On the Growth of Averaged Weyl Sums for Rigid Rotations

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Abstract

Let $\omega \in \mathbf{R} \setminus \mathbf{Q}$ and $f \in L^2(\mathbf{S}^1)$ of zero average. We study the asymptotic behaviour of the Weyl sums $S(m,\omega)f(x) = \sum_{k=0}^{m-1} f(x+k\omega)$ and their averages $\hat{S}(m,\omega)f(x) = \frac{1}{m}\sum_{j=1}^{m} S(j,\omega)f(x)$, in the L^2-norm. In particular, for a suitable class of Liouville rotation numbers $\omega \in \mathbf{R} \setminus \mathbf{Q}$, we are able to construct examples of functions $f \in H^s(\mathbf{S}^1)$, $s > 0$, such that, for all $\epsilon > 0$, $||\hat{S}(m,\omega)f||_2 \geq C_\epsilon m^{\frac{1}{1+s}-\epsilon}$ as $m \to \infty$. We show in addition that, for all $f \in H^s(\mathbf{S}^1)$, $\liminf m^{-\frac{1}{1+s}}(\log m)^{-1/2} \parallel \hat{S}(m,\omega)f \parallel_2 < \infty$, for all $\omega \in \mathbf{R} \setminus \mathbf{Q}$.

1 Introduction and statement of the results

We study the asymptotic behaviour of the following skew products on the cylinder:

$$(x_0, v_0) \in \mathbf{S}^1 \times \mathbf{R} \equiv \mathbf{T}^*\mathbf{S}^1 \to (\mathbf{x_1} = \mathbf{x_0} + \omega, \mathbf{v_1} = \mathbf{v_0} + \mathbf{f}(\mathbf{x_0})). \tag{1.1}$$

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Here $\omega \in \mathbf{R} \setminus \mathbf{Q}$ and $f : \mathbf{S}^1 \to \mathbf{R}$, $\quad \mathbf{S^1} = \mathbf{R}/\mathbf{Z}$. After m iterations of the map we have

$$x_m = x_0 + m\omega, \quad v_m = v_0 + S(m, \omega)f(x_0),$$

where the *Weyl sums* $S(m,\omega)f$ of the function f for the rigid rotation by ω are defined to be

$$S(m, \omega)f(x) = \sum_{k=0}^{m-1} f(x + k\omega) \ . \tag{1.2}$$

The asymptotic behaviour of the v-variable is therefore completely determined by the asymptotic behaviour of the Weyl sums.

It is well known [KN] that, if $\{p_k/q_k\}_{k\geq 1}$ denotes the sequence of approximants of the *continued fraction expansion* [K] of the irrational number ω, then, for any $f \in BV(\mathbf{S}^1)$ of zero mean,

$$\| \ S(q_k, \omega)f \ \|_\infty < \ \mathrm{Var}(f) \quad \text{for all } k \in \mathbf{N}\,. \tag{1.3}$$

The upper bound (1.3), known as the Denjoy-Koksma inequality, rules out the possibility of having growth to infinity of Weyl sums for smooth functions of zero mean. However, the set of "times" where the bound (1.3) holds is rather small since the q_k's grow at least exponentially. Perhaps then $S(m,\omega)f$ still gets to be sufficiently large at intermediate times resulting in growth of the averaged Weyl sums

$$\hat{S}(m, \omega)f(x) = \frac{1}{m}\sum_{j=1}^{m} S(j, \omega)f(x). \tag{1.4}$$

We will show this picture is essentially correct and obtain bounds on the growth of the $\hat{S}(m,\omega)f$. We are interested in the L^2 growth, as $m \to +\infty$, of the averaged Weyl sums (1.4) for L^2-functions f. More precisely, we answer the following question. Given $s > 0$ and a function $f \in H^s(\mathbf{S}^1)$ (see (1.5)), what is the largest possible exponent $\tau(s) > 0$ for which there exists an irrational number ω so that

$$\| \ \hat{S}(m, \omega)f \ \|_2 \geq Cm^{\tau(s)},$$

for all $m \in \mathbf{N}$? We find that essentially $\tau(s) = \frac{1}{1+s}$ (see (1.14)$-$(1.15)). Our results show that the growth of $\| \ \hat{S}(m,\omega)f \ \|_2$ is worsened by smoothness. In particular, if $f \in C^\infty(\mathbf{S}^1)$, then $\forall\omega$, $\forall\epsilon$, $\exists C > 0$ such that

$$\| \ \hat{S}(m_k, \omega)f \ \|_2 \leq Cm_k^\epsilon$$

along a diverging sequence $\{m_k\}_{k\in\mathbf{N}}$ of natural numbers.

We write f_n for the Fourier coefficients of the function $f \in L^2(\mathbf{S}^1)$ and recall that one can express the smoothness properties of f in terms of its Fourier coefficients using the usual Sobolev spaces $H^s(\mathbf{S}^1)$:

$$H^s(\mathbf{S}^1) = \{f \in L^2(\mathbf{S}^1) \mid \| f \|_s^2 := \sum_{n\in\mathbf{Z}} n^{2s} |f_n|^2 < +\infty\}. \tag{1.5}$$

To state and interpret our main results, we need some notation. We will consider classes $\mathcal{R}_\gamma \subset \mathcal{S}_\gamma \subset \mathbf{R}\setminus\mathbf{Q}$ of irrational numbers defined as follows: $\omega \in \mathcal{S}_\gamma$ iff there exists a constant $R > 0$ and an infinite subset $\mathcal{K} \subset \mathbf{N}$ such that

$$\| q_k\omega \| \leq \frac{R}{q_k^{1+\gamma}} \qquad \forall k \in \mathcal{K}. \tag{1.6}$$

The set $\mathcal{R}_\gamma$ is defined by the property that in the above definition it is possible to take $\mathcal{K} = \mathbf{N}$.

By definition, $\mathcal{R}_{\gamma'} \subset \mathcal{R}_\gamma$ and $\mathcal{S}_{\gamma'} \subset \mathcal{S}_\gamma$ if $\gamma' \geq \gamma$ and, by basic properties of the continued fraction expansion, $\mathcal{R}_0 = \mathcal{S}_0 = \mathbf{R}\setminus\mathbf{Q}$. It is possible to show that the intersection $\mathcal{R}_\infty$ of all sets $\mathcal{R}_\gamma$, for $\gamma \geq 0$, is an uncountable subset of the set of Liouville irrationals. In addition, the set $\mathcal{S}_\gamma$ is the complementary set of the set of Diophantine irrationals having Diophantine exponent strictly less than γ (which is a G_δ dense set) and the intersection $\mathcal{S}_\infty$ of all sets $\mathcal{S}_\gamma$ is the set of all Liouville irrationals.

It will finally be useful to consider, for $\omega \in \mathcal{S}_\gamma$ ($\gamma \geq 0$) and for $r \geq 1$ the set

$$A_r(\omega) = \{m \in \mathbf{N} \mid \exists \ell \in \mathcal{K} \text{ so that } \mathbf{2q_{\ell+1} \leq m \leq 2rq_{\ell+1}}\}, \tag{1.7}$$

where $\mathcal{K} \subset \mathbf{N}$ is an infinite set such that (1.6) holds. The points in $A_r(\omega)$ form a divergent subsequence of $\mathbf{N}$.

We can now state our main results. To simplify the formulations, we will always assume that

$$f_0 = \int_{\mathbf{S}^1} f dx = 0. \tag{1.8}$$

Theorem 1.1 *(i) Suppose* $|f_n| \geq c|n|^{-\nu}$ *for some* $\nu > 1/2$. *Let* $\omega \in \mathcal{S}_\gamma$ *for some* $\gamma \geq 0$. *Then for all* $r \geq 1$ *there exists a* $C_r > 0$ *so that for all* $m \in A_r(\omega)$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_r m^{(1-\frac{\nu}{1+\gamma})}. \tag{1.9}$$

If in addition $\omega \in \mathcal{R}_\gamma$, *there exists a constant* $C_\gamma > 0$ *so that* $\forall m \in \mathbf{N}$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_\gamma m^{(1-\frac{\nu}{1+\gamma})\frac{1+\gamma}{1+\gamma+\gamma\nu}}. \tag{1.10}$$

(ii) Let $\omega \in \mathcal{S}_\gamma$ *and* $s \geq 0$. *Then there exists a function* $f \in H^s(\mathbf{S}^1)$ *with the following properties. First, for all* $r \geq 1$ *and for all* $\epsilon > 0$, *there is a constant* $C_{r,\epsilon} > 0$ *so that for all* $m \in A_r(\omega)$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_{r,\epsilon}\, m^{1-\frac{s}{1+\gamma}-\epsilon}\ . \tag{1.11}$$

If in addition $\omega \in \mathcal{R}_\gamma$, *there exists a* $C_{\gamma,\epsilon} > 0$ *so that* for all $m \in \mathbf{N}$

$$\| \hat{S}(m,\omega)f \|_2 \geq C_{\gamma,\epsilon}\, m^{(1-\frac{s}{1+\gamma})\frac{1+\gamma}{1+\gamma+\gamma s}-\epsilon}. \tag{1.12}$$

Note that (1.10) is optimised, for $\nu > 1/2$ fixed, as $\gamma \to +\infty$, so that for any Liouville irrational number $\omega \in \mathcal{R}_\infty$ you get, $\forall \epsilon > 0$:

$$\| \hat{S}(m,\omega)f \|_2 \geq C_\epsilon m^{\frac{1}{1+\nu}-\epsilon}, \quad \forall m \in \mathbf{N}. \tag{1.13}$$

Similarly, for $s > 0$ fixed, (1.12) is optimised as $\gamma \to +\infty$. Hence we get the following statement. For all $s \geq 0$ and for any Liouville rotation number $\omega \in \mathcal{R}_\infty$ there exists an $f \in H^s(\mathbf{S}^1)$ such that, for all $\epsilon > 0$,

$$\| \hat{S}(m,\omega)f \|_2 \geq C_\epsilon m^{\frac{1}{1+s}-\epsilon}, \quad \forall m \in \mathbf{N}. \tag{1.14}$$

Note that the lower bounds get smaller as f is taken smoother. To see that this is not due to a bad choice of rotation number, we prove the following upper bound, which shows that the above lower bounds are close to optimal.

Theorem 1.2 *Let* $\omega \in \mathbf{R} \setminus \mathbf{Q}$.

(i) Let $s \geq 0$. *Then there exists a diverging sequence of natural numbers* $\{m_k\}_{k\in\mathbf{N}}$ *such that, for any* $f \in H^s(\mathbf{S}^1)$

$$\| \hat{S}(m_k,\omega)f \|_2 \leq C \| f \|_s\ m_k^{\frac{1}{1+s}} \sqrt{\log m_k}\ . \tag{1.15}$$

In addition, if $\omega \in \mathbf{R} \setminus \mathbf{Q}$ *satisfies* $\| q_k\omega \| \geq r/q_k^{1+\gamma}, \quad \forall k \in \mathbf{N}$ *and for some* $r > 0$, *then for all* $m \in \mathbf{N}$

$$\| \hat{S}(m,\omega)f \|_2 \leq C \| f \|_s\ m^{1-\frac{s}{1+\gamma}} \sqrt{\log m}\,. \tag{1.16}$$

(ii) Let $\nu > 1/2$ and $\epsilon > 0$. Then there exists a diverging sequence of natural numbers $\{m_k\}_{k\in\mathbf{N}}$ such that, if $| f_n |\leq C | n |^{-\nu}$, then

$$\| \hat{S}(m_k,\omega)f \|_2 \leq C_\epsilon\, m_k^{\frac{1}{1/2+\nu}+\epsilon}. \tag{1.17}$$

In addition, if $\omega \in \mathbf{R}\setminus\mathbf{Q}$ satisfies $\| q_k\omega \|\geq r/q_k^{1+\gamma}$, $\quad \forall k \in \mathbf{N}$ and for some $r > 0$, then $\forall\epsilon$, $\exists C_\epsilon$, so that for all $m \in \mathbf{N}$

$$\| \hat{S}(m,\omega)f \|_2 \leq C_\epsilon\, m^{1-\frac{\nu-1/2}{1+\gamma}+\epsilon}. \tag{1.18}$$

Remark After finishing this paper, we have been informed by D.Volny that bounds similar to (1.9) and (1.16) were independently obtained in [LV].

We can now explain the general picture emerging from these results. To fix ideas, let $|f_n| = |n|^{-\nu}(\nu > 1/2)$. Then $f \in H^s(\mathbf{S}^1)$ for any $s < \nu - 1/2$. We wish to find irrational ω so that $\| \hat{S}(m,\omega)f \|_2$ goes to infinity as fast as possible. The best lower bound we were able to get is (1.10), which is optimized as $\gamma \to +\infty$, yielding (1.13). Note that the exponent tends to zero as ν tends to infinity. On the other hand, the upper estimate (1.17) shows that (1.13) is close to optimal, hence the growth of $\| \hat{S}(m,\omega)f \|_2$ is increasingly slow as f is taken smoother and smoother. In fact, as the proofs will show, the behaviour of $\| \hat{S}(m,\omega)f \|_2$ is controlled by two competing effects. Note first that, if $\omega \in \mathbf{Q}$, then both $\| S(m,\omega)f \|_2$ and $\| \hat{S}(m,\omega)f \|_2$ will typically grow like m. On the other hand, if $\omega \in \mathbf{R}\setminus\mathbf{Q}$, the ergodic theorem tells us that $\| S(m,\omega)f \|_2= o(m)$, hence $\| \hat{S}(m,\omega)f \|_2= o(m)$. This suggests that irrationals well approximated by rationals (γ large) are the best candidates for producing growth in $\| \hat{S}(m,\omega)f \|_2$. This is further corroborated by (1.9) and (1.18), which show there is a competing effect between the smoothness of f and the Diophantine properties of the rotation number ω and that for γ sufficiently large the growth exponent approaches 1. However, as γ becomes larger, so do the gaps between successive q_k, since q_{k+1} is at least of the order of $q_k^{1+\gamma}$. This is at the origin of the exponent of (1.10) which gives a weaker growth than expected.

In fact, our lower bounds predict superdiffusive behaviour, i.e.

$$\| \hat{S}(m,\omega)f \|_2\geq Cm^\tau$$

for some $\tau > 1/2$, for $1/2 < \nu < 1$ and for suitable rotation numbers $\omega \in \mathcal{R}_\gamma$, for γ sufficiently large. On the other hand, the upper bounds in Theorem

1.2, in particular (1.15), rule out diffusive or superdiffusive behaviour for all $f \in H^s(S^1)$ with $s > 1$ and for any choice of an irrational rotation number.

There are several reasons for our interest in the model (1.1). Note that it is a symplectic transformation of $T^*\mathbf{S}^1$ with symplectic form $dx \wedge dv$, that can be seen as a perturbation of

$$(x, v) \in T^*\mathbf{S}^1 \to (x + \alpha, v) \in T^*\mathbf{S}^1$$

which is completely integrable with invariant tori $v = c$ [G][Be][B]. If f is in $C^k(\mathbf{S}^1)$ and ω sufficiently poorly approximated by the rationals (i.e. Diophantine of sufficiently high order), it is easy to see that those tori are preserved. It suffices to solve the *cohomological equation* $f(x) = g(x + \omega) - g(x)$ for a smooth g.

Our results above deal with the opposite extreme, when ω is sufficiently well approximated by the rationals, i.e. weakly Diophantine or Liouville. The tori are then broken and the motion is unbounded. Actually, unboundedness of the Weyl sums $\| S(m, \omega)f \|_2$ (i.e. $\sup_m \| S(m, \omega)f \|_2 = \infty$) is known to be equivalent to the non-solvability of the cohomological equation for $g \in L^2(\mathbf{S}^1, dx)$ (see e.g. [L]). The only previous work that we are aware of discussing the asymptotic behaviour of v_m is [B]. There an argument is proposed claiming to show that the Weyl sums $\| S(m, \omega)f \|_2$ themselves, before averaging, with $f_n \cong |n|^{-\nu}$ satisfy the lower bound (1.9) *for all values of* m and $\nu > 1/2$. As we already pointed out, this can not be true for $\nu > 1$ because of Denjoy-Koksma. As it turns out, Theorem 1.2 shows that the result is not true for the averaged Weyl sums either, as one might have hoped. Only when $\nu < 1$, when Denjoy-Koksma is no longer an obstruction, does the following result indeed give a lower bound valid for all $m \in \mathbf{N}$:

Proposition 1.3 *Suppose* $|f_n| \geq c|n|^{-\nu}$ *for some* $\nu > 1/2$. *Then there exists a* $C > 0$ *so that for all* $\omega \in \mathbf{R} \setminus \mathbf{Q}$ *and for all* $m \in \mathbf{N}$

$$\| S(m, \omega)f \|_2 \geq Cm^{1-\nu}. \tag{1.19}$$

Once the invariant tori are broken, it can be asked whether the map (1.1) is ergodic on the cylinder $T^*\mathbf{S}^1$ with respect to Lebesgue measure. In [Pa], it is shown that the answer is affirmative if $f \in C^1([0, 1])$, $f(0) \neq f(1)$ and if ω is irrational, so that $S(m, \omega)f(x)$ is unbounded for almost all x. In this case, the Fourier coefficients of f satisfy $|f_n| > c|n|^{-1}$ so that Theorem 1.1(i) as well as (1.13) apply with $\nu = 1$.

A particular example that has been much studied is

$$f_\beta(x) = \chi_{[0,\beta]}(x) - \beta,$$

for some $0 < \beta < 1$. It is a classical result of Kesten [Ke][P2] that $S(m,\omega)f_\beta$ is bounded iff $\beta \in \mathbf{Z}\omega \pmod 1$. The asymptotic behaviour of $\hat{S}(m,\omega)f_\beta$ is important in the study of incommensurate one-dimensional structures, where the following model was proposed [AG]. With u_m denoting the m-th atomic position, define the sequence u_m recursively by $(0 < \xi\beta < a)$

$$u_{m+1} - u_m = a + \xi f_\beta(m\omega + x_0),$$

so that

$$u_m = ma + \xi S(m,\omega) f_\beta(x_0) + x_0.$$

The model is said to have no average lattice whenever $S(m,\omega)f_\beta(x_0)$ is unbounded. For $x_0 = 0$, $\beta = 1/2$ and $\omega = \tau^{-2}$, $\tau = \frac{1+\sqrt{5}}{2}$, it was shown in [GLV] and [D] that the averaged Weyl sums diverge logarithmically: this behaviour is expected to persist for a large class of quadratic irrationals ω [CMPS][D]. Since the Fourier coefficients $f_{\beta\ 2k} = 0$, $f_{\beta\ 2k+1} = \frac{1}{i\pi(2k+1)}$ when $\beta = 1/2$, it follows from our work that, if $\omega \in \mathcal{S}_\gamma$ is chosen such that all q_k with $k \in \mathcal{K}$ are odd, where $\mathcal{K}$ is an infinite set of natural numbers for which (1.6) holds, then (see (1.9)):

$$\| \hat{S}(m,\omega) f_\beta \|_2 \geq C_r m^{(1-\frac{1}{1+\gamma})}, \quad \forall m \in A_r(\omega). \tag{1.20}$$

The possibility of having such behaviour for suitable rotation numbers was predicted in a comment at the end of [GLV]. In addition, if $\omega \in \mathcal{R}_\gamma$, we now have (see (1.10))

$$\| \hat{S}(m,\omega) f_\beta \|_2 \geq C_\gamma m^{\frac{\gamma}{(1+2\gamma)}}, \quad \forall m \in \mathbf{N}. \tag{1.21}$$

In particular, taking $\omega \in \mathcal{R}_\infty$, (1.21) implies that the averaged fluctuations behave almost diffusively, as in the case of a random structure. Indeed, if $\omega \in \mathcal{R}_\infty$, then for all ϵ, there exists a C_ϵ so that

$$\| \hat{S}(m,\omega) f_\beta \|_2 \geq C_\epsilon m^{\frac{1}{2}-\epsilon}, \quad \forall m \in \mathbf{N}.$$

Further applications of our results will be given elsewhere [DBF].

2 Proofs

The proofs of our results are based on Fourier series estimates by using the following basic properties of the continued fraction expansion of irrational numbers. Let $\{p_k/q_k\}_{k\geq 1}$ be the sequence of approximants of the continued fraction expansion of $\omega \in \mathbf{R} \setminus \mathbf{Q}$. Then the following holds [K]:

$$\| j\omega \| \geq \| q_k\omega \| \quad \text{for all } j < q_{k+1}\,, \tag{2.1}$$

and

$$\frac{1}{2q_{k+1}} < \frac{1}{q_k + q_{k+1}} < \| q_k\omega \| < \frac{1}{q_{k+1}}\,. \tag{2.2}$$

Note that it follows from (1.6) and (2.2) that if $\omega \in \mathcal{S}_\gamma$, then for a diverging sequence of natural numbers

$$q_{k+1} \geq \frac{1}{2R} q_k^{1+\gamma}\,. \tag{2.3}$$

It is easy to establish that

$$\| S(m,\omega)f \|_2^2 = \sum_{n\in\mathbf{Z}} |f_n|^2 \frac{\sin^2 \pi m n\omega}{\sin^2 \pi n\omega},$$

and

$$\| \hat{S}(m,\omega)f \|_2^2 = \sum_{n\in\mathbf{Z}} |f_n|^2 G_m(n\omega)\,,$$

where the function $G_m : \mathbf{R} \to \mathbf{R}^+$ is defined as

$$G_m(x) := \frac{1}{4\sin^2 \pi x} \left|1 - e^{\pi i(m+1)x} \frac{\sin \pi m x}{m \sin \pi x}\right|^2\,. \tag{2.4}$$

We need the following estimates on the functions G_m. We would like to point out that, throughout this paper, we will use the common practice of letting constants change their value from line to line, without necessarily changing their name.

Lemma 2.1 *The functions $G_m : \mathbf{R} \to \mathbf{R}^+$ are 1-periodic and satisfy the symmetry property $G_m(1-x) = G_m(x)$ for $0 \leq x \leq 1$, hence*

$$\| \hat{S}(m,\omega)f \|_2^2 = \sum_{n\in\mathbf{Z}} |f_n|^2 G_m(\| n\omega \|)\,, \tag{2.5}$$

where $\|\cdot\|$ denotes the distance from the nearest integer. Moreover, there exist constants $C > c > 0$ such that the following inequalities hold:

$$c/\sin^2 \pi x \le G_m(x) \le C/\sin^2 \pi x \qquad \forall\ 1/m \le x \le 1/2. \tag{2.6}$$

and

$$cm^2 \le G_m(x) \le Cm^2 \qquad \forall\ 0 \le x \le 1/m \tag{2.7}$$

In addition, the upper bounds in (2.6) *and* (2.7) *hold for all $x \in \mathbf{R}$.*

Proof. We know from basic calculus that

$$\frac{2}{\pi} \le \frac{\sin \pi x}{\pi x} \le 1 \quad \text{for all } 0 \le x \le 1/2\ , \tag{2.8}$$

where the upper bound holds for all $x \in \mathbf{R}$. It follows from (2.8) that

$$\frac{2}{\pi} \le \left|\frac{\sin \pi m x}{m \sin \pi x}\right| = \left|\frac{\sin \pi m x}{m x} \frac{x}{\sin \pi x}\right| \le \frac{\pi}{2}\ , \tag{2.9}$$

where the lower inequality holds for $0 \le x \le 1/2m$ and the upper inequality for all $x \in \mathbf{R}$, by evenness and periodicity.

Thus the inequality

$$G_m(x) \le C/\sin^2 \pi x \quad \forall\, x \in \mathbf{R}$$

holds by choosing C large enough:

$$\frac{1}{4}\Big(1 + \max_{x\in\mathbf{S}^1} \left|\frac{\sin \pi m x}{m \sin \pi x}\right|\Big)^2 \le \frac{1}{4}\Big(1 + \frac{\pi}{2}\Big)^2 \le C.$$

Secondly, the inequality

$$G_m(x) \ge c/\sin^2 \pi x \quad \forall 1/2 \ge x \ge 1/m \tag{2.10}$$

holds since by (2.8),

$$\left|\frac{\sin \pi m x}{m \sin \pi x}\right| \le \frac{1}{m \sin \pi x} \le \frac{1}{2} \quad \forall\, \frac{1}{m} \le x \le 1/2\ ,$$

which implies

$$\left|1 - e^{\pi i(m+1)x} \frac{\sin \pi m x}{m \sin \pi x}\right|^2 \ge \Big(1 - \left|\frac{\sin \pi m x}{m \sin \pi x}\right|\Big)^2 \ge 1/4\ ,$$

so that we can take $c \leq 1/16$ in (2.10). This proves (2.6). In order to prove (2.7), write

$$u_m(x) := 1 - \frac{\sin \pi m x}{m \sin \pi x}\, e^{\pi i(m+1)x}.$$

There exist $C > c > 0$ such that,

$$cmx \leq |u_m(x)| \leq Cmx \qquad \forall 0 \leq x \leq 1/m.$$

In fact, the estimate from above can be obtained as follows. Since

$$u_m(x) = 1 - \frac{\sin \pi m x}{\pi m x} + \frac{\sin \pi m x}{\pi m x}\,(1 - \frac{\pi x}{\sin \pi x}) + \frac{\sin \pi m x}{m \sin \pi x}\,(1 - e^{\pi i(m+1)x}),$$

the conclusion follows by basic calculus, (2.8) and (2.9). As to the estimate from below, we will consider separately the two intervals $0 \leq x \leq 1/2m$ and $1/2m \leq x \leq 1/m$. In the first case, since $u_m(0) = 0$ and, by a straigthforward computation,

$$u_m'(x) = \left(\pi \frac{\cos \pi x}{\sin \pi x} - \pi m \frac{\cos \pi m x}{\sin \pi m x} - i\pi(m+1)\right) \frac{\sin \pi m x}{m \sin \pi x}\, e^{\pi i(m+1)x}\,,$$

the conclusion follows by the mean value theorem and (2.9). In the second case, it can be noticed that the above argument proving (2.10) holds true for $x \geq 1/2m$. Since $G_m(x) = |u_m(x)|^2/4\sin^2 \pi x$, (2.7) follows. □

The functions $\frac{\sin^2 \pi m x}{\sin^2 \pi x}$ and $G_m(x)$ are traced in Fig. 1 Note that the behaviour of both functions is similar in the region $0 \leq x \leq \frac{1}{4m}$, where they are both of order m^2. For bigger values of x the difference manifests itself in that unlike $\frac{\sin^2 \pi m n \omega}{\sin^2 \pi n \omega}$, $G_m(x)$ has no zeros (see (2.6)). This eventually explains the lower bounds of Theorem 1.1 which we can now start proving.

Proof of Theorem 1.1. (i) Let $\omega \in \mathcal{S}_\gamma$ and denote as before by $\{p_k/q_k\}_{k \in \mathbf{N}}$ the sequence of truncations of its continued fraction expansion. We can assume that $1 + \gamma - \nu > 0$, since otherwise the bounds in (1.9) – (1.10) are trivially satisfied. By Lemma 2.1 a lower bound for the averaged Weyl sum $\hat{S}(m, \omega)f$ can be obtained as follows:

$$\| \hat{S}(m,\omega)f \|_2^2 \geq c\Big(\sum_{\|n\omega\| \geq 1/m} |f_n|^2 \parallel n\omega \parallel^{-2} + m^2 \sum_{\|n\omega\| < 1/m} |f_n|^2 \Big)\,. \tag{2.11}$$

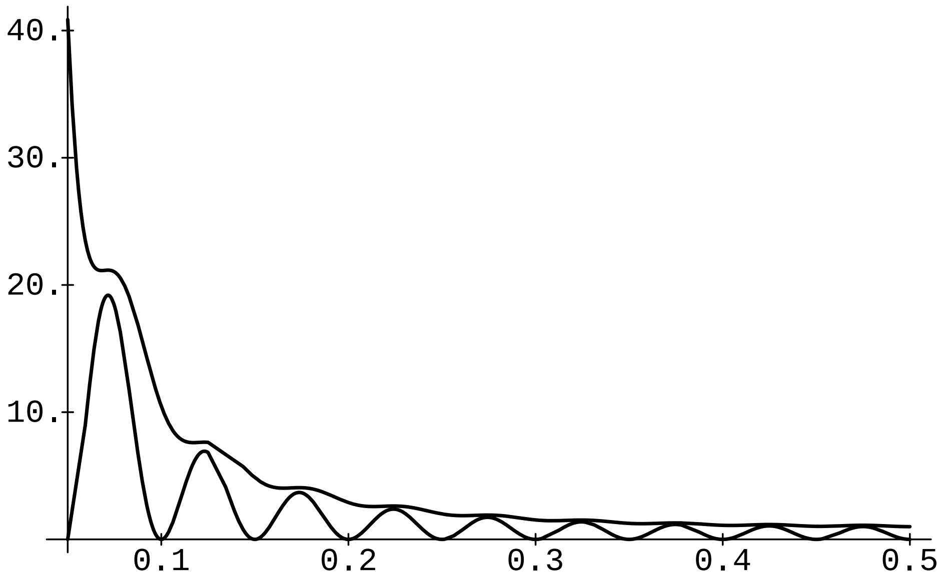

Figure 1: The grahps of $|\frac{\sin \pi m x}{\sin \pi x}|$ and $G_m^{1/2}(x)$ for $m = 20$

Since $\omega \in \mathcal{S}_\gamma$, there exists an infinite set $\mathcal{K} \subset \mathbf{N}$ such that (2.3) holds for $k \in \mathcal{K}$. Suppose now that $m \in A_r(\omega)$. Then for some $\ell \in \mathcal{K}$

$$\frac{1}{2rq_{\ell+1}} \leq \frac{1}{m} \leq \frac{1}{2q_{\ell+1}}.$$

Hence (2.2) implies

$$\frac{1}{m} \leq \parallel q_\ell \omega \parallel \leq \frac{2r}{m}.$$

Using (2.11) we therefore get

$$\parallel \hat{S}(m,\omega) f \parallel_2^2 \geq c |f_{q_\ell}|^2 \parallel q_\ell \omega \parallel^{-2},$$

from which (1.9) follows upon using (2.3).

Assume $\omega \in \mathcal{R}_\gamma$. To prove (1.10), we proceed as follows. Let $(k_m)_{m \in \mathbf{N}}$ be the sequence of natural numbers defined in the following way:

$$k_m = \max\{k \in \mathbf{N} \mid \parallel q_k \omega \parallel \geq 1/m\}\,. \tag{2.12}$$

Then

$$\parallel \hat{S}(m,\omega) f \parallel_2^2 \geq c\Big(|f_{q_{k_m}}|^2 q_{k_m+1}^2 \, + \, m^2 |f_{q_{k_m}+1}|^2 \Big) \tag{2.13}$$

and, by the hypothesis on the Fourier coefficients of f,

$$\| \hat{S}(m,\omega)f \|_2 \geq c\Big(q_{k_m}^{-\nu} q_{k_m+1} \,+\, m q_{k_m+1}^{-\nu}\Big)\,. \tag{2.14}$$

Consider now for all $k \in \mathbf{N}$ the functions

$$S_k(x) = q_k^{-\nu} q_{k+1} + x q_{k+1}^{-\nu} \quad \text{and} \quad S_{k,\alpha}(x) = x^{-\alpha} S_k(x)\,, \quad x \in \mathbf{R}^+\,. \tag{2.15}$$

The idea is to find the exponent $\alpha \in (0,1)$ so that all the functions $S_{k,\alpha}$ have a common strictly positive lower bound on $\mathbf{R}^+$. In fact, basic calculus shows that $S_{k,\alpha}$ has a unique minimum on $\mathbf{R}^+$ and that

$$\min S_{k,\alpha} = c_\alpha q_k^{-\nu(1-\alpha)} q_{k+1}^{1-(1+\nu)\alpha}\,. \tag{2.16}$$

where $c_\alpha > 0$ and depends only on $\alpha \in (0,1)$. Since $\omega \in \mathcal{R}_\gamma$, by (2.3),

$$\min S_{k,\alpha} \geq c_\alpha q_{k+1}^{(1-\alpha)(1-\frac{\nu}{1+\gamma})-\nu\alpha}\,. \tag{2.17}$$

Thus, we may choose $\alpha \in (0,1)$ such that

$$(1-\alpha)(1-\frac{\nu}{1+\gamma}) - \nu\alpha = 0 \tag{2.18}$$

thereby obtaining

$$S_{k,\alpha}(x) \geq c_\alpha > 0 \quad \forall x \in \mathbf{R}^+\,. \tag{2.19}$$

A short computation of the exponent given by (2.18) yields (1.10) since, by (2.14) and (2.19),

$$\| \hat{S}(m,\omega)f \|_2 \geq c S_{k_m}(m) \geq c m^\alpha S_{k_m,\alpha}(m) \geq c'_\alpha m^\alpha\,. \tag{2.20}$$

(ii) Consider

$$f(x) = \sum_{k\in\mathbf{N}} \frac{1}{k} q_k^{-s} \exp 2\pi i q_k x. \tag{2.21}$$

Clearly $f \in H^s(\mathbf{S}^1)$. Also, for all $s' > s$, $\exists c_{s'} > 0$ so that $|f_{q_k}| \geq c_{s'}\, q_k^{-s'}$ for all k. The reasoning is now identical to the one in (i), with s' replacing ν. $\square$

We now turn to the proof of Theorem 1.2 which deals with upper bounds along subsequences for the averaged Weyl sums. We start with a lemma.

Lemma 2.2 *Suppose that $\omega \in \mathcal{S}_\gamma$ for some $\gamma > 0$ (see (1.6)). Define, for any $0 < \gamma^- \leq \gamma$, $m_k = [q_k^{1+\gamma^-}/R]$ for $k \in \mathcal{K}$. Then:*

(i) If $f \in H^s(\mathbf{S}^1)$ for some $s \geq 0$, then

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \| f \|_s \left(m_k^{1-\frac{s}{1+\gamma^-}} + m_k^{\frac{1}{1+\gamma^-}} \right) ; \tag{2.22}$$

(ii) If, in particular, $| f_n | \leq C | n |^{-\nu}$, then

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \left(m_k^{1-\frac{\nu-1/2}{1+\gamma^-}} + m_k^{\frac{1}{1+\gamma^-}} \right) . \tag{2.23}$$

Proof. We split the series (2.5) as follows:

$$\| \hat{S}(m, \omega) f \|_2^2 = \sum_{n < q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) + \sum_{n \geq q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) , \tag{2.24}$$

By the definition (2.12) of k_m and by property (2.1),

$$\| n\omega \| \geq \| q_{k_m} \omega \| \geq 1/m \quad \text{for all } n < q_{k_m+1} . \tag{2.25}$$

Hence, (2.6) and (2.2) imply

$$\sum_{n < q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) \leq C \| f \|_0^2 \, q_{k_m+1}^2 . \tag{2.26}$$

Clearly, the estimate obtained in Lemma 2.1 and the bound on the Fourier coefficients of any $f \in H^s(\mathbf{S}^1)$ given by (1.5) yield

$$\sum_{n \geq q_{k_m+1}} |f_n|^2 G_m(\| n\omega \|) \leq C \| f \|_s^2 \; m^2 {q_{k_m+1}}^{-2s} . \tag{2.27}$$

Suppose $\ell \in \mathcal{K}$ is sufficiently large and choose $m_\ell = [q_\ell^{1+\gamma^-}/R]$ with $0 < \gamma^- \leq \gamma$. Then it follows from (2.2) and (1.6) that

$$\| q_\ell \omega \| < \frac{1}{m_\ell} \leq \| q_{\ell-1} \omega \| \; .$$

Hence, by (2.24) – (2.27), and since $k_{m_\ell} = \ell - 1$ (see (2.12)), it follows that

$$\| \hat{S}(m_\ell, \omega) f \|_2 \leq C \| f \|_s \left(m_\ell q_\ell^{-s} + q_\ell \right) ,$$

which yields (2.22). The proof of (2.23) follows by noticing that if the Fourier coefficients of f satisfy the estimates $|f_n| \le Cn^{-\nu}$, $\nu > 1/2$, then $f \in H^{\nu^-}(S^1)$, for all $\nu^- < \nu - 1/2$. $\Box$

Proof of Theorem 1.2. Lemma 2.2 already contains the main part of the proof of Theorem 1.2. In fact, suppose $f \in H^s(\mathbf{S}^1)$. Then, Lemma 2.1 and (2.1) imply

$$\| \hat{S}(m,\omega)f \|_2^2 \le C\Big(\sum_{j<q_{k_m+1}} |f_j|^2 \| j\omega \|^{-2} + m^2 \sum_{j \ge q_{k_m+1}} |f_j|^2 \Big), \tag{2.28}$$

where k_m is defined as in (2.12). The first summand on the right of (2.28) can be estimated, using 2.1, (2.2) and the defining property (1.5) of the Fourier coefficients of $f \in H^s(S^1)$, by

$$\sum_{k \le k_m} \| q_k\omega \|^{-2} \sum_{q_k \le j < q_{k+1}} |f_j|^2 \le C \| f \|_s^2 \sum_{k \le k_m} \frac{q_{k+1}^2}{q_k^{2s}}. \tag{2.29}$$

Assume that the irrational number ω is such that for some $\gamma \ge 0$,

$$\| q_k\omega \| \ge 1/q_k^{1+\gamma}, \tag{2.30}$$

for all but a finite number of $k \in \mathbf{N}$. Hence by (2.2)

$$\sum_{k \le k_m} \frac{q_{k+1}^2}{q_k^{2s}} \le \sum_{k \le k_m} q_{k+1}^{2(1-\frac{s}{1+\gamma})} \le k_m q_{k_m+1}^{2(1-\frac{s}{1+\gamma})}, \tag{2.31}$$

provided $1+\gamma \ge s$, since the sequence $(q_k)_{k\in\mathbf{N}}$ is increasing. We recall also that, since by $q_k \ge 2^{k/2}$ for all $k \in \mathbf{N}$, by its definition (2.12), $k_m \le C \log m$.

The second summand on the right of (2.28) can be estimated by

$$\| f \|_s^2 \, m^2 q_{k_m+1}^{-2s}. \tag{2.32}$$

Finally,

$$\| \hat{S}(m,\omega)f \|_2 \le C \| f \|_s \Big((\log m)^{1/2} q_{k_m+1}^{1-\frac{s}{1+\gamma}} + m q_{k_m+1}^{-s} \Big). \tag{2.33}$$

Take $m_\ell = q_{\ell+1} + q_\ell$. Then it is easily checked that $k_{m_\ell} = \ell$ so that

$$\| \hat{S}(m_\ell,\omega)f \|_2 \le C \| f \|_s \Big((\log m_\ell)^{1/2} m_\ell^{1-\frac{s}{1+\gamma}} + m_\ell^{1-s} \Big). \tag{2.34}$$

Now choose $\gamma = s$. Then

$$1 - \frac{s}{1+\gamma} = \frac{1}{1+s} \quad \text{and} \quad 1 - s < \frac{1}{1+s}. \tag{2.35}$$

Hence,

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \| f \|_s \left((\log m_k)^{1/2} m_k^{\frac{1}{1+s}} \right). \tag{2.36}$$

The above agument settles the case of a rotation number satisfying (2.30) for $\gamma = s$, i.e. sufficiently strongly Diophantine compared to the degree of smoothness of the function $f \in H^s(S^1)$.

Finally, we are left with the case that

$$\| q_k \omega \| < \frac{1}{q_k^{1+s}} , \tag{2.37}$$

for $k \in \mathcal{K}$ an infinite set of natural numbers. We are therefore under the hypothesis of Lemma 2.2 (for $\gamma = s$), which gives, taking $\gamma^- = s$,

$$\| \hat{S}(m_k, \omega) f \|_2 \leq C \| f \|_s \left(m_k^{1 - \frac{s}{1+s}} + m_k^{\frac{1}{1+s}} \right), \tag{2.38}$$

for a suitable choice of the sequence $(m_k)_{k \in \mathbf{N}}$. From (2.36) and (2.38), equation (1.15) now follows. The proof of (1.17) follows from (1.15) by noticing that if the Fourier coefficients of f satisfy the estimates $|f_n| \leq Cn^{-\nu}$, $\nu > 1/2$, then $f \in H^{\nu^-}(S^1)$, for all $\nu^- < \nu - 1/2$. Finally, to prove (1.16) and (1.18), it is enough to use the hypothesis on ω in (2.33). $\square$

Proof of Theorem 1.3. We have

$$\begin{aligned} \| S(m,\omega) f \|_2^2 \ &\geq \ c \sum_{n \in \mathbf{Z}} \frac{1}{|n|^{2\nu}} \frac{\sin^2 \pi m n \omega}{\sin^2 \pi n \omega} \\ &\geq \ c \sum_k \frac{1}{{q_k}^{2\nu}} \frac{\sin^2 \pi m \| q_k \omega \|}{\sin^2 \pi \| q_k \omega \|} . \end{aligned}$$

Let $k'_m = \min\{k \mid \| q_k \omega \| < \frac{1}{2m}\}$, then

$$\| S(m,\omega) f \|_2 \geq C \ m \ q_{k'_m}^{-\nu}.$$

But, since $\| q_{k'_m-1}\omega \| \geq \frac{1}{2m}$, (2.2) implies $2m \geq q_{k'_m}$, so that

$$\| S(m,\omega)f \|_2 \geq C_\nu \, m^{1-\nu},$$

proving the result. $\square$

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