Quantum chaos: a brief first visit

Stephan De Bièvre

Abstract. These notes contain a slightly expanded version of six lectures I delivered at the Cuernavaca Summer School on Analysis and Mathematical Physics (June 2000). They provide an elementary and self-contained introduction to some aspects of quantum chaos that is adapted – it is hoped – to the audience of the school, which consisted of advanced undergraduate and beginning graduate students in mathematics having had little or no exposure to physics in general and to classical or quantum mechanics in particular. The lectures therefore include brief self-contained introductions to the Hamiltonian and quantum mechanics of a point particle. The basic problematics of quantum chaos is first explained using billiards as an example. Chaotic maps on the torus are then treated in some detail and a complete and self-contained proof of the Schnirelman theorem is given in this context.

1. Introduction

Quantum chaos, or quantum chaology, as Michael Berry prefers to call it, is a part of semi-classical analysis. It is concerned with the study of the semi-classical properties of quantum systems that have a chaotic Hamiltonian system as a classical limit. One of the goals of these lectures is to provide an introduction to this field lying on the intersection of pure mathematics and theoretical physics, and in particular to explain each of the words used above. To set the stage, I will begin by discussing some of the typical questions and conjectures in quantum chaos in the context of "billiards", where this can be done quite simply since both the billiard flow and the Dirichlet Laplacian are easily described (but not easily studied!) mathematical objects. I will in particular explain what is meant by "regular", "stable" or "predictable" motion, as well as by "irregular", "unstable" or "unpredictable" motion and will discuss some notions of "spectral statistics". Unfortunately, proving theorems on chaotic billiard flows and on their relation to the spectral properties of the corresponding Dirichlet Laplacian is difficult and well beyond the scope of these lectures. This can however be done much more easily for chaotic maps on the torus and these notes provide you with a complete proof of the so-called Schnirelman theorem in this context. To that end, I first need to develop some preparatory material. I will in particular give a brief introduction to Hamiltonian mechanics and show in which sense billiard flows are Hamiltonian dynamical systems. This

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will allow me to give a precise meaning to the idea of "regular" motion through the notion of completely integrable Hamiltonian systems. Some simple ideas and definitions of ergodic theory will then be used to capture the concept of "chaotic motion". After a short introduction to quantum mechanics, I will furthermore show how Weyl quantization associates to each Hamiltonian dynamical system a spectral problem thereby generalizing the link between the billiard flow and the Dirichlet Laplacian. With these tools at my disposal, I study in the last part of the lectures some chaotic maps on the torus, show how to associate a spectral problem to them (the so-called *quantum map*) and finally prove the one universally valid result that is known in quantum chaos, i.e. the Schnirelman theorem.

The only mathematical prerequisites for understanding these notes are: a good grasp of multi-variable calculus and in particular of the implicit function theorem; some familiarity with the rudiments of the geometrical theory of ordinary differential equations, with the basic facts about (bounded self-adjoint operators on) Hilbert spaces, measure and probability; a good working knowledge of the Fourier transform and some familiarity with tempered distributions. No prior knowledge of physics is assumed. New concepts or terminology are systematically italicized whenever they appear for the first time so if you see a word you don't know you should either consult a dictionary or be able to find it in italics somewhere earlier in the text. The exercises are an integral part of the text and the results of many of them are used later on. For the most part, they require only rather straightforward, occasionally somewhat lengthy, calculations. The ones that do require some more thought are marked with a star.

2. Billiards

Billiards are bounded planar domains Ω with a piecewise smooth boundary. Two typical objects of study concerning such domains are the "billiard flow" and the Laplacian with Dirichlet boundary conditions on the boundary of Ω. The billiard flow is simply the (frictionless!) motion of a particle (the billiard ball) in the domain, bouncing off its boundaries through specular reflection (angle of incidence = angle of reflection, see Fig. 1). Note that, at corners, this rule is

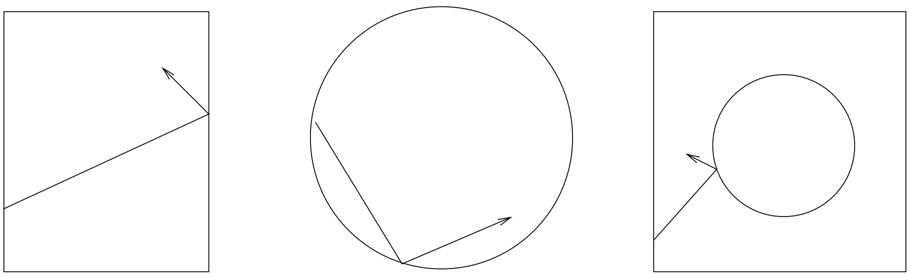

FIGURE 1. The rectangular, circular and Sinai billiards

generally not unambiguously defined, but inside a rectangular domain it implies that the ball backtracks when it hits a corner. The billiard motion depends clearly on the shape of the boundary and can be very complicated, even when the latter is relatively simple, as we shall see. The Laplacian with Dirichlet boundary conditions is the partial differential operator

$$\Delta = \frac{\partial^2}{\partial {y_1}^2} + \frac{\partial^2}{\partial y_2^2},$$

acting on twice differentiable functions in $L^2(\Omega, dy)$ vanishing on the boundary $\partial\Omega$. It is well known [**RS**] that there exists an orthonormal basis of eigenfunctions of Δ:

$$-\Delta\varphi_j = \lambda_j \varphi_j, \quad \varphi_j(y) = 0, \quad \forall y \in \partial\Omega \quad (j \in \mathbb{N}).$$

One can order the eigenvalues so that

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \ldots$$

Moreover, $\lim_{j\to\infty} \lambda_j = \infty$, so that each eigenvalue is at most finitely degenerate [**RS**]. If these facts are not familiar to you, you should immediately do the following exercises. They will give you some feeling as to why they have a chance to be true.

EXERCISE 2.1. Compute explicitly the eigenvalues and eigenfunctions of the Dirichlet Laplacian when $\Omega = [0, a] \times [0, b] (a, b > 0)$. Is it easy to order the eigenvalues and to find an explicit formula for λ_j, for any $j \in \mathbb{N}$? What if $a = b = 1$? Find a necessary and sufficient condition on a and b so that the eigenvalues are all non-degenerate.

EXERCISE 2.2. * Compute (as) explicitly (as possible) the eigenvalues and eigenfunctions of the Dirichlet Laplacian when Ω is the disc of radius a. Is it easy to order the eigenvalues and to find an explicit formula for λ_j, for any $j \in \mathbb{N}$? **Hint:** Use the method of separation of variables in polar coordinates and if you don't know what that is, consult [**Arf**] or any other book with a similar title.

Note that the collection of the eigenvalues is called the *spectrum* of the billiard and that physicists often say "the energy levels" rather than "the eigenvalues," for reasons that will become clear later on in these notes. Of the billiards presented in Fig. 1, only the first two are simple enough that the $\{\lambda_j, \varphi_j\}$ can be computed explicitly. Finding explicit expressions for the $\{\lambda_j, \varphi_j\}$ is for general billiard shapes an impossible task and the method of separation of variables used in the above exercises only applies in a few cases. Faced with this situation, one may ask the (apparently!) considerably more modest question:

> Is it possible to obtain information (possibly of a statistical nature) about the asymptotic behaviour as $j \to \infty$ of the λ_j and the φ_j in terms of information about the billiard flow?

As such, this question probably means little to you. Hopefully its meaning will become clearer in the next two sections. At any rate, the question is natural from a physicist's point of view and it finds its origin in geometrical optics and (semi-classical) quantum mechanics (as explained in section 11). It turns out that the answer to the question is "Yes" and the study of the question and of its answer, in particular for billiards with chaotic flows, is the essence of quantum chaos as applied to billiards. There are in fact two main conjectures in this context, that can be very loosely paraphrased as follows:

> If the billiard flow is regular (predictable), then its spectrum is irregular (random, uncorrelated).
> If the billiard flow is irregular (chaotic, unpredictable), then its spectrum is regular (correlated).

The conjectures above are called, respectively, the Berry-Tabor [**BT**] and the Bohigas-Giannoni-Schmit [**BGS**] conjectures. They were stated in a much more precise form in the original papers of course, but the above rather vague, non-technical and rather paradoxical statements give a first idea of what they say (and are easy to remember!). Although there is a wealth of numerical evidence in favour of these conjectures, there are only very few theorems supporting them, so that the conjectures are still largely open to date. This explains in part why quantum chaos remains such an active field of research.

Concerning the asymptotic behaviour of the eigenfunctions φ_j as $j \to \infty$, there exists one important result, the so-called Schnirelman theorem, of which I will now state a simplified version, in the context of billiards. In section 14, I will prove it for quantized toral automorphisms.

> The Schnirelman theorem: If the billiard flow is chaotic, then there exists a density one sequence of integers $(j_k)_{k\in\mathbb{N}}$, so that, for each Borel subset $B \subset \Omega$, one has

$$\lim_{k\to\infty} \int_B \mid \varphi_{j_k}(y) \mid^2 dy = \frac{\mid B \mid}{\mid \Omega \mid}. \tag{2.1}$$

What does this mean? I first need to explain some terminology. A strictly monotonic sequence of integers $(j_k \in \mathbb{N})_{k\in\mathbb{N}}$ is said to be of *density* $a (0 \leq a \leq 1)$ if the following is true:

$$\lim_{J\to\infty} \frac{1}{J} \sharp\{k \in \mathbb{N} \mid j_k \leq J\} = a. \tag{2.2}$$

Loosely stated, saying that a sequence is density a means therefore that the fraction of integers that belong to the sequence equals a. Here are some examples:

EXERCISE 2.3. Show that the sequence $j_k = k^2$ is density zero and that the sequence $j_k = 2k$ is density 1/2. What about $j_k = k + \text{Int}\sqrt{k}$, where Int indicates the integer part? It is instructive to compute the first few terms of each of these sequences to get a feeling for what happens.

Now, recalling that the φ_j were chosen normalized, so that in particular $\int_\Omega \mid \varphi_{j_k}(y) \mid^2 dy = 1$, it is clear that each eigenfunction defines a probability measure μ_j with density $\mid \varphi_j(y) \mid^2$ on Ω as follows:

$$\mu_j(B) = \int_B \mid \varphi_j(y) \mid^2 dy.$$

The Schnirelman theorem therefore states that there exists a density one sequence of integers j_k so that the measures μ_{j_k} converge to the normalized Lebesgue measure on Ω. Since

$$\frac{\mid B \mid}{\mid \Omega \mid} = \int_B \frac{1}{\mid \Omega \mid} dy$$

one could also very loosely rewrite (2.1) as

$$\lim_{k\to\infty} \mid \varphi_{j_k}(y) \mid^2 = \frac{1}{\mid \Omega \mid}, \qquad \text{a.e. } y \in \Omega.$$

One therefore often says that the eigenfunctions "equidistribute": the probability densities $\mid \varphi_{j_k}(y) \mid^2$ tend to a constant, independent of $y \in \Omega$.

The Schnirelman theorem holds under a weaker condition on the flow, called *ergodicity*, that will be explained precisely in section 10. For now, let me just mention that saying the billiard flow is ergodic implies that the average time spent by a typical billiard trajectory $t \to y_t$ in any Borel set $B \subset \Omega$, is proportional to the Lebesgue measure $| B |$ of B. In other words:

$$\lim_{T\to\infty} \frac{1}{T} \mid \{t \le T \mid y_t \in B\} \mid = \frac{| B |}{| \Omega |}.$$

The Schnirelman theorem therefore states that the probability measures μ_j associated to the eigenfunctions of the Dirichlet Laplacian converge (when $j \to \infty$ along a density one sequence) to the same measure as the one obtained from the dwelling time of a typical billiard orbit, provided the billiard flow is ergodic.

For billiards, the Schnirelman theorem was proven in **[GL]** and **[ZZ]**. It was first stated and proved in the context of geodesic flows and Laplace-Beltrami operators on Riemannian manifolds **[Sc][Z1][CdV]**.

In addition to the Schnirelman theorem, there are several more detailed conjectures on the behaviour of the eigenfunctions φ_j as $j \to \infty$, but it is more difficult even to paraphrase them without additional mathematical tools so I won't go into this aspect of the problem any further.

In the next section I will start by giving a more precise idea of what is meant by "regular" and "irregular" or "chaotic" motion.

3. Regular and chaotic motion: an example

In order to get an intuitive idea of the kind of properties of billiard flows we are interested in, let us study the flows on the circular and Sinai billiards in some detail. I claim that there is an important qualitative difference between those

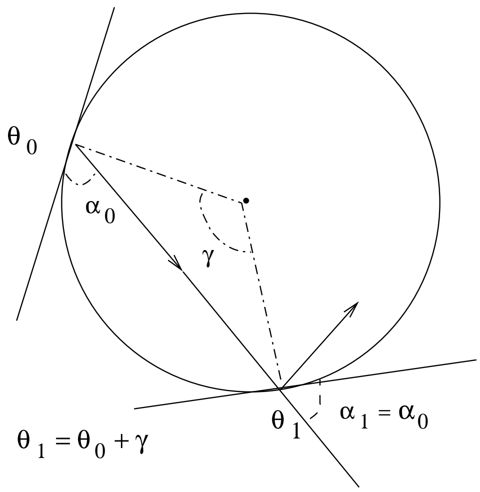

FIGURE 2. Billiard flow on the circular billiard

two flows. What is it? First of all, note that the motion on the circular billiard is extremely easy to describe. Consider a particle starting at some point on the boundary with angular coordinate θ_0, having an initial velocity making an angle α_0 with the tangent line to the circle (see Fig. 2). Then, clearly, $\theta_1 = \theta_0 + 2\alpha_0, \alpha_1 = \alpha_0$ so that, in obvious notations, $\theta_n = \theta_0 + 2n\alpha_0 \pmod{2\pi}$. In other words, it is easy to keep track of the trajectory of the particle by recording the angular coordinate θ_n of the points on the boundary where it successively hits and we have an explicit simple formula for θ_n. You can try to do the same thing for the Sinai billiard, but this is considerably nastier. First of all, clearly, $\alpha_n \neq \alpha_0$ in this case. Moreover, after some trial and error you will easily convince yourself that there is no simple formula expressing θ_n, α_n in terms of $\theta_{n-1}, \alpha_{n-1}$ and even less a formula expressing θ_n, α_n directly in terms of θ_0, α_0. This is related to (I did not say "equivalent to"!) the following observation: whereas the motion on the circular billiard is very stable, the one on the Sinai billiard is highly unstable. To see what I mean by this, take ϵ small and consider the trajectory on the circular billiard with initial conditions $\theta_0' = \theta_0, \alpha_0' = \alpha_0 + \epsilon$. Then $\theta_n' = \theta_n + \epsilon n (\mathrm{mod}\ 2\pi)$. So the distance between the two orbits θ_n' and θ_n grows only linearly with n. Now look what happens in a similar situation on the Sinai billiard. Consider the periodic orbit $\theta_n = \theta_0 = 0, \alpha_n = \alpha_0 = \frac{\pi}{2}$ and investigate the nearby orbit θ_n', α_n' indicated in Fig. 3). The ball starts at S, deviating from SB by an angle ϵ_0. Since $SB = BO$, $\theta_0' > \epsilon_0$ and $\alpha_0' = \frac{\pi}{2} - (\epsilon_0 + \theta_0')$ so that $\epsilon_1 = \epsilon_0 + 2\theta_0' > 3\epsilon_0$, provided, of course, ϵ_0 is chosen small enough so that the ball does not miss the circular obstacle after two bounces off the outer billiard edge (as happens in the figure). Repeating the argument yields

$$\epsilon_n > 3\epsilon_{n-1} > \cdots > 3^n \epsilon_0,$$

so that $\theta_n' > 3^n \epsilon_0$. Of course, after a finite number of bounces this estimate fails

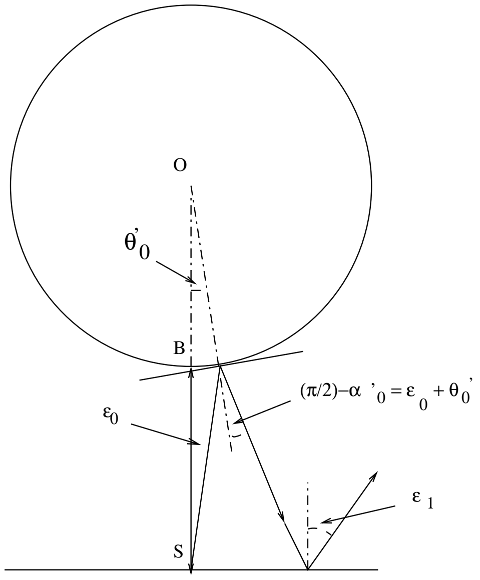

FIGURE 3. Exponential instability on the Sinai billiard

since then the ball misses the circular obstacle and hits the side walls once or several times before hitting the obstacle again. As a result the precise statement is:

$$\forall N \in \mathbb{N}, \exists \epsilon_0 > 0, \text{ so that } \forall n \leq N, \theta_n' \geq 3^n \epsilon_0.$$

So what we have just shown is that the periodic orbit $\theta_n = \theta_0 = 0, \alpha_n = \alpha_0 = \frac{\pi}{2}$ is exponentially unstable: although we investigated only one particular nearby orbit, it seems safe to assume (and it is true) that a typical nearby orbit will separate from it exponentially fast, and not just at a linear rate, as in the circular billiard.

EXERCISE 3.1. Find two nearby orbits on the circular billiard that don't separate at all.

Another useful way to think about the difference in the behaviour of the flows on the two billiards is as follows. Suppose you know the initial position and velocity of your billiard ball only with a finite precision ϵ, and not precisely. (This is the case when you play billiard, by the way.) This means you know θ_0, α_0 belong to an interval I_ϵ of size ϵ. How long does it last, then, before you have essentially no information about the position and velocity of the ball? Since in the circular billiard two initial conditions inside the interval I_ϵ will typically separate linearly, the corresponding trajectories will be a distance of order 1 apart after a time of order $1/\epsilon$. This means that two balls starting a distance ϵ apart can be at opposite ends of the circular billiard after a time $1/\epsilon$. But this implies that at such times you have lost all useful information about the position of your ball, since all you can say is that, having started in the initial interval I_ϵ, it is now somewhere in the billiard, which is not saying much indeed! For example, if $\epsilon = 10^{-4}$ (which would be a pretty good precision for a billiard player, if you think about it!), this time is of the order of 10^4 bounces (Remember the motion is frictionless). But the same reasoning on the Sinai billiard leads to an estimate of order $\ln \epsilon$ for this time scale, which is exceedingly short: of the order of 8 bounces for $\epsilon = 10^{-4}$! So the information loss is very rapid in this case.

Exponential instability of this kind shows up in many dynamical systems and when defined precisely goes by the name of hyperbolicity. It is considered the hallmark of chaos. I will present another example of a hyperbolic dynamical system in section 10 and we will study at some length one of its important properties, called *exponential mixing*.

The motion on the circular billiard is a typical example of what could be called a "regular", "stable" or "predictable" dynamics, the one on the Sinai billiard being "irregular", "unstable", "unpredictable" or "chaotic". More precise definitions will follow later (Sections 8 and 9).

Further reading: For a survey and further references to the vast literature on billiard flows, see [**T**]. (None of this is likely to improve your game, though.)

4. Spectral statistics

I have pointed out before that it is rarely possible to find the spectrum and eigenfunctions of a Dirichlet Laplacian explicitly. One does however have some hopes of obtaining statistical and asymptotic information about the spectrum in terms of properties of the billiard flow. The simplest object of study is the *integrated (or cumulative) density of states* or *counting function* or *mode number*, defined as

$$N(\lambda) = \#\{j \mid \lambda_j \leq \lambda\}.$$

Its asymptotic behaviour is given by "Weyl's law" [**Ho**] (valid in this form if $\partial\Omega$ is smooth enough):

$$N(\lambda) = \frac{1}{4\pi}(\text{area } \Omega)\ \lambda + \frac{1}{4\pi}(\text{length } \partial\Omega)\ \sqrt{\lambda} + \mathrm{o}(\sqrt{\lambda}).$$

This formula implies that the average number of λ_j per unit length, the *mean level density* behaves like

$$\bar\rho(\lambda) = \frac{N(\lambda)}{\lambda} \sim \frac{1}{4\pi}(\text{area } \Omega) \equiv \bar\rho,$$

so that, for large j, the successive eigenvalues are separated by a fixed average distance (the mean level spacing) equal to $4\pi/$ area Ω. This means in particular that the mean level density encodes some very rough information about the underlying billiard, namely its area. To put it differently, if I know the area of the billiard, then I know the asymptotic mean level density of its spectrum. This is perhaps not very exciting at first sight, since the mean level density is only a very rough piece of information about the spectrum. It is nevertheless quite surprising and intellectually pleasing to note that this information is equivalent to a very simple geometric property of the billiard, namely its area. Note also that the area of a billiard is easy to determine, whereas $\bar\rho$ is a priori not at all. The next question that imposes itself is:

> "Can one obtain more detailed information about the asymptotic spectral statistics by using some dynamical properties of the billiard flow?"

Having studied the mean level spacing, it seems reasonable to study the fluctuations around this mean. For that purpose, introduce, for $s \in \mathbb{R}, \lambda > 0$

$$\frac{\#\{j \mid \lambda_j \le \lambda, \lambda_{j+1} - \lambda_j \le s\}}{N(\lambda)}.$$

This quantity counts the fraction of eigenvalues λ_j below λ that are at a distance less than s from the next eigenvalue λ_{j+1}. Assuming the limit exists (which is anything

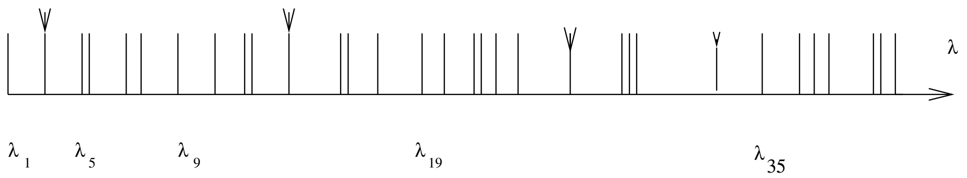

FIGURE 4. Some levels of a fictitious spectrum.

but obvious in any given spectrum), define the *integrated nearest-neighbour spacing distribution*

$$P(s) = \lim_{\lambda\to\infty} \frac{\#\{j \mid \lambda_j \le \lambda, \lambda_{j+1} - \lambda_j \le s\}}{N(\lambda)}.$$

Clearly, $P(s) = 0$ for $s < 0$, P is increasing and $P(\infty) = 1$ so P is the distribution function of a probability measure on $\mathbb{R}$. You should think of $P(s)$ as the probability that a nearest-neighbour spacing picked at random in the spectrum is less than s. Also,

$$P(b) - P(a) = \int_a^b P'(s)ds = \int_a^b dP(s)$$

is the probability that such a spacing lies between a and b.

EXERCISE 4.1. Prove that

$$\int_{-\infty}^{\infty} s dP(s) = \bar\rho^{-1},$$

and interpret this formula.

It is important that you develop some feeling for what $P(s)$ represents. For that purpose it is good to realize that the quantities $N(\lambda)$ and $P(s)$ can be defined for any increasing and diverging sequence $(\lambda_j)_{j\in\mathbb{N}}$ of positive numbers, whether it is the spectrum of a billiard or not (see Fig. 4) and to do the following exercises.

EXERCISE 4.2. Compute $N(\lambda), \bar\rho$ and $P(s), P'(s)$ for the following sequences $(\delta > 0)$:
(i) $\lambda_j = \delta j$;
(ii) $\lambda_{2j} = \lambda_{2j+1} = \delta j$;
(iii)* Let ω be an irrational and define $\lambda_{j+1} = \lambda_j + \delta \text{ Frac } (\omega j),\ \lambda_0 = 0$ where Frac x denotes the fractional part of the number x. **Hint:** Have a look at Exercises A.3 and A.4.

Let us now explain how $P(s)$ can be used to measure the "regularity" of the spectrum. For that purpose note that if the mean level density $\bar\rho$ is prescribed, then the most regular spectrum with such level density is of course given by $\lambda_j = \bar\rho^{-1}j$: all nearest-neighbour spacings are equal to the mean spacing. In particular, in such an imaginary "picket fence" spectrum the levels are never degenerate or even close together, which is in turn reflected in the fact that $P'(0) = 0$. If, on the contrary $\lambda_{j+1} - \lambda_j = \frac{2}{\bar\rho} \text{ Frac } (\omega j)$ (with ω irrational), then $P'(0) = \bar\rho/2$, indicating that very small level spacings do occur. This suggests that the *nearest-neighbour spacing distribution* $P'(s)$ can be used as a measure of the regularity of the spectrum. If $P'(0) = 0$, there is said to be "level repulsion" in the spectrum since then the probability of finding successive levels close together is very small. The degree of regularity in the spectrum can then further be measured by the way in which $P'(s)$ vanishes as $s \to 0$. If $P'(0)$ does not vanish, one says there is "level bunching" or "level clustering" in the spectrum, making the spectrum less regular, or less rigid.

The original Berry-Tabor and Bohigas-Giannoni-Schmit conjectures are very strong statements about the level statistics that imply the following claims for the nearest neighbour spacing distribution:

If the billiard flow is regular (predictable), then $P'(s) = \bar\rho\exp(-\bar\rho s)$ so that in particular $P'(0) \neq 0$.
If the billiard flow is irregular (chaotic, unpredictable), $P'(s) = \frac{\pi}{2}\bar\rho s\exp(-\frac{\pi}{4}(\bar\rho s)^2)$ so that in particular $P'(0) = 0$.

The above discussion gives us a fair idea of what is understood by "regular" or "irregular" spectra. It remains to explain more clearly what "regular" and "irregular" motions are. In the next four sections I will try to give a more precise meaning to these concepts. For that purpose, I will give a short introduction to Hamiltonian mechanics, explain in which sense a billiard flow is a Hamiltonian flow and show that the most regular Hamiltonian flows possible are the so-called completely integrable ones. I will finally try to convince you that the notion of unpredictability is rather well captured in the definition of "mixing".

EXERCISE 4.3. Use the Weyl law to derive an asymptotic expression for the λ_n as a function of n. Why can the result not be used to study $P(s)$?

5. Hamiltonian mechanics

I hope you remember Newton's second law from your high school days. It says that the force experienced by a point particle is equal to its mass times its acceleration:

$$ma = F, \quad \text{or} \quad m\frac{d^2q}{dt^2}(t) = F(q(t), \dot{q}(t)), \tag{5.1}$$

where $t \in \mathbb{R} \mapsto q(t) \in \mathbb{R}^3$ is the particle trajectory and $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ the *force*. From a physical point of view "force" is a very complicated concept, but mathematically, it is just a function from $\mathbb{R}^6$ to $\mathbb{R}^3$. Similarly, from a mathematical point of view, Newton's second law is just a second order differential equation. One says the force F is *conservative* if there exists a function $V : \mathbb{R}^3 \to \mathbb{R}$, called the *potential*, so that $F(q, v) = -\nabla V(q)$. In particular, a conservative force does not depend on the velocity of the particle. This terminology comes from the fact that, in that case, the *energy*, defined by

$$E : (q, v) \in \mathbb{R}^6 \mapsto \frac{1}{2}mv^2 + V(q) \in \mathbb{R},$$

is a *constant of the motion*, *i.e.*

$$\frac{d}{dt}E(q(t), \dot{q}(t)) = 0. \tag{5.2}$$

EXERCISE 5.1. Prove that every solution of (5.1) satisfies (5.2) if the force is conservative.

There is an elegant way to rewrite (5.1), called the Hamiltonian formalism. Introducing the *linear momentum* $p(t) = m\frac{dq}{dt}(t)$, it is clear that the second order differential equation (5.1) is equivalent to the set of coupled first order equations

$$\dot{q}(t) = \frac{p(t)}{m}, \qquad \dot{p}(t) = -\nabla V(q(t)). \tag{5.3}$$

Introducing furthermore the *Hamiltonian*

$$H : (q, p) \in \mathbb{R}^6 \mapsto \frac{p^2}{2m} + V(q) \in \mathbb{R}, \tag{5.4}$$

equation (5.3) can be rewritten

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)). \tag{5.5}$$

That all this rewriting is of any use can not be clear to you yet, but there is an intriguing elegance to it that can be felt as follows. Let $f : \mathbb{R}^6 \to \mathbb{R}$ and compute $(x(t) = (q(t), p(t)))$

$$\begin{aligned}
\frac{d}{dt}f(x(t)) &= \frac{\partial f}{\partial q}(x(t))\dot{q}(t) + \frac{\partial f}{\partial p}(x(t))\dot{p}(t) \\
&= \frac{\partial f}{\partial q}(x(t))\frac{\partial H}{\partial p}(x(t)) - \frac{\partial f}{\partial p}(x(t))\frac{\partial H}{\partial q}(x(t)) \\
&= \{f, H\}, \qquad (5.6)
\end{aligned}$$

where I introduced, for any pair of functions $f, g \in C^\infty(\mathbb{R}^6)$, the *Poisson bracket* of f and g *via*

$$\{f, g\}(x) = \frac{\partial f}{\partial q}(x)\frac{\partial g}{\partial p}(x) - \frac{\partial f}{\partial p}(x)\frac{\partial g}{\partial q}(x). \tag{5.7}$$

Note that I use the shorthand

$$\frac{\partial f}{\partial q}(x)\frac{\partial g}{\partial p}(x) = \sum_{i=1}^{3}\frac{\partial f}{\partial q_i}(x)\frac{\partial g}{\partial p_i}(x), \text{ etc.}$$

and that the above definition immediately generalizes to $\mathbb{R}^{2n}$. The Poisson bracket then defines a product on $C^\infty(\mathbb{R}^{2n})$ that has the following very important and easily checked properties:

$$\{f,g\} = -\{g,f\} \quad \text{(Anti-symmetry)} \tag{5.8}$$

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad \text{(Jacobi identity)} \tag{5.9}$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \quad \text{(Derivation)} \tag{5.10}$$

$$\{f,g\} = 0, \forall g \in C^\infty(\mathbb{R}^{2n}) \Rightarrow f \text{ is a constant (Non-degeneracy)} \tag{5.11}$$

EXERCISE 5.2. (i) Check, through direct calculations, the above properties.

(ii) Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Define on $\mathcal{L}(\mathcal{H})$ the following product (called the commutator):

$$[\cdot,\cdot] : (A,B) \in \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \mapsto [A,B] = AB - BA \in \mathcal{L}(\mathcal{H}).$$

Prove that the commutator has properties completely analogous to the four properties listed above for the Poisson bracket.

In conclusion, the Poisson bracket has some nice properties and they have a surprising similarity to the properties of the commutator. This is quite intriguing, and we will see in the next sections that it has some important consequences.

For now, let me point out how the Poisson bracket can be used to define some associated structures. Given any $g \in C^\infty(\mathbb{R}^{2n})$, one can first of all construct the *Hamiltonian vector field* X_g by

$$X_g : x \in \mathbb{R}^{2n} \mapsto X_g(x) = (\frac{\partial g}{\partial p}(x), -\frac{\partial g}{\partial q}(x)) \in \mathbb{R}^{2n},$$

or, for those familiar with the rudiments of differential geometry, by

$$X_g = \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}.$$

One can then consider the associated system of first order differential equations

$$\dot{x}(t) = X_g(x(t)) \quad \text{or } \dot{q}(t) = \frac{\partial g}{\partial p}(x(t)), \quad \dot{p}(t) = -\frac{\partial g}{\partial q}(x(t)), \tag{5.12}$$

and study the corresponding *Hamiltonian flow* Φ_t^g, defined by

$$\Phi_t^g(x) = x(t),$$

where $t \mapsto x(t)$ is the solution of (5.12) with initial condition $x(0) = x$. Note that the rewritten form of Newton's law, equation (5.5) is precisely of this form, with a particular choice for the function g. I will always assume the flows are *complete*, which means that all differential equations considered are supposed to have global solutions for all $t \in \mathbb{R}$. For precise technical conditions guaranteeing this, you may consult [**AM**], for example.

We will need a few properties of Hamiltonian flows that I shall not prove:

PROPOSITION 5.3. Let $f, g, h \in C^\infty(\mathbb{R}^{2n})$. We have:

(i) the Hamiltonian flows preserve the Poisson bracket, *i.e.* for all $t \in \mathbb{R}$,

$$\{f \circ \Phi_t^g, h \circ \Phi_t^g\} = \{f,h\} \circ \Phi_t^g; \tag{5.13}$$

(ii) the Hamiltonian flows preserve the volume, *i.e.*, for all $t \in \mathbb{R}$ and for all Borel subsets B of $\mathbb{R}^{2n}$, one has

$$\int_B dx = \int_{\Phi_t^g(B)} dx.$$

(iii) If $\{f, g\} = 0$, then, $\forall s, t \in \mathbb{R}$, $\Phi_t^f \circ \Phi_s^g = \Phi_s^g \circ \Phi_t^f$.

It is very instructive to compute a few simple Hamiltonian flows, so I urge you strongly to do the following exercise, the results of which I will use extensively later on.

EXERCISE 5.4. (i) Let $H(x) = \frac{p^2}{2m}$. Compute Φ_t^H explicitly.

(ii) Compute Φ_t^H for $H(x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$, $(x = (q, p) \in \mathbb{R}^2, \omega > 0)$.

(iii) Let $L : \mathbb{R}^6 \to \mathbb{R}^3$ be defined by $L(x) = x \wedge p$. This vector is called the *angular momentum* in physics texts. Compute $\{L_i, L_j\}$ for $i, j = 1, 2, 3$. Compute also $\Phi_t^{L_i}$ and interpret the result geometrically. Explain the statement, found in all physics texts on classical mechanics, that "angular momentum is the generator of rotations."

(iv) Let $g(x) = q \cdot p$. Compute Φ_t^g.

(v) Let g be a homogeneous quadratic polynomial in the variables $q, p \in \mathbb{R}$, *i.e.*:

$$g(x) = \frac{1}{2}\alpha q^2 + \beta qp + \frac{1}{2}\gamma p^2, \qquad (\alpha, \beta, \gamma \in \mathbb{R}).$$

Show that, for each t, Φ_t^g is a linear map on $\mathbb{R}^2$ given explicitly by

$$\Phi_t^g = \exp Bt, \qquad B = \begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix}$$

and that $\det \Phi_t^g = 1$. Compare this to what you found in (i)-(iv).

(vi) Compute $\Phi_t^{p_i}, \Phi_t^{q_i}$, for $i = 1 \dots n$.

(vii) Let $f_1, f_2 \in C^\infty(\mathbb{R}^{2n})$. Show that $\Phi_t^{f_1} = \Phi_t^{f_2}$ for all $t \in \mathbb{R}$ iff $f_1 - f_2 = c \in \mathbb{R}$ iff $X_{f_1} = X_{f_2}$.

For further reference, let me note that a diffeomorphism Φ of $\mathbb{R}^{2n}$ that preserves the Poisson bracket as in (5.13) is called a *canonical transformation* or a *symplectomorphism*. The linear canonical transformations will play a particular role further on in these notes and are the subject of the following exercise.

EXERCISE 5.5. (i) Let A be a two by two matrix and Φ_A the associated linear map of $\mathbb{R}^2$. Show that Φ_A is a canonical transformation if and only if $\det A = 1$ (*i.e.* iff $A \in \mathrm{SL}(2, \mathbb{R})$).

(ii) Show that, if $\mathrm{Tr} A > 2$, and if $A \in \mathrm{SL}(2, \mathbb{R})$, then there exists a homogeneous quadratic polynomial g so that $\Phi_A = \Phi_1^g$. **Hint:** Diagonalize A, then use (iv) and (v) of Exercise 5.4.

(iii) Show that, if $\mathrm{Tr} A < -2$, such a polynomial does not exist. **Hint:** Show that, if $\mathrm{Tr} B = 0$, then $\mathrm{Tr} \exp B \geq -2$.

(iv) What happens if $-2 \leq \mathrm{Tr}\ A \leq 2$?

Hamiltonians in physical problems do not always have the form (5.4) because forces are not always of the form $F = -\nabla V$. An important exception is the case of a particle of mass m and charge e moving in a *magnetic field* $B : \mathbb{R}^3 \to \mathbb{R}^3$.

Newton's second law then reads

$$m\ddot{q}(t) = e\dot{q}(t) \wedge B(q(t)), \tag{5.14}$$

where the force $F(q, v) = ev \wedge B(q)$ is called the Lorentz force. The question arises whether this second order equation can be poured into Hamiltonian form. This is indeed the case. First, we need to recall that electromagnetic theory teaches us that a magnetic field is a divergence free vector field (*i.e.* $\mathrm{div}B = 0$) so that there exists a vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ for which $B = curl A = \nabla \wedge A$. A is called the *vector potential.* It then follows from a straightforward and instructive calculation that, if $x(t) = (q(t), p(t))$ is a flow line of the Hamiltonian flow generated by

$$H(x) = \frac{(p - eA(q))^2}{2m},$$

then $q(t)$ satisfies (5.14).

EXERCISE 5.6. (i) Prove this last statement.

(ii)* Let $A(x) = (0, Bq_1, 0)$ for some $B > 0$. Show $B(q) = (0, 0, B)$ and prove that the solutions of (5.14) describe a helicoidal motion along the vertical axis.

A crucial notion in all what follows is the one of *constant of the motion.* A function f is said to be a constant of the motion for a Hamiltonian flow Φ_t^g if and only if

$$\frac{d}{dt} f \circ \Phi_t^g(x) = 0, \ \forall t \in \mathbb{R}, x \in \mathbb{R}^{2n}.$$

Since, as we already checked,

$$\frac{d}{dt} f \circ \Phi_t^g(x) = \{f, g\}(\Phi_t^g(x)), \tag{5.15}$$

f is a constant of the motion for Φ_t^g iff it has vanishing Poisson bracket with g.

EXERCISE 5.7. Show that the set of constants of the motion of a given Hamiltonian flow Φ_t^g is closed under the Poisson bracket (*i.e.* if f_1, f_2 are constants of the motion, then so is $\{f_1, f_2\}$).

Note that, clearly, $\{g, g\} = 0$ so that g is always a constant of the motion for the flow it generates. As a result, the level surfaces of g:

$$\Sigma_c = \{x \in \mathbb{R}^{2n} \mid g(x) = c\} \ (c \in \mathbb{R})$$

are invariant surfaces for the flow Φ_t^g. In particular, given an initial point x in *phase space* $\mathbb{R}^{2n}$, the curve $t \mapsto \Phi_t^g(x)$ does not wander throughout all of phase space, but stays on the $2n - 1$-dimensional surface Σ_c with $c = g(x)$. This constitutes a simplification with respect to the case of an arbitrary (non-Hamiltonian) flow, for which such a priori information is not available. The situation further simplifies if there are several constants of the motion $f_1, f_2, \ldots f_k$, since the motion is then confined to one of the surfaces

$$\Sigma_c = \{x \in \mathbb{R}^{2n} \mid f_i(x) = c_i, i = 1 \ldots k\} \equiv F^{-1}(c) \ (c \in \mathbb{R}^k).$$

which are $2n - k$ dimensional smooth surfaces if the Jacobian of the map

$$F : x \in \mathbb{R}^{2n} \to F(x) = (f_1(x), \ldots f_k(x)) \in \mathbb{R}^k$$

has rank k. This reduces the set of differential equations determining the flow effectively from a problem in $2n$ variables to one in $2n - k$ variables. One may be tempted to think that the optimal case is the one where $k = 2n-1$ (Why?), but this is asking a bit too much. There are only a few interesting Hamiltonians in applications that have so many constants of the motion. The most interesting systems having a large number of constants of the motion turn out to be the *completely integrable* ones. They have exactly n constants of the motion (and some beautiful additional structure, as explained in section 7). Here is an important example of such a system, that I will come back to several times. Let

$$H(x) = \frac{p^2}{2m} + V(r) \in C^\infty(\mathbb{R}^4),$$

where $x = (q,p) \in \mathbb{R}^4$ and $r = \sqrt{q_1^2 + q_2^2}$. Then the angular momentum

$$L_3(x) = q_1 p_2 - q_2 p_1$$

is easily checked to be a constant of the motion (*i.e.* $\{L_3, H\} = 0$).

EXERCISE 5.8. * Show that $H \in C^\infty(\mathbb{R}^4)$ implies that $V \in C^\infty([0,\infty))$ and that $V'(0) = 0$.

Further reading: There exists a vast literature on Hamiltonian mechanics. The standard textbook used in advanced physics courses is [**G**] or [**LL1**]. The Poisson bracket structure hides a lot of beautiful geometry, referred to as "symplectic geometry". Mathematically oriented treatments of the subject exploit the geometric point of view to varying degrees. Consult [**A**] or [**AM**] if you are interested.

6. Billiard flows are Hamiltonian

The Hamiltonian formalism derives its main interest from the observation that it provides a framework that, once suitably generalized, incorporates practically all of physics. Billiard flows are Hamiltonian and this is part of the reason why they can be claimed to be a suitable toy model for more realistic physical systems. In this section I want to briefly show how billiard flows fit in the Hamiltonian framework, using the circular billiard as an example. Let $D_a = \{q \in \mathbb{R}^2 \mid \| q \| < a\}$ and consider, for each $\epsilon > 0$, the Hamiltonian

$$H_\epsilon : x \in D_{a+\epsilon} \times \mathbb{R}^2 \mapsto \frac{p^2}{2} + V_\epsilon(\| q \|),$$

where $V_\epsilon \in C^\infty(D_{a+\epsilon})$ has the profile of Fig. 5. Note that, given E, the trajectories $t \mapsto x_\epsilon(t) = (q_\epsilon(t), p_\epsilon(t))$ of the Hamiltonian flow with energy $H(x_\epsilon(t)) = E$ generated by H_ϵ are confined to the region $\| q \| \leq r_*(\epsilon)$ (see Fig. 5) since $E = H_\epsilon(q_\epsilon(t), p_\epsilon(t)) \geq V(q_\epsilon(t))$. Moreover, inside D_a, $\nabla V_\epsilon(q) = 0$, so there $\ddot{q}_\epsilon(t) = 0$ and the trajectory is a straight line. In the region $a < \| q_\epsilon(t) \| < r_*(\epsilon)$, $-\nabla V_\epsilon(q)$ is pointing radially inward forcing the trajectory to bend back as indicated on the figure. The resulting motion is therefore much like the one of a marble rolling in a soup plate. It is easy to convince oneself that, in the limit $\epsilon \to 0$ the trajectory $q_\epsilon(t)$ converges to a billiard trajectory of the circular billiard of radius a. Note that for circular billiards phase space is reduced from all of $\mathbb{R}^{2n}$ to only $D_a \times \mathbb{R}^n$.

EXERCISE 6.1. (i) Prove the above statement with a degree of rigour to your liking. The important statement is that, in the limit, the trajectory reflects specularly from the boundary at $r = a$. You may find it helpful to use conservation of angular momentum.

(ii) Generalize the preceding discussion to billiards with arbitrary shapes. Do corners pose problems?

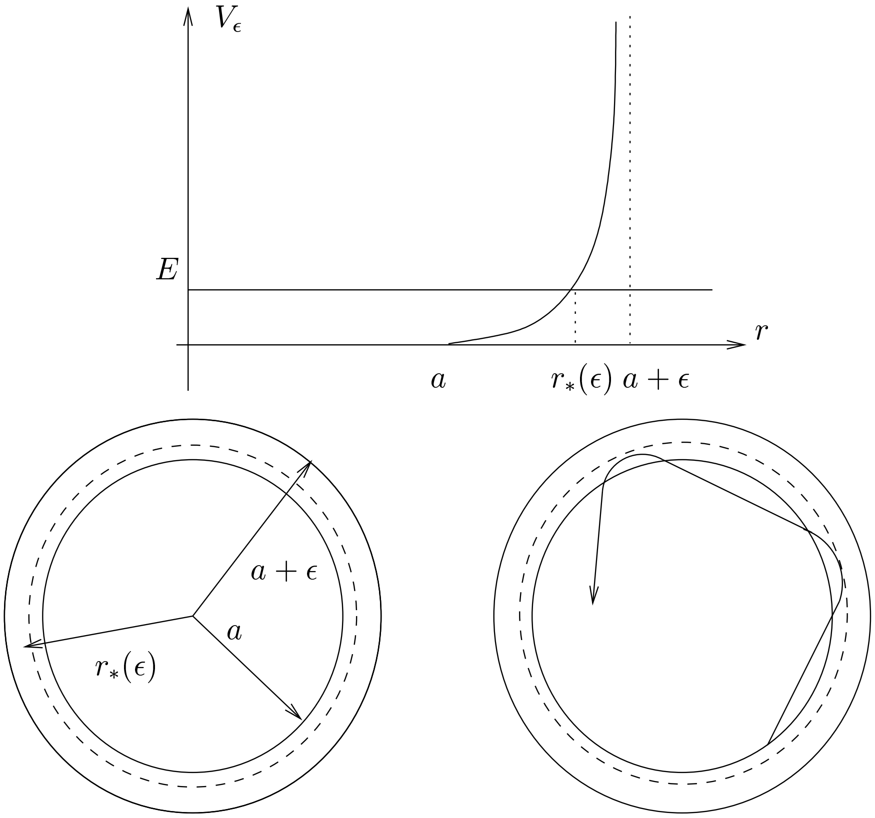

FIGURE 5. A potential for a soft-walled billiard and the corresponding motion

7. Completely integrable systems: the definition and an example

DEFINITION 7.1. Let U be an open subset of $\mathbb{R}^{2n}$ and let $H \in C^\infty(U)$. Then H is said to be completely integrable (on U) iff there exists $F \in C^\infty(U, \mathbb{R}^n)$, $(F(x) = (f_1(x), f_2(x), \dots f_n(x)))$ such that $H = f_1$ and

(i) $\{f_i, f_j\} = 0$;
(ii) The rank of $J(F)$, the Jacobian of F, is n.

In other words, a completely integrable Hamiltonian has n functionally independent constants of the motion that Poisson commute. While the latter condition is the crucial one, condition (ii) is of a technical nature and serves to assure that the surfaces $F^{-1}(c)$ are smooth submanifolds of $U \subset \mathbb{R}^{2n}$.

Before stating the general theorem that explains the importance of completely integrable systems, let me study in some detail the example introduced at the end of section 5, where $F(x) = (H(x), L_3(x))$. It is convenient to introduce polar coordinates first. Setting

$$p_r = p_1 \cos\theta + p_2 \sin\theta, \quad p_\theta = r(-p_1 \sin\theta + p_2 \cos\theta),$$

one easily sees that

$$H(x) = \frac{p_r^2}{2m} + V_{eff}(r), \quad V_{eff}(r) = \frac{L_3^2}{2mr^2} + V(r), \quad L_3(x) = p_\theta.$$

For convenience, we will always assume $V(0) = 0$.

EXERCISE 7.2. (i) Prove that $\{\theta, p_\theta\} = 1 = \{r, p_r\}, \{r, \theta\} = 0 = \{p_r, p_\theta\}$.

(ii) Show that the Jacobian of $F(x) = (H(x), L_3(x))$ is of rank 2 unless either $x = (0,0)$ or $p_r = 0$ and $V'(r) = \frac{p_\theta^2}{mr^3}$. **Hint:** Use exercise 5.8.

(iii) Suppose V is a strictly increasing, convex function of its argument. Set $L_3 = \ell$ and show that, for each $\ell > 0$, $V_{eff} = \frac{\ell^2}{2mr^2} + V(r)$ has a unique minimum at a radius $r_*(\ell)$ and that, for all $E > V_{eff}(r_*(\ell))$ the level surfaces

$$\Sigma_{E,\ell} = \{x \in \mathbb{R}^4 \mid H(x) = E, L_3(x) = \ell\}$$

are smooth submanifolds of $\mathbb{R}^{2n}$ (see Fig. 6). Show they are compact.

(iv) Compute $\dot{r}, \dot{p}_r, \dot{\theta}, \dot{p}_\theta$.

(v) Still under the hypothesis of (iii), find all circular trajectories of the Hamiltonian flow (*i.e.* solutions of Hamilton's equations for which $\dot{r}(t) = 0$) and compare your result with what you found in (ii)-(iii).

Let's place ourselves in the conditions of part (iii) of the above exercise. We will show that the $\Sigma_{E,\ell}$ are tori. For that purpose, note that, since $L_3 = p_\theta \neq 0$, and since the constraint $H(x) = E$ defines a closed curve in the (r, p_r) plane, there is a one-to-one correspondence between $\Sigma_{E,\ell}$ and the surface in the (q_1, q_2, p_r)-space drawn in Fig. 6. The latter is clearly a torus. The flow lines of Φ_t^H wrap around this torus in a way that can be quite explicitly described as follows from the following exercise and section 8.

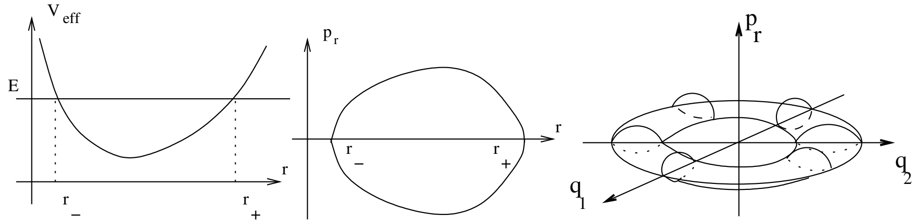

FIGURE 6. The surface $\Sigma_{E,\ell}$.

EXERCISE 7.3. * (i) Show that the trajectories of the Hamiltonian flow on $\Sigma_{E,\ell}$ are all periodic iff

$$\frac{\ell}{m\pi} \int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{2m(E - V_{eff}(r))}} \in \mathbb{Q}.$$

What happens if this condition is not satisfied? Describe the trajectories $x(t) = (q(t), p(t))$ of the Hamiltonian flow in this case and draw a qualitative picture of their projection onto the q-plane (Compare your picture to Fig. 5).

(ii) How should the pictures of Fig. 6 be modified when $\ell = 0$? Are the surfaces $\Sigma_{E,0}$ still tori? Describe the trajectories $x(t) = (q(t), p(t))$ of the Hamiltonian flow in this case.

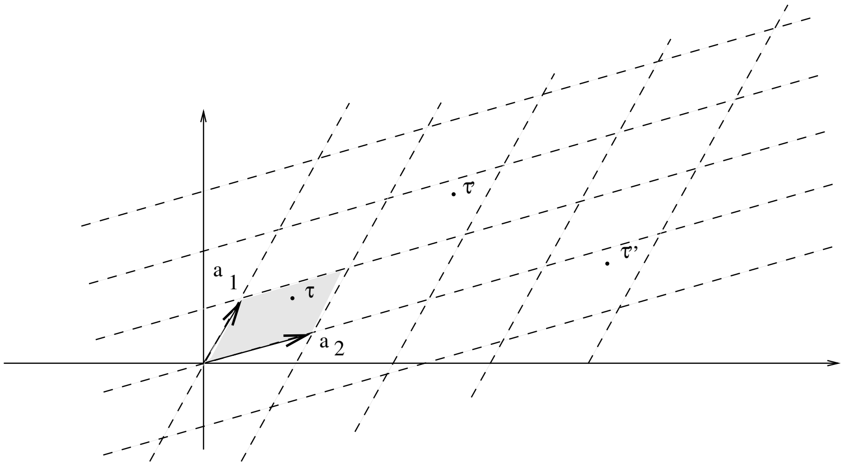

FIGURE 7. A lattice and fundamental domain in $\mathbb{R}^2$: $\tau \sim \tau' \sim \tau''$.

(iii) Analyze the above questions for the circular billiard. Show that the surfaces $\frac{p^2}{2m} = E$, $L_3(x) = \ell$ in $D_a \times \mathbb{R}^2$ are tori if, on the boundary of D_a, you identify the incident and reflected momentum vectors. Also, relate E and ℓ to the angle α (see section 3); show that $r_- = \frac{\ell}{\sqrt{2mE}}$; compute the expression in part (i) of this exercise (with $V_{eff} = \frac{\ell^2}{2mr^2}$, $r_+ = a$, and r_- as before) and prove it equals $\frac{\alpha}{\pi}$. Interpret the result geometrically.

8. Completely integrable systems: the general case

Having studied one example of a completely integrable system, we can now turn to the general case. The theorem below will establish that, surprisingly, the previous example actually describes the general situation! We need some notations first.

Let $a_i, i = 1 \dots n$ be a basis in $\mathbb{R}^n$ and write $\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$ for the lattice

$$\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n = \left\{ \sum_{i=1}^{n} n_i a_i \mid n_i \in \mathbb{Z}, i = 1 \dots n \right\}.$$

This is clearly an additive subgroup of $\mathbb{R}^n$. Consider then the n-torus $\mathbb{T}^n = \mathbb{R}^n/(\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n)$. As a set, this is just the collection of equivalence classes in $\mathbb{R}^n$ under the equivalence relation

$$\tau \sim \tau' \Leftrightarrow \tau' - \tau \in \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n.$$

It is sometimes helpful to identify it with the *fundamental domain* in $\mathbb{R}^n$ defined by

$$\{\tau \in \mathbb{R}^n \mid \tau = \sum_{i=1}^{n} \alpha_i a_i, 0 \leq \alpha_i < 1\}.$$

I will denote the equivalence class of $\tau \in \mathbb{R}^n$ by $[\tau]$. Note that $\mathbb{T}^n$ is a group under the law $[\tau] + [\tau'] = [\tau + \tau']$, a fact I invite you to check should you not find it obvious. It is also helpful to draw some pictures for the case $n = 2$ (Fig. 7).

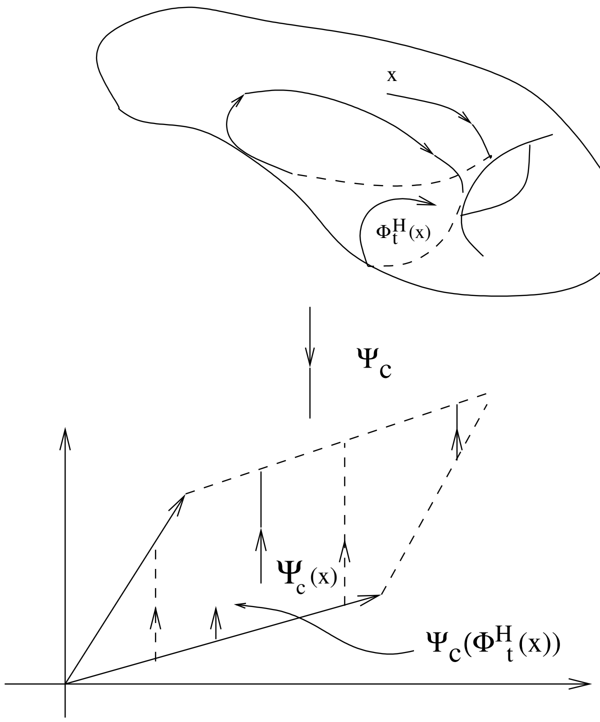

FIGURE 8. Ψ_c intertwines Φ_t^H with a flow by translations on the torus

Now pick $\omega \in \mathbb{R}^n$ and define, for all $t \in \mathbb{R}$

$$T_t^\omega : [\tau] \in \mathbb{T}^n \mapsto [\tau + t\omega] \in \mathbb{T}^n.$$

Note that $T_{t+t'}^\omega = T_t^\omega \circ T_{t'}^\omega$. The family T_t^ω is called a flow by translations on $\mathbb{T}^n$. Such flows are quite simple and some of their properties are studied in Appendix A. Their interest in the context of completely integrable systems derives from the following theorem.

THEOREM 8.1. *Suppose H is completely integrable on $U \subset \mathbb{R}^n$. Let $c \in \mathbb{R}^n$ be such that the level surface $\Sigma_c = F^{-1}(c)$ is compact and connected. Then there exists a basis $a_i(c), (i = 1, \ldots n)$ of $\mathbb{R}^n$ and a diffeomorphism (i.e. a smooth invertible map)*

$$\Psi_c : \Sigma_c \subset \mathbb{R}^{2n} \to \mathbb{T}^n = \mathbb{R}^n/(\mathbb{Z}a_1(c) + \cdots + \mathbb{Z}a_n(c))$$

with the following additional property, for all $t \in \mathbb{R}$ (see Fig. 8):

$$\Psi_c \circ \Phi_t^H \circ \Psi_c^{-1} = T_t^{e_1}.$$

Here $e_1 = (1, 0, \ldots 0) \in \mathbb{R}^n$.

The theorem states two important facts. The first is that each level surface Σ_c can be identified with a torus. That is interesting and surprising by itself: after all, a priori we only know the Σ_c are n-dimensional smooth surfaces in $\mathbb{R}^{2n}$. They could have any topology at all. It turns out that the condition $\{f_i, f_j\} = 0$ imposes considerable restrictions on the topology and forces the Σ_c to be tori. The second part of the theorem, even more important for our purposes, states that, under this identification of Σ_c with a torus, the Hamiltonian flow becomes very simple: it is just a translation. To appreciate this result one needs of course to understand translations on tori a bit better: their essential features are briefly discussed in Appendix A.

Let me now sketch the proof of Theorem 8.1. You will see that the construction of the map Ψ_c is actually quite straightforward and explicit so that the theorem is not as abstract as it may at first seem. The details of the proof make up nice exercises or can be found in [**A**] and [**AM**], where the theory of completely integrable systems is further developed.

Proof: Define, for each $\tau \in \mathbb{R}^n$

$$\Xi_\tau : x \in \Sigma_c \mapsto \Xi_\tau(x) \equiv \Phi^{f_1}_{\tau_1} \circ \Phi^{f_2}_{\tau_2} \cdots \circ \Phi^{f_n}_{\tau_n}(x) \in \Sigma_c.$$

First of all, it follows from Proposition 5.3 (iii) that the flows $\Phi^{f_i}_t$ all commute so that

$$\Xi_\tau \circ \Xi_\sigma = \Xi_{\tau+\sigma}, \quad \forall \tau, \sigma \in \mathbb{R}^n. \tag{8.1}$$

Now fix $x_* \in \Sigma_c$ and define

$$\Xi_* : \tau \in \mathbb{R}^n \mapsto \Xi_\tau(x_*) \in \Sigma_c.$$

This map has the following properties:

(i) It is a local diffeomorphism: this follows from condition (ii) of Definition 7.1;

(ii) It is surjective: this follows from (i) above and from the connectedness of Σ_c together with (8.1);

Now define, for each $x \in \Sigma_c$, $G_x = \{\tau \in \mathbb{R}^n \mid \Xi_\tau(x) = x\}$. Clearly:

(iii) G_x is a subgroup of $\mathbb{R}^n$;

(iv) G_x does not depend on x (use property (ii) above as well as (8.1)!).

Let's denote this group by G_*. One has:

(v) G_* has no accumulation points: this follows almost immediately from (i) above.

It is possible to show with a little work that the only subgroups of $\mathbb{R}^n$ that have no accumulation points are lattices *i.e.* G_* is of the form $G_* = \mathbb{Z}a_1(c) + \cdots + \mathbb{Z}a_k(c)$ for some $a_i(c), i = 1, \ldots k \le n$. But now we are almost finished. Define $Q = \mathbb{R}^n/G_*$ and note that $Q \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$. Then consider

$$\hat{\Xi}_* : [\tau] \in Q \mapsto \Xi_*(\tau) \in \Sigma_c.$$

But this is now, in view of all that precedes, a diffeomorphism. As a result, and since Σ_c is compact, so is Q, from which one concludes $k = n$ and $Q = \mathbb{T}^n$ so that Q is a torus. Taking $\Psi_c = \hat{\Xi}_*^{-1}$ the first part of the theorem is proven. Note furthermore that, for all $\tau \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$, one has

$$\hat{\Xi}_*([\tau + te_1]) = \Phi^H_t(\hat{\Xi}_*([\tau])$$

which concludes the proof. □

EXERCISE 8.2. * Did you get the right answer in Exercise 7.3 (ii)?

It remains to explain in which sense completely integrable systems have regular or stable motion. I will not give a very precise argument, but just give an idea of why this is true. As we just saw, if a system is completely integrable, then its phase space is foliated by invariant tori and the Hamiltonian flow consists of translations on these tori. Two nearby initial conditions will be on nearby tori and the orbits of the Hamiltonian flow emanating from them will correspond to translations in slightly different directions. They will therefore separate slowly, linearly in time. In this sense, completely integrable systems exhibit stable motion.

Note that the circular billiard is clearly integrable. That should be clear from section 6 and from Exercise 7.3. The same is true for the rectangular billiard, which is, paradoxically perhaps, a little bit harder to study (see Exercise A.6).

A more precise statement of the Berry-Tabor conjecture is then finally that the spectrum of a (typical) completely integrable billiard has a nearest neighbour spacing distribution $P'(s)$ given by $P'(s) = \bar{\rho}\exp(-\bar{\rho}s)$. As I mentioned before, there are very few theorems supporting this conjecture. You may wish to consult the original paper of Berry and Tabor for a heuristic argument explaining the origin of the conjecture. For a recent review and some mathematically rigorous results supporting the Berry-Tabor conjecture in certain particular cases (rectangular billiards and similar systems), you may consult [**Ma**].

Many of the basic examples of completely solvable Hamiltonian systems found in textbooks are completely integrable. In addition, many interesting systems can be seen as perturbations of completely integrable ones, so that much effort has been expended on understanding the perturbations of completely integrable systems, a vast subject well beyond the scope of these lectures [**A**]. Instead, I will turn my attention in section 9 to systems that have a totally different dynamical behaviour, and in particular have no constants of the motion other than the Hamiltonian itself.

EXERCISE 8.3. * (i) Show that there does not exist a map $F:\mathbb{R}^{2n}\to\mathbb{R}^{n+1}$ so that the Jacobian of F has rank $n+1$ and $\{f_i, f_j\} = 0$, for all $i, j = 1\dots n$. **Hint:** This is not so easy, except in the case $n = 1$. For the general case, check the literature on Hamiltonian mechanics and symplectic geometry.

(ii) Let Ω be a positive definite n by n matrix. Show that the Hamiltonian

$$H(x) = \frac{p^2}{2m} + \frac{1}{2}mq\cdot\Omega q \tag{8.2}$$

is completely integrable. **Hint:** Ω can be diagonalized. This is called the *harmonic oscillator* Hamiltonian. It describes the motion of a system close to a stable equilibrium point. Indeed, an equilibrium point of a Hamiltonian $H = \frac{p^2}{2m} + V(q)$ is a point x_* so that $x(t) = x_*$ is a solution of the Hamilton equations. Then, clearly, $p_* = 0$ and $V'(q_*) = 0$. Writing the Taylor series for V, the Hamiltonian reduces, after a shift in the q-variables, to (8.2) with $\Omega_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}(q_*)$. The equilibrium point is stable if this matrix is positive definite: for more on this, consult any book on classical mechanics.

(iii) Let $H(x) = \frac{p^2}{2m} + V(r)$, $x \in \mathbb{R}^6$. Show H is completely integrable (on an open subset of $\mathbb{R}^6$).

9. Mixing as a measure of irregularity

Having understood the simplest possible Hamiltonian systems, which are the completely integrable ones, and having seen they are stable, regular and predictable, it is time for us to give a precise mathematical meaning to the rather vague notion of irregular, chaotic or unpredictable motion, that I illustrated with the Sinai billiard.

For that purpose, I propose the following thought experiment. Fix E and consider on the energy surface Σ_E a (small) set B. Imagine the flow Φ_t^H is totally

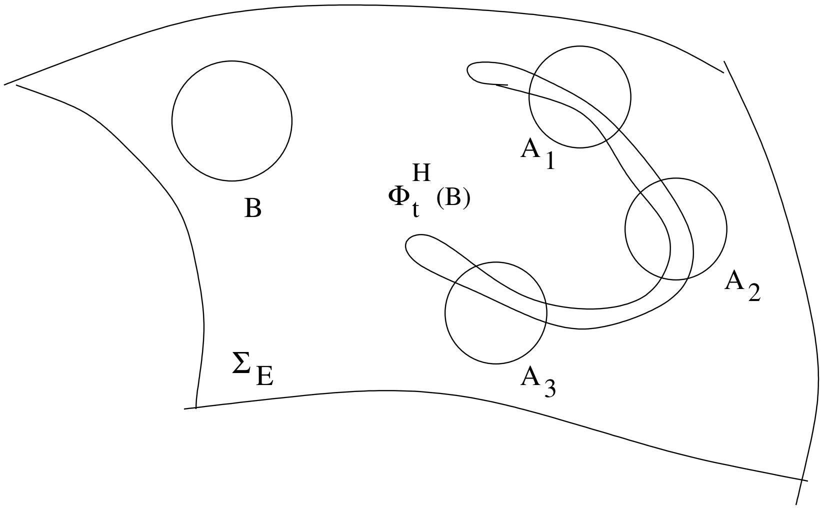

FIGURE 9. What is unpredictability? The probability of reaching any of the A_i starting from B must eventually be the same!

unpredictable in whatever vague intuitive sense you wish to attach to this term and ask yourself the following question (Fig. 9):

> "Given a set A on Σ_E, and a time t, what is the probability for a trajectory that started in B at $t = 0$ to end up in A at time t?"

My claim is that, if the motion is unpredictable, then you should not know at all where the trajectories are going and so the answer should not depend on where A is located on the energy surface. Similarly, it should not depend on where B is located, nor on the shape or size of B! You may find this strange, but the point here is that saying the motion is unpredictable is saying that knowing where the trajectory starts gives you no information on where it ends up. To put it differently: having started in B, the trajectory could be anywhere on Σ_E at time t. So what should the answer depend on? Clearly on how big A is: indeed, if $A = \Sigma_E$, then this probability is obviously 1 and if A is small this probability should be small! I therefore claim that the intuitive idea of unpredictable motion is well captured by the requirement that the answer to the above question should be equal to the "size of A." To make this into a mathematical definition we need to work a little more. First of all, how to express mathematically the "probability" showing up in the question? For that purpose, I will use the *Liouville measure*, which is a flow-invariant probability measure on Σ_E, defined as follows. I'll suppose ∇H does not vanish anywhere on Σ_E, which is furthermore assumed to be compact. Let dS_E be the surface element on Σ_E (Recall that Σ_E is a smooth surface in the Euclidean space $\mathbb{R}^{2n}$ and hence we can define its surface area). Then it is not too hard to show that

$$\frac{\partial}{\partial \epsilon} \int_{E \leq H(x) \leq E+\epsilon} f(x) dx_{\ |\epsilon=0} = \int_{\Sigma_E} f(x) \frac{dS_E(x)}{\| \nabla H(x) \|}. \tag{9.1}$$

It follows that $dS_E(x)/ \| \nabla H(x) \|$ defines a flow-invariant measure on Σ_E.

EXERCISE 9.1. Establish (9.1) and prove the statement following it. **Hint:** Revise your elementary Riemannian geometry (the theory of surfaces) and use Proposition 5.3(ii).

Normalizing this measure, one obtains the Liouville measure μ_E:

$$\mu_E(B) = \frac{\int_{B\subset\Sigma_E} dS_E(x)/\parallel \nabla H(x) \parallel}{\int_{\Sigma_E} dS_E(x)/\parallel \nabla H(x) \parallel}$$

which is a probability measure on Σ_E. We can then reformulate the question as follows:

> "Given a set A on Σ_E, and a time t, what is the fraction of all the trajectories starting in B at $t = 0$ that ends up in A at time t?"

which has a precise meaning provided we define:

> the fraction of all the trajectories starting in B at $t = 0$ that ends up in A at time $t = \frac{\mu_E(\Phi_t^H(B)\cap A)}{\mu_E(B)}$.

Finally, we can not expect this fraction to be equal to $\mu_E(A)$ (*i.e.* to the size of A) at any given finite time t, so let us define the motion to be unpredictable if this is asymptotically true:

$$\lim_{t\to\infty} \frac{\mu_E(\Phi_t^H(B)\cap A)}{\mu_E(B)} = \mu_E(A).$$

In fact, this definition can be found in all books on dynamical systems under the term *mixing*:

DEFINITION 9.2. The Hamiltonian flow Φ_t^H is said to be *mixing* with respect to the Liouville measure μ_E on Σ_E iff for all measurable sets $A, B \subset \Sigma_E$ one has

$$\lim_{t\to\infty} \mu_E(\Phi_t^H(B)\cap A) = \mu_E(B)\mu_E(A).$$

In other words, I will consider a flow to be unpredictable if it is mixing. Of course, how unpredictable the motion really is depends on how fast the limit is reached. Very roughly speaking, systems that are exponentially unstable in the sense described in section 3 are exponentially mixing, which means the above limit is reached exponentially fast. To get some vague and intuitive idea of this link between the exponential instability discussed in section 3 and exponential mixing note that, if the flow is mixing, then any set B must eventually "cover" all of Σ_E (see Fig. 9). But at the same time, since μ_E is an invariant measure, the volume of $\Phi_t^H(B)$ must be equal to the one of B. This implies that the flow must stretch the set B in some directions, squeeze it in others and bend it around in such a way that its filaments end up wrapping all around Σ_E. Now *if* the motion is exponentially unstable, then the set B will be stretched exponentially fast in some directions at least and so there is hope that the required spreading happens exponentially fast. Although this intuition is partially corroborated by theorems, I should warn you that the link between exponential instability (for which the technical term is hyperbolicity) and exponential mixing is considerably more subtle than the above discussion may lead you to believe (see **[Ba1][Ba2][D]** for details).

Definition 9.2 can be rewritten in terms of the characteristic functions χ_A, χ_B of A and B as follows:

$$\lim_{t\to\infty} \int_{\Sigma_E} \chi_B \circ \Phi_t^H(x)\chi_A(x)d\mu_E = \int_{\Sigma_E} \chi_B(x)d\mu_E \int_{\Sigma_E} \chi_A(x)d\mu_E.$$

A simple density argument then shows that mixing is equivalent to:

(9.2)
$$\forall f, g \in L^2(\Sigma_E, d\mu_E), \lim_{t\to\infty} \int_{\Sigma_E} (f \circ \Phi_t^H)(x) g(x) d\mu_E = \int_{\Sigma_E} f(x) d\mu_E \int_{\Sigma_E} g(x) d\mu_E.$$

Requiring a flow to be (exponentially) mixing is a very strong condition and there are not many Hamiltonian systems that have been proven to be mixing. For some billiard flows, and in particular for the Sinai billiard, this is known to be the case, but the proof is far from easy. In fact, it is to some extent the absence of large classes of examples of Hamiltonian flows that are (exponentially) mixing that has spurned the interest in the study of discrete Hamiltonian dynamical systems (and their quantum equivalents), to which I will come in the next section.

EXERCISE 9.3. * Show that a Hamiltonian system that is mixing on Σ_E can not have a smooth constant of the motion (functionally independent of H itself). **Hint:** You could show that if the flow is mixing on Σ_E, then any measurable invariant subset of Σ_E is of Liouville measure 0 or 1. Would you agree to claim that a Hamiltonian system is more predictable if it has more constants of the motion?

Further reading: The statistical properties of dynamical systems are the subject of ergodic theory. Here are some of the many good introductions to this beautiful field: [**CFS**] [**P**] [**W**].

10. Hyperbolic automorphisms of the two-torus

A *continuous dynamical system* is a collection $(\Phi_t)_{t\in\mathbb{R}}$ of measure-preserving transformations of a probability space (M, μ) such that $\Phi_t \circ \Phi_s = \Phi_{t+s}, \Phi_0 = \mathrm{id}_M$. Similarly, one obtains a *discrete dynamical system* when the collection is indexed by $t \in \mathbb{Z}$ so that $\Phi_t = (\Phi_1)^t$ and hence a discrete dynamical system is completely determined by a single invertible measure preserving map $\Phi = \Phi_1$ that is iterated to obtain the dynamics. We have so far been studying continuous Hamiltonian dynamical systems $\left(\Phi_t^H\right)_{t\in\mathbb{R}}$ on energy surfaces (Σ_E, μ_E) and now turn to discrete Hamiltonian dynamical systems on the two-torus.

First, let us consider the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the quotient of phase space $\mathbb{R}^2$ by $\mathbb{Z}^2$ and recall that a smooth function $f \in C^\infty(\mathbb{T}^2)$ on the torus is nothing but a smooth periodic function on $\mathbb{R}^2$, $f(x+n) = f(x), \forall x \in \mathbb{R}^2, n \in \mathbb{Z}^2$. The first remark to make is that the Poisson bracket of two periodic functions is still periodic so that the Poisson bracket restricts naturally to a product on $C^\infty(\mathbb{T}^2)$ conserving the properties listed in section 5, including the non-degeneracy, which now reads ($f \in C^\infty(\mathbb{T}^2)$):

$$\{f, g\} = 0, \ \forall g \in C^\infty(\mathbb{T}^2) \Rightarrow f = 0.$$

As a result, we can study Hamiltonian mechanics on the torus in the same manner as on $\mathbb{R}^2$. Indeed, given $H \in C^\infty(\mathbb{T}^2)$, viewed as a periodic function on $\mathbb{R}^2$, one can consider the corresponding Hamiltonian flow $\Phi_t^H : \mathbb{R}^2 \to \mathbb{R}^2$, which is easily seen to be invariant under translations by $\mathbb{Z}^2$, *i.e.*

$$\Phi_t^H(x+n) = \Phi_t(x) + n, \ \forall x \in \mathbb{R}^2, n \in \mathbb{Z}^2.$$

As a result, Φ_t^H defines a map on the torus that I shall denote by the same symbol.

EXERCISE 10.1. Prove all the claims made in the previous paragraph.

Of course, since we are in two dimensions, such flows are automatically completely integrable. The situation is very different when we consider *discrete Hamiltonian dynamical systems.* Those are defined as the iterates $\Phi^t (t \in \mathbb{Z})$ of a fixed canonical transformation Φ of the torus, the latter being a smooth invertible map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ such that, for all $f, g \in C^\infty(\mathbb{T}^2)$,

$$\{f \circ \Phi, g \circ \Phi\} = \{f, g\} \circ \Phi.$$

The examples I will be interested in are of the following very simple form. Let A be a two by two matrix and consider the associated linear map $x \in \mathbb{R}^2 \mapsto Ax \in \mathbb{R}^2$. You will undoubtedly have no trouble checking that this defines a map on the two-torus, *i.e.* $[Ax] = [Ax']$ if $[x] = [x']$, if and only if A has integer entries. On the other hand, we saw (Exercise 5.5) that A defines a canonical transformation if and only if $\det A = 1$ so that we can conclude that for any $A \in SL(2, \mathbb{Z})$, the map $\Phi_A : [x] \in \mathbb{T}^2 \mapsto [Ax] \in \mathbb{T}^2$ defines a canonical transformation of the torus. We shall be interested in the discrete dynamical system generated by Φ_A for the following reason:

Proposition 10.2. Let $A \in SL(2, \mathbb{Z})$ and suppose $\mid \mathrm{Tr}\ A \mid > 2$. Then the discrete Hamiltonian dynamical system generated by Φ_A is exponentially mixing on $\mathbb{T}^2$. More precisely, there exists a constant C_A and $\gamma > 0$ so that, for all $f, g \in C^1(\mathbb{T}^2)$ one has

(10.1)

$$\left| \int_{\mathbb{T}^2} (f \circ \Phi_A^t)(x) g(x) dx - \left(\int_{\mathbb{T}^2} f(x) dx \right) \left(\int_{\mathbb{T}^2} g(x) dx \right) \right| \leq C_A \parallel f \parallel_{2,1} \parallel g \parallel_{2,1} e^{-\gamma t}$$

Here

$$\parallel g \parallel_{2,1}^2 = \int_{\mathbb{T}^2} \mid g(x) \mid^2 dx + \int_{\mathbb{T}^2} \mid \nabla g(x) \mid^2 dx.$$

To simplify the notations I have simply written $x \in \mathbb{T}^2$ rather than $[x] \in \mathbb{T}^2$. The decay rate γ in the exponential is directly related to the matrix A as follows. The hypotheses on A imply that A has two eigenvectors $v_\pm$ and two eigenvalues $\lambda^{\pm 1}$ with $\lambda > 1$. One has $\gamma = \ln \lambda$.

Proof: The proof of this well known result is taken from [**DB**]. The essence of the proof is contained in its last few lines, which you might wish to read first. I start however with some preliminaries. We can take $v_\pm$ to be normalized, so that $v_\pm = (\cos\theta_\pm, \sin\theta_\pm)$ and $s_\pm = \tan\theta_\pm$ for some angles $\theta_\pm$. Note for later purposes that the slopes $s_\pm$ are quadratic irrationals, which means they are zeroes of a quadratic polynomial with integer coefficients (Prove this!).

It will obviously be enough to show (10.1) for f, g of mean zero, so from now on I work in the subspace of $L^2(\mathbb{T}^2)$ orthogonal to the constant functions. Define there, on the obvious domain

$$\{\phi \in L^2(\mathbb{T}^2) \mid \int_{\mathbb{T}^2} \phi(x) dx = 0,\ D_\pm \phi \in L^2(\mathbb{T}^2)\},$$

the self-adjoint operators

$$D_\pm = \frac{1}{2\pi i} (v_\pm, \nabla) = \frac{1}{2\pi i} (\cos\theta_\pm \partial_q + \sin\theta_\pm \partial_p)$$

with eigenfunctions $\chi_n(x) = \exp i2\pi(n_2 q - n_1 p)$: $D_\pm \chi_n = \langle v_\pm, n \rangle \chi_n$. Here, $\langle a, b \rangle = a_1 b_2 - a_2 b_1$. The irrationality of $s_\pm$ guarantees that the eigenvalues $\langle v_\pm, n \rangle$ are a countable dense subset of $\mathbb{R}$ and that 0 is not an eigenvalue (since $n \neq 0$). Indeed,

$$\langle v_\pm, n\rangle = \cos\theta_\pm(n_2 - s_\pm n_1),$$

and $\{\ \mathrm{Frac}(s_\pm n_1) \mid n_1 \in \mathbb{Z}\}$ is dense in $[0,1]$ since $s_\pm$ is irrational. (See Appendix A, if you did not know this last fact.) Also

$$D_\pm(f \circ \Phi_A^t) = e^{\pm\gamma t}(D_\pm f) \circ \Phi_A^t, \tag{10.2}$$

and

$$D_\pm^{-1} g = \sum_{n \in \mathbb{Z}^{2*}} g_n(\langle v_\pm, n\rangle)^{-1}\chi_n. \tag{10.3}$$

The only technical ingredient of the proof is the following estimate, valid for all $g \in C^1(\mathbb{T}^2)$ of mean zero:

$$\| D_\pm^{-1} g \|_2 \leq C_A \| g \|_{2,1} . \tag{10.4}$$

It is enough to write the proof of (10.4) for D_-. As already remarked, this operator has dense pure point spectrum covering the entire real axis, but 0 is not an eigenvalue. As a result, D_-^{-1} is a well-defined self-adjoint operator as well, but it is unbounded. The estimate (10.4) shows that any $g \in C^1$ of mean zero is in its domain. To prove (10.4), let us suppose, without loss of generality, $\cos\theta_- > 0, \sin\theta_- > 0$ and write

$$\| D_-^{-1} g \|^2 = \sum_{n_1 n_2 > 0} | \langle v_-, n\rangle |^{-2} | g_n |^2 + \sum_{n_1 n_2 < 0} | \langle v_-, n\rangle |^{-2} | g_n |^2 .$$

The second sum poses no problem since there exists a constant C so that $| \langle v_-, n\rangle | \geq C$ for each $n \in \mathbb{Z}^2, n_1 n_2 < 0$. For the first sum, we use the following result of the theory of continued fractions [**K**]: since s_- is a quadratic irrational,

$$\exists C > 0, \text{ so that } \quad \forall m, n > 0, \ \frac{1}{2Cm} < | n - m s_- | .$$

Inserting this into the first sum, (10.4) follows almost immediately. The proof of (10.1) is now easily completed:

$$\begin{aligned}
\int_{\mathbb{T}^2} (f \circ A^t)(x) g(x)\, dx &= (f \circ A^t, g)_{L^2} \\
&= (D_-(f \circ A^t), D_-^{-1} g)_{L^2} \\
&\leq e^{-\gamma t} \| D_-^{-1} g \|_2 \| D_- f \|_2,
\end{aligned}$$

from which (10.1) follows. □

EXERCISE 10.3. I cheated a little. To prove Φ_A is mixing I have to show (9.2) holds for all $f, g \in L^2(\mathbb{T}^2, dx)$. Instead, I showed the stronger statement (10.1), but only for $f, g \in C^1(\mathbb{T}^2)$. Show that the latter statement implies the former.

EXERCISE 10.4. Show that, if $A \in SL(2,\mathbb{Z}), |\mathrm{Tr}A| > 2$, then there does *not* exist $g \in C^\infty(\mathbb{T}^2), t \in \mathbb{R}$ so that $\Phi_t^g = \Phi_A$. In other words, Φ_A does not belong to a Hamiltonian flow. Contrast this result with your findings in Exercises 5.4 and 5.5. The situation is very different on the plane from what it is on the torus! **Hint:** Remember Exercise 9.3.

Maps satisfying the hypotheses of the Proposition are called *hyperbolic toral automorphisms* in the mathematics literature and "cat maps"(!) by physicists. Typical examples are

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 2g & 1 \\ 4g^2-1 & 2g \end{pmatrix},\ g \in \mathbb{N}$$

They are clearly exponentially unstable in the same manner as the Sinai billiard.

EXERCISE 10.5. Prove this by studying the distance between $A^t x$ and $A^t x'$ as a function of t and of the distance between x and x'.

There is a second property of the Φ_A that we shall need:

PROPOSITION 10.6. Let $f \in C(\mathbb{T}^2)$ and define, for all $x_0 \in \mathbb{T}^2$:

$$\langle f \rangle_T (x_0) = \frac{1}{T} \sum_{t=1}^{T} (f \circ A^t)(x_0). \tag{10.5}$$

Then, for almost all $x_0 \in \mathbb{T}^2$, we have

$$\lim_{T\to\infty} \langle f \rangle_T (x_0) = \int_{\mathbb{T}^2} f(x)dx. \tag{10.6}$$

Equation (10.6) says that the average value of f along almost every orbit $(\Phi_A^t(x_0))_{t\in\mathbb{Z}}$ of the dynamical system (*i.e.* the *time average* of f) equals the average of f over the torus (which is the so-called *ensemble average* of f, in the language of statistical physics). This expresses the idea that (almost) each orbit spends on average equal times in equal regions of the torus. This property is called *ergodicity* and it is a well known result in ergodic theory that any mixing dynamical system is automatically ergodic. The converse is certainly not true, as the following exercise shows.

EXERCISE 10.7. Show that flows by translations on tori are never mixing. **Hint:** You may find Appendix A useful.

Rather than relying on the above general result to prove Proposition 10.6, I will give a direct proof of a similar statement that will be sufficient for our purposes:

PROPOSITION 10.8. Let $f \in L^2(\mathbb{T}^2)$. Then

$$\lim_{T\to\infty} \| \langle f \rangle_T - f_0 \|_{L^2(\mathbb{T}^2)} = 0. \tag{10.7}$$

where $f_0 = \int_{\mathbb{T}^2} f(x)dx$.

This means that the time average $\langle f \rangle_T$, which is itself a function on the torus for each T, converges in the L^2-norm to a constant function, namely the average of f over the torus. The two propositions make basically the same statement, the only difference being that in the first the pointwise limit of the functions $\langle f \rangle_T$ is studied, whereas in the second the L^2 limit is investigated.

Proof of Proposition 10.8: Let us first treat the case where $f(x) = \chi_n(x) = \exp i2\pi\langle x, n\rangle$, $n \neq 0$. Then

$$\begin{aligned} \int_{\mathbb{T}^2} | \langle \chi_n \rangle_T (x) |^2 \ dx &= \frac{1}{T^2} \sum_{t=1}^{T} \sum_{t'=1}^{T} \int_{\mathbb{T}^2} \exp i2\pi \langle x, A^{-t}n - A^{-t'}n \rangle \ dx \\ &= \frac{1}{T^2} \sum_{t=1}^{T} \sum_{t'=1}^{T} \delta_{A^{-t}n, A^{-t'}n}, \end{aligned}$$

from which the result follows for $f(x) = \chi_n(x)$ since $A^{-t}n = A^{-t'}n$ iff $t = t'$ (Remember that $\mid \mathrm{Tr}A \mid > 2$). This implies immediately the result for trigonometric polynomials and a standard approximation argument then proves it for all L^2 functions. □

It is important to note that the "almost every" in Proposition 10.6 can not be omitted. Indeed, Φ_A has many periodic orbits:

EXERCISE 10.9. (i) Let $x = (q,p) \in [0,1[\times[0,1[\cong \mathbb{T}^2$. Show that there exists $T \in \mathbb{N}$ such that $\Phi_A^T(x) = x$ if and only if $q, p \in \mathbb{Q}$.

(ii) Show that

$$\#\{x \in \mathbb{T}^2 \mid \exists t \leq T, A^t x = x\} \sim e^{\gamma T}$$

as $T \to \infty$.

(iii) Show that, if $x_0 \in \mathbb{T}^2$ is a periodic point for Φ_A, then (10.6) can not hold.

In conclusion, hyperbolic toral automorphisms are perfect examples of "chaotic" Hamiltonian dynamical systems that have the added virtue of being extremely simple, making the analysis of their dynamical properties (exponential mixing, periodic orbits, ...) quite straightforward, as the previous pages showed. They are in particular much simpler than chaotic billiard flows. Although they do not model real life physical systems, they are considered useful toy-models for investigating chaos-related problems, and in particular quantum chaos. There is nevertheless a problem: to talk about quantum chaos, we need a spectral problem naturally related to the toral automorphisms that can play the role played by the Dirichlet Laplacian in the case of billiards. There exists indeed such a spectral problem, as I will show in section 13, but to understand its origin and to see why it is the natural analog of the Dirichlet Laplacian, you need to understand some notions of quantum mechanics. This is the subject of the next two sections in which I will explain a set of ideas coming from quantum mechanics that permit to relate a Hamiltonian system to certain spectral problems, generalizing in this manner the link between Dirichlet Laplacians and the corresponding billiard flows.

11. Quantum mechanics

11.1. Wave packets and the Schrödinger equation. In this section I wish to introduce the basic ingredients of (non-relativistic) quantum mechanics. I will not be rigorous in my discussion, but I will instead try to bring out some of the main ideas that underly the theory and explain its mathematical structure. The quantum mechanics that we use today was born in the 1920's after about a quarter of a century of hard work by some of the best physicists of the day. They had come to realize that the very successful description of nature based on the ideas that I briefly presented in section 5 was inadequate for the understanding of physical phenomena on the atomic scale. The predictions of classical mechanics were simply not in agreement with a certain number of experiments performed at the beginning of the century. It took some time to develop the new theory, especially since it turned out to require a radically new approach, having a number of philosophical and epistemological implications that remain puzzling to this day. I shall not go into any of this but instead content myself in this section to present quantum mechanics as a "finished" product. By this I mean that I will not motivate its formulation with experimental evidence, as is often done in physics books, nor base it on an axiomatic framework, as mathematicians sometimes like to do. Instead, I will simply explain its structure and analyze (some of) its predictions.

We have seen how the motion of a point particle is modeled in classical mechanics: at each instant of time t, all information about the particle is obtained from the knowledge of its location $(q(t), p(t))$ in phase space. The quantum mechanical model is very different. It describes the particle at each time t by a function ψ_t in $L^2(\mathbb{R}^3)$. The *wave function* or *wave packet* as it is called, is required to be normalized:

$$\| \psi_t \|^2 \equiv \int_{\mathbb{R}^3} | \psi_t |^2 (y)\, d^3y = 1. \tag{11.1}$$

As a result, $| \psi_t |^2 (y)$ is a probability density on $\mathbb{R}^3$. It is interpreted by saying that for each measurable set $B \subset \mathbb{R}^3$

$$0 \leq \int_B | \psi_t |^2 (y) d^3y \leq 1 \tag{11.2}$$

is the probability of finding the particle inside the set B at the time t. Equation (11.1) can then be interpreted as saying that the particle is sure to be somewhere. As a result, the mean position of the point particle at time t is given by the point in $\mathbb{R}^3$ with coordinates

$$< Q_i >_{\psi_t} \equiv (\psi_t, Q_i \psi_t) = \int_{\mathbb{R}^3} y_i \, | \psi_t |^2 (y) d^3y, \; i = 1, 2, 3 \tag{11.3}$$

whenever the integral converges. Here $(\cdot, \cdot)$ denotes the standard inner product on $L^2(\mathbb{R}^3)$ and Q_i is the linear operator defined by $(Q_i\phi)(y) = y_i\phi(y)$, $\forall \phi \in L^2(\mathbb{R}^3)$ about which I will say more later. Physicists often refer to $\psi_t(y)$ as a *probability amplitude*. By this they only mean that $\psi_t(y)$ is a complex-valued function the modulus squared of which is a genuine probability density.

It was the great discovery of Schrödinger that, just as the evolution of the phase space point $(q(t), p(t))$ is governed by Newton's second law or, equivalently, by Hamilton's equations of motion, the evolution of the wave packet ψ_t describing a particle of mass m subject to a potential V is determined by the partial differential equation

$$i\hbar \partial_t \psi_t(y) = -\frac{\hbar^2}{2m} (\Delta \psi_t)(y) + V(y)\psi_t(y), \tag{11.4}$$

since called the Schrödinger equation. Here $i = \sqrt{-1}$, Δ is the Laplacian on $\mathbb{R}^3$, *i.e.*

$$\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}, \tag{11.5}$$

and $\hbar$ is Planck's constant, given by $\hbar = 10^{-34} \frac{kgm^2}{s}$. If, in addition, a magnetic field $B = \mathrm{rot} A$ is present,the equation becomes

$$i\hbar \partial_t \psi_t(y) = \frac{1}{2m} (\frac{\hbar}{i} \partial_y - eA(y))^2 \psi_t(y) + V(y)\psi_t(y).$$

Note that $\hbar$ has the same physical dimension as angular momentum, which is also the dimension of an energy multiplied by a time. With this observation, one checks easily that all terms in (11.4) have the same physical dimension. In the same way as the main task of classical mechanics is to understand the behaviour of the solutions of Hamilton's equations, the principal problem of quantum mechanics is the study of the solutions to (11.4), and to understand how their behaviour depends on the properties of the potential V and on the initial condition ψ_0.

To be fair, I should point out that some of the problems of quantum mechanics can not quite be reduced to that last statement. One of them is the so-called measurement problem. To get a glimpse of what that is, recall that the sharp position $q(t)$ of the particle in classical mechanics is replaced here by a mean position, so that the notion of trajectory looses its meaning in quantum mechanics. To be complete, I should explain how this mean position is measured. That is actually a difficult thing to explain and it is at the origin of the different interpretations and philosophical problems raised by quantum mechanics. Let me just say that, in the standard interpretation of quantum mechanics, one imagines that if many particles are all described by the same wave function ψ_t, and if one measures the position of each of them, and then averages over the results of those measurements, then the result will be the number $< \psi_t, Q\psi_t >$. Note that this implies that one admits that the outcome of an individual experiment on just one particle can not be predicted with certainty. Should you find this puzzling and unsatisfactory, I can only offer you one consolation: you are not alone in this. The rest of these lectures will not be concerned with this very interesting problem.

Should you find the preceding two paragraphs strange and unwieldy, you are on the right track: they are. After all, to describe a point particle by a probability density ($\mid \psi_t \mid^2 (y)$) rather than by a trajectory, as in classical mechanics, is perhaps not so strange. But to then come up with the idea of writing down a dynamical equation (equation (11.4)) not for this probability density itself, but rather for a complex valued function ($\psi_t(y)$), the modulus squared of which gives the probability density in question, that is sorcery (or genius). The big surprise is that this sorcery gives a very accurate description of the behaviour of the microscopic world, which is one good reason to try to live with it.

To rigorously study the solutions to the partial differential equation (11.4), it would first be necessary to give a precise meaning to the different terms appearing in it, then to establish when a unique solution $t \in \mathbb{R} \to \psi_t \in L^2(\mathbb{R}^3)$ exists. In particular, we would have to give a precise meaning to $\partial_t \psi_t$ and worry about the fact that not for every $\psi \in L^2(\mathbb{R}^3)$, $\Delta\psi \in L^2(\mathbb{R}^3)$. I will not deal with these questions here but instead I will put mathematical rigour to the side, and help you to develop some feeling for the behaviour of the solutions to (11.4). This will bring out some striking similarities between quantum mechanics and the Hamiltonian formulation of classical mechanics and allow me to show how these ideas naturally lead one to associate a Hamiltonian flow on $\mathbb{R}^{2n}$ to certain spectral problems on $L^2(\mathbb{R}^n)$, thereby generalizing the link between $-\Delta_D$ on Ω and the corresponding billiard flow. We will then finally be ready to return to the hyperbolic toral automorphisms and the spectral problem to which they are naturally associated.

11.2. The Ehrenfest equations. Let us first establish that the Schrödinger equation is compatible with the probabilistic interpretation of the wave function, given in (11.1)-(11.2). Indeed, if ψ_t is a solution of (11.4), then

$$\frac{d}{dt} \parallel \psi_t \parallel^2 = \frac{d}{dt}\left(\int_{\mathbb{R}^3} \mid \psi_t \mid^2 (y) d^3y\right) = 0. \tag{11.6}$$

Equation (11.6) follows from a simple (formal) computation upon using (11.4) and a partial integration. Hence, if $\| \psi_0 \| = 1$, then $\| \psi_t \| = 1$, $\forall t \in \mathbb{R}$, so that we can indeed interpret $| \psi_t |^2 (y)$ as a probability density at all times t.

Next, let me rewrite (11.4) as follows:

$$i\hbar\partial_t\psi_t = H\psi_t, \tag{11.7}$$

where I introduced the linear partial differential operator

$$(H\phi)(y) = \frac{1}{2m}(P^2\phi)(y) + V(y)\phi(y) = (\frac{P^2}{2m} + V(Q))\phi(y), \tag{11.8}$$

where

$$P^2 = P_1^2 + P_2^2 + P_3^2, \quad P_j = \frac{\hbar}{i}\frac{\partial}{\partial y_j}. \tag{11.9}$$

One calls H the *Hamiltonian* (operator) or the *Schrödinger operator* and $P = (P_1, P_2, P_3)$ the *momentum* operators. Note that I am not shying away from the widespread abuse of notation that consists of using the same letter H to designate the function on phase space $H(x) = \frac{p^2}{2m} + V(q)$ and the operator in (11.8); you better try not to be bothered by it. Let me now justify this analogy with classical mechanics, both in notation and in terminology. For that purpose, I will show that, if ψ_t is a solution to (11.4), then

$$\frac{d}{dt} < Q_j >_{\psi_t} = \tfrac{1}{m} < P_j >_{\psi_t}, \tag{11.10}$$

$$\frac{d}{dt} < P_j >_{\psi_t} = - < \tfrac{\partial V}{\partial y_j} >_{\psi_t}, \tag{11.11}$$

where

$$< \frac{\partial V}{\partial y_j} >_{\psi_t} = \int_{\mathbb{R}^3} \frac{\partial V}{\partial y_j}(y) \mid \psi_t \mid^2 (y) d^3y, \tag{11.12}$$

and

$$< P_j >_{\psi_t} \equiv (\psi_t, P_j\psi_t) = \int_{\mathbb{R}^3} \overline{\psi_t}(y)\frac{\hbar}{i}\frac{\partial\psi_t}{\partial y_j}(y) d^3y. \tag{11.13}$$

Equations (11.10-11.11) are called *the Ehrenfest equations.* Comparing them to (5.3), the similarities are striking. If one interprets $< P_j >_{\psi_t}$ as the coordinates of the mean momentum of the particle, then the quantum mechanical mean position and momentum satisfy equations that are very close to the classical equations of motion. The difference is that in general

$$< \frac{\partial V}{\partial y_j} >_{\psi_t} \neq \frac{\partial V}{\partial y_j}(< Q >_{\psi_t}), \tag{11.14}$$

so that (11.10-11.11) are not ordinary differential equations satisfied by $t \in \mathbb{R} \to (< Q >_{\psi_t}, < P >_{\psi_t}) \in \mathbb{R}^6$. Here I use the short hand notation $Q = (Q_1, Q_2, Q_3)$, and similarly for P. Equality is obtained in (11.14) for arbitrary ψ_t only if $\mathrm{grad}_y V$ is linear in the coordinates y_j, such as when V is the harmonic oscillator potential $\frac{1}{2}m\omega^2 q^2$. At any rate, equations (11.10-11.11), apart from suggesting a further interpretation of the wave function through the introduction of the mean momentum $< P_j >_{\psi_t}$, also give a first indication on the possibility of describing the motion of a point particle through the use of the Schrödinger equation. The Ehrenfest equations suggest the existence of a natural link between the solutions of the Schrödinger equation and of the Hamiltonian equations of motion, that we shall further explore below.

Let me now prove (11.10-11.11). I will only give a formal proof, but if you suppose throughout that ψ_t belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, then the calculation below is completely justified. Recall that $\psi \in \mathcal{S}(\mathbb{R}^3)$ if $\psi \in C^\infty(\mathbb{R}^3)$ and ψ, as well as all its derivatives, decreases faster than any inverse power of $||y||$. Typical examples are $\psi(y) = p(y)\exp -||y||^2$ where $p(y)$ is a polynomial.

$$\frac{d}{dt} < Q_j >_{\psi_t} = \int y_j \overline{\partial_t \psi_t}(y)\psi_t(y)d^3y + \int y_j\overline{\psi_t}(y)(\partial_t\psi_t)(y)d^3y. \tag{11.15}$$

Now use (11.4) to replace $\partial_t\psi_t$:

$$\begin{aligned}
\frac{d}{dt} < Q_j >_{\psi_t} &= \int y_j \left[-i\frac{\hbar}{2m}(\Delta\overline{\psi_t}(y) + \frac{i}{\hbar}V(y)\overline{\psi_t}(y)\right]\psi_t(y)d^3y \\
&\quad + \int y_j\overline{\psi_t}(y)\left[i\frac{\hbar}{2m}(\Delta\psi_t)(y) - \frac{i}{\hbar}V(y)\psi_t(y)\right]d^3y \\
&= -i\frac{\hbar}{2m}\int\left[y_j(\Delta\overline{\psi_t})(y)\psi_t(y) - y_j\overline{\psi_t}(y)(\Delta\psi_t)(y)\right]d^3y \\
&= -i\frac{\hbar}{2m}\int 2\overline{\psi_t}(y)(\partial_j\psi_t)(y)d^3y = \frac{1}{m} < P_j >_{\psi_t}.
\end{aligned}$$

Here I used partial integration (several times) in the last step; note that there are no boundary terms because $\psi_t \in \mathcal{S}(\mathbb{R}^3)$. The proof of (11.11) is similar.

The mean position introduced above is associated to a probability density. The same is true for the mean momentum, as I will now explain. For that purpose, let me introduce the Fourier transform of $\psi \in L^2(\mathbb{R}^3)$:

$$\widehat{\psi}(p) = \frac{1}{(2\pi\hbar)^{3/2}}\int \psi(y)\exp\left(-i\frac{y\cdot p}{\hbar}\right)\, d^3y. \tag{11.16}$$

This definition differs from the one you are used to by the appearance of the factor $\hbar$ in certain places but you can easily check that all the usual properties of the Fourier transform still hold. In particular, the Fourier inversion formula now reads as follows:

$$\psi(y) = \frac{1}{(2\pi\hbar)^{3/2}}\int \widehat{\psi}(p)\exp\left(i\frac{y\cdot p}{\hbar}\right)\, d^3p. \tag{11.17}$$

The Plancherel identity holds unaltered:

$$\int |\psi(y)|^2\, dy = \int |\hat{\psi}(p)|^2\, dp.$$

Using the latter and

$$\begin{aligned}
\widehat{P_j\psi}(p) &= \frac{\hbar}{(2\pi\hbar)^{3/2}i}\int\frac{\partial\psi}{\partial y_j}(y)\exp\left(-i\frac{y\cdot p}{\hbar}\right)d^3y \\
&= -\frac{\hbar}{(2\pi\hbar)^{3/2}i}\int\psi(y)\frac{\partial}{\partial y_j}\exp\left(-i\frac{y\cdot p}{\hbar}\right)\, d^3y \\
&= p_j\widehat{\psi}(p),
\end{aligned}$$

it is then easy to see that we can rewrite (11.13) as

$$< P_j >_{\psi_t} = \int p_j\, |\widehat{\psi}_t(p)|^2\, d^3p. \tag{11.18}$$

This strongly suggests that we should interpret $| \widehat{\psi}_t |^2 (p)$ as the probability density of linear momentum, meaning that for all measurable subsets B of $\mathbb{R}^3$, we think of

$$\int_B | \widehat{\psi}_t |^2 (p) d^3p \tag{11.19}$$

as the probability of finding the momentum of the particle at time t inside B. Remark that the Plancherel identity and (11.1) guarantee that $\int_{\mathbb{R}^3} | \widehat{\psi}_t |^2 (p) d^3p = 1$, making this probabilistic interpretation consistent.

It is important to remark that the perhaps somewhat surprising factors of $\hbar$ appearing in the definition of the Fourier transform I adopted here are indispensable for the above interpretations of $| \psi_t |^2 (y)$ and $| \widehat{\psi}_t |^2 (p)$ to make sense. Indeed, since y is a position and p a momentum, $y \cdot p$ has the dimension of angular momentum. Dividing it by $\hbar$ one obtains a dimensionless quantity that can be exponentiated. On the other hand, if $\int_B | \psi_t |^2 (y) d^3y$ is to be a probability, and hence dimensionless, $\psi_t(y)$ has to have the dimension of $(\text{volume})^{-1/2} = (\text{meter})^{-3/2}$. Similarly, since $p = mv$, the probabilistic interpretation of $| \widehat{\psi}_t |^2 (p)$ makes sense only if $\widehat{\psi}_t(p)$ has the dimension of $(\frac{\text{kg m}}{\text{s}})^{-3/2}$. These remarks explain the appearance of the factor $\hbar^{-3/2}$ in the right hand side of (11.16).

Although $\hbar$ is a physical *constant*, one can consider it, from a mathematical point of view, as a parameter in the Schrödinger equation (11.4). Since, moreover, the numerical value of $\hbar$ is, in the usual units, very small, it is often justified to study the solutions of the Schrödinger equation asymptotically for small $\hbar$. This is what is called "semi-classical analysis": it is in this regime of "small $\hbar$" that the properties of the solutions of the Schrödinger equation can be fruitfully related to properties of the Hamiltonian flow of $H(x) = \frac{p^2}{2m} + V(q)$, as suggested already by the Ehrenfest equations and as I shall show in at least one instance below.

11.3. Observables, commutators and unitary flows. So far I explained how the classical notions of position and momentum appear in quantum mechanics. To each is associated an operator and the diagonal matrix element of this operator on the wave function ψ_t is interpreted as the mean value of the corresponding physical quantity (see (11.3) and (11.13)). Note that the "sharp" position $q(t)$ and momentum $p(t)$ of classical mechanics are replaced by the probability densities $| \psi_t |^2 (y)$ and $| \widehat{\psi}_t |^2 (p)$, so that the notion of trajectory disappears in quantum mechanics. In view of the suggestive form of (11.8), I suppose you won't be surprised if I now tell you that one interprets

$$< H >_{\psi_t} \equiv (\psi_t, H\psi_t) \tag{11.20}$$

as the mean energy of the particle at time t. Here also, there is a probability measure hidden in the background, but it is harder (in fact impossible) to compute in general, and I will come back to this point later on. A calculation similar to the one leading to (11.10-11.11) shows

$$\frac{d}{dt} < H >_{\psi_t} = 0 \tag{11.21}$$

if ψ_t is a solution to the Schrödinger equation. You will recognize here the law of conservation of energy.

EXERCISE 11.1. Prove (11.21).

Quantities such as position, momentum, and energy are examples of what physicists call "observables". In classical mechanics, observables are represented by real valued functions f on phase space. As we saw on some examples, in quantum mechanics they are represented by self-adjoint operators F on $L^2(\mathbb{R}^3)$. Recall that self-adjoint operators have the property that

$$(\phi, F\psi) = (F\phi, \psi) \tag{11.22}$$

which implies that their diagonal matrix elements are real. Note that (11.22) does indeed hold for Q_j, P_j, and for H, as you can check easily with a few more partial integrations as in the previous subsection. This allows for the interpretation of the matrix element $(\psi_t, F\psi_t)$ as the mean value of the corresponding observable. As a last example, the operators corresponding to the components of the angular momentum vector are, to nobody's surprise, I hope,

$$L_1 = Q_2P_3 - Q_3P_2,\ L_2 = Q_3P_1 - Q_1P_3,\ L_3 = Q_1P_2 - Q_2P_1. \tag{11.23}$$

There is a beautiful analog to (5.15) in quantum mechanics. Suppose F is a self-adjoint operator representing some physical observable and compute

$$\begin{aligned}\frac{d}{dt}(\psi_t, F\psi_t) &= ((\partial_t\psi_t), F\psi_t) + (\psi_t, F\partial_t\psi_t)\\ &= (-\frac{i}{\hbar}H\psi_t, F\psi_t) + (\psi_t, F(-\frac{i}{\hbar}H)\psi_t)\\ &= \frac{1}{i\hbar}(\psi_t, [F, H]\psi_t),\end{aligned}$$

where I introduced *the commutator*

$$[F, H] = FH - HF \tag{11.24}$$

and used the fact that H satisfies (11.22). So

$$\frac{d}{dt} < F >_{\psi_t} = < \frac{[F, H]}{i\hbar} >_{\psi_t} . \tag{11.25}$$

Comparing this to (5.15), one notices that the Poisson bracket of Hamiltonian mechanics is "replaced" in quantum mechanics by the commutator divided by $i\hbar$. Remark also that the Ehrenfest equations as well as (11.21) are special cases of (11.25), in the same way as in classical mechanics. In view of (11.25), a self-adjoint operator is called a *constant of the motion* for H if $[F, H] = 0$. In analogy with the result of Exercise 5.7 we also have: if F and G are constants of the motion, then so is $[F, G]$.

EXERCISE 11.2. Show that $(1 \leq j, k \leq n)$

$$[Q_j, P_k] = i\hbar\delta_{j,k}, \tag{11.26}$$

These are the so-called *canonical commutation relations* (CCR). Compute also $[L_1, L_2], [Q_j^2, P_k^2]$ and compare the result to $\{L_1, L_2\}, \{q_j^2, p_k^2\}$. Does this look interesting?

In section 5, I pointed out that Hamilton's equations of motion can be generalized by replacing H with any function g on phase space $\mathbb{R}^{2n}$, which leads to the notion of a Hamiltonian flow on $\mathbb{R}^{2n}$. What would be the corresponding generalization of the Schrödinger equation (11.7)? I start with a definition.

DEFINITION 11.3. A map $t \in \mathbb{R} \to U_t \in \mathcal{B}(L^2(\mathbb{R}^3))$ is called a *unitary one-parameter group* iff

$$\begin{array}{llll} (i) & U_0 & = \mathrm{id}_{L^2(\mathbb{R}^3)}, & \\ (ii) & \| U_t\psi \| & = \| \psi \| & \forall \psi \in L^2(\mathbb{R}^3), \\ (iii) & U_{t+t'} & = U_t \circ U_{t'} & \forall t, t' \in \mathbb{R}^2. \end{array}$$

I will now argue that this is the proper analog of the classical notion of Hamiltonian flow. To see this, remark that for any self-adjoint operator G on $L^2(\mathbb{R}^n)$, we can write

$$i\hbar\partial_t\psi_t = G\psi_t, \tag{11.27}$$

in analogy with (11.7). From this, it follows as above that, if F is a self-adjoint operator, then

$$\frac{d}{dt} < F >_{\psi_t} = < \frac{[F, G]}{i\hbar} >_{\psi_t} . \tag{11.28}$$

It is tempting to solve (11.27) formally by writing

$$\psi_t = \left(\exp -\frac{i}{\hbar}Gt\right)\psi_0, \tag{11.29}$$

where

$$\exp -\frac{i}{\hbar}Gt = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-it}{\hbar}\right)^n G^n. \tag{11.30}$$

Writing $U_t^G = \exp -\frac{i}{\hbar}Gt$, a further formal calculation shows that U_t^G is a unitary one-parameter group. All of this can be turned into properly formulated mathematics. In fact, when F and G are bounded self-adjoint operators, the above computations are perfectly rigorous (and, in particular, the series defining $\exp -\frac{i}{\hbar}Gt$ converges in operator norm). The mathematical problems stem from the fact that we need to apply all this to operators such as $G = Q_j$ or $G = -\Delta$, that are not bounded. For example $Q_j\psi(y) = y_j\psi(y)$ does not necessarily belong to $L^2(\mathbb{R}^n)$ when ψ does, so that Q_j is not even defined as a linear map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. I will not worry about such problems here, but for now, I hope that I have convinced you that, if an analog to (5.12-5.15) exists, it must be (11.27-11.28), and that therefore the unitary one-parameter groups are the proper quantum analogs of the Hamiltonian flows.

It is instructive to compute some examples. If $G = P_j$ then (11.27) is readily solved to yield

$$\psi_t(y) = [(\exp -\frac{i}{\hbar}tP_j)\psi_0](y) = \psi_0(y_1, \ldots, y_j - t, \ldots, y_n).$$

As a result, we see that the self-adjoint operator P_j defined in (11.9) generates translations in the j-direction. This analogy with classical mechanics (see Exercise 5.4 (vi)) gives yet another indication that we are justified in associating P_j with the momentum observable. I leave it up to you to show that the operators L_i introduced in (11.23) generate rotations of the wave function and to prove that

$$[(\exp -\frac{i}{\hbar}tQ_j)\psi_0\widehat{]}(p) = \widehat{\psi}_0(p_1, \ldots, p_j + t, \ldots, p_n)$$

(and to compare all this to Exercise 5.4 (iii)).

EXERCISE 11.4. Show that, for $b \in \mathbb{R}^n, \psi_0 \in L^2(\mathbb{R}^n)$,

$$[(\exp -\frac{i}{\hbar} b \cdot P)\psi_0](y) = \psi_0(y-b) \tag{11.31}$$

and

$$[(\exp \frac{i}{\hbar} b \cdot Q)\psi_0]\widehat{\ }(p) = \widehat{\psi}_0(p-b) \tag{11.32}$$

EXERCISE 11.5. Check using a formal computation that there is also an analog to Proposition 5.3 (iii):

$$[F, G] = 0 \iff U_t^F U_{t'}^G = U_{t'}^G U_t^F, \quad \forall t, t' \in \mathbb{R}. \tag{11.33}$$

To summarize, in quantum mechanics all information about a point particle is contained in its wave function ψ_t. This information is extracted with expressions such as (11.3),(11.13), and (11.20). The evolution of the wave function is determined completely by the Schrödinger equation (11.4) or (11.7). I have also pointed out some amazing, beautiful and rather intriguing analogies between quantum mechanics and classical mechanics in its Hamiltonian formulation that will be further exploited in the following section.

11.4. Semi-classical analysis of the Schrödinger operator. One fruitful way to study the solutions of the Schrödinger equation

$$i\hbar\partial_t\psi_t = H(\hbar)\psi_t, \psi_0 = \psi,$$

is through the spectral analysis of the Schrödinger operator

$$H(\hbar) = -\frac{\hbar^2}{2m}\Delta + V(Q)$$

itself. To illustrate this point, let me restrict myself to the case where V is a positive, smooth function, tending to infinity at infinity. It can then be shown (see [**RS**]) that there exists, for each $\hbar > 0$, an orthogonal basis $\varphi_n^\hbar$ of $L^2(\mathbb{R}^n)$ and a sequence

$$0 \leq E_1(\hbar) \leq E_2(\hbar) \leq \cdots \leq E_n(\hbar) \leq \ldots$$

such that $\lim_{n\to\infty} E_n(\hbar) = \infty$ and

$$H(\hbar)\varphi_n^\hbar = E_n(\hbar)\varphi_n^\hbar.$$

In terms of the eigenvalues $E_n(\hbar)$ and eigenfunctions $\varphi_n^\hbar$ the solutions of the Schrödinger equation can then be written

$$\psi_t = \sum_n (\varphi_n^\hbar, \psi) e^{-\frac{itE_n(\hbar)}{\hbar}} \varphi_n^\hbar.$$

In this sense, studying the solutions of the Schrödinger equation reduces to studying the eigenfunctions and eigenvalues of the Schrödinger operator. Except for some very simple cases (such as $V(q) = \frac{1}{2}m\omega^2 q^2$), it is however hard to obtain much precise information about those objects. As pointed out before, one approach is then to study the problem asymptotically as $\hbar$ tends to zero and to analyze the properties of the $E_n(\hbar), \varphi_n^\hbar$ in this limit, called the *semi-classical* regime. It turns out that, for small $\hbar$, the properties of those quantities are closely related to properties of the Hamiltonian flow generated by $H(x) = \frac{p^2}{2m} + V(q)$ on the energy surface $H(x) = E_n(\hbar)$. There is no way I can go into this here, or even to give a simple example of this phenomenon in some detail, but I will illustrate one aspect of it in the context of the hyperbolic toral automorphisms by proving the Schnirelman theorem (section

14). For more information on the enormous field that semi-classical analysis has become over the last thirty years you may consult **[H][I][R]** (as a start!).

11.5. Billiards again. We have seen, in section 6, that billiard flows are Hamiltonian systems. In the previous subsections, I explained how quantum mechanics associates to each Hamiltonian of the type $H(x) = \frac{p^2}{2m} + V(q)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ a self-adjoint operator $H = -\frac{\hbar^2}{2m}\Delta + V$ on $L^2(\mathbb{R}^3)$. Applying this precept to the family of Hamiltonians H_ϵ introduced in section 6, yields the family of self-adjoint operators $H_\epsilon = -\frac{\hbar^2}{2m}\Delta + V_\epsilon$ on $L^2(D_{a+\epsilon})$. As in the previous subsection, it can be shown **[RS]** that there exists an orthonormal basis φ_n^ϵ of $L^2(D_{a+\epsilon})$ such that

$$H_\epsilon \varphi_n^\epsilon = E_n(\epsilon)\varphi_n^\epsilon$$

for a sequence

$$0 \leq E_0(\epsilon) \leq E_1(\epsilon) \leq \cdots \leq E_n(\epsilon) \leq \ldots, \qquad \lim_{n\to\infty} E_n(\epsilon) = \infty.$$

Moreover, $\lim_{\epsilon\to 0} E_n(\epsilon) = \frac{\hbar^2}{2m}\lambda_n$, where λ_n is the n-th eigenvalue of the Dirichlet Laplacian on D_a. These results generalize, as in the classical case, in a natural way to more general billiard shapes. We won't need these results and I will not prove them, but they show how the Dirichlet Laplacian makes its entry into quantum mechanics and why the question asked in section 2 is natural from this point of view. A moment's reflection also allows to conclude that the limit $\lambda_n \to \infty$ discussed there corresponds precisely to the limit $\hbar \to 0$ of the previous subsection.

In the following two sections I will use the ideas developed here to construct a spectral problem that is naturally related to the hyperbolic toral automorphisms in the same way as the Schrödinger operator is linked to a Hamiltonian flow. The last section of these notes will then be used to study this spectral problem in the semi-classical regime.

Further reading: If this section has aroused your interest and you want to know more, you will just have to learn quantum mechanics. However, be warned that quantum mechanics can not really be learned without reading physics books: **[CTDL][LL2][M]** are some typical examples. This is a tough job for mathematically trained people since physicists don't shy away from a priori illegal operations: for them all limits may be interchanged, all series converge, all functions are infinitely differentiable, or so it seems, and this tends to scare mathematicians. Formal computations are nevertheless a good first approach to any subject and should not be despised. They often allow you to feel what ought to be true and to appreciate the beauty of a subject before getting into the nitty-gritty of actually proving things. The mathematics underpinning (non-relativistic) quantum mechanics has been extensively developed and can be learned in **[RS][CFKS][HS]**, for example. In addition, the analogies between symplectic geometry and the theory of self-adjoint and unitary operators on Hilbert space have spurred on much beautiful mathematics that has often nothing to do with quantum mechanics as such. To give just one example, you can appreciate this by reading the introductions to **[K][V]** which are books on the representation theory of Lie groups.

12. Weyl quantization

12.1. Introduction. It is clear from the previous section that there are some rather amazing structural similarities between classical and quantum mechanics.

The mathematics of classical mechanics deals with functions on $\mathbb{R}^{2n}$ and their Poisson brackets, and with Hamiltonian flows. The mathematics of quantum mechanics with operators on $L^2(\mathbb{R}^n)$ and their commutators, and with unitary flows. We have noticed that certain operators seem to "naturally" correspond to certain simple functions and that this correspondence maps Poisson brackets into commutators (see Exercise 11.2). An obvious (mathematical) question that presents itself (and that was asked by the British physicist P.A.M. Dirac in the early days of quantum mechanics) is whether this correspondence can be extended (uniquely perhaps) to a linear map associating to each function f in a suitable (preferably large) vector space of functions on phase space $\mathbb{R}^{2n}$ a (possibly unbounded, but densely defined) operator $Op\ f$ on $L^2(\mathbb{R}^n)$ in such a way that

$$(12.1) \qquad (A) \qquad \frac{[Op\ f, Op\ g]}{i\hbar} = Op\ \{f, g\},$$

and such that

$$(B) \qquad Op\ q_i = Q_i, \quad Op\ p_i = P_i,$$

and

$$(C) \qquad Op\ \bar{f} = (Op\ f)^*.$$

If such a map existed and were unique, it would associate to *each* real function of the vector space considered a self-adjoint operator and hence a unitary flow (by exponentiation) in a very natural way. If, in addition,

$$(D)\ Op\ [\frac{(p - A(q))^2}{2m} + V(q)] = \frac{1}{2m}(P - A(Q))^2 + V(Q).$$

then it would neatly generalize the prescriptions of quantum mechanics and give them a sound mathematical underpinning.

We will investigate the (non)existence of such a map Op in this section. Note first that we can not be too naïve. One can not just replace in any function $f(q,p)$, q par Q and p par P because, for example $qp = pq$, but $QP \neq PQ$. This is called the "ordering problem" and any choice for the map Op is sometimes said to correspond to a choice of ordering. Note also that, *if* a map Op exists satisfying (A)-(C), then the canonical commutation relations (see Exercise 11.2) imply that

$$(12.2) \qquad Op\ 1 = Op\ \{q_1, p_1\} = \frac{[Q_1, P_1]}{i\hbar} = \mathrm{id}_{L^2(\mathbb{R}^n)}.$$

Furthermore, under some additional technical conditions, one can show (see Proposition 12.1 and Exercise 12.6 (i)) that, if f is a function of q or p alone, then a map satisfying (A) and (B) also satisfies

$$(E) \qquad Op\ f(q) = f(Q), \quad Op\ f(p) = f(P).$$

This is looking pretty good, because the conditions (A)-(C) imply therefore (D), at least for $A(q) = 0$! So if Op existed and if it were unique, then it would go a long way in "explaining" in mathematical terms the origin of the Schrödinger equation.

12.2. Quadratic Hamiltonians. To simplify the discussion, I will restrict it to $\mathbb{R}^2$, but the results easily extend to $\mathbb{R}^{2n}$. Let $\mathcal{P}_2$ denote the set of polynomials in q and p of degree at most 2. Note that, for all $f, g \in \mathcal{P}_2$, $\{f, g\} \in \mathcal{P}_2$.

PROPOSITION 12.1. There exists a unique linear map

$$Op : \mathcal{P}_2 \to \mathcal{L}(\mathcal{S}(\mathbb{R}))$$

satisfying conditions (A)-(B)-(C) above. In particular,

$$Op\ q^2 = Q^2, Op\ p^2 = P^2, Op\ qp = \frac{1}{2}(PQ + QP). \tag{12.3}$$

Note that condition (C) is here to be interpreted as saying that

$$\forall f \in \mathcal{P}_2, \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}), (\varphi, Op\ f\psi) = (Op\ \bar{f}\varphi, \psi).$$

Proof: Existence does not pose a problem since it is easily checked that the choice $Op\ q = Q, Op\ p = P, Op\ 1 = \mathrm{id}_{\mathcal{S}(\mathbb{R})}$, together with (12.3) defines (after linear extension) a map with all the desired properties. As for uniqueness, let us suppose there exists a map $\widetilde{Op}$ with the desired properties. We need to show

$$\widetilde{Op}\ q^2 = Q^2, \widetilde{Op}\ p^2 = P^2, \widetilde{Op}\ qp = \frac{1}{2}(PQ + QP).$$

Since $\{q^2, q\} = 0$ and $\{q^2, p\} = 2q$ one easily concludes from (A)-(B) that

$$[\widetilde{Op}\ q^2 - Q^2, P] = 0, [\widetilde{Op}\ q^2 - Q^2, Q] = 0,$$

and similarly that

$$[\widetilde{Op}\ p^2 - P^2, P] = 0, [\widetilde{Op}\ p^2 - P^2, Q] = 0,$$

so that there exist $T_1, T_2 \in \mathcal{L}(\mathcal{S}(\mathbb{R}))$ with

$$\widetilde{Op}\ q^2 = Q^2 + T_1, \widetilde{Op}\ p^2 = P^2 + T_2, [T_i, Q] = 0 = [T_i, P],$$

and $T_i^* = T_i (i = 1, 2)$. We first show that there exist constants $c_i \in \mathbb{C}$ so that $T_i = c_i \mathrm{id}_{\mathcal{S}(\mathbb{R})}$. Using (C), it is easy to extend T_1 from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ as follows. For $\chi \in \mathcal{S}'(\mathbb{R}), \psi \in \mathcal{S}(\mathbb{R})$, define

$$(T_1\chi)(\psi) = \chi(\overline{T_1\bar{\psi}}).$$

EXERCISE 12.2. Recall that the imbedding of $\phi \in \mathcal{S}(\mathbb{R})$ in $\mathcal{S}'(\mathbb{R})$ is given by $\chi_\phi(\psi) = \int_{\mathbb{R}} \phi(y)\psi(y)dy, \forall \psi \in \mathcal{S}(\mathbb{R})$. Show that, with the above definition, one has indeed

$$T_1\chi_\phi = \chi_{T_1\phi},$$

and that, as operators on $\mathcal{S}'(\mathbb{R})$, one still has $[T_i, Q] = 0 = [T_i, P]$.

One then has immediately, for each $a \in \mathbb{R}$,

$$(Q - a)T_1\delta_a = T_1(Q - a)\delta_a = 0,$$

from which one concludes that the tempered distribution $T_1\delta_a$ is necessarily a multiple of δ_a, *i.e.* that $T_1\delta_a = c_1(a)\delta_a$ for some $c_1(a) \in \mathbb{C}$. Consequently, for all $\psi \in \mathcal{S}(\mathbb{R})$,

$$\overline{(T_1\psi)}(a) = \delta_a(\overline{T_1\psi}) = (T_1\delta_a)(\bar{\psi}) = c_1(a)\bar{\psi}(a),$$

which proves that T_1 is a multiplication operator by a function $\overline{c_1(a)}$. Since, finally, $[T_1, P] = 0$, it follows that $c_1(a)$ does not depend on a. The proof for T_2 is similar. Now

$$4\ \widetilde{Op}\ qp = \widetilde{Op}\ \{q^2, p^2\} = (i\hbar)^{-1}[Q^2 + T_1, P^2 + T_2] = 2(QP + PQ),$$

since obviously $[T_1, T_2] = 0$. Finally

$$-2\widetilde{Op}\ q^2 = \widetilde{Op}\ \{qp, q^2\} = (i\hbar)^{-1}[\frac{1}{2}(QP + PQ), Q^2 + T_1] = -2Q^2,$$

which shows $T_1 = 0$. A similar computation shows $T_2 = 0$ as well, concluding the proof. □

EXERCISE 12.3. Extend the above result to higher dimensions and show in particular that

$$Op\ q_j p_k = \frac{1}{2}(Q_j P_k + P_k Q_j), \forall j, k \in \{1, \ldots, n\}.$$

These results are rather encouraging. In particular, in view of what we learned in the previous section, we expect there to be a natural correspondence between the linear Hamiltonian flows generated by homogeneous quadratic polynomials $g(x) = \frac{1}{2}\alpha q^2 + \beta qp + \frac{1}{2}\gamma p^2$ (see Exercise 5.4) and the unitary one-parameter groups

$$\exp -\frac{i}{\hbar}(Op\ g)\ t = \exp -\frac{i}{\hbar}\left(\frac{1}{2}\alpha Q^2 + \beta\frac{1}{2}(QP + PQ) + \frac{1}{2}\gamma P^2\right) t.$$

Those objects will be crucial when we return to the hyperbolic toral automorphisms in section 13. A closed expression for the operators $\exp -\frac{i}{\hbar}(Op\ g)\ t$ is rather difficult to obtain **[F]**, but some special cases are easily computed:

EXERCISE 12.4. Show that, for all $\psi \in L^2(\mathbb{R})$,

$$\left(\exp -\frac{i}{\hbar}Q^2 t\ \psi\right)(y) = e^{-\frac{i}{\hbar}y^2 t}\ \psi(y); \tag{12.4}$$

$$\left(\exp -\frac{i}{\hbar}P^2 t\ \psi\right)\hat{}\,(p) = e^{-\frac{i}{\hbar}p^2 t}\ \hat{\psi}(p); \tag{12.5}$$

$$\left(\exp -\frac{i}{2\hbar}(QP + PQ)t\ \psi\right)(y) = e^{-\frac{t}{2}}\ \psi(e^{-t}y). \tag{12.6}$$

Note that each of those operators is unitary and leaves $\mathcal{S}(\mathbb{R})$ invariant.

12.3. The Groenewold-Van Hove theorem. Encouraged by the result of the previous subsection, one may hope to extend the map Op to the vector space $\mathcal{P}$ of all polynomials in q and p. This can however *not* be done, as the following theorem shows. (There exist several versions of this result, see for example **[AM]** or **[F]**.)

THEOREM 12.5. *(Groenewold-Van Hove) There does not exist a map* $Op : \mathcal{P} \to \mathcal{L}(\mathcal{S}(\mathbb{R}))$ *satisfying (A)-(B)-(C).*

EXERCISE 12.6. Prove the above result, along the following lines. Suppose a map Op with the desired properties exists and proceed as follows:
(i) Show that, for all $n \in \mathbb{N}$, $Op\ q^n = Q^n, Op\ p^n = P^n$;
(ii) Compute $\{q^3, p^2\}$ and deduce $Op\ q^2 p$ from it.
(iii) Compute $\{q^2, p^3\}$ and deduce $Op\ qp^2$ from it.
(iv) Check that

$$\{qp^2, q^2 p\} + \frac{1}{3}\{q^3, p^3\} = 0.$$

(v) Compute

$$[Op\ qp^2, Op\ q^2 p] + \frac{1}{3}[Q^3, P^3]$$

and show you obtain a contradiction implying the theorem.

Faced with this obstacle, much effort has been expended to find a set of natural conditions in the spirit of (A)-(E) (but weaker, of course) that would guarantee the existence of a (unique) map Op satisfying them. Any such map is usually called a "quantization", a notion for which no generally agreed upon formal definition exists.

The oldest, and still in many ways the most satisfactory such quantization, is the Weyl quantization that I shall now describe since it is an important tool in semi-classical analysis and since we will need it in the formulation and the proof of the Schnirelman theorem.

12.4. Weyl quantization for Schwartz functions. I will continue to work on $\mathbb{R}^2$, the extension to $\mathbb{R}^{2n}$ being immediate. Let $f \in \mathcal{S}(\mathbb{R}^2)$ (the Schwartz space, defined previously) and define the *symplectic Fourier transform* $\tilde{f}$ of f as follows:

$$\tilde{f}(a) = \int_{\mathbb{R}^2} f(x) \exp \frac{i}{\hbar}\langle a, x\rangle \frac{dx}{2\pi\hbar} \tag{12.7}$$

Recall that $\langle a, x\rangle = a_1 p - a_2 q$. This is of course just the Fourier transform in a slightly funny notation. Note that the function $\tilde{f}$ depends on $\hbar$, but I shall not explicitly indicate this dependence. The Fourier inversion formula now yields:

$$f(x) = \int_{\mathbb{R}^2} \tilde{f}(a) \exp -\frac{i}{\hbar}\langle a, x\rangle \frac{da}{2\pi\hbar} \tag{12.8}$$

EXERCISE 12.7. (i) Show that $\tilde{f}$ belongs to $\mathcal{S}(\mathbb{R}^2)$. (ii) Show (12.8).

The idea of Weyl was to apply to this formula the simple rule: "replace q by Q and p by P" to yield

$$Op^W\ f = \int_{\mathbb{R}^2} \tilde{f}(a) \exp -\frac{i}{\hbar}(a_1 P - a_2 Q) \frac{da}{2\pi\hbar}. \tag{12.9}$$

Let us make sense of this formula. It is convenient to introduce

$$U(a) = \exp -\frac{i}{\hbar}(a_1 P - a_2 Q). \tag{12.10}$$

These are called the *phase space translation operators.* We already studied $U(a_1, 0)$ and $U(0, a_2)$ in Exercise 11.4, and the results there, together with those of Exercise 12.8 below explain this terminology. A simple way to give a precise definition of the $U(a)$ (without resorting to the theory of unbounded self-adjoint operators), is as follows. Let $a \in \mathbb{R}^2$ and let $\psi \in \mathcal{S}(\mathbb{R})$. Consider the first order partial differential equation

$$\partial_s \psi(s, y) = -\frac{i}{\hbar}(a_1 P - a_2 Q)\psi(s, y) = \left(-a_1\partial_y + \frac{ia_2}{\hbar} y\right)\psi(y, s), \tag{12.11}$$

with initial condition $\psi(0, y) = \psi(y)$. This equation is easily solved (by the method of characteristics, for example) to yield

$$\psi(s, y) = \exp(-\frac{i}{2\hbar}s^2 a_1 a_2)\ \exp(\frac{i}{\hbar} s a_2 y)\ \psi(y - sa_1).$$

On the other hand, (12.11) is solved formally by the expression

$$\psi(s, y) = [\exp -\frac{i}{\hbar}s(a_1 P - a_2 Q)]\psi(y).$$

This leads one to *define* ($s = 1$)

(12.12)

$$U(a)\psi(y) = [\exp -\frac{i}{\hbar}(a_1 P - a_2 Q)\psi]\psi(y) = (\exp -\frac{i}{2\hbar}a_1 a_2)(\exp \frac{i}{\hbar}a_2 y)\ \psi(y - a_1).$$

EXERCISE 12.8. Prove that, for all $a \in \mathbb{R}^2$,

$$U(-a)PU(a) = P - a_2, \qquad U(-a)QU(a) = Q - a_1.$$

LEMMA 12.9. *Let $f \in \mathcal{S}(\mathbb{R}^2)$. There exists a unique bounded operator on $L^2(\mathbb{R})$, denoted by $Op^W\ f$, such that, for all $\phi, \psi \in L^2(\mathbb{R})$,*

$$(\phi, Op^W\ f\psi) = \int_{\mathbb{R}^2} \tilde{f}(a)(\phi, U(a)\psi)\frac{dx}{2\pi\hbar}.$$

The proof is an immediate consequence of the Riesz lemma. Actually, with $f \in \mathcal{S}(\mathbb{R}^2)$, $Op^W\ f$ is a Hilbert-Schmidt operator, a fact you are invited to prove in Exercise 12.12.

In the following lemma, I collect some of the basic properties of $Op^W\ f$ and of the $U(a)$ that we shall be needing.

LEMMA 12.10. *(i) For all $\psi \in \mathcal{S}(\mathbb{R})$,*

$$(Op^W\ f)\psi(y) = \int_{\mathbb{R}^2} f(\frac{y+s}{2}, p) \exp i\frac{(y-s)p}{\hbar}\ \psi(s)\frac{ds\ dp}{2\pi\hbar}; \tag{12.13}$$

(ii) If $Op^W f = 0$, then $f = 0$;
(iii) For all $a, a' \in \mathbb{R}^2$,

$$U(a)U(a') = \exp -\frac{i}{2\hbar}\langle a, a'\rangle U(a+a'); \tag{12.14}$$

(iv) $\| Op^W f \|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 2\int_{\mathbb{R}^2} \mid f(x) \mid \frac{dx}{2\pi\hbar}$.

The proof of this lemma is again generously left as an exercise for you:

EXERCISE 12.11. * To prove (i), use (12.12), the definition of Op^W and compute, using basic properties of the Fourier transform.

To prove (ii), use (i) and think.

To prove (iii), use (12.12) and compute.

For (iv), compute $(\phi, Op^W\ f\psi)$, $\phi, \psi \in \mathcal{S}(\mathbb{R})$ using (ii), the Schwartz inequality and the Riesz lemma.

EXERCISE 12.12. * (i) Let $k \in \mathcal{S}(\mathbb{R}^2)$ and define $K\psi(y) = \int_{\mathbb{R}} k(y, y')\psi(y')dy'$. Show that $K = Op^W\ f$ with

$$f(q, p) = \int_{\mathbb{R}} k(q + \frac{s}{2}, q - \frac{s}{2}) \exp -\frac{i}{\hbar}sp\ ds.$$

(ii) Show that

$$\int_{\mathbb{R}^2} \mid k(y, y') \mid^2 dydy' = \int_{\mathbb{R}^2} \mid f(x) \mid^2 \frac{dx}{2\pi\hbar}.$$

(iii) Show that, if $f \in \mathcal{S}(\mathbb{R}^2)$, then $Op^W\ f$ is a Hilbert-Schmidt operator and that, for all $g \in \mathcal{S}(\mathbb{R}^2)$,

$$\mathrm{Tr}\ (Op^W\ f)^*\ Op^W\ g = \int_{\mathbb{R}^2} \bar{f}(x)g(x)\frac{dx}{2\pi\hbar}.$$

12.5. The Wigner function. Part (ii) of the last lemma above guarantees that the map $f \in \mathcal{S}(\mathbb{R}^2) \mapsto Op^W\ f \in \mathcal{L}(L^2(\mathbb{R}))$ is injective. It is however not surjective (not even onto the Hilbert-Schmidt operators). If $F \in \mathcal{L}(L^2(\mathbb{R}))$ and if there exists f so that $F = Op^W f$, one says that f is the *Weyl symbol* of F. Although not all bounded operators have a Weyl symbol, the space $Op^W(\mathcal{S}(\mathbb{R}^2))$ is rich enough to "separate points" in $L^2(\mathbb{R})$, in the following sense:

LEMMA 12.13. *Let $\phi, \psi \in L^2(\mathbb{R})$ and suppose that, for all $f \in \mathcal{S}(\mathbb{R}^2)$*

$$(\phi, Op^W\ f\ \phi) = (\psi, Op^W f\ \psi).$$

Then $\exists \lambda \in \mathbb{C}, |\lambda| = 1$, so that $\phi = \lambda\psi$.

Proof: From the definition of Op^W one sees immediately that the condition implies that $(\phi, U(a)\phi) = (\psi, U(a)\psi)$, for all $a \in \mathbb{R}^2$. Using (12.12) one concludes furthermore that

$$\bar{\phi}(y)\phi(y - a_1) = \bar{\psi}(y)\psi(y - a_1),$$

from which the result follows easily. □

It follows from Lemma 12.10 (iv) that for each $\psi \in L^2(\mathbb{R})$ fixed, the map

$$W_\psi : f \in \mathcal{S}(\mathbb{R}^2) \mapsto (\psi, Op^W f\ \psi) \in \mathbb{C}$$

is a tempered distribution *on phase space*. It is called the *Wigner function of* ψ in the physics literature. The above lemma guarantees that the map $\psi \mapsto W_\psi$ is injective so that the Wigner function (which is a function on phase space $\mathbb{R}^2$) contains all the information contained in ψ itself (which is a function on configuration space $\mathbb{R}$). The Wigner function derives its interest from the observation that the properties of the Hamiltonian flow generated by $H(x) = \frac{p^2}{2m} + V(q)$ are reflected in the properties of the Wigner functions of the eigenfunctions $\varphi_n^\hbar$ of the Schrödinger operator, in the limit $\hbar \to 0$. This will be illustrated in section 14. Note that, in the usual notations of the theory of distributions, one writes

$$(\psi, Op^W f\ \psi) = \int_{\mathbb{R}^2} f(x) W_\psi(x) dx,$$

so that, through the use of the Wigner function, one expresses a "quantum expectation value" (*i.e.* the left hand side of the above equation) as a "classical expectation value" (the right hand side). As a result, in analogy with probability theory, physicists think of W_ψ as a kind of probability density, although it is not necessarily positive. Here are some exercises that will help you to familiarize yourself with the Weyl quantization and the Wigner function and that will further corroborate this point of view.

EXERCISE 12.14. Let $\psi \in \mathcal{S}(\mathbb{R})$. Show that $W_\psi \in \mathcal{S}(\mathbb{R}^2)$ and that

$$W_\psi(x) = \int_{\mathbb{R}} \overline{\psi(q + \frac{\eta}{2})}\psi(q - \frac{\eta}{2}) \exp i\frac{\eta p}{\hbar}\ \frac{d\eta}{2\pi\hbar}.$$

Show furthermore that

$$\int_{\mathbb{R}} W_\psi(q,p) dp = \mid \psi \mid^2 (q), \qquad \int_{\mathbb{R}} W_\psi(q,p) dq = \mid \hat{\psi} \mid^2 (p).$$

Note that these relations justify to some extent the interpretation of the Wigner function as a joint probability distribution for q and p: its marginals give indeed the correct quantum mechanical probability distributions for these quantities.

EXERCISE 12.15. Let $\eta(y) = \left(\frac{1}{\pi\hbar}\right)^{1/4} \exp -\frac{y^2}{2\hbar}$ and define, for each $x \in \mathbb{R}^2$, $\eta_x = U(x)\eta$. The η_x are the so-called *coherent states* that pop up in various places throughout the semi-classical literature.

(i) Compute the Wigner function $W_x \equiv W_{\eta_x}$ of η_x and prove it belongs to $\mathcal{S}(\mathbb{R}^2)$. **Hint:** Use (12.13) and review your Gaussian integrals.

(ii) Show that $\lim_{\hbar\to 0}(\eta_x, Op^W\ f\ \eta_x) = f(x)$.

(iii) Stare at the results of (i) and (ii) for a while and convince yourself that the statement "the coherent state $\eta_x = U(x)\eta$ represents a state of the system which is localized in phase space in a region of linear size $\sqrt{\hbar}$ about the point x" makes perfect sense.

EXERCISE 12.16. * There are several other ways besides Lemma 12.13 in which to see that the vector space $Op^W(\mathcal{S}(\mathbb{R}))$ is rather large. Here are four of them:

(i) Prove that, for all $0 \neq \psi \in L^2(\mathbb{R})$,

$$\overline{\{Op^W f\ \psi \mid f \in \mathcal{S}(\mathbb{R}^2)\}}^{\|\cdot\|} = L^2(\mathbb{R});$$

Hint: Show that the orthogonal complement of $\{Op^W f\ \psi \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ is trivial.

(ii) Let $\mathcal{K} \neq \{0\}$ be a closed subspace of $L^2(\mathbb{R})$ and suppose that, for all $f \in \mathcal{S}(\mathbb{R}^2), Op^W\ f\ \mathcal{K} \subset \mathcal{K}$. Then $\mathcal{K} = L^2(\mathbb{R})$. **Hint:** Use (i).

(iii) Let $B \in \mathcal{L}(L^2(\mathbb{R}))$. Show that

$$[B, Op^W\ f] = 0,\ \forall f \in \mathcal{S}(\mathbb{R}^2) \Rightarrow \exists c \in \mathbb{C}, B = c\ \mathrm{Id}_{L^2(\mathbb{R})}.$$

In other words, any bounded operator B that commutes with all $Op^W\ f$ is a multiple of the identity. **Hint:** Prove that $[B, U(a)] = 0$ for all $a \in \mathbb{R}^2$, then study $(\phi, [B, U(a)]\psi)$ using the explicit expression for the $U(a)$.

(iv) Prove that $\{Op^W f \mid f \in \mathcal{S}(\mathbb{R}^2)\}$ is a dense subspace of the compact operators. **Hint:** Use Exercise 12.12

Property (ii) above is referred to in the mathematics literature as the *irreducibility* of the vector space $Op^W\ \mathcal{S}(\mathbb{R}^2)$. The four properties (i)-(iv) express the physical idea that "all quantum observables" (*i.e.* all operators on $L^2(\mathbb{R})$) are "functions of Q and P."

12.6. The Egorov theorem. The final ingredient from the Weyl symbol calculus that I will need for the proof of the Schnirelman theorem (Section 14) is the so-called semi-classical Egorov theorem of which I will state a very simple version. For the general case, you may consult **[R]**. For the original formulation, which was not semi-classical, I refer to Professor Egorov's notes in this volume.

THEOREM 12.17. *For $f \in \mathcal{S}(\mathbb{R}^2)$ and g a quadratic polynomial in q, p, we have*

$$Op^W(f \circ \Phi_t^g) = e^{i\frac{tOp\ g}{\hbar}}\ Op^W\ f\ e^{-i\frac{tOp\ g}{\hbar}}. \tag{12.15}$$

Recall that Op is defined in Proposition 12.1.

Proof: I will ignore domain questions and compute formally. A complete proof would require showing that $Op\ g$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ and that $e^{i\frac{tOp\ g}{\hbar}}$ leaves $\mathcal{S}(\mathbb{R})$ invariant. Proposition 12.1 guarantees that, for all $a \in \mathbb{R}^2$,

$$i\hbar\ Op\ \{g, \langle a, x\rangle\} = [Op\ \ g, \langle a, X\rangle],$$

where $X = (Q, P)$. The crucial equality is then, in the notations of Exercise 5.4 (v),

$$e^{i\frac{tOp\ g}{\hbar}}\ \langle a, X\rangle\ e^{-i\frac{tOp\ g}{\hbar}} = \langle a, e^{tB}\ X\rangle.$$

Indeed, writing L_t(respectively R_t) for the left-hand side (respectively right-hand side) of the above equation, repeated use of Proposition 12.1 yields:

$$\begin{aligned}\frac{dR_t}{dt} &= \langle a, Be^{tB}X\rangle = Op\ \ \langle a, Be^{tB}x\rangle \\ &= Op\ \frac{d}{dt}\langle a, e^{tB}x\rangle = Op\ \{\langle a, e^{tB}x\rangle, g\} \\ &= -\frac{i}{\hbar}[\langle a, e^{tB}X\rangle, Op\ g] = \frac{i}{\hbar}[Op\ g, R_t],\end{aligned}$$

and

$$\begin{aligned}\frac{dL_t}{dt} &= \frac{i}{\hbar}e^{i\frac{tOp\ g}{\hbar}}[Op\ g, \langle a, X\rangle]e^{-i\frac{tOp\ g}{\hbar}} \\ &= \frac{i}{\hbar}[Op\ g, L_t].\end{aligned}$$

So L_t and R_t satisfy the same differential equation and the same initial condition $L_0 = R_0 = \langle a, X\rangle$, from which one concludes $L_t = R_t$. From this, one immediately has by exponentiation

$$e^{i\frac{tOp\ g}{\hbar}}\ U(a)\ e^{-i\frac{tOp\ g}{\hbar}} = U(\Phi^g_{-t}a), \tag{12.16}$$

so that the result follows from the definition of $Op^W\ f$. □

EXERCISE 12.18. Give a direct proof through explicit computation of (12.15) and (12.16) for $g(x) = q^2, p^2, qp$ respectively, using the results of Exercise 12.4.

The above result, and Exercises 5.4 and 5.5 suggest the following question. Given $A \in \mathrm{SL}(2, \mathbb{R})$, does there exist a unitary operator $M(A)$ on $L^2(\mathbb{R})$ so that

$$M(A)^{-1}(Op^W\ f)M(A) = Op^W\ (f \circ A), \tag{12.17}$$

and is it unique? Uniqueness is easily settled:

LEMMA 12.19. *If $M_1(A)$ and $M_2(A)$ are two unitaries satisfying (12.17), then there exists $c \in \mathbb{C}, \mid c \mid = 1$ so that $M_2(A) = cM_1(A)$.*

EXERCISE 12.20. Prove this lemma, which is an almost immediate consequence of Exercise 12.16 (iii).

What about existence? In view of Exercise 5.5 and Theorem 12.17, the answer is affirmative when $\mathrm{Tr}A \geq -2$, since there exists in that case a homogeneous quadratic polynomial g so that $A = \Phi^g_t$ so that we can set $M(A) = \exp -\frac{i}{\hbar}Op\ gt$. The general case is treated in the following exercise. We won't need it, but you may want to study it anyway, since it does not rely on the incompletely proven Theorem 12.17.

EXERCISE 12.21. (i) Define $\Pi\psi(y) = \psi(-y)$ and show that

$$\Pi U(a)\Pi = U(-a).$$

(ii) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in\ \mathrm{SL}\ (2, \mathbb{R}), a \neq 0$. Check that

$$A = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix},$$

(iii) If $a > 0$, define

$$M(A) = \left[\exp \frac{i}{\hbar}\frac{Q^2c}{2a}\right]\ \left[\exp -\frac{i}{2\hbar}(QP + PQ)\ln a\right]\ \left[\exp -\frac{i}{\hbar}\frac{P^2b}{2a}\right],$$

and use the result of Exercise 12.18 to show (12.17).

(iv) If $a < 0$, use (i)-(iii) above and find a unitary operator $M(A)$ satisfying (12.17).

(v) Find in the same way as above a suitable operator $M(A)$ for the case when $d \neq 0$.

(vi) Let $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$(\mathcal{F}\psi)(y) = \int_{\mathbb{R}} f(y') \exp -i\frac{yy'}{\hbar}\psi(y')\frac{dy'}{\sqrt{2\pi\hbar}}.$$

Show that

$$\mathcal{F}U(a)\mathcal{F}^* = U(\Omega a).$$

(vii) If $a = 0 = d$, one has

$$A = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & -b^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Construct an appropriate $M(A)$.

(viii) So far, we showed that, for each $A \in \mathrm{SL}(2, \mathbb{R})$, there exists a unitary operator $M(A)$ satisfying (12.17). In view of Lemma 12.19, the choice is not quite unique, but you will easily show that, whatever choices we make, we always have

$$M(A)M(B) = c(A, B)M(AB),$$

for some $c(A, B) \in \mathbb{C}, \mid c(A, B) \mid = 1$.

12.7. The Weyl quantization for periodic functions. I have introduced the Weyl quantization only for functions of the Schwartz class, but one can make sense of (12.9) for much larger classes of functions, and in particular for all smooth functions that are, together with all their derivatives, of at most polynomial growth [**R**]. Once so extended, Op^W coincides with Op (see Proposition 12.1) on quadratic polynomials, satisfies conditions (B), (C) and (E), and has many other nice properties but not, of course (A). A weaker, semi-classical form of (A) does however hold, as we will see below (Proposition 12.22). For our purposes here, it will be sufficient to have a definition of $Op^W f$ and establish some of these properties for $f \in C^\infty(\mathbb{T}^2)$ (*i.e.* for smooth periodic functions on the plane). This is very easily obtained. Indeed, for such f we can write

$$f(x) = \sum_{n \in \mathbb{Z}^2} f_n \exp -2\pi i(n_1 p - n_2 q), \tag{12.18}$$

and so, in complete agreement with (12.9) we define

$$Op^W\ f = \sum_{n \in \mathbb{Z}^2} f_n \exp -2\pi i(n_1 P - n_2 Q) = \sum_{n \in \mathbb{Z}^2} f_n U(2\pi\hbar n). \tag{12.19}$$

Clearly this sum converges in operator norm and

$$\| Op^W\ f \| \leq \sum_{n \in \mathbb{Z}^2} \mid f_n \mid,$$

so that Op^W defines a linear map from $C^\infty(\mathbb{T}^2)$ to the bounded operators on $L^2(\mathbb{R})$. Note also that it is totally obvious that (C) and (E) are satisfied! The further crucial properties of the Weyl quantization that we shall need are collected in the following proposition.

PROPOSITION 12.22. Let $f, g \in C^\infty(\mathbb{T}^2)$. Then there exists a constant C (depending on g and f) such that

$$(i) \parallel Op^W\ f\ Op^W\ g - Op^W\ fg \parallel \ \leq \ C\hbar \tag{12.20}$$

$$(ii) \parallel \frac{[Op^W\ f, Op^W\ g]}{i\hbar} - Op^W\ \{f,g\} \parallel \ \leq \ C\hbar \tag{12.21}$$

Proof: (i) One has, using (12.14),

$$Op^W\ fg - Op^W\ f\ Op^W\ g = \sum_{n,m\in\mathbb{Z}^2} f_n g_m \left(e^{\frac{i}{2\hbar}\langle n,m\rangle(2\pi\hbar)^2} - 1\right) U(2\pi\hbar n)U(2\pi\hbar m).$$

The simple estimate

$$|\exp is - 1| \leq |s|,$$

then allows to conclude that

$$\parallel Op^W\ fg - Op^W\ f\ Op^W\ g \parallel_{L^2(\mathbb{R})} \leq \sum_{n,m\in\mathbb{Z}^2} | f_n g_m || n || m | 2\pi^2\hbar.$$

Now, since $f, g \in C^\infty(\mathbb{T}^2)$, one easily shows that

$$\sum_{n\in\mathbb{Z}^2} | f_n || n |^\ell$$

converges for any $\ell > 0$, from which the result follows. Part (ii) is proved similarly. □

EXERCISE 12.23. * (i) Clearly, $C^\infty(\mathbb{T}^2)$ is a commutative algebra under multiplication of functions. Show that $Op^W(C^\infty(\mathbb{T}^2))$ is also an algebra under multiplication of operators, *i.e.* that for all $f, g \in C^\infty(\mathbb{T}^2)$, there exists (a unique) $h \in C^\infty(\mathbb{T}^2)$ such that $(Op^W\ f)(Op^W\ g) = Op^W\ h$. Remark that the function h depends on $\hbar$. This shows that the above proposition says that, although the map Op^W does not intertwine the natural algebra and Lie-algebra structures on $C^\infty(\mathbb{T}^2)$ and $Op(C^\infty(\mathbb{T}^2))$, it does so up to terms that are at least of first order in $\hbar$.

(ii) Show that the family $\{Op^W\ f \mid f \in C^\infty(\mathbb{T}^2)\}$ does not separate the points of $L^2(\mathbb{R})$ in the sense of Lemma 12.13 and that the conclusions of Exercise 12.16 also no longer hold if $\mathcal{S}(\mathbb{R}^2)$ is replaced by $C^\infty(\mathbb{T}^2)$.

EXERCISE 12.24. Check that the Egorov theorem (Theorem 12.17) still holds for $f \in C^\infty(\mathbb{T}^2)$, $A \in \mathrm{SL}\ (2,\mathbb{Z})$.

Further reading: The links between symbol calculus and quantization, and the analysis behind it, has been extensively explained in many places from different points of view. For a nice overview of and comparison between the different quantizations most commonly encountered (Weyl, Wick, anti-Wick, Kohn-Nirenberg) and their merits, you may consult [**F**]. The detailed analysis of the control of the error terms in $\hbar$ is given in [**R**]. The lecture notes of Prof. Egorov and Prof. Vasilevski in this volume also deal in part with certain aspects of these issues.

13. Quantum mechanics on the torus

13.1. The Hilbert spaces. We now dispose of all the ingredients to describe the spectral problem naturally associated to the hyperbolic toral automorphisms. First, we need an appropriate Hilbert space $\mathcal{H}_\hbar$. When the classical phase space is $\mathbb{R}^2$, the corresponding Hilbert space used in quantum mechanics is $L^2(\mathbb{R})$ as we saw in the preceding sections. I will use the insights gained there to construct a suitable Hilbert space $\mathcal{H}_\hbar$ when the phase space is $\mathbb{T}^2$. I want to use the operators $Op^W\ f$, for f belonging to $C^\infty(\mathbb{T}^2)$ as quantum observables on $\mathcal{H}_\hbar$. Exercise 12.23 (ii) gives some indications as to why choosing $\mathcal{H}_\hbar$ equal to $L^2(\mathbb{R})$ is not the right idea: $L^2(\mathbb{R})$ is "too big" for this purpose. In more technical terms, one wants to find a Hilbert space on which the family $Op^W\left(C^\infty(\mathbb{T}^2)\right)$ acts irreducibly. Recall from Exercise 12.16 that this means that the space $\mathcal{H}_\hbar$ must not have any closed subspaces that are invariant under $Op^W\left(C^\infty(\mathbb{T}^2)\right)$. Since you undoubtedly did Exercise 12.23, you probably noticed that

$$[U(n), Op^W\ f] = 0, \qquad \forall f \in C^\infty(\mathbb{T}^2),\ \forall n \in \mathbb{Z}^2.$$

This suggests we should be looking for a Hilbert space on which all the $U(n)$ act as multiples of the identity operator! It is obvious that such a space can not be constructed using L^2-functions, but it is easy to see that the $U(a)$, $a \in \mathbb{R}^2$, originally defined on $\mathcal{S}(\mathbb{R})$, extend continuously to $\mathcal{S}'(\mathbb{R})$.

EXERCISE 13.1. Prove this. **Hint:** Remember Exercise 12.2.

With this in mind, let us therefore define

$$\mathcal{H}_\hbar = \{\psi \in \mathcal{S}'(\mathbb{R}) \mid U(1,0)\psi = \psi = U(0,1)\psi\}, \tag{13.1}$$

so that for all $\psi \in \mathcal{H}_\hbar$, one has

$$U(n)\psi = e^{\frac{i}{2\hbar}n_1 n_2}\psi \tag{13.2}$$

in view of (12.14). Since

$$U(1,0)U(0,1) = \exp -\frac{i}{\hbar}\ U(0,1)U(1,0),$$

it is easy to see that such a non-zero ψ can only exist if $\exists N \in \mathbb{N}$ so that $2\pi\hbar N = 1$. The following lemma completes the description of $\mathcal{H}_\hbar$:

LEMMA 13.2. *Either $\mathcal{H}_\hbar = \{0\}$ or $\exists N \in \mathbb{N}$ so that $2\pi\hbar N = 1$. In the latter case one has $dim\mathcal{H}_\hbar = N$ and a basis of $\mathcal{H}_\hbar$ is given by*

$$e_j = \sqrt{\frac{1}{N}}\sum_{n\in\mathbb{Z}} \delta_{\frac{j}{N}+n}, \tag{13.3}$$

for $j = 1 \dots N$.

Note that the e_j are defined for any $j \in \mathbb{Z}$ and that clearly $e_j = e_{j+N}$, for all $j \in \mathbb{Z}$.

Proof: If $2\pi\hbar N = 1$, then $\psi \in \mathcal{H}_\hbar$ if and only if

$$\psi(y-n) = \psi(y),\ e^{i\frac{my}{\hbar}}\psi(y) = \psi(y), \forall m, n \in \mathbb{Z}.$$

From the second of these conditions one infers that

$$\psi = \sum_{k\in\mathbb{Z}} c_k \delta_{2\pi\hbar k} = \sum_{k\in\mathbb{Z}} c_k \delta_{\frac{k}{N}},$$

and from the first that $c_{k+N} = c_k$, which proves the result. □

EXERCISE 13.3. The requirement that the $U(n)$ act as a multiple of the identity does not impose that $U(1,0)\psi = U(0,1)\psi = \psi$ but only that there exists $(\kappa_1, \kappa_2) \in [0, 2\pi[\times[0, 2\pi[$ so that

$$U(1,0)\psi = e^{i\kappa_1}\psi, \quad U(0,1)\psi = e^{i\kappa_2}\psi.$$

Study the space $\mathcal{H}_\hbar(\kappa)$ of tempered distributions satisfying these conditions, along the lines of the above lemma.

From now on, I will always suppose that $\hbar$ is chosen so that there exists an integer N for which $2\pi\hbar N = 1$ and whenever in one formula both $\hbar$ and N appear, they will be related by this equality.

13.2. The Weyl calculus. We are now in a position to develop an appropriate Weyl calculus on $\mathcal{H}_\hbar$, thanks to the following remark:

LEMMA 13.4. *(i)* $U(a)\mathcal{H}_\hbar \subset \mathcal{H}_\hbar$ *iff* $a = \frac{n}{N}, n \in \mathbb{Z}^2$;
(ii) If $f \in C^\infty(\mathbb{T}^2)$, *then* $Op^W\ f\ \mathcal{H}_\hbar \subset \mathcal{H}_\hbar$.

Proof: Let $a \in \mathbb{R}^2$. Suppose that for all $\phi \in \mathcal{H}_\hbar$, $U(a)\phi \in \mathcal{H}_\hbar$, so that $U(n)U(a)\phi = U(a)\phi$. A simple computation using (12.14) now completes the proof of (i). Part (ii) is then obvious from definition (12.19). □

So far, $\mathcal{H}_\hbar$ is only a vector space, not yet a Hilbert space. The following result shows that $\mathcal{H}_\hbar$ carries an essentially unique Hilbert space structure:

LEMMA 13.5. *There exists a Hilbert space structure* $\langle\cdot,\cdot\rangle_{\mathcal{H}_\hbar}$ *on* $\mathcal{H}_\hbar$ *– unique up to a global normalization factor – so that* $U(\frac{n}{N})$ *is unitary for each* $n \in \mathbb{Z}^2$.

Proof: One has

$$U(\frac{1}{N},0)e_j = e_{j+1}, \qquad U(0,\frac{1}{N})e_j = e^{-i2\pi\frac{j}{N}}e_j. \tag{13.4}$$

It follows from the first of these relations that, if a Hilbert space structure with the required properties exists, then it satisfies $\langle e_j, e_j\rangle_{\mathcal{H}_\hbar} = \langle e_k, e_k\rangle_{\mathcal{H}_\hbar}$ for all $j,k \in \{1\ldots N\}$ and from the second that, for $j \neq k$,

$$\langle e_j, e_k\rangle_{\mathcal{H}_\hbar} = \langle U(0,\frac{1}{N})e_j, U(0,\frac{1}{N})e_k\rangle_{\mathcal{H}_\hbar} = \exp i\frac{2\pi}{N}(j-k)\langle e_j, e_k\rangle_{\mathcal{H}_\hbar},$$

so that the e_j are mutually orthogonal. The result now follows easily. □

In the following, I will pick the normalization of the Hilbert space structure (which has no bearing on the rest of the theory) so that the e_j defined in (13.3) form an orthonormal basis of $\mathcal{H}_\hbar$.

The definition of the Wigner function of $\psi \in \mathcal{H}_\hbar$ is now completely analogous to the one in section 12. W_ψ is the distribution on $\mathbb{T}^2$ given by

$$W_\psi : f \in C^\infty(\mathbb{T}^2) \mapsto \langle\psi, Op^W\ f\psi\rangle_{\mathcal{H}_\hbar} \in \mathbb{C}.$$

The space $\mathcal{H}_\hbar$ has, with respect to the operators $Op^W\ f, f \in C^\infty(\mathbb{T}^2)$, precisely the right "size", as the following result suggests:

LEMMA 13.6. *(i) Let* $\phi, \psi \in \mathcal{H}_\hbar$ *and suppose that, for all* $f \in C^\infty(\mathbb{T}^2)$,

$$(\phi, Op^W\ f\ \phi) = (\psi, Op^W f\ \psi).$$

Then $\phi = \psi$.

(ii) For all non-zero $\psi \in \mathcal{H}_\hbar$,

$$\{Op^W f\ \psi \mid f \in \mathcal{C}^\infty(\mathbb{T}^2)\} = \mathcal{H}_\hbar;$$

(iii) Let $\mathcal{K} \neq \{0\}$ *be a subspace of* $\mathcal{H}_\hbar$ *and suppose that, for all* $f \in C^\infty(\mathbb{T}^2)$, $Op^W\ f\ \mathcal{K} \subset \mathcal{K}$. *Then* $\mathcal{K} = \mathcal{H}_\hbar$.

(iv) Let $B \in \mathcal{L}(\mathcal{H}_\hbar)$. *Then*

$$[B, Op^W\ f] = 0,\ \forall f \in C^\infty(\mathbb{T}^2) \Rightarrow \exists c \in \mathbb{C}, B = c\ Id_{\mathcal{H}_\hbar}.$$

In other words, any operator B that commutes with all $Op^W\ f$ is a multiple of the identity.

These results are of course completely analogous to those of Lemma 12.13 and Exercise 12.16 and indicate we found the right Hilbert space to work on.

EXERCISE 13.7. Prove this lemma. **Hint:** First show that, if $\mu \in \mathbb{C}^N$ is such that

$$\tilde{\mu}_\ell \equiv \sum_{k=0}^{N-1} e^{-2\pi i \ell \frac{k}{N}} \mu_k = 0, \forall \ell \in \{1 \dots N\},$$

then $\mu = 0$. **Hint:** $\tilde{\mu}$ is called the discrete Fourier transform of μ.

EXERCISE 13.8. * Show that the map $f \in C^\infty(\mathbb{T}^2) \mapsto Op^W\ f \in \mathcal{L}(\mathcal{H}_\hbar)$ is surjective, but not injective. Describe its kernel.

We will need the following estimates:

LEMMA 13.9. *Let $f, g \in C^\infty(\mathbb{T}^2)$. Then there exists a constant C (depending on g and f) such that*

$$(i)\ \|\ Op^W\ f\ Op^W\ g - Op^W\ fg\ \|_{\mathcal{H}_\hbar} \ \leq\ C\hbar \tag{13.5}$$

$$(ii)\ \|\ \frac{[Op^W\ f, Op^W\ g]}{i\hbar} - Op^W\ \{f,g\}\ \|_{\mathcal{H}_\hbar} \ \leq\ C\hbar \tag{13.6}$$

$$(iii)\ |\ \frac{1}{N}\ Tr\ Op^W\ f^*\ Op^W g - \int_{\mathbb{T}^2} \bar{f}(x) g(x) dx\ | \ \leq\ C\hbar \tag{13.7}$$

Proof: Parts (i)-(ii) are completely analogous to Proposition 12.22. Part (iii) results from a simple calculation using (13.4). □

The Hilbert spaces $\mathcal{H}_\hbar$ being finite dimensional, they can of course be identified with $\mathbb{C}^N$ through the choice of an orthonormal basis. Using the basis $e_j, j = 1, \dots N$, defined in (13.3), one has

$$\psi \in \mathcal{H}_\hbar \to (\psi_1, \dots \psi_N) \in \mathbb{C}^N,$$

where $\psi_j = \langle e_j, \psi \rangle_{\mathcal{H}_\hbar}$. This is called the *position representation* by physicists. This particular choice has some special properties. They will be of interest for a better understanding of the Schnirelman theorem in section 14, and are spelled out in the following simple exercise.

EXERCISE 13.10. (i) Let $f \in C^\infty(\mathbb{T}^2)$ be a function of q alone (*i.e.* $\frac{\partial f}{\partial p} = 0$). Show that

$$Op^W\ f\ e_j = f(\frac{j}{N}) e_j.$$

so that, for each $\psi \in \mathcal{H}_\hbar$,

$$\left(Op^W\ f\ \psi\right)_j = f(\frac{j}{N}) \psi_j.$$

In other words, on $\mathbb{C}^N$, $Op^W\ f$ acts as a multiplication operator by f.

(ii) Given $\psi \in \mathcal{H}_\hbar$, define a probability measure on the interval $[0, 1]$ through

$$\lambda_\psi(I) = \sum_{\frac{j}{N} \in I, 0 \leq j < N} |\ \langle e_j, \psi \rangle\ |^2\ .$$

Let $f \in C^\infty(\mathbb{T}^2)$ be a function of q alone. Show that

$$\langle \psi, Op^W f, \psi \rangle_{\mathcal{H}_\hbar} = \int_0^1 f(y) d\lambda_\psi(y).$$

Recall that for $\psi \in L^2(\mathbb{R})$, we have a probability measure $|\psi(y)|^2 dy$ on $\mathbb{R}$. For $\psi \in \mathcal{H}_\hbar$, the analogous measure is λ_ψ, which is supported on $[0,1]$. This corroborates the idea that the $\mathcal{H}_\hbar$ are the correct Hilbert spaces associated to $\mathbb{T}^2$. For $\psi \in L^2(\mathbb{R})$, we also have the measure $\mid \hat{\psi}(p) \mid^2 dp$, giving the momentum probability density. This also has an analog for $\psi \in \mathcal{H}_\hbar$:

(iii)* Find an orthonormal basis $\tilde{e}_j$ of $\mathcal{H}_\hbar$ so that, if $f \in C^\infty(\mathbb{T}^2)$ depends on p only, then

$$Op^W\ f\ \tilde{e}_j = f(\frac{j}{N})\tilde{e}_j.$$

This is called the *momentum representation.* **Hint:** Think (discrete) Fourier.

13.3. Quantizing the toral automorphisms. From the definition of the operators $M(A), A \in \mathrm{SL}(2,\mathbb{R})$ in section 12.6, it is again clear that they extend from maps of $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ to continuous maps from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. As a result, we can define, for each $\psi \in \mathcal{H}_\hbar$, $M(A)\psi \in \mathcal{S}'(\mathbb{R})$. In general, $M(A)\psi$ does not belong to $\mathcal{H}_\hbar$, but I will restrict the discussion to matrices $A \in \mathrm{SL}(2,\mathbb{Z})$ of the form

$$A = \begin{pmatrix} 2g & 1 \\ 4g^2-1 & 2g \end{pmatrix}, \quad g \in \mathbb{Z},$$

for which one has:

LEMMA 13.11. *If A is of the form above, then $M(A)\mathcal{H}_\hbar = \mathcal{H}_\hbar$ and $M(A)$ is unitary on $\mathcal{H}_\hbar$.*

The general case, when A is an arbitrary element of $\mathrm{SL}(2,\mathbb{Z})$, is dealt with easily as well, but at the cost of some additional notational complications using the spaces $\mathcal{H}_\hbar(\kappa)$, introduced in Exercise 13.3. I refer to [**BDB1**] for details.

Proof: For $\psi \in \mathcal{H}_\hbar$, compute

$$U(n)M(A)\psi = M(A)[M(A)^{-1}U(n)M(A)]\psi = M(A)U(A^{-1}n)\psi,$$

where I used (12.17). Since, in view of (13.2),

$$U(An)\psi = e^{\frac{i}{2\hbar}[An]_1[An]_2}\psi,$$

the first part of the statement follows from an easy calculation. The unitarity of $M(A)$ can be checked for example by computing the matrix of $M(A)$ in the basis e_j explicitly. The proof is however long and messy this way, since the definition of $M(A)$ in section 12 is not very explicit and I will omit it. An alternative proof is given in [**BDB1**]. □

The operator $M(A)$ on $\mathcal{H}_\hbar$ is often called the *quantum map* associated to A. They were first constructed and studied in the context of quantum chaos in [**HB**].

14. The Schnirelman theorem

In this section, I finally return to quantum chaos. The question I want to address is how – in the semi-classical limit – the properties of the eigenfunctions and eigenvalues of the unitary operators $M(A)$ on the finite dimensional Hilbert spaces $\mathcal{H}_\hbar$ are influenced by the chaotic nature of the dynamics induced by A on $\mathbb{T}^2$, for $A \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{Tr}A > 2$. This is the same question as the one posed concerning billiards in section 2 as I hope the intervening sections convinced you. Let me write

$$M(A)\varphi_j^{(N)} = e^{i\theta_j^{(N)}}\varphi_j(N), \qquad j = 1, \ldots, N, \tag{14.1}$$

for the eigenvalues and eigenfunctions of $M(A)$ on $\mathcal{H}_\hbar$. Note that, since $M(A)$ is unitary, the eigenvalues lie on the unit circle in the complex plane. Do not be tempted to think the above problem is trivial because the Hilbert spaces are finite dimensional and hence finding the $\varphi_j^{(N)}, \theta_j^{(N)}$ reduces to diagonalizing an N by N matrix: we are interested in the behaviour of these quantities when $N \to \infty$!

14.1. Eigenfunction behaviour: the Schnirelman theorem. I already gave an approximate statement of the Schnirelman theorem for billiards in section 2. The precise statement in the present context goes as follows:

THEOREM 14.1. *For each $N \in \mathbb{N}$, there exists a subset $E(N) \subset \{1, \ldots, N\}$ such that:*

(i) $\displaystyle\lim_{N\to\infty} \frac{\sharp E(N)}{N} = 1$;

(ii) For any $f \in C^\infty(\mathbb{T}^2)$, for any sequence $(j_N \in E(N))_{N\in\mathbb{N}}$, one has

$$\lim_{N\to\infty} \langle \varphi_{j_N}^{(N)}, Op^W\ f\ \varphi_{j_N}^{(N)}\rangle_{\mathcal{H}_\hbar} = \int_{\mathbb{T}^2} f(x)dx.$$

To get a feeling for what this means, it is helpful to look at it in various ways. First, note that, in obvious notations,

$$\langle \varphi_j^{(N)}, Op^W\ f\ \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar} = \int_{\mathbb{T}^2} W_j^{(N)}(x) f(x)dx,$$

so that the Schnirelman theorem states that $\lim_{N\to\infty} W_{j_N}^{(N)}(x) = 1$. As mentioned in section 2, this is referred to as the "equidistribution" of the eigenfunctions. The following simple exercise makes the link between this formulation of the Schnirelman theorem and the one given in section 2 for billiards.

EXERCISE 14.2. Show that, under the hypotheses of the Schnirelman theorem and with the notations of Exercise 13.10, we have that, for all intervals $I \subset [0, 1]$,

$$\lim_{N\to\infty} \lambda_{\varphi_{j_N}^{(N)}}(I) = \mid I \mid.$$

The proof of the Schnirelman theorem requires two simple lemmas:

LEMMA 14.3. *Let B be a bounded operator on some Hilbert space $\mathcal{K}$ and let $\phi \in \mathcal{K}, \parallel \phi \parallel_{\mathcal{K}} = 1$. Then*

$$\mid \langle \phi, B\phi\rangle_{\mathcal{K}} \mid^2 \leq \langle \phi, B^*B\phi\rangle_{\mathcal{K}}.$$

EXERCISE 14.4. Prove this.

The second one is a little messier.

LEMMA 14.5. *Let $0 \leq a_j^{(N)} \leq 1$ be a family indexed by (j, N), with $N \in \mathbb{N}$ and $1 \leq j \leq N$. Then the following two statements are equivalent:*

(i) There exist sets $E(N) \subset \{1, \ldots, N\}$ such that $\displaystyle\lim_{N\to\infty} \frac{\sharp E(N)}{N} = 1$ and such that for any sequence $(j_N \in E(N))_{N\in\mathbb{N}}$ one has

$$\lim_{N\to\infty} \sup_{j\in E(N)} a_j^{(N)} = 0;$$

$$(ii) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} a_j^{(N)} = 0$$

Proof: That (i) implies (ii) is obvious. To prove (ii) implies (i), define for each $k \in \mathbb{N}^*$ the sets

$$B_k(N) = \{1 \le j \le N \mid a_j^{(N)} \ge \frac{1}{k}\}.$$

Since

$$\frac{1}{N} \sum_{j=1}^{N} a_j^{(N)} \ge \frac{1}{k} \frac{\sharp B_k(N)}{N},$$

it follows from (ii) that

$$\lim_{N\to\infty} \frac{\sharp B_k(N)}{N} = 0.$$

Consequently, there exists a sequence of integers N_k with the following property:

$$\forall k \in \mathbb{N}^*, \ \exists N_k \in \mathbb{N}^*, \forall N \ge N_k, \quad \frac{\sharp B_k(N)}{N} \le \frac{1}{k}.$$

We can and will suppose the sequence N_k is strictly increasing. As a result, any N lies between two successive N_k:

$$\forall N \in \mathbb{N}, \exists ! k_N, N_{k_N} \le N < N_{k_N+1}.$$

Moreover, $\lim_{N\to\infty} k_N = \infty$. Now define

$$E(N) = \{1 \le j \le N \mid a_j(N) < \frac{1}{k_N}\}.$$

Clearly, since $N \ge N_{k_N}$,

$$1 \ge \frac{\sharp E(N)}{N} = 1 - \frac{\sharp B_{k_N}(N)}{N} \ge 1 - \frac{1}{k_N},$$

so that

$$\lim_{N\to\infty} \frac{\sharp E(N)}{N} = 1.$$

Since $\sup_{j\in E(N)} a_j^{(N)} \le \frac{1}{k_N}$, (i) follows immediately. □

Proof of Theorem 14.1: The proof will be very short, but uses more or less every result we showed so far. It is sufficient to consider the case where $f_0 = \int_{\mathbb{T}^2} f(x)dx = 0$. We will first prove that

$$(14.2) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} | \langle \varphi_j^{(N)}, Op^W \ f \ \varphi_j^{(N)} \rangle_{\mathcal{H}_\hbar} |^2 = 0.$$

Using (14.1), we immediately have, for each $t \in \mathbb{N}$

$$\langle \varphi_j^{(N)}, Op^W \ f \ \varphi_j^{(N)} \rangle_{\mathcal{H}_\hbar} = \langle M(A)^t \varphi_j^{(N)}, Op^W \ f \ M(A)^t \varphi_j^{(N)} \rangle_{\mathcal{H}_\hbar},$$

so that, for each $T \in \mathbb{N}$,

$$\langle \varphi_j^{(N)}, Op^W \ f \ \varphi_j^{(N)} \rangle_{\mathcal{H}_\hbar} = \langle \varphi_j^{(N)}, Op^W \ < f >_T \ \varphi_j^{(N)} \rangle_{\mathcal{H}_\hbar},$$

where I used the notation $< f >_T$ introduced in Proposition 10.6 as well as the Egorov theorem. Lemma 14.3, together with Lemma 13.9 (i) and (iii) then imply that, for every $T \in \mathbb{N}$, there exists constants C'_T, C_T, independent of N, so that

$$\frac{1}{N}\sum_{j=1}^{N} \mid \langle \varphi_j^{(N)}, Op^W\ f\ \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar} \mid^2 \leq$$

$$\begin{aligned}
&\leq \frac{1}{N}\sum_{j=1}^{N}\langle \varphi_j^{(N)}, \left(Op^W\ < f >_T\right)^* \left(Op^W\ < f >_T\right) \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar}\\
&\leq \frac{1}{N}\sum_{j=1}^{N}\langle \varphi_j^{(N)}, \left(Op^W\ \overline{< f >_T} < f >_T\right) \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar} + \frac{C'_T}{N}\\
&\leq \int_{\mathbb{T}^2} \overline{< f >_T (x)} < f >_T (x) dx + \frac{C_T}{N}
\end{aligned}$$

Since we don't know a priori if the limit as $N \to \infty$ of the left hand side exists, we take the limsup of both sides with respect to N to obtain

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \mid \langle \varphi_j^{(N)}, Op^W\ f\ \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar} \mid^2 \leq \int_{\mathbb{T}^2} \mid < f >_T (x) \mid^2 dx.$$

Remarking that the left hand side does not depend on T, and using the ergodicity of A as expressed in Proposition 10.8, (14.2) follows immediately. Lemma 14.5 then implies the existence of a set $E(N, f) \subset \{1, \dots, N\}$ having properties (i) and (ii) of the theorem. This basically ends the proof, except for the dependence of the set $E(N, f)$ on the function f. The latter can be eliminated by using the same tricks as those used to prove Lemma 14.5. I will merely sketch the argument, leaving you to fill in the details. Introduce $\mathcal{G}$, the set of all trigonometric polynomials on $\mathbb{T}^2$ with rational coefficients. This is a countable set the elements of which we denote by $f_\ell, \ell \in \mathbb{N}$. Applying the arguments of the first part of the proof to

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{\ell=1}^{L} \mid \langle \varphi_j^{(N)}, Op^W\ f_\ell\ \varphi_j^{(N)}\rangle_{\mathcal{H}_\hbar} \mid^2$$

one shows there exist sets $E(N, L) \subset \{1, \dots, N\}$ for which (i) and (ii) of the theorem hold with $f = f_\ell$ for each $1 \leq \ell \leq L$. One can moreover choose the $E(N, L)$ so that $E(N, L) \subset E(N, L - 1)$. Then

$$\forall L \in \mathbb{N}^*,\ \exists N_L \in \mathbb{N}^*, \forall N \geq N_L, \quad \frac{\sharp E(N, L)}{N} \geq 1 - \frac{1}{L}.$$

Choosing N_L strictly increasing, we can define, for each N, the integer L_N by $N_{L_N} \leq N < N_{L_N+1}$ and construct

$$E(N) = E(N, L_N).$$

Since $L_N \to \infty$ as $N \to \infty$, the theorem follows from the observation that

$$\forall f \in C^\infty(\mathbb{T}^2), \forall \epsilon > 0, \exists \ell \in \mathbb{N}, \sum_{n\in\mathbb{Z}^2} \mid f_n - f_{\ell,n} \mid \leq \epsilon,$$

together with the observation that

$$\parallel Op^W\ f \parallel_{\mathcal{H}_\hbar} \leq \sum_{n\in\mathbb{Z}^2} \mid f_n \mid .$$

□

The three essential ingredients of this proof (and of all existing proofs of the Schnirelman theorem) are clearly some basic semi-classical estimates, the Egorov theorem and the ergodicity of A. In particular, the result does not depend on any stronger dynamical properties, such as the (exponential) mixing for example, of the underlying dynamics. As such, the Schnirelman theorem is clearly only a first step in the semi-classical analysis of spectral problems associated to classically chaotic systems. Mixing does play a role when investigating the behaviour of off-diagonal matrix elements of the form ($k \neq j$)

$$\langle \varphi_j^{(N)}, Op^W\ f\ \varphi_k^{(N)} \rangle_{\mathcal{H}_\hbar}$$

as explained in [**Z2**][**Z3**][**CR**][**Bo**]. The role of exponential mixing is explored in [**BonDB**].

An interesting question raised by the Schnirelman theorem is whether there are any "exceptional" sequences of eigenfunctions φ_{j_N}, for which the limit

$$\lim_{N\to\infty} \langle \varphi_{j_N}^{(N)}, Op^W\ f \varphi_{j_N}^{(N)} \rangle_{\mathcal{H}_\hbar} = \mu(f)$$

exists but does not equal $\int_{\mathbb{T}^2} f(x)dx$.

EXERCISE 14.6. Show that, if μ exists, then $\mu(f \circ A) = \mu(f)$.

A particularly interesting possibility would be the existence of such sequences where

$$\mu(f) = \frac{1}{T} \sum_{t=0}^{T-1} f(A^t x_0);$$

here x_0 is a periodic point of the dynamics generated by A, of period T, so that $A^T x_0 = x_0$. In other words, μ is the natural A-invariant measure supported on a periodic orbit. Such sequences, if they exist, are called (strong) "scars". For the case at hand, various strengthenings of the Schnirelman theorem as stated above indicate strongly that such sequences should not exist [**DEGI**][**KR1**][**KR2**]. The existence or absence of some (possibly weaker) form of scarring is in general an interesting question in quantum chaos, that can be asked also in the context of billiards, for example. For more information on the question of scarring, you may consult [**S1**] and references therein.

14.2. Eigenvalue behaviour. The attentive reader should be disconcerted to see the end of these notes approach without me returning to the problems of eigenvalue statistics and the Berry-Tabor and Bohigas-Giannoni-Schmit conjectures with which I started them. Did I not claim that the hyperbolic automorphisms of the torus were much simpler than chaotic billiards and did I not use this claim to motivate their study? As it turns out, those systems have a rather special spectrum in some respects (Exercise 14.7) and the (properly reformulated) Bohigas-Giannoni-Schmit conjecture is *not* verified by them [**Ke**]. Numerical evidence indicates that *perturbed* hyperbolic automorphisms, on the other hand, do have eigenvalue statistics compatible with the BGS conjecture [**MO**]. Unfortunately, whereas the proof of the Schnirelman theorem given in the previous section goes through almost unaltered for those systems [**BDB1**], proving anything about their eigenvalue statistics seems to remain quite out of reach for the moment. The only rigorous result (which holds both for the hyperbolic toral automorphisms and their perturbations) I am aware of is a Weyl-type law saying that the eigenvalues equidistribute on the unit circle, *i.e.*

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{1 \leq j \leq N \mid \theta_j^{(N)} \in I\} = \mid I \mid,$$

for any interval I on the unit circle [**BDB1**], but this does of course not say anything about the statistics of the level spacings.

EXERCISE 14.7. * (i) Show that, for all $N \in \mathbb{N}$, there exists a $T_N \in \mathbb{N}$ so that

$$A^{T_N} n = n \mod N,$$

for all $n \in \mathbb{Z}^2$. The smallest such integer, that we shall continue to denote by T_N is called the period of A modulo N.

(ii) Show that $M(A)^{2T_N}$ is a multiple of the identity on $\mathcal{H}_\hbar$. **Hint:** Remember Lemma 13.6 (iv), as well as the Egorov theorem.

(iii) Let $n(N)$ be the smallest integer so that $M(A)^{n(N)}$ is a multiple of the identity on $\mathcal{H}_\hbar$. This is called the *quantum period* of A. Investigate the relation between T_N and $n(N)$.

(iii) Show that the eigenvalues $\theta_j^{(N)}$ of $M(A)$ are necessarily of the form $2\pi\frac{k}{n(N)} + \phi_N$, for some ϕ_N.

15. Cabos sueltos

My visit to Mexico has been very short, but Carlos and Salvador have been such excellent guides that I certainly hope to come back here some day. I also hope I have been equally good a guide to the sights of quantum chaos of which the present lectures gave you only a glimpse. Actually, most of the main issues in this field have simply not been touched upon here: neither geodesic flows and Laplace-Beltrami operators on constant negative curvature surfaces, nor trace formulas have even been mentioned and the links between quantum chaos and random matrix theory or number theory have been completely ignored. To explore the subject further I recommend you the literature already cited, as well as some of the many review articles available: [**B**][**S1**][**S2**][**GMW**]. I hope these notes have at least furnished an adequate introduction to those more advanced treatments of the subject.

Appendix A. Torus translations

In the notations of section 8, consider $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. For notational simplicity, I choose $a_i = e_i, i = 1\dots n$, where $\{e_i\}_{i=1\dots n}$ is the canonical basis of $\mathbb{R}^n$. Let $\omega = (\omega_1, \dots \omega_n) \neq 0$ and consider the flow $T_t^\omega(\tau) = \tau + t\omega \mod \mathbb{Z}^n$. We want to understand the *orbits* of any $\tau_0 \in \mathbb{T}^n$, *i.e.* the curves

$$t \in \mathbb{R} \mapsto T_t^\omega(\tau_0) \in \mathbb{T}^n.$$

The behaviour of these curves depends on ω, of course. The easiest situation is the following:

LEMMA A.1. *The following statements are equivalent:*
(i) the flow T_t^ω has one periodic orbit;
(ii) all orbits of T_t^ω are periodic with the same period;
(iii) $\forall 1 \leq i < j \leq n$, ω_i and ω_j are commensurate.

Proof: Clearly $T_t^\omega(\tau_0) = \tau_0$ if and only if there exists $m \in \mathbb{Z}^n$ so that $\tau_0 + t\omega = \tau_0 + m$ from which the lemma easily follows. □

Note that in this case each orbit is a closed one-dimensional submanifold of the torus. At the other extreme, one has the following result:

LEMMA A.2. *If for all* $m \in \mathbb{Z}^{n*}$, $\omega \cdot m \neq 0$, *then, for all* $f \in C(\mathbb{T}^n)$, *for all* $\tau_0 \in \mathbb{T}^n$, *one has*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(T_t^\omega(\tau_0))dt = \int_{\mathbb{T}^n} f(\tau)d\tau.$$

where the limit is reached uniformly in τ_0.

In other words, under the hypothesis on ω stated, the flow is ergodic (see section 9). Note also that the condition on ω is fulfilled for a set of full measure.
Proof: First prove the result for $f(\tau) = \exp i2\pi m\cdot\tau, m \in \mathbb{Z}^n, m \neq 0$ through a direct computation. Conclude then through an approximation argument. □

EXERCISE A.3. Let B be an open subset of $\mathbb{T}^n$. Show that, under the hypothesis of the last lemma, one has

$$\frac{\mid \{0 \le t \le T \mid T_t^\omega(\tau_0) \in B\} \mid}{T} \to \mid B \mid .$$

This means that the fraction of time spent in the set B converges to the size of $\mid B \mid$.

One can equally well consider discrete translations, *i.e.* $(T_t^\omega)_{t\in\mathbb{Z}}$:

EXERCISE A.4. (i) Let $\omega \in \mathbb{R}$ and consider $T_t^\omega(\tau) = \tau + t\omega \bmod 1$ with $t \in \mathbb{Z}$ and $\tau \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Then, for all $f \in C(\mathbb{T}^1)$,

$$\frac{1}{T}\sum_{t=0}^{T-1} f(T_t^\omega(\tau) = \frac{1}{T}\sum_{t=0}^{T-1} f(\tau + t\omega) \to \int_0^1 f(\tau)dt,$$

iff $\omega \in \mathbb{R}\setminus\mathbb{Q}$.
(ii) Show that, if $\omega \in \mathbb{R}\setminus\mathbb{Q}$, then the orbit $(T_t^\omega(\tau))_{t\in\mathbb{Z}}$ is a dense subset of $\mathbb{T}^1$.

EXERCISE A.5. Can you detect any qualitative difference between two trajectories in a circular billiard, one with α rational, one with α irrational? Let $t \to \gamma(t)$ be the trajectory of a ball in a circular billiard and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Do you expect

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\gamma(t))\ dt$$

to exist? If so, what do you expect the answer to be? Does it depend on whether α is rational or irrational for the trajectory considered? Can you prove your conjectures?

EXERCISE A.6. * Study the motion in a square billiard. Prove that, if the slope of the trajectory with respect to one of the sides of the billiard is rational, then the trajectory is periodic. Show that, if the slope is irrational, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\gamma(t))\ dt = \int_0^1\int_0^1 f(x,y)dxdy.$$

Adapt the above statements to the case of a rectangular billiard with sides of length a and b. **Hint:** You may wish to consult [**CFS**] or [**T**] for inspiration.

References

[Arf] Arfken G., *Mathematical methods for physicists,* Academic Press (1985).

[A] Arnold, V., *Mathematical methods in classical mechanics,* New-York-Berlin, Springer-Verlag (1973)

[AM] Abraham R., Marsden J., *Foundations of mechanics,* Benjamin-Cummings (1978)

[B] Berry M., *Semiclassical mechanics of regular and irregular motion*, in "Chaotic behaviour of deterministic systems", Les Houches Lectures vol XXXVI, ed. G. Iooss, R.H.G. Helleman and R. Stora, pp. 171-271, Amsterdam North-Holland (1983).

[Ba1] Baladi, V., *Positive transfer operators and decay of correlations,* World Scientific, Singapore (2000).

[Ba2] Baladi, V. *Decay of correlations-Minicourse-Handout notes*, 1999 AMS Summer Research Institute, Seattle Washington "Smooth ergodic theory and applications".

[BonDB] Bonechi F., De Bièvre S., *Exponential mixing and* $|\ln \hbar|$ *time scales in quantized hyperbolic maps on the torus,* Commun. Math. Phys, **211**, 3, 659-686 (2000).

[BDB1] Bouzouina A., De Bièvre S., *Equipartition of the eigenfunctions of quantized ergodic maps on the torus*, Commun.Math.Phys. **178**, 83-105 (1996).

[BDB2] Bouzouina A., De Bièvre S., *Equidistribution des valeurs propres et ergodicité semi-classique de symplectomorphismes du tore quantifiés*, C. R. Acad. Sci. Paris, Série I **t.326**, 1021-1024 (1998).

[BGS] Bohigas O., Giannoni M.-J., Schmit C., *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52**, 1-4 (1984).

[Bo] Bouzouina A., *Comportement semi-classique de symplectomorphismes du tore quantifiés*, doctoral dissertation, Université Paris 9 (1997).

[BT] Berry M., Tabor M., *Level clustering in the regular spectrum*, Proc. Roy. Soc. A **356**, 375-394 (1977).

[CdV] Colin de Verdière Y., *Ergodicité et fonctions propres du Laplacien*, Commun. Math. Phys. **102**, 497-502 (1985).

[CFKS] Cycon H.L., Froese R.G., Kirsch W., Simon B., *Schrödinger Operators*, Springer-Verlag (1987).

[CFS] Cornfeld I.P., Fomin S.V., Sinai Y.G., *Ergodic theory*, Springer-Verlag (1982).

[CR] Combescure M., Robert D., *Distribution of matrix elements and level spacings for classically chaotic systems*, Ann. Inst. H. Poincaré **61**,4 (1994), 443-483.

[CTDL] Cohen-Tannoudji C., Diu B., Laloë F., Quantum mechanics, volumes I and II, John Wiley (1977).

[D] Dolgopyat D., *On decay of correlations in Anosov flows,* Annals of Math. **147**, 357-390 (1998).

[DB] De Bièvre S., *Chaos, quantization and the classical limit on the torus*, Proceedings of the XIVth Workshop on Geometrical Method in Physics – Bialowieza (1995), mp_arc 96-191, Polish Scientific Publishers PWN (1998).

[DEGI] Degli Esposti, M., Graffi, S., Isola, S., *Stochastic properties of the quantum Arnold cat in the classical limit*, Comm. Math. Phys **167** , 471-509 (1995).

[F] Folland G., *Harmonic analysis in phase space*, Princeton University Press, Princeton (1988).

[G] Goldstein, H., *Classical mechanics*, Addison-Wesley (1959).

[GMW] Guhr T., Müller-Groeling H., Weidenmüller H., *Random matrix theories in quantum physics: common concepts*, Phys. Rep. **299**, 189-425 (1998).

[GL] Gérard P., Leichtnam E., *Ergodic properties of the eigenfunctions for the Dirichlet problem*, Duke. Math. J. **71** (1993), 559-607.

[H] Helffer B., *Semi-classical analysis for the Schrödinger operator and applications*, LNM 1336, Springer Verlag (1988).

[Ho] Hörmander, L., *The analysis of linear partial differential operators IV*, Springer-Verlag (1985).

[HB] Hannay J.H., Berry M.V., *Quantization of linear maps-Fresnel diffraction by a periodic grating*, Physica D **1**, 267 (1980).

[HS] Hislop P., Sigal I.M., Introduction to spectral theory, Springer-Verlag (1996).

[I] Ivrii V, *Microlocal analysis and precise spectral asymptotics*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[Ke] Keating J., *The cat maps: quantum mechanics and classical motion,* Nonlinearity **4**, 309–341 (1991).

[Kh] Khinchin A., *Continued Fractions,* The University of Chicago Press, Chicago and London (1964).

[K] Kirillov A., *Elements of the theory of representations*, Springer (1976).

[KR1] Kurlberg, P., Rudnick, Z. *Hecke theory and equidistribution for the quantization of linear maps of the torus*, preprint 1999, chao-dyn/9901031, Duke Math. J., to appear.

[KR2] Kurlberg, P., Rudnick, Z. *On quantum ergodicity for linear maps of the torus*, preprint 1999, math.NT/9910145.

[LL1] Landau L.D., Lifshitz E.M., *Mechanics*, Pergamon Press, Oxford (1960).

[LL2] Landau L.D., Lifshitz E.M., *Quantum mechanics, nonrelativistic theory*, Pergamon Press, Oxford (1965).

[M] Messiah A., Quantum mechanics, North Holland, Amsterdam (1961).

[Ma] Marklof J., Pair correlation densities of inhomogeneous quadratic forms, preprint may 2000.

[MO] de Matos M.B., Ozorio de Almeida A.M., *Quantization of Anosov maps,* Ann. Phys. **237**, 46-65 (1995).

[P] Petersen K., *Ergodic theory*, Cambridge UP, (1983).

[R] Robert D., *Autour de l'approximation semi-classique*, Birkhäuser (1987).

[RS] Reed M., Simon B. *A course in mathematical physics*, volume IV, Academic Press, London (1978).

[S1] Sarnak P., *Arithmetic quantum chaos*, Israel Math. Conf. Proc. **8**, 183-236 (1995).

[S2] Sarnak P., *Quantum chaos, symmetry and zeta functions. Lectures I and II*, Current developments in mathematics, 1997 (Cambridge, MA), 127–159, Int. Press, Boston, MA (1999).

[Sc] Schnirelman A., *Ergodic properties of eigenfunctions*, Usp. Math. Nauk. **29**, 181-182 (1974).

[T] Tabachnikov S., *Billiards*, Panoramas et Synthèses 1, SMF (1995).

[V] Vogan D., *Unitary representations of reductive Lie groups*, Princeton University Press (1987).

[W] Walters P., *An introduction to ergodic theory*, Graduate Texts in Mathematics 79, Springer Verlag (1982).

[Z1] Zelditch S., *Uniform distribution of the eigenfunctions on compact hyperbolic surfaces*, Duke Math. J **55** (1987), 919-941.

[Z2] Zelditch S., *Quantum transition amplitudes for ergodic and for completely integrable systems*, J. Funct. Anal **94** (1990), 415-436.

[Z3] Zelditch S., *Quantum mixing*, J. Funct . Anal **140** (1996), 68-86.

[ZZ] Zelditch S., Zworski M., *Ergodicity of eigenfunctions for ergodic billiards*, Comm. Math. Phys. **175**, 3, 673-682 (1996).

UFR de Mathématiques et UMR AGAT, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq Cedex France

E-mail address: `debievre@agat.univ-lille1.fr`