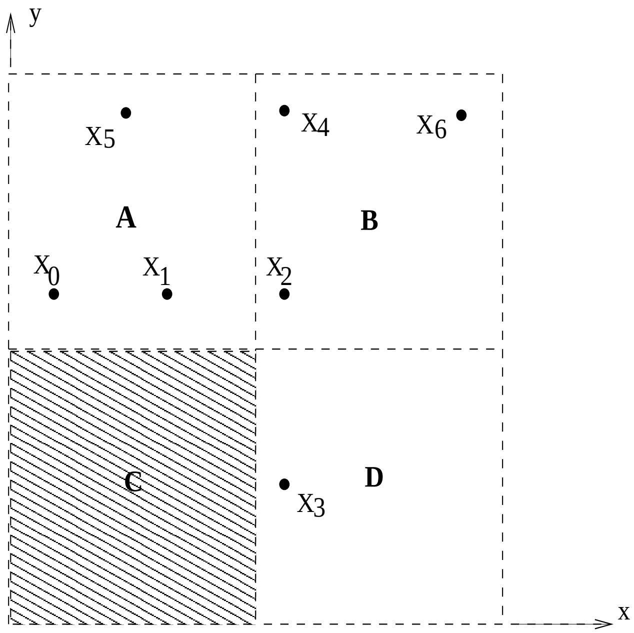
  yields a linear hyperbolic map

$$T_2 \circ T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & a \\ b & 1 + ab \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{if} \quad |2 + ab| > 2.$$

- A generalization of this are linked toral twist maps (Burton-Easton (1980) and Przytycki (1983)). They proved ergodicity for maps $T = T_2 \circ T_1$, with, for example, T_1 and T_2 of the following type $(x, y \in [0, 1])$:

$$T_1(x, y) = (x + 2y\chi_{[1/2,1]}(y), y), \quad T_2(x, y) = (x, y + 4x\chi_{[1/2,1]}(x)).$$

Here $\chi_{[1/2,1]}$ is the characteristic function of $[1/2, 1]$.



In A, apply T_1 (T_2 is trivial).

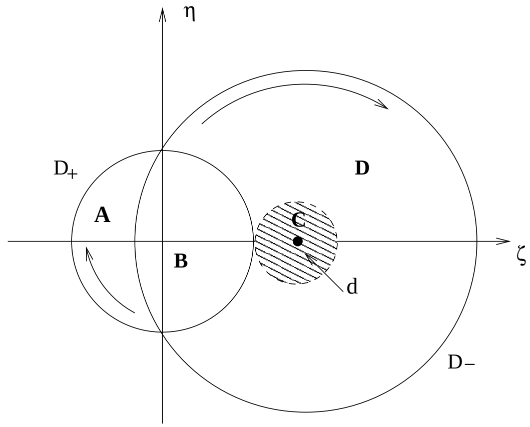
In D, apply T_2 (T_1 is trivial).

In B alternate them.

In C, T is the identity.

On $A \cup B \cup C$, T is ergodic.

- And this is our situation:



In $A = D_+ \setminus D_-$, apply T_+.

In $D = D_- \setminus D_+$, apply T_-.

In $B = D_+ \cap D_-$ alternate them.

In C, T is completely integrable.

Here, in scaled coordinates $\zeta = \frac{\Phi}{\sqrt{2E+\alpha^2}}, \eta = \frac{\Pi}{\sqrt{2E+\alpha^2}}$, one has

$$T_+(r_+, \theta_+) = (r_+, \theta_+ + \varphi_+(r_+)), \quad T_-(r_-, \theta_-) = (r_-, \theta_- + \varphi_-(r_-)),$$

where

$$a_- = \frac{2}{\sqrt{2E + \alpha^2}}, \quad a_- = \frac{L}{\sqrt{2E + \alpha^2}}, \quad d = \frac{\alpha}{\sqrt{2E + \alpha^2}}$$

$$\varphi_-(r_-) = \frac{a_-}{\sqrt{1 - r_-^2}}, \quad \varphi_+(r_+) \frac{a_+}{\sqrt{1 - d^2 - r_+^2}}$$

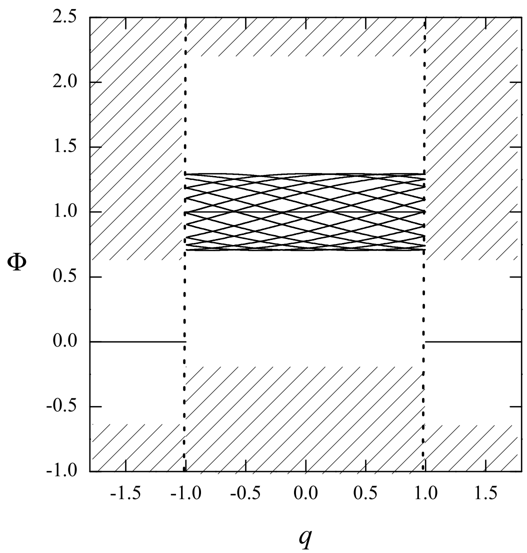
RULES OF THUMB:

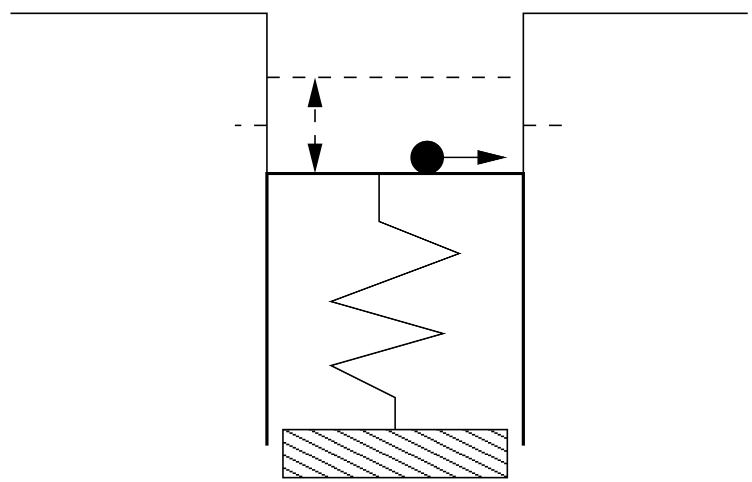
- One expects chaos when a_- and a_+ are large, since then the shears are large. So in particular L large should yield chaos.
- On the edge of either disc, the shears are always large. So trajectories there should be unstable.
- In physical terms, this means one expects instability in the motion when the particle is moving slowly. This means the time it takes to cross the circle is long compared to the oscillator period. Small changes in particle speed lead to large changes in the crossing time, hence to a large uncertainty on the oscillator position: hence this yields the unpredictability of the motion.

Void I: a region of complete integrability

SOURCE : There exists an open set of initial conditions with orbits forever trapped in the interaction region $q \in [-1, 1]$ iff

$$E < \frac{1}{2}\alpha^2 \Leftrightarrow d \geq \frac{1}{\sqrt{2}} \Leftrightarrow (d, 0) \notin D_+.$$

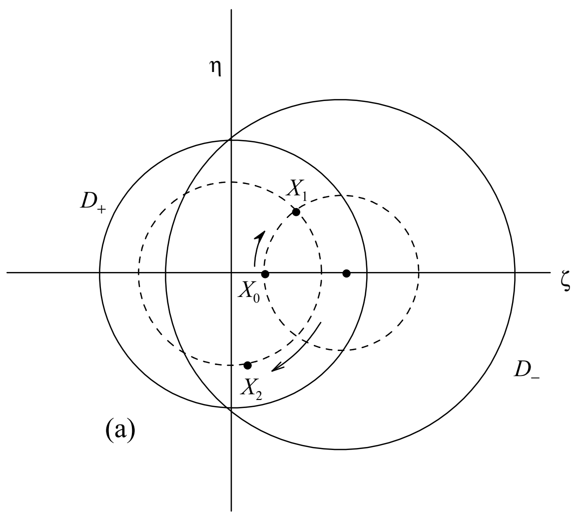


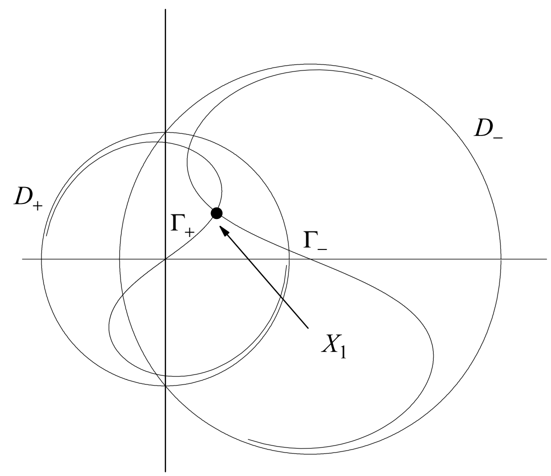


The edge is clean: no KAM structures (up to numerical accuracy)

Void II: a prominent elliptic island

SOURCE : The simplest imaginable periodic orbits in which the particle goes around the full circle exactly once per period, crossing $q = \pm 1$ without ever bouncing back.

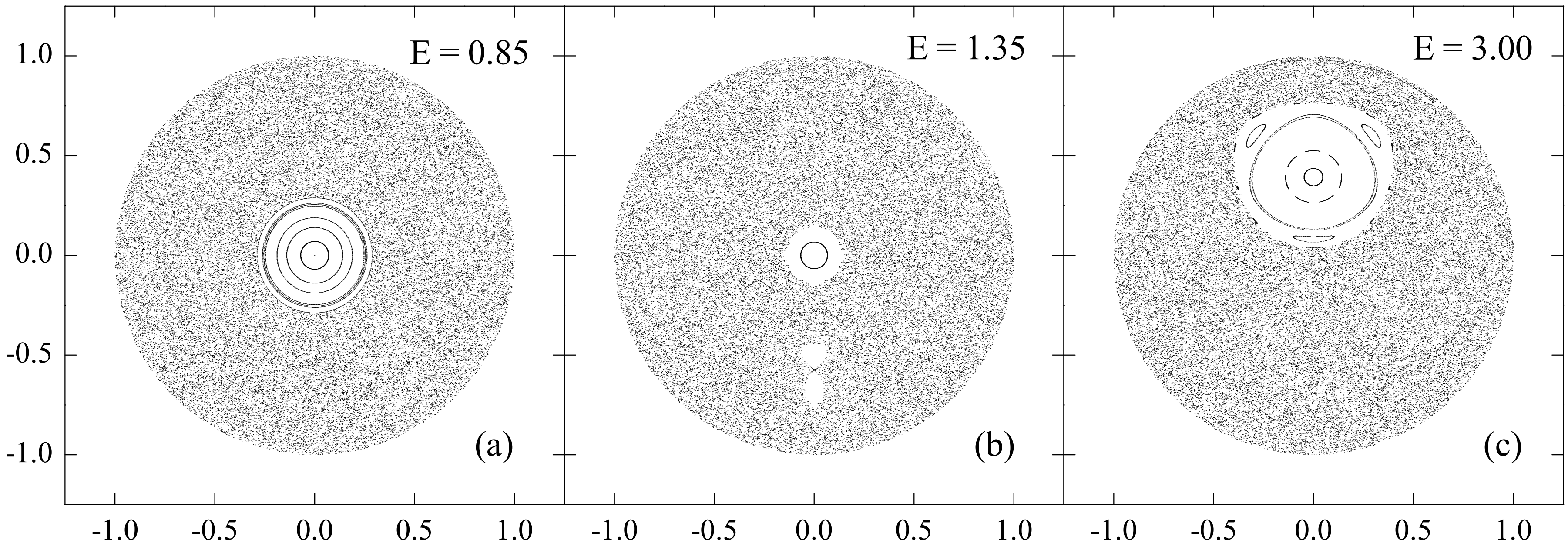




There is an infinite number of such orbits, giving rise to fixed points in our phase diagrams, accumulating at $\zeta = d \pm 1$, $\eta = 0$. There stability can be explicitly determined.

Some more examples

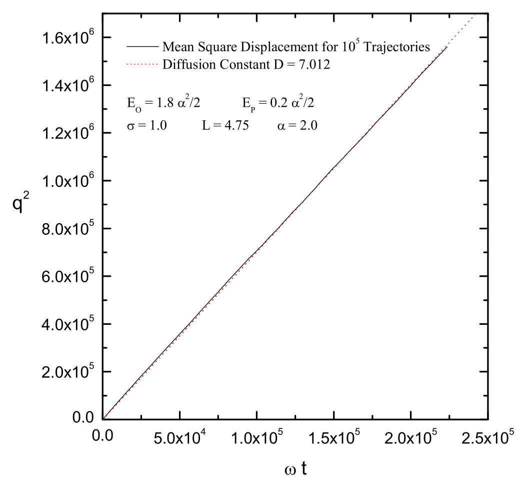
$$\alpha = 2 \qquad L = 2.105$$



In the second of these figures, there is a fixed point of the "lozenge" type in between the two small elliptic islands. The trace of its stability matrix was computed to be -2.026.

Outlook

- Can one prove the claims on ergodicity, and the existence of a mixed phase space with two clear components, one integrable, one chaotic?
- Can one use some of the insights gained to understand the behaviour of a particle in the 1-d pinball machine?



Work in progress. . .