Linear friction and diffusion in Hamiltonian open systems

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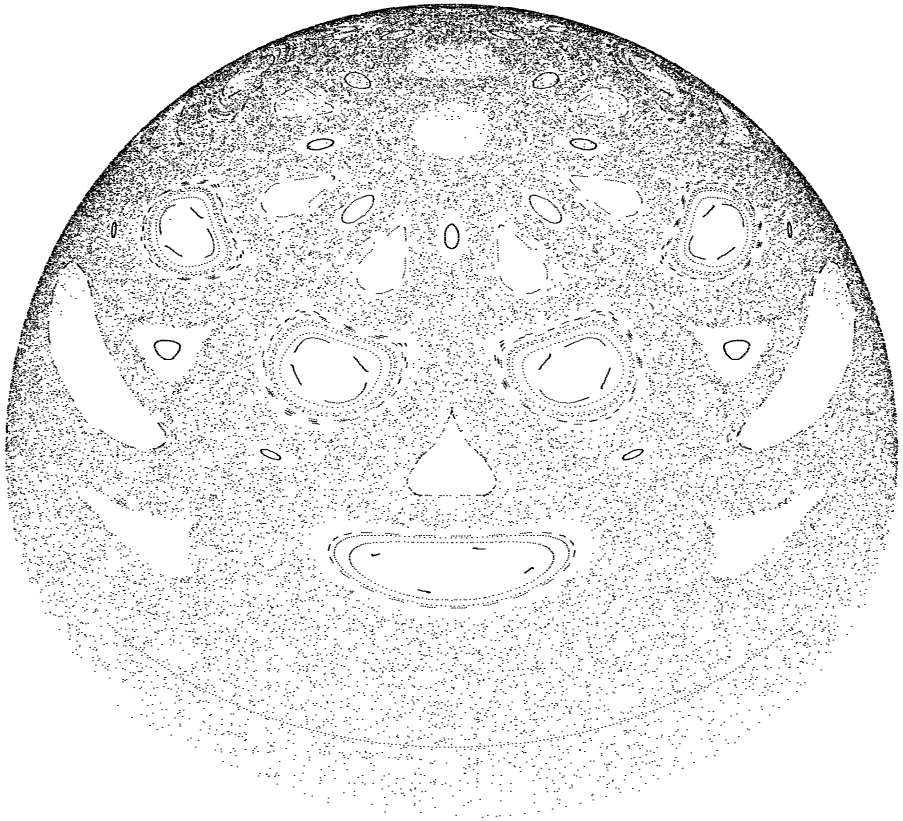
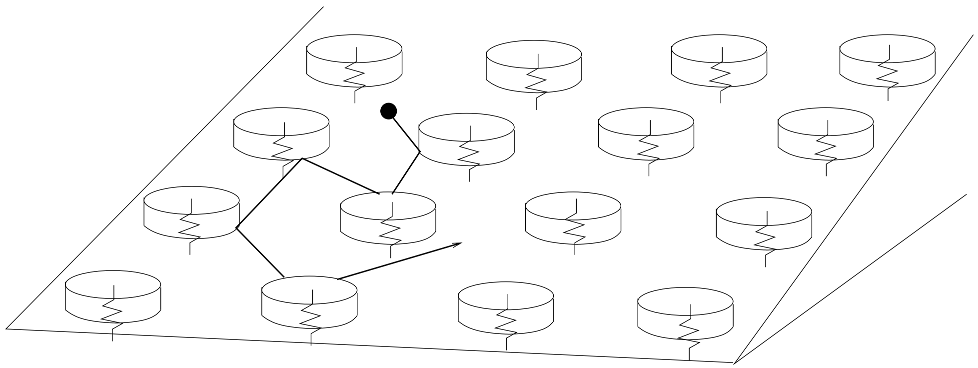


Ohm's law and the pinball machine

- Ohm's law: $V = RI$ or $\vec{E} = \rho\vec{j}$ or $\vec{v} = \frac{\mathrm{q}\tau}{m}\vec{E}$.

$$m\frac{d\vec{v}}{dt} = \mathrm{q}\vec{E} - \frac{m}{\tau}\vec{v}, \qquad \vec{v}(t) \sim \frac{\mathrm{q}\tau}{m}\vec{E} \quad (t \to \infty).$$

- The pinball machine (or the inelastic Lorentz gas)



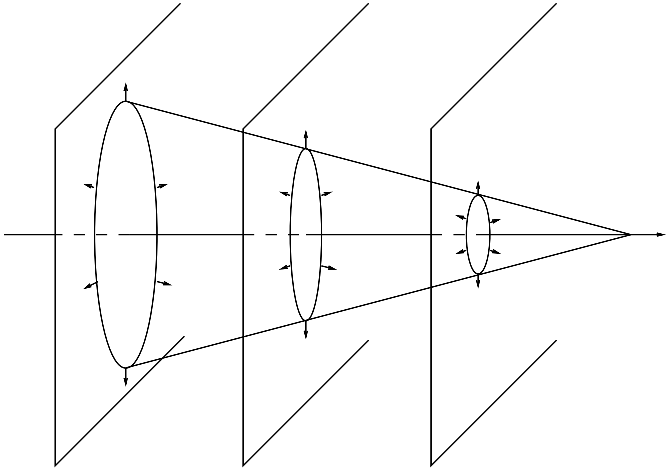
Towards a Hamiltonian model for Ohm's law?

A continuum model: friction

L. Bruneau and S.D.B.

A Hamiltonian model for linear friction in a homogeneous medium, Commun. Math. Phys. **229**, 511-542, 2002

THE MODEL



$$H(X) = H_{\rm S} + H_{\rm R} + \int_{\mathbb{R}^n} \rho(x - q_{\rm S}, y) q_{\rm R}(x,y)\, {\rm d}x{\rm d}y.$$

$$H_{\rm S} = \frac{p_{\rm S}^2}{2} + V(q_{\rm S}).$$

$$H_{\rm R} = \frac{1}{2}\int_{\mathbb{R}} \left(p_{\rm R}^2(x,y) + |\nabla_y q_{\rm R}(x,y)|^2\right)\, {\rm d}x{\rm d}y.$$

$$V = 0 \Rightarrow \lim_{t\to+\infty} \dot{q}_{\rm S}(t) = 0.$$

THE RESULTS

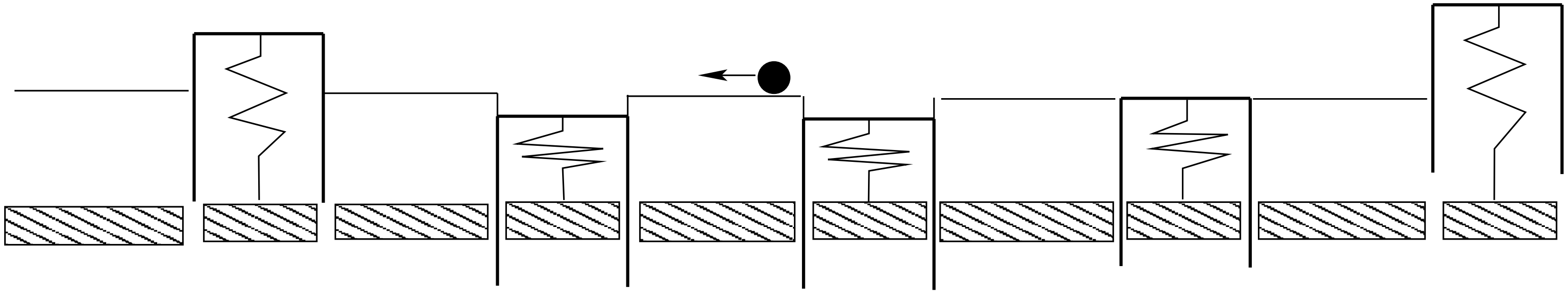
$$V = -Fq \Rightarrow \lim_{t\to+\infty} \dot{q}_{\rm S}(t) = v_F \sim \frac{F}{\gamma}.$$

$$\ddot{q}_{\rm S} = -\nabla V(q_{\rm S}) - \gamma \dot{q}_{\rm S}.$$

The particle experiences a friction force linear in its velocity, due to the presence of the field. Its behaviour is therefore Ohmic.

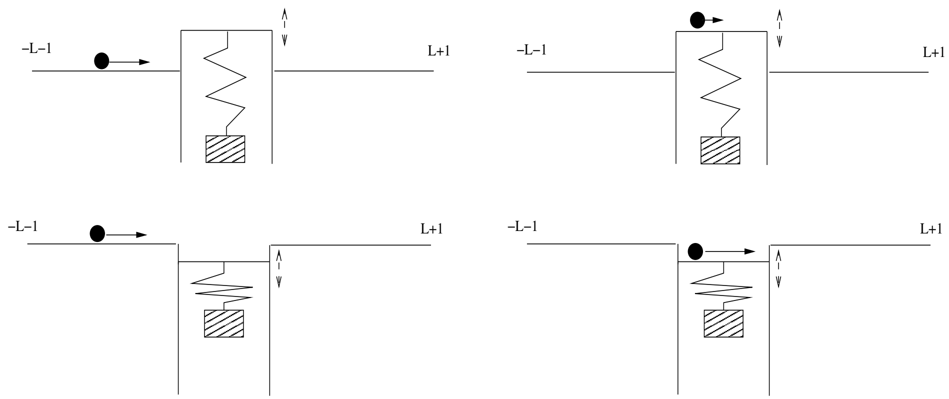
A 1-d inelastic Lorentz gas: fluctuations

S.D.B., P. Parris, A. Silvius, Physica D, **208**, 96-114, 2005 and cond-math/0507181



One can imagine putting the system on a slope, or that the particle is charged and an electric field is applied.

Two examples of what may happen in one cell:



THE MODEL A one-dimensional periodic array (with period a) of identical oscillators of frequency ω. The particle interacts with the oscillator at ma if it is within a distance $\sigma < \frac{a}{2}$.

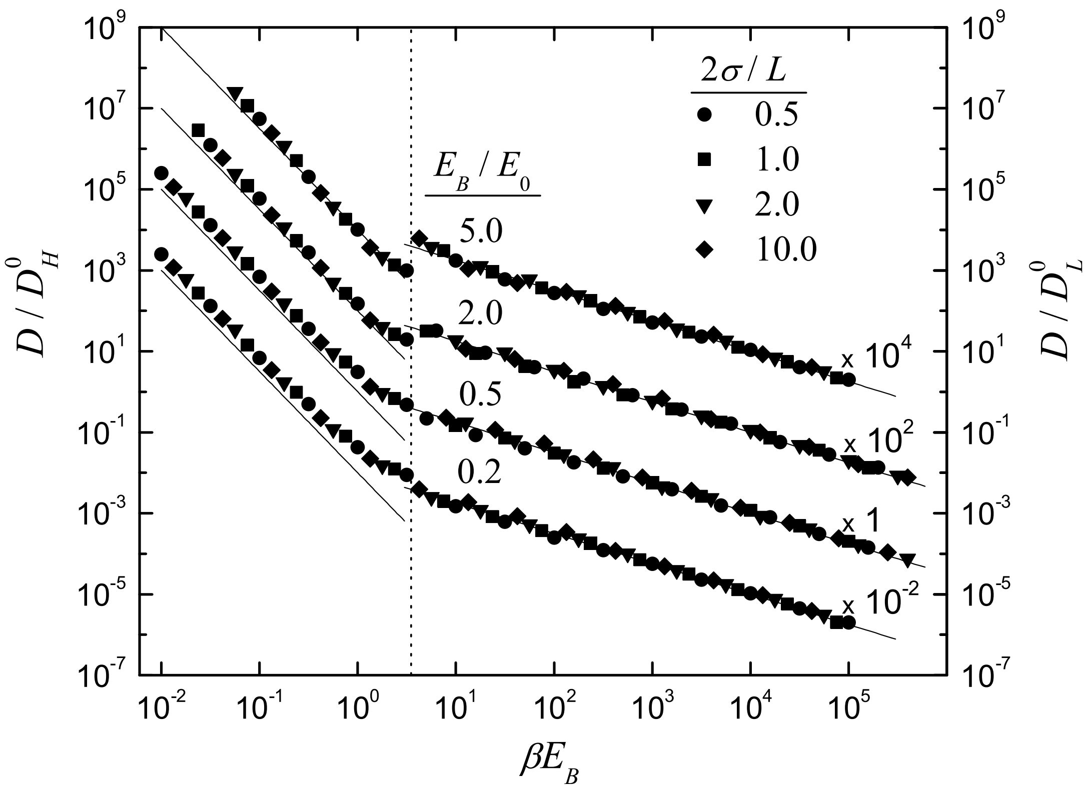
$$H = \frac{1}{2}p_{\mathrm{S}}^2 + \sum_{m\in\mathbb{Z}} \frac{1}{2}\left(p_m^2 + \omega^2 q_m^2\right) + \alpha \sum_m q_m n_m\left(q_{\mathrm{S}}\right). \quad (1)$$

where $n_m\left(q_{\mathrm{S}}\right)$ vanishes outside the interaction region associated with the oscillator at ma and is equal to unity inside it.

TWO DIMENSIONLESS PARAMETERS

- E_B/E_0: here $E_B = \frac{\alpha^2}{2\omega^2}$ is the binding energy and $E_0 = \sigma^2\omega^2$.
- $2\sigma/L$: here $L = a - 2\sigma$ is the size non-interacting region in a cell.

Diffusion constants as a function of temperature and system parameters



High temperature: $D \sim D_H^0 (\beta E_B)^{-5/2}$

$$D_H^0 = \sqrt{\frac{9 E_B a^2}{32\pi} \frac{E_B}{E_0}}$$

Low temperature: $D \sim D_L^0 (\beta E_B)^{-3/4}$

$$D_L^0 = \frac{a}{2\sigma} \Gamma(3/4) \sqrt{\frac{E_B a^2}{2\pi^2}}$$

TWO REGIMES – TWO DYNAMICS :

- **High temperatures** Traversal time $<<<$ oscillator period.

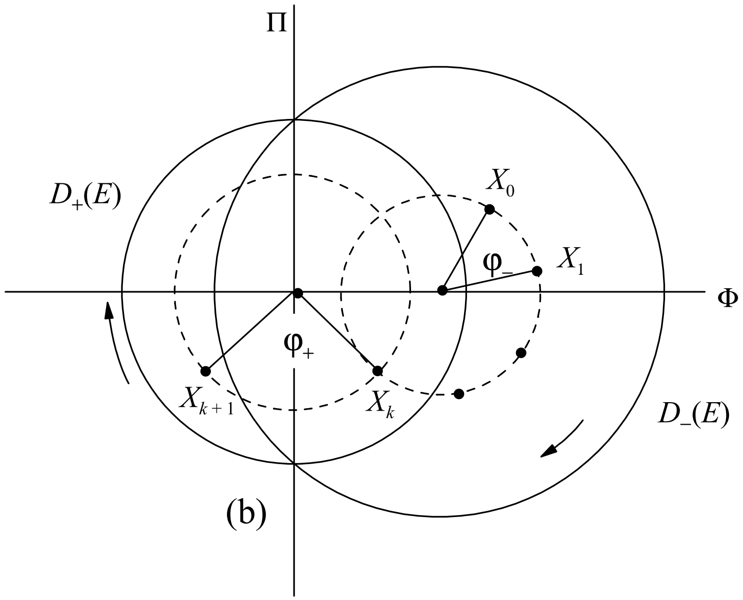
Typical potential energy barrier $\Delta \sim \sqrt{2E_B kT} <<<$ particle energy. A thermalized particle will pass through many interaction regions in succession before slowing down and undergoing a velocity reversing (or randomizing) kick back up to thermal velocities. Adiabatic regime: the random potential seen by the particle typically changes adiabatically with respect to the particle's net motion (*cfr.* polaron).

- **Low temperatures**: Traversal time $>>>$ oscillator period.

Random walk, hopping motion.

The dynamics in one cell

- Phase space : $q_{\mathrm{S}} \in]-1-L/2, 1+L/2[,\ p_{\mathrm{S}}, q_0, p_0 \in \mathbb{R}\ (\sigma = 1)$
- Hamiltonian: $H_\alpha(q_{\mathrm{S}}, p_{\mathrm{S}}, q_0, p_0) = \frac{1}{2}\left(p_{\mathrm{S}}^2 + p_0^2 + q_0^2\right) - \alpha q_0 n_0(q_{\mathrm{S}}(q)$, where n_0 is the characteristic function of $[-1, 1]$. Three parameters: α, L and E.
- The dynamics from the oscillator's point of view, at fixed $E > 0$:

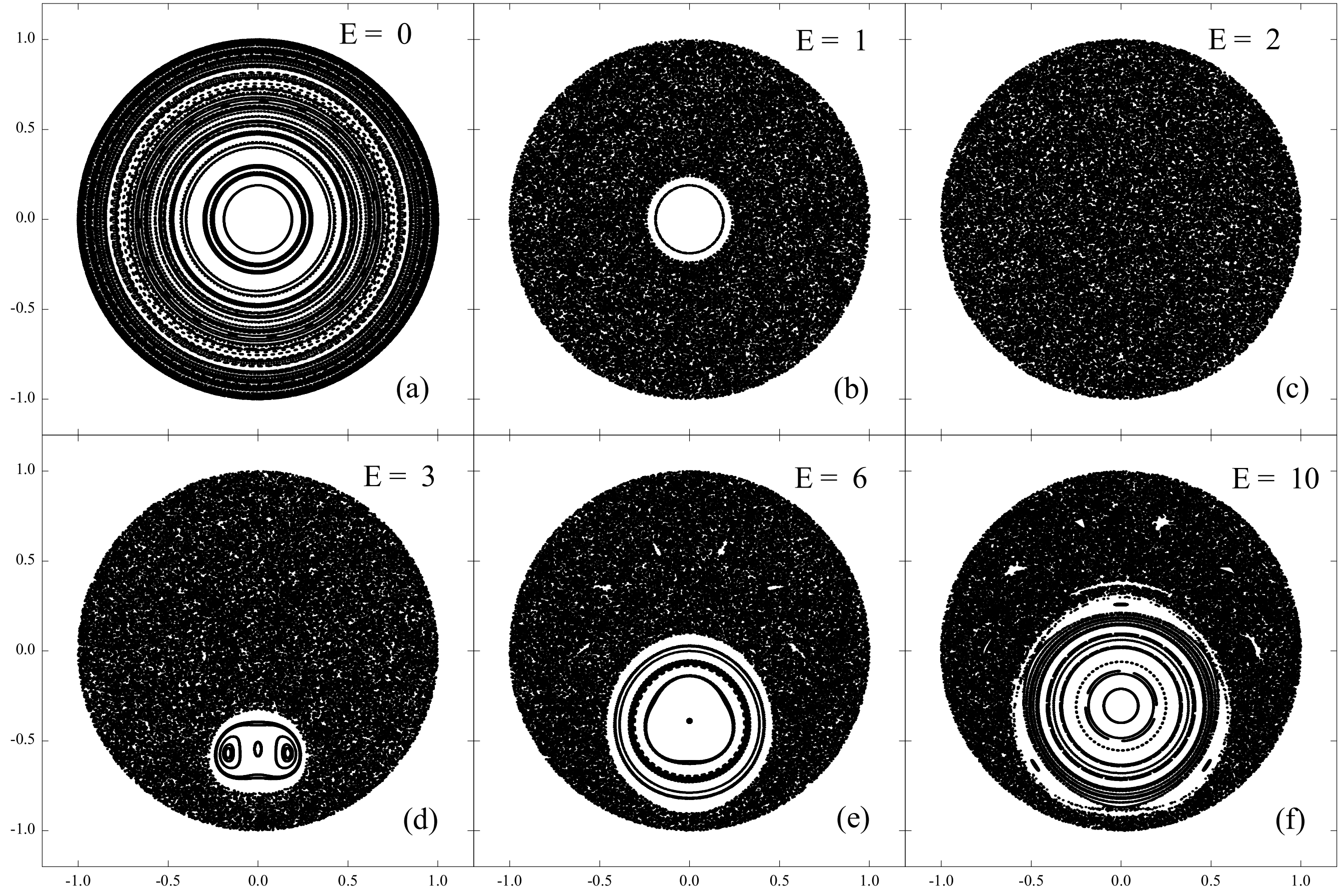


$D_-(E)$: disc at $q_0 = \alpha, p_0 = 0$, of radius $\sqrt{2E + \alpha^2}$.

$D_+(E)$: disc at origin, of radius $\sqrt{2E}$

Numerical observations : varying E

Oscillator phase diagrams for a system with $\alpha = 2, L = 2.37$. Shown are 100 randomly chosen orbits, each visiting 500 times the Poincaré section at $q = 0$.



Main features of the dynamics

Depending on the values of the three system parameters α, L, E one observes the following types of behaviour:

- The phase space has one ergodic, chaotic component.
- Coexistence of a chaotic with a totally integrable component. The boundary between the two is “clean”, without KAM type structures.
- One large, prominent elliptic island along the Π-axis.
- One-parameter families of parabolic periodic orbits, similar to bouncing ball modes in billards.

CONCLUSION: In physical terms, one expects instability in the motion when the particle is moving slowly. This means the time it takes to cross the circle is long compared to the oscillator period. Small changes in particle speed lead to large changes in the crossing time, hence to a large uncertainty on the oscillator position: hence this yields the unpredictability of the motion. This is observed both in the one-cell dynamics and in the low temperature behaviour of the diffusion constant for the oscillator chain.

Outlook

- Can one prove the claims on ergodicity, and the existence of a mixed phase space with two clear components, one integrable, one chaotic?
- What about linear response?
- What about non-equilibrium statistical mechanics?

Work in progress. . .