Quantum chaos: what's that?

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Nato Summer School on Equidistribution in Number Theory

Montreal – July 2005

INTRODUCTION

GOAL (i) Provide some (physics) background to the lectures of Lindenstrauss, Rudnick and Venkatesh on “quantum equidistribution.”

(ii) Explain in which sense the questions they address are different sides of the same coin.

READING

S. De Bièvre, *Quantum chaos: a brief first visit*, “Cuernavaca Summer School in Harmonic Analysis and Mathematical Physics”, juin 2000, (58 pages), mp_arc 01-207, Contemporary Mathematics **289**, 161-218 (2001).

S. De Bièvre, *Recent results on quantum map eigenstates*, Proceedings of QMATH 9, Giens, september 2004.

Both available at : **http://euler.dms.umontreal.ca/∼debievre/**

PREREQUISITES

Multivariable calculus, matrix algebra and some imagination.

THE PLAN

LECTURE 1 : A CRASH COURSE IN CLASSICAL MECHANICS

LECTURE 2: A CRASH COURSE IN QUANTUM MECHANICS

LECTURE 3: TWO WORDS ON SEMICLASSICAL ANALYSIS

LECTURE 4: QUANTUM CHAOS ON THE TORUS,

EQUIPARTITION AND NUMBER THEORY

LECTURE 1 : A CRASH COURSE IN CLASSICAL MECHANICS

NEWTON $\uparrow$ Remember "mass times acceleration equals force"?

$$m\ddot{q}(t) = F(q(t)),\ q(0) = q,\ \dot{q}(0) = v. \tag{1}$$

Given: the **force** $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the **mass** m and the **initial data** $q, v \in \mathbb{R}^n$.

Unknown: the motion of the system $t \in \mathbb{R} \mapsto q(t) \in \mathbb{R}^n$.

SO: Classical mechanics is about solving coupled non-linear second order ordinary differential equations. . . **of a special type**:

A CONSERVATIVE FORCE is a force $F(q) = -\nabla V(q)$ for a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ (the **potential**). Why "conservative"?

PROPOSITION *Energy conservation* Let

$$E : (q, v) \in \mathbb{R}^n \times \mathbb{R}^n \to \frac{1}{2}mv^2 + V(q) \in \mathbb{R}. \quad \leftarrow \ \mathbf{ENERGY}$$

Let $t \in \mathbb{R} \mapsto q(t) \in \mathbb{R}^n$ be a solution to (1), then, for all $t \in \mathbb{R}$,

$$E(q(t), \dot{q}(t)) = E(q(0), \dot{q}(0)). \quad \leftarrow \ \mathbf{ENERGY\ CONSERVATION}$$

EXAMPLES to keep in mind.

- The Kepler problem: $d = 3$, $V(q) = -\frac{GmM}{\|q\|}$.
- Central potentials: $V(q) = W(\| q \|)$.
- Harmonic systems: $V(q) = \frac{1}{2} m q^T \Omega^2 q$, where Ω^2 is a positive definite n by n matrix. Now $\ddot{q} = -\Omega^2 q$ and

$$q(t) = \cos \Omega t q + \frac{\sin \Omega t}{\Omega} v.$$

In general, the behaviour of the solutions depends on the type of potential one considers. In particular, if $V(q) \to +\infty$ when $|q| \to +\infty$, the motion is bounded:

$$\sup_{t \in \mathbb{R}} |q(t)| \leq C < +\infty.$$

Indeed, for all t

$$V(q(t)) \leq \frac{1}{2m} \dot{q}(t)^2 + V(q(t)) = E(q(0), \dot{q}(0)).$$

NEWTON $\downarrow$ HAMILTON$\uparrow$

Introduce the **Hamiltonian**

$$H : x = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \frac{p^2}{2m} + V(q) \in \mathbb{R},$$

and observe that Newton's equation (1) is equivalent to the first order system of differential equations called Hamilton's equations

$$\dot{q}(t) = \frac{p(t)}{m} = \frac{\partial H}{\partial p}(x(t)), \quad \dot{p} = -\nabla V(q(t)) = -\frac{\partial H}{\partial q}(x(t)), \tag{2}$$

with initial conditions $x(0) = (q, mv)$. One defines the corresponding flow $\Phi_t^H : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ by $\Phi_t^H(x) = (q(t), p(t))$, where $q(0) = q, p(0) = p$.

Question: "What is the big deal?"

Answer 1 "It's pretty! Look!"

$$\begin{aligned}\frac{d}{dt}f(q(t),p(t)) &= \partial_q f(x(t))\dot{q}(t) + \partial_p f(x(t))\dot{p}(t)\\ &= \partial_q f(x(t))\partial_p H(x(t)) - \partial_p f(x(t))\partial_q H(x(t))\\ &:= \{f,H\}(x(t)),\end{aligned}$$

where I introduced the **Poisson bracket**

$$\{\cdot,\cdot\} : (f,g) \in C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \mapsto \{f,g\} \in C^\infty(\mathbb{R}^{2n}),$$

$$\text{with} \qquad \{f,g\}(x) = \partial_q f(x)\partial_p g(x) - \partial_p f(x)\partial_q g(x).$$

Properties: thanks to (3)-(4) below, $C^\infty(\mathbb{R}^{2n})$ is a Lie-algebra.

$$\{f,g\} = -\{g,f\} \quad \text{(Anti-symmetry)} \tag{3}$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad \text{(Jacobi identity)} \tag{4}$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \text{(Derivation)} \tag{5}$$

Answer 2 "It's also useful, look!"

DEFINITION A **constant of the motion** is a function $f \in C^\infty(\mathbb{R}^{2n})$ which is constant along the solutions of (2): $f \circ \Phi_t^H = f$ for all $t \in \mathbb{R}$.

Clearly f is a constant of the motion iff $\{f, H\} = 0$. Consequently H itself is always a constant of the motion. Constants of the motion are a Good Thing: the more, the merrier! Indeed, if $f_1, f_2, \dots f_k$ are constants of the motion, and $c \in \mathbb{R}^k$, then the flow Φ_t^H leaves their common level surface

$$\Sigma(c) = \{x \in \mathbb{R}^{2n} | f_j(x) = c_j, j = 1 \dots k\}$$

invariant. So, if they are functionally independent, the flow takes place on a $(2n-k)$-dimensional surface in $\mathbb{R}^{2n}$.

PROPOSITION If f and g are constants of the motion, then so is $\{f, g\}$.

So, the Poisson bracket is a machine for producing constants of the motion. . . .

EXAMPLE Central potentials: $V(q) = W(\| q \|)$, $q \in \mathbb{R}^3$. Then the three components $\ell_i(x) = (q \wedge p)_i$, $i = 1 \dots 3$ of the **angular momentum** vector are constants of the motion. They satisfy $\{\ell_1, \ell_2\} = \ell_3$ plus cyclic permutation. Noticed the appearance of $\mathrm{so}(3)$?

Answer 3: It has nice generalizations in various directions. (i) First, any function $f \in C^\infty(\mathbb{R}^{2n})$ generates a flow $\Phi_t^f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as follows: $\Phi_t^f(x) = x(t)$ where $t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}^{2n}$ solves

$$\dot{q}(t) = \frac{\partial f}{\partial p}(x(t)), \quad \dot{p} = -\frac{\partial f}{\partial q}(x(t)), \quad x(0) = x = (q,p). \tag{6}$$

EXAMPLES • On $\mathbb{R}^6$, let $f(x) = q_1 p_2 - q_2 p_1 = \ell_3(x)$. Then

$$\dot{q}_1(t) = -q_2(t),\ \dot{q}_2(t) = q_1(t), \dot{p}_1(t) = -p_2(t),\ \dot{p}_2(t) = p_1(t),\ \dot{q}_3 = 0 = \dot{p}_3.$$

Then $\Phi_t^f(x) = (R_t q, R_t p)$, where $R_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$

• On $\mathbb{R}^2$, let $f(q,p) = q^2/2$, then $\dot{q} = 0, \dot{p} = -q$, so $q(t) = q, p(t) = p - tq$ and

$$\Phi_t^{q^2/2}(q,p) = (q, p - tq) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

The maps Φ_t^f are **symplectic**:

DEFINITION: a diffeomorphism Φ of $\mathbb{R}^{2n}$ is symplectic if, for all $f, g \in C^\infty(\mathbb{R}^{2n})$,

$$\{f \circ \Phi, g \circ \Phi\} = \{f, g\} \circ \Phi.$$

(ii) Second, it is the start of symplectic geometry.... Roughly, a symplectic manifold is a manifold N so that the vector space $C^\infty(N)$ is equipped with a composition law, called a Poisson bracket, satisfying (3)-(5), as well as the non-degeneracy condition: $\{f, g\} = 0, \forall g \in C^\infty(N) \Rightarrow f$ is a constant

FACT Locally, there always exist coordinates (called Darboux coordinates) on N so that the Poisson bracket takes on the form it has on $\mathbb{R}^{2n}$.

EXAMPLES (a) Every **cotangent bundle** has a natural symplectic structure. The geodesic flow on a Riemannian manifold can be viewed as a Hamiltonian flow on the cotangent bundle of the manifold. ($\longrightarrow$ **Lindenstrauss, Venkatesh, Marklof**)

Example: The Poincaré half plane. This is the upper half plane $q = (q_1, q_2) \in \mathbb{R} \times \mathbb{R}^+$, viewed as a Riemannian manifold with line element:

$$\mathrm{d}s^2 = q_2^{-2}(\mathrm{d}q_1^2 + \mathrm{d}q_2^2).$$

This means that, if $\gamma : t \in [a, b] \mapsto \gamma(t) = (q_1(t), q_2(t)) \in \mathbb{R} \times \mathbb{R}^+$ is a smooth curve, then the length of this curve is defined by

$$s(t) = \int_a^t \frac{\mathrm{d}s}{\mathrm{d}t'}(t')\mathrm{d}t' = \int_a^t q_2^{-1}(t')\sqrt{\dot{q}_1^2(t') + \dot{q}_2(t')^2}\mathrm{d}t'.$$

Now let $p = (p_1, p_2) \in \mathbb{R}^2$ and define $H(q, p) = \frac{1}{2}q_2^2(p_1^2 + p_2^2)$. Then

$$\dot{q}_1 = q_2^2 p_1, \quad \dot{q}_2 = q_2^2 p_2, \quad \dot{p}_1 = 0, \quad \dot{p}_2 = -q_2(p_1^2 + p_2^2).$$

Exercise: Taking second derivatives, show that you obtain the geodesic equations of motion on the Poincaré half plane. Solve them. Use $\sigma = \ln q_2$. Note that

$$H(q(t), p(t)) = \frac{1}{2}\frac{\dot{q}_1(t)^2 + \dot{q}_2(t)^2}{q_2(t)^2}.$$

(b) The **torus** $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Write $x = (q,p) \in [0,1[^2$ and define the Poisson bracket as on $\mathbb{R}^2$. For example,

$$\{\cos q \sin p, \sin q \cos p\} = \sin^2 q \sin^2 p - \cos^2 q \cos^2 p.$$

For $n = 1$, the group SL$(2,\mathbb{Z})$ acts on $\mathbb{T}^2$ by symplectic automorphisms. These maps can have rich behaviour, despite their apparent simplicity ($\longrightarrow$ **SDB, Rudnick**).

(c) The two-sphere with coordinates (θ, ϕ). Then

$$\{f,g\}(\theta,\phi) = -\frac{1}{\sin\theta}\left(\partial_\theta f \partial_\phi g - \partial_\phi f \partial_\theta g\right)(\theta,\phi).$$

Two extremes: complete integrability and ergodicity

Let Φ_t^H be a Hamiltonian flow on $\mathbb{R}^{2n}$. It is said to be **completely integrable** if there exist n functionally independent constants of the motion $f_1 \ldots f_n$, with $\{f_i, f_j\} = 0$. Supposing they are compact, the level surfaces $\Sigma(c) = \{x \in \mathbb{R}^{2n} | f_j(x) = c_j\}$ are then **n-dimensional** tori on which the Hamiltonian flow acts as a translation flow (Liouville-Arnold).

Example $H(x) = \frac{p^2}{2m} + W(\| q \|)$, $q, p \in \mathbb{R}^3$. Take $f_1 = H, f_2 = \ell^2, f_3 = \ell_z$.

It is said to be **ergodic** if a typical trajectory explores the entire $\mathbf{2n-1}$**-dimensional energy surface**. More precisely, given a typical trajectory on the energy surface Σ_E, the time it spends in any subset B of Σ_E is asymptotically equal to the relative size of that set in the full energy surface:

$$\lim_{T\to\infty} \frac{|\{0 \leq t \leq T | x(t) \in B\}|}{T} = \frac{|B|}{|\Sigma_E|}.$$

SUMMARY

For later purposes, remember the following populist slogans:

“The dynamics of many physical systems is given by a Hamiltonian flow Φ_t^H on $\mathbb{R}^{2n}$. The maps Φ_t^H are symplectic, which just means they leave the Poisson bracket invariant.”

“It is therefore of great importance to study Hamiltonian flows Φ_t^f on arbitrary symplectic manifolds and to understand them thoroughly.”

“Since in such generality, only rather soft general statements can be made, it is of interest to understand relevant examples, such as geodesic flows on Riemannian manifolds, which display a rich variety of behaviour.”

“Since this is often still very hard, one can hope to get insight in various issues by studying discrete dynamical systems, meaning iterations of a fixed map Φ, where Φ is a symplectic map on a symplectic manifold (Example: an element of SL$(2, \mathbb{Z})$ on $\mathbb{T}^2$).”

LECTURE 2: A CRASH COURSE IN QUANTUM MECHANICS

NEWTON $\downarrow$ **HAMILTON** $\leftrightarrow$ **SCHRÖDINGER** $\uparrow$

SCHRÖDINGER Newton's second law iznogood. To study the motion of a particle in a potential $V : \mathbb{R}^3 \rightarrow \mathbb{R}$, solve not Newton's equation nor Hamilton's equation, but my equation:

$$\mathrm{i}\hbar \frac{\partial \psi_t}{\partial t}(y) = -\frac{\hbar^2}{2m}\Delta\psi_t(y) + V(y)\psi_t(y), \qquad \psi_0 = \phi, \qquad \| \phi \| = 1.$$

Here $\hbar$ is a physical constant called Planck's constant and the unknown is the function $t \in \mathbb{R} \mapsto \psi_t \in L^2(\mathbb{R}^3, \mathbb{C}; \mathrm{d}y)$. One calls ψ_t **the wavefunction** of the particle at time t. It contains all information about the particle's state. It "replaces" $(q(t), p(t))$, which played this role in classical mechanics.

To extract information about the particle from the wavefunction, proceed as follows:

The probability to find the particle at time t inside a set $B \subset \mathbb{R}^3$ is

$$\int_B \mid \psi_t \mid^2 (y)\mathrm{d}y$$

and the probability that its momentum falls inside some set $C \subset \mathbb{R}^3$ is

$$\int_C \mid \tilde{\psi}_t \mid^2 (p)\mathrm{d}p,$$

where $\tilde{\psi}_t$ is the Fourier Transform of ψ_t:

$$\tilde{\psi}_t(p) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\frac{px}{\hbar}} \psi_t(y)\mathrm{d}y.$$

In particular, the mean position and momentum of the particle are

$$\int_{\mathbb{R}^3} y_j \mid \psi_t \mid^2 (y)\mathrm{d}y, \qquad \int_{\mathbb{R}^3} p_j \mid \tilde{\psi}_t \mid^2 (p)\mathrm{d}p.$$

EHRENFEST Brilliant! However, people may find this strange, so let's try to demystify this. Introduce the operators on $L^2(\mathbb{R}^3, \mathrm{d}y)$

$$P_j\psi_t(y) = \frac{\hbar}{\mathrm{i}}\frac{\partial\psi_t}{\partial y_j}(y), \quad Q_j\psi_t(y) = y_j\psi_t(y).$$

Then

$$\langle Q_j\rangle_t := \langle Q_j\psi_t, \psi_t\rangle = \langle \psi_t, Q_j\psi_t\rangle = \int_{\mathbb{R}^3} y_j \mid \psi_t \mid^2 (y)\mathrm{d}y$$

and

$$\langle P_j\rangle_t := \langle P_j\psi_t, \psi_t\rangle = \langle \psi_t, P_j\psi_t\rangle = \int_{\mathbb{R}^3} p_j \mid \tilde{\psi}_t \mid^2 (p)\mathrm{d}p$$

are the mean position and momentum. Here

$$\langle \phi, \psi\rangle := \int_{\mathbb{R}^3} \overline{\phi}(y)\psi(y)\mathrm{d}y.$$

Moreover, the "Canonical Commutation Relations" hold, namely

$$[Q_j, P_k] = \mathrm{i}\hbar\delta_{jk} \quad \Rightarrow \textbf{Heisenberg uncertainty principle}.$$

Now, if one defines $\hat{H} = \frac{P^2}{2m} + V(Q)$ one can rewrite the Schrödinger equation as

$$i\hbar\partial_t\psi_t = \hat{H}\psi_t, \quad \psi_0 = \phi.$$

Since, for all $\phi, \psi \in L^2(\mathbb{R}^3, \mathrm{d}y)$, $\langle \psi, \hat{H}\phi \rangle = \langle \hat{H}\psi, \phi \rangle$, you can solve it through $\psi_t = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t\hat{H}}\phi$. Consequently

$$\begin{aligned} \frac{d}{dt}\langle \psi_t, Q_j\psi_t \rangle &= \langle \partial_t\psi_t, Q_j\psi_t \rangle + \langle \psi_t, Q_j\partial_t\psi_t \rangle \\ &= \frac{1}{i\hbar}\langle \psi_t, [Q_j, \hat{H}]\psi_t \rangle = \frac{1}{m}\langle \psi_t, P_j\psi_t \rangle, \end{aligned}$$

and

$$\frac{d}{dt}\langle \psi_t, P_j\psi_t \rangle = \frac{1}{i\hbar}\langle \psi_t, [P_j, \hat{H}]\psi_t \rangle = -\langle \psi_t, \nabla_j V(Q)\psi_t \rangle.$$

Remember Hamilton's formulation of Newton's equation:

$$\dot{q} = p/m, \dot{p} = -\nabla V(q).$$

WARNING Whereas Hamilton's equations are a system of ordinary differential equations for the unknown $t \mapsto (q(t), p(t))$, the Ehrenfest equations

$$\frac{d}{dt}\langle\psi_t, Q_j\psi_t\rangle = \frac{1}{m}\langle\psi_t, P_j\psi_t\rangle, \qquad \frac{d}{dt}\langle\psi_t, P_j\psi_t\rangle = -\langle\psi_t, \nabla_j V(Q)\psi_t\rangle.$$

are not in general a system for $t \mapsto (\langle Q\rangle_t, \langle P\rangle_t)$!! Indeed, in general

$$\langle\psi_t, \nabla_j V(Q)\psi_t\rangle = \int_{\mathbb{R}^3} \nabla_j V(y)|\psi_t(y)|^2 \mathrm{d}y \neq \nabla_j V(\langle\psi_t, Q_j\psi_t\rangle).$$

A notable exception is the case of a harmonic potential $V(q) = \frac{1}{2}q^T\Omega^2 q$. Then

$$\frac{d}{dt}\langle\psi_t, Q_j\psi_t\rangle = \frac{1}{m}\langle\psi_t, P_j\psi_t\rangle, \qquad \frac{d}{dt}\langle\psi_t, P_j\psi_t\rangle = -\Omega^2_{jk}\langle\psi_t, Q_k\psi_t\rangle.$$

The mean position and momentum then follow the classical trajectories of the system in phase space. In particular

$$\langle Q\rangle_t = \cos\Omega t\langle Q\rangle_0 + \frac{\sin\Omega t}{\Omega}\langle P\rangle_0.$$

MORAL OF THE STORY: QUADRATIC HAMILTONIANS ARE PARTICULARLY SIMPLE

QUESTION How to solve the Schrödinger equation $\mathrm{i}\hbar\partial_t\psi_t = \hat{H}\psi_t,\ \psi_0 = \phi$?

ANSWER Look for an orthonormal basis of eigenfunctions of $\hat{H}$:

$$\hat{H}(\hbar)\psi_n^\hbar = E_n(\hbar)\psi_n^\hbar, \quad \text{then} \quad \psi_t = \sum_n \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n(\hbar)t}\langle\psi_n^\hbar, \phi\rangle\psi_n^\hbar.$$

Such a basis exists if $V(q) \to +\infty$ as $q \to \infty$. Studying the behaviour of the spectrum $E_n(\hbar)$ and of the eigenfunctions $\psi_n^\hbar$ leads to a wealth of interesting and hard problems. The eigenvalue equations are partial differential equations and explicit solutions are hardly ever available, except in some special cases, such as the Kepler problem and for harmonic potentials.

One interesting question is the asymptotic behaviour along sequences $E_n(\hbar)$ that converge to a fixed value E_c as $\hbar$ goes to zero: this is part of a field called **semi-classical analysis**, for which specific techniques have been developed (see Lecture 3).

SUMMING UP

In quantum mechanics the time evolution of a system is no longer given by a Hamiltonian flow Φ_t^H on a symplectic manifold (the **classical phase space**), but by a unitary flow $U_t = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$ on a complex Hilbert space $\mathcal{H}$ (the **quantum state space**). A typical example, beyond the ones given, is

$$\mathcal{H} = L^2(M, \mathrm{dvol}_g),\ \hat{H} = -\hbar^2\Delta_g \rightarrow \textbf{Lindenstrauss, Venkatesh}.$$

But one can also consider simpler examples, where the Hilbert space is finite-dimensional and the flow is replaced by the iteration of a fixed unitary map (so you obtain a $\mathbb{Z}$-action, rather than an $\mathbb{R}$-action) $\rightarrow$**Lecture 4, Rudnick**.

(Incidentally, in quantum computing, quantum cryptography and quantum information theory, one almost exclusively deals with finite dimensional Hilbert spaces (the N-fold tensor product of $\mathbb{C}^2$ with itself): a computation is then a certain product of unitaries.)

LECTURE 3: SEMICLASSICAL ANALYSIS

DIRAC How exciting. Did you notice the amazing analogy between

$\{q_j, p_k\} = \delta_{jk}$ and $\frac{1}{i\hbar}[Q_j, P_k] = \delta_{jk}$?

It looks as if in quantum mechanics the Lie-algebra of operators on the Hilbert space $L^2(\mathbb{R}^n)$ replaces the Lie-algebra of smooth functions on phase space $\mathbb{R}^{2n}$ of Hamiltonian mechanics. In classical mechanics the **observables** are represented by functions on phase space, in quantum mechanics by operators on a Hilbert space. Can anyone come up with a Lie-algebra homomorphism between these two? I mean, a map

$$\mathrm{O}p : f \in C^\infty(\mathbb{R}^{2n}) \to \mathrm{O}p(f) : \mathcal{D} \subset L^2(\mathbb{R}^n) \to \mathcal{D} \subset L^2(\mathbb{R}^n),$$

such that

$$\frac{1}{i\hbar}[\mathrm{O}p(f), \mathrm{O}p(g)] = \mathrm{O}p(\{f, g\}),$$

and such that $\qquad \mathrm{O}p(q_j) = Q_j, \mathrm{O}p(p_j) = P_j.$

VAN HOVE No, that does not exist (1957).

WEYL Well, it almost exists ($\pm$ 1930).

For

$$f(q,p) = \int_{\mathbb{R}^{2n}} \tilde{f}(a) e^{-\frac{i}{\hbar}(a_1 p - a_2 q)} \frac{da}{(2\pi\hbar)^n},$$

set

$$\mathrm{Op}^W(f) = \int_{\mathbb{R}^{2n}} \tilde{f}(a) e^{-\frac{\mathrm{i}}{\hbar}(a_1 P - a_2 Q)} \frac{\mathrm{d}a}{(2\pi\hbar)^n} \leftarrow \textbf{WEYL QUANTIZATION}$$

Then

$$\frac{1}{i\hbar}[\mathrm{Op}^W(f), \mathrm{Op}^W(g)] = \mathrm{Op}^W(\{f,g\}) + O(\hbar), \leftarrow \textbf{SEMICLASSICAL}$$

and there is no error term as long as f and g are polynomials of degree at most two in the q_j and the p_j. Quite explicitly, one has for example for $f(q,p) = h(q)$, respectively $g(q,p) = k(p)$

$$\mathrm{Op}^W(f) = h(Q) \quad \mathrm{Op}^W(g) = k(P).$$

and

$$\mathrm{Op}^{\mathrm{W}} q_j p_j = \frac{1}{2}(Q_j P_j + P_j Q_j) \neq Q_j P_j \leftarrow \textbf{ORDERING PROBLEM}.$$

A crucial, all important property of the Weyl quantization is the following theorem:

THEOREM (EGOROV) For all $f, g \in C^\infty(\mathbb{R}^{2n})$, one has, for all $t \in \mathbb{R}$

$$e^{\frac{i}{\hbar}\mathrm{Op}^W(g)t}\mathrm{Op}^W(f)e^{-\frac{i}{\hbar}\mathrm{Op}^W(g)t} = \mathrm{Op}^W(f \circ \Phi_t^g) + O_t(\hbar)$$

Moreover, if g is a quadratic function, the error term vanishes.

QUESTION Why is this useful?

ANSWER Note that $\mathrm{Op}^{\mathrm{W}} H = \hat{H}$ if $H(q,p) = \frac{p^2}{2m} + V(q)$. So, since the solution ψ_t of the Schrödinger equation $\mathrm{i}\hbar\partial_t\psi_t = \hat{H}\psi_t$ with initial condition $\psi_0 = \phi$ is $\psi_t = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}\,\phi$, we find that

$$\langle \psi_t, \mathrm{Op}^{\mathrm{W}}(f)\psi_t\rangle = \langle \phi, \mathrm{Op}^{\mathrm{W}}(f \circ \Phi_t^H)\phi\rangle + O_t(\hbar).$$

TO CONCLUDE If we know enough about the classical evolution Φ_t^H appearing in the right hand side, we can infer from it information about the quantum evolution in the left hand side, in the limit of small $\hbar$.

LECTURE 4 : QUANTUM CHAOS ON THE TORUS

QUANTUM CHAOS The semi-classical analysis of quantum systems having a chaotic Hamiltonian system as their classical limit.

TWO FAMILIES OF EXAMPLES

1. *Quantum* : Δ on $L^2(M, \mathrm{dvol}(g))$; $\Delta\psi_n = \lambda_n\psi_n$;
   Chaotic system : the geodesic flow on (M, g), a compact negatively curved manifold. This is a Hamiltonian flow on T^*M!
   Semi-classical : $\lambda_n \to +\infty$. $\leftarrow$ **Lindenstrauss, Venkatesh**

2. *Quantum* : quantum maps $=$ unitary maps on N dimensional spaces.
   Chaotic system : symplectic Anosov maps on the torus.
   Semi-classical : $N \to +\infty$. $\leftarrow$ **Rudnick, I**

THE BASIC QUESTION Relate the asymptotic behaviour of the eigenvalues and eigenfunctions to the statistical properties of the underlying classical dynamical system. In particular, what is the signature of chaos on the eigenfunctions and on the eigenvalues? I will talk only about the eigenfunctions.

THE CLASSICAL DYNAMICS: Continuous Automorphisms of the Torus (CAT)

Consider $A \in \mathrm{SL}(2,\mathbb{Z}),\ |\ \mathrm{Tr}A\ | > 2\ \Rightarrow\ Av_{\pm} = \mathrm{e}^{\pm\gamma_0} v_{\pm}.$

A acts as a symplectic map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. It defines a discrete dynamical system by iteration. It is

- Hyperbolic : a.e. $x, x' \in \mathbb{T}^2,\ t \in \mathbb{N}$ (not too large),

$$\mathrm{d}(x,x') \sim \epsilon \Rightarrow \mathrm{d}(A^t x, A^t x') \sim \epsilon \mathrm{e}^{\gamma_0 t}.$$

- Exponentially mixing : $\forall f, g \in C^\infty(\mathbb{T}^2)$,

$$\left| \int_{\mathbb{T}^2} (f \circ A^t)(x) g(x) dx - \int_{\mathbb{T}^2} f(x) dx \int_{\mathbb{T}^2} g(x) dx \right| \leq C_{A,f} \parallel \nabla g \parallel_1\ \mathrm{e}^{-\gamma_0 t}.$$

- Ergodic: for all $f \in C^\infty(\mathbb{T}^2)$, for almost all $x_0 \in \mathbb{T}^2$

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} f(A^t x_0) = \int_{\mathbb{T}^2} f(x) \mathrm{d}x.$$

Some typical examples

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \leftarrow \textbf{THE ARNOLD CAT MAP}$$

or, more generally, for $a, b \in \mathbb{N}_*$

$$A_{a,b} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \Phi_a^{p^2/2} \circ \Phi_{-b}^{q^2/2} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}.$$

Also, for $g \in \mathbb{N}_*$, $A_g = \begin{pmatrix} 2g & 1 \\ 4g^2 - 1 & 2g \end{pmatrix}$.

Of course, they all act linearly on $\mathbb{R}^2$ and pass through the quotient by $\mathbb{Z}^2$ since they have integer entries. They are area-preserving (because $\det A = 1$) and hence symplectic, meaning

$$\{f \circ A, g \circ A\} = \{f, g\} \circ A.$$

THE CORRESPONDING QUANTUM MAP From Lectures 1 and 2, we know that if a particle moves in one dimension, its phase space is $\mathbb{R}^2$ and the quantum Hilbert space is $L^2(\mathbb{R}, \mathrm{d}y)$, so that the wavefunctions are functions $\psi(y)$ of one variable. The classical dynamical system is then a Hamiltonian flow Φ_t^H on $\mathbb{R}^2$ (*i.e.* an $\mathbb{R}$-action), and the quantum dynamics is a unitary flow $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t\mathrm{Op}^{\mathrm{W}}(H)}$ on $L^2(\mathbb{R}, \mathrm{d}y)$.

The link between these two theories is provided by semi-classical analysis, through the Weyl quantization of the classical observables and the limit $\hbar \longrightarrow 0$.

In the present situation, the classical dynamics is a $\mathbb{Z}$ action on $\mathbb{T}^2$, obtained by iterating a fixed element $A \in \mathrm{SL}(2, \mathbb{Z})$.

QUESTION What is the quantum Hilbert space of states? And the dynamics? And the quantization of observables?

The Hilbert spaces Since the system has a two-dimensional phase space, it is reasonable to expect to describe the quantum states with wavefunctions $\psi(y)$ of one variable. But since the phase space is a torus, one expects that the wavefunctions must be periodic $\psi(y-1)=\psi(y)$, as well as their Fourier transforms: $\tilde{\psi}(p-1)=\tilde{\psi}(p)$. This intuition leads to the following definition.

With

$$U(a)\psi(y)=\mathrm{e}^{-\frac{i}{\hbar}(a_1P-a_2Q)}\psi(y)=\mathrm{e}^{-\frac{i}{2\hbar}a_1a_2}\mathrm{e}^{\frac{i}{\hbar}a_2y}\psi(y-a_1),$$

define

$$\mathcal{H}_\hbar=\{\psi\in\mathcal{S}'(\mathbb{R})\mid U(1,0)\psi=\psi=U(0,1)\psi\},\quad \mathbf{2\pi\hbar N=1\Rightarrow\ dim\mathcal{H}_\hbar=N}.$$

Then

$$\psi\in\mathcal{H}_\hbar\ \Rightarrow\ \psi(y)=\sum_{\ell\in\mathbb{Z}}c_\ell\delta(y-\frac{\ell}{N});\quad c_{\ell+N}=c_\ell.$$

Weyl quantization For $f \in C^\infty(\mathbb{T}^2)$, $x = (q,p) \in \mathbb{T}^2$, write

$$f(x) = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 p - n_2 q)}$$

and define

$$\mathrm{Op}^{\mathrm{W}} f = \hat{f} = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 P - n_2 Q)} = \sum_{n \in \mathbb{Z}^2} f_n U(\frac{n}{N}) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

The quantum dynamics Let's treat an example. For $a, b \in \mathbb{N}_*$, we have

$$A_{a,b} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \Phi_a^{p^2/2} \circ \Phi_{-b}^{q^2/2} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}$$

Defining (following Schrödinger!)

$$M(A) = \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar} a P^2} \mathrm{e}^{\frac{\mathrm{i}}{2\hbar} b Q^2},$$

it is easy to check that, provided a and b are even, for all $t \in \mathbb{Z}$,

$$M(A)\mathcal{H}_\hbar = \mathcal{H}_\hbar \quad \text{and} \quad M(A)^{-t} \operatorname{Op}^{\mathrm{W}} f \, M(A)^t - \operatorname{Op}^{\mathrm{W}}(f \circ A^t) = 0.$$

This is the EGOROV theorem, and there is no error term in $\hbar$ because the dynamics is linear. A similar construction works for all hyperbolic elements of $\mathrm{SL}(2, \mathbb{Z})$.

$M(A)$ is the quantum map we wish to study. It is naturally related to the discrete Hamiltonian dynamics on $\mathbb{T}^2$ obtained by iterating A. It acts on the N dimensional spaces $\mathcal{H}_\hbar$ and we are interested in the behaviour of its eigenfunctions and eigenvalues in the $N \to \infty$ limit:

$$M(A)\psi_j^{(N)} = \mathrm{e}^{i\theta_j^{(N)}}\psi_j^{(N)}, \quad j = 1 \dots N.$$

The basic result : the Schnirelman theorem. EQUIDISTRIBUTION AT LAST

THEOREM 1 (Bouzouina-DB 96) For “almost all” sequences $\psi_N \in \mathcal{H}_\hbar$, so that $M(A)\psi_N = \mathrm{e}^{i\theta_N}\psi_N$,

$$\langle \psi_N, \mathrm{Op}^{\mathrm{W}} f \psi_N \rangle \stackrel{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2) \tag{7}$$

COMMENTS: (i) This is proven by adapting known arguments, which is why it holds also for $\epsilon \neq 0$ and on higher dimensional tori.

(ii) The result can be adapted for maps that are not continuous such as the Baker and sawtooth maps (De Bièvre, Degli Esposti 1997) and to systems with a mixed phase space (Marklof, O’Keefe 2004).

SOME SIMPLE COMPUTATIONS

Notation $\quad a, b \in \mathbb{R}^n \;\Rightarrow\; ab = \sum_{j=1}^n a_j b_j$

$$m\ddot{q}(t) = -\nabla V(q(t)) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}m\dot{q}(t)^2 \;=\; \sum_{j=1}^n m\ddot{q}_j(t)\dot{q}_j(t) \;=\; m\dot{q}(t)\ddot{q}(t) \;=\; -\dot{q}(t)\nabla V(q(t)) \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}V(q(t)).$$

Suppose $V(q) = W(\| q \|)$, $\| q \| = \sqrt{q_1^2 + q_2^2 + q_3^2}$. Recall that $\ell_3(x) = q_1 p_2 - q_2 p_1$. Then

$$\begin{aligned}\{\ell_3(x), V(q)\} &= -\partial_p \ell_3(x)\partial_q V(q) = -\partial_p \ell_3(x)\frac{q}{\| q \|}W'(\| q \|)\\ &= -(q_1\partial_{p_1}\ell_3(x) + q_2\partial_{p_2}\ell_3(x))\frac{1}{\| q \|}W'(\| q \|)\\ &= -(q_1(-q_2) + q_2(q_1))\frac{1}{\| q \|}W'(\| q \|) = 0.\end{aligned}$$

SOME MORE SIMPLE COMPUTATIONS

$$\langle \phi, P_j \psi \rangle = \frac{\hbar}{\mathrm{i}} \int_{\mathbb{R}^3} \overline{\phi}(y) \partial_j \psi(y) \mathrm{d}y = -\frac{\hbar}{\mathrm{i}} \int_{\mathbb{R}^3} \overline{\partial_j \phi}(y) \psi(y) \mathrm{d}y = \langle P_j \phi, \psi \rangle$$

$$\langle \phi, P^2 \psi \rangle = \sum_{j=1}^{3} \langle \phi, P_j^2 \psi \rangle = \sum_{j=1}^{3} \langle P_j \phi, P_j \psi \rangle = \sum_{j=1}^{3} \langle P_j^2 \phi, \psi \rangle = \langle P^2 \phi, \psi \rangle$$

$$[Q_1, P_1] \psi(y) = \frac{\hbar}{\mathrm{i}} \left(y_1 \partial_1 \psi(y) - \partial_1 (y_1 \psi)(y) \right) = \mathrm{i} \hbar \psi(y)$$

$$[Q_j, P^2] = \sum_{k=1}^{3} [Q_j, P_k^2] = \sum_{k=1}^{3} P_k [Q_j, P_k] + [Q_j, P_k] P_k = 2 \mathrm{i} \hbar P_j .$$

If A is an operator on $L^2(\mathbb{R}^3)$, then, since $\mathrm{i}\hbar \partial_t \psi_t = \hat{H} \psi_t$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_t, A \psi_t \rangle &= \langle \partial_t \psi_t, A \psi_t \rangle + \langle \psi_t, A \partial_t \psi_t \rangle \\ &= -\frac{1}{i\hbar} \langle \hat{H} \psi_t, A \psi_t \rangle + \frac{1}{i\hbar} \langle \psi_t, A \hat{H} \psi_t \rangle = \frac{1}{\mathrm{i}\hbar} \langle \psi_t, [A, \hat{H}] \psi_t \rangle . \end{aligned}$$

STILL SOME MORE SIMPLE COMPUTATIONS

If $f(q,p) = \cos p \cos q$, $(q,p) \in \mathbb{R}^2$, then

$$f(q,p) = \frac{1}{4}(\mathrm{e}^{\mathrm{i}(p+q)} + \mathrm{e}^{\mathrm{i}(p-q)} + \mathrm{e}^{-\mathrm{i}(p-q)} + \mathrm{e}^{-\mathrm{i}(p+q)})$$
$$= \int_{\mathbb{R}^2} \tilde{f}(a_1, a_2) e^{-\frac{i}{\hbar}(a_1 p - a_2 q)} \frac{\mathrm{d}a_1 \mathrm{d}a_2}{(2\pi\hbar)}$$

$$\text{with } \tilde{f}(a_1,a_2) = \frac{2\pi\hbar}{4}(\delta(a_1+\hbar)\delta(a_2-\hbar) + \delta(a_1+\hbar)\delta(a_2+\hbar)$$
$$+\delta(a_1-\hbar)\delta(a_2-\hbar) + \delta(a_1-\hbar)\delta(a_2+\hbar)).$$

$$\text{So} \quad \mathrm{Op}^{\mathrm{W}}(f) = \frac{1}{4}((\mathrm{e}^{\mathrm{i}(P+Q)} + \mathrm{e}^{\mathrm{i}(P-Q)} + \mathrm{e}^{-\mathrm{i}(P-Q)} + \mathrm{e}^{-\mathrm{i}(P+Q)})$$
$$\neq \cos P \cos Q.$$

$$\text{Indeed} \quad \cos P \cos Q = \frac{1}{4}(\mathrm{e}^{\mathrm{i}P}\mathrm{e}^{\mathrm{i}Q} + \mathrm{e}^{\mathrm{i}P}\mathrm{e}^{-\mathrm{i}Q} + \mathrm{e}^{-\mathrm{i}P}\mathrm{e}^{\mathrm{i}Q} + \mathrm{e}^{-\mathrm{i}P}\mathrm{e}^{-\mathrm{i}Q})$$

and

$$[Q,P] = \mathrm{i}\hbar \Rightarrow \mathrm{e}^{\mathrm{i}Q}\mathrm{e}^{\mathrm{i}P} = \mathrm{e}^{\mathrm{i}(Q+P)}\underline{\mathrm{e}^{\frac{\mathrm{i}}{2}\hbar}} \neq \mathrm{e}^{\mathrm{i}(Q+P)}.$$

Exercise Let $f(q,p) = \sin q$ and $g(q,p) = \cos p$. Prove that

$$\| \frac{1}{\mathrm{i}\hbar}[\mathrm{Op}^{\mathrm{W}} f, \mathrm{Op}^{\mathrm{W}} g] - \mathrm{Op}^{\mathrm{W}}(\{f,g\}) \| \leq C\hbar.$$

Generalize the result to $f, g \in C^\infty(\mathbb{T}^2)$.

- **INGREDIENT** One symplectic manifold N with a Poisson bracket $\{f, g\}$ for $f, g \in C^\infty(N)$.

- **CONSTRUCT** $h \in C^\infty(N); t \in \mathbb{R} \longrightarrow \Phi_t^h \in \mathrm{Diff}_{\mathrm{sympl}}(N)$, an $\mathbb{R}$-action on N (*i.e.* $\Phi_{t_1}^h \circ \Phi_{t_2}^h = \Phi_{t_1+t_2}^h : N \to N$) by symplectic diffeomorphisms (meaning $\{f \circ \Phi_t^h, g \circ \Phi_t^h\} = \{f, g\} \circ \Phi_t^h$).

- **PROPERTIES** $\frac{\mathrm{d}}{\mathrm{d}t} f \circ \Phi_t^h = \{f, h\} \circ \Phi_t^h$.

EXAMPLES • $N = \mathbb{R}^{2n}$, $h(x) = H(x) = \frac{p^2}{2m} + V(q)$, with $x = (q, p)$. The flow $\Phi_t^H(x) = (q(t), p(t))$ then describes the motion of a particle in the potential V since Hamilton's equations for this H imply that $m\ddot{q}(t) = -\nabla V(q(t))$.

• $N = \mathbb{H} \times \mathbb{R}^2$ and $h(x) = H(x) = \frac{1}{2} q_2^2 (p_1^2 + p_2^2)$, with $x = (q, p)$. The flow $\Phi_t^H(x) = (q(t), p(t))$ then describes geodesic motion on $\mathbb{H}$ since the curve $t \in \mathbb{R} \to q(t) \in \mathbb{H}$ is a geodesic. What Jens (I mean Professor Marklof) calls the unit (co)tangent bundle is $\Sigma_E := \{x \in N | H(x) = E\}$ with $E = 1$.

VARIATIONS – GENERALIZATIONS • Take $(N, \{\cdot, \cdot\})$ and a fixed symplectic diffeomorphism Φ. Consider $t \in \mathbb{Z} \to \Phi^t \in \mathrm{Diff}_{\mathrm{sympl}}(N)$. This is a $\mathbb{Z}$-action on N: a discrete dynamical system. Example: $N = \mathbb{T}^2, \Phi = A \in \mathrm{SL}(2, \mathbb{Z})$.

• Let $(N, \{\cdot, \cdot\})$ be a symplectic manifold and G a group. A symplectic action of G on N is a group homomorphism $\phi : g \in G \mapsto \phi_g \in \mathrm{Diff}_{\mathrm{sympl}}(N)$ (*i.e.* $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$). Example: $G = \mathrm{SL}(2, \mathbb{R})$ and $N = \mathbb{H} \times \mathbb{R}^2$ (*cfr.* Jens!). The action ϕ is transitive if there are no non-trivial G-invariant subsets of N. In the previous example the action is NOT transitive since each surface Σ_E is $\mathrm{SL}(2, \mathbb{R})$-invariant; the action is transitive on each Σ_E, though (Jens again!).

QUESTION Given a Lie-group G, can you classify all symplectic transitive G-actions?

THE STRUCTURES OF QUANTUM MECHANICS

- **INGREDIENT** One Hilbert space $\mathcal{H}$. Note that the vectorspace of linear operators on $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$ comes with a bilinear antisymmetric form, namely the commutator $[A, B] = AB - BA$ which is a derivation ($A, B, C \in \mathcal{L}(\mathcal{H})$)

$$[A, BC] = [A, B]C + B[A, C]$$

  and satisfies the Jacobi identity

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

- **CONSTRUCT** Take $C = C^* \in \mathcal{L}(\mathcal{H})$. Make $U_t^C := \exp -\mathrm{i}Ct,\ t \in \mathbb{R} \to U_t^C \in \mathcal{U}(\mathcal{H})$, a unitary representation of $\mathbb{R}$ on $\mathcal{H}$ (*i.e.* $U_{t_1}^C U_{t_2}^C = U_{t_1+t_2}^C$) (Venkatesh!).

- **PROPERTIES** $\frac{\mathrm{d}}{\mathrm{d}t} U_{-t}^C A U_t^C = \mathrm{i} U_{-t}^C [C, A] U_t^C = U_{-t}^C \frac{[A,C]}{\mathrm{i}} U_t^C$.

EXAMPLES • $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}y)$. $C = \hat{H} = \frac{P^2}{2m} + V(Q)$.

• $\mathcal{H} = L^2(\mathbb{H}, \mathrm{dvol}_g)$, $C = -\Delta_g$.

VARIATIONS – GENERALIZATIONS • Take a Hilbert space $\mathcal{H}$ and a fixed unitary map U. Consider $t \in \mathbb{Z} \to U^t \in \mathcal{U}(\mathcal{H})$. This is a unitary representation of $\mathbb{Z}$ on $\mathcal{H}$: a discrete quantum dynamical system. Example: Lecture 4 and Rudnick's quantum equidistribution lectures.

• Let $\mathcal{H}$ be a Hilbert space and G a group. A unitary representation of G on $\mathcal{H}$ is a group homomorphism $U : g \in G \mapsto U_g \in \mathcal{U}(\mathcal{H})$ (*i.e.* $U_{g_1} \circ U_{g_2} = U_{g_1 g_2}$) (Venkatesh! He wrote ρ where I write U, but it's the same thing). Example: $G = \mathrm{SL}(2, \mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{H}, \mathrm{dvol}_g)$.

A representation is irreducible if there are no non-trivial G-invariant closed subspaces of $\mathcal{H}$. In the previous example the representation is NOT irreducible since $-\Delta_g$ has invariant subspaces; the representation is irreducible when restricted to one of those, though (Venkatesh again!).

QUESTION Given a Lie-group G, can you classify all its irreducible unitary representations?

On the unusual choice of words in mathematics

or

Why hardly anybody takes us seriously

In mathematics, the following is true.

THEOREM Most real **numbers** are irrational. Most irrational **numbers** are normal. Most real **numbers** are therefore normal. But for any real **number** you may come across, it is impossible to tell if **it** is normal. Unless, of course, **it** is rational, in which case **it** is not normal.

Is the following true in Real Life?

Most real **people** are irrational. Most irrational **people** are normal. Most real **people** are therefore normal. But for any real **person** you may come across, it is impossible to tell if **he** is normal. Unless, of course, **he** is rational, in which case **he** is not normal.