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Renormalization of quantum field theory on curved space-times, a causal approach.

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Introduction.

In this thesis, we study and solve the problem of the renormalization of a perturbative quantum field theory of interacting scalar fields on curved space times following the causal approach.

Quantum field theory is one of the greatest and most successfull achievements of modern physics, since its numerical predictions are probed by experiments with incredible accuracy. Furthermore, QFT can be applied to many fields ranging from condensed matter theory, solid state physics to particle physics. One of the greatest challenges for modern mathematical physics is to unify quantum field theory and Einstein's general relativity. This program seems today out of reach, however we can address the more recent question to first try to **define and construct** quantum field theory on curved Lorentzian space times. This problem was solved in the groundbreaking work of Brunetti and Fredenhagen [26] in 2000.

Their work was motivated by the observation that both the conventional axiomatic approach to quantum field theory following Wightman's axioms or the usual textbook approach in momentum space failed to be generalized to curved space-times for several obvious reasons:
- there is no Fourier transform on curved space time
- the space time is no longer Lorentz invariant.

Indeed, the starting point of the work [26] was to follow one of the very first approach to QFT due to Stueckelberg, which is based on the concept of causality.

The ideas of Stueckelberg were first understood and developed by Bogoliubov ([7]) and then by Epstein-Glaser ([21], [22]) (on flat space time). In these approaches, one works directly in spacetime and the renormalization is formulated as a problem of extension of distributions. Somehow, this point of view based on the S-matrix formulation of QFT was almost completely forgotten by people working on QFT at the exception of few people as e.g. Stora, Kay, Wald who made important contributions to the topic ([57],[71]). However, in 1996, a student of Wightman, M. Radzikowsky revived the subject. In his thesis, he used microlocal analysis for the first time in this context and introduced the concept of *microlocal spectrum condition*, a condition on the wave front set of the distributional two-point function which represents the quantum states, which characterizes the quantum states of positive energy (named Hadamard states) on curved space times. In 2000, in a breakthrough paper, Brunetti and Fredenhagen were able to generalize the Epstein-Glaser theory on curved space times by relying on the fundamental contribution of Radzikowski. These results were further extended by Fredenhagen, Brunetti, Hollands, Wald, Rejzner, etc. to Yang-Mills fields and the gravitation.

Let us first explain what do we mean by “a quantum field theory”.

The input data of a quantum field theory. Our data are a smooth globally hyperbolic oriented and time oriented manifold (M, g) and an algebra bundle $\underline{H}$ (called bundle of local fields) over M. Smooth sections of $\underline{H}$ represent polynomials of the scalar fields with coefficients in $C^\infty(M)$. $\underline{H}$ has in fact the structure of a Hopf algebra bundle, i.e. a vector bundle the fibers of which are Hopf algebras. The natural causality structure on M induces a natural partial order relation for elements of M: $x \leqslant y$ if y lives in the causal future of x. The metric g gives a natural d'Alembertian operator $\Box$ and we choose some distribution $\Delta_+ \in \mathcal{D}'(M^2)$ in such a way that:

- the distribution Δ_+ is a bisolution of $\Box$,
- the wave front set and the singularity of Δ_+ satisfy some specific constraints (actually, $WF(\Delta_+)$ satisfies the microlocal spectrum condition).

From the input data to modules living on configuration spaces and the $\star$ product. For each finite subset I of the integers, we define the configuration space M^I as the set of maps from I to M figuring a cluster of points in M labelled by indices of I. From the algebra bundle $\underline{H}$, we construct a natural infinite collection of $C^\infty(M^I)$-modules $(\mathcal{H}^I)_I$ (each $\mathcal{H}^I$ containing products of fields at points labelled by I) and define a collection of subspaces $(V^I)_I$ of distributions on M^I indexed by finite subsets I of $\mathbb{N}$ (each V^I contains the Feynman amplitudes). The collections $(M^I)_I, (\mathcal{H}^I)_I, (V^I)_I$ enjoy the following simple property: for each inclusion of finite sets of integers $I \subset J$ we have a corresponding projection $M^J \mapsto M^I$ and inclusions $\mathcal{H}^I \hookrightarrow \mathcal{H}^J$, $V^I \hookrightarrow V^J$. We can define a product $\star$ (“operator product of fields”), which to a pair of elements A, B in a subset of $\left(\mathcal{H}^I \otimes_{C^\infty(M^I)} V^I\right) \times \left(\mathcal{H}^J \otimes_{C^\infty(M^J)} V^J\right)$ where I, J are disjoint finite subsets of $\mathbb{N}$, assigns an element in $\mathcal{H}^{I\cup J} \otimes_{C^\infty(M^{I\cup J})} V^{I\cup J}$. The product $\star$ is defined by some combinatorial formula (which translates the “Wick theorem” and is equivalent to a Feynman diagrammatic expansion) which involves powers of Δ_+. The partial order on M induces a partial order $\leqslant$ between elements A, B in $\mathcal{H}^I \times \mathcal{H}^J$ for all I, J.

The definition of a quantum field theory. A quantum field theory is a collection T_I of morphisms of $C^\infty(M^I)$-modules:

$$T_I : \mathcal{H}^I \otimes_{C^\infty(M^I)} V^I \mapsto \mathcal{H}^I \otimes_{C^\infty(M^I)} V^I,$$

which satisfies the following axioms

1. $\forall |I| \leqslant 1, T_I$ is the identity map,
2. *the Wick expansion property* which generalizes the Wick theorem,
3. **the causality equation** which reads $\forall A, B$ s.t. $B \not\leq A$

$$T(AB) = T(A) \star T(B). \quad (1)$$

The maps T_I can be interpreted as the time ordering operation of Dyson. The main problem is to find a solution of the equation (1). This solution turns out to be non unique, actually all solutions of this equation are related by the renormalization group of Bogoliubov ([7],[10]).

Renormalization as the problem of making sense of the above definition. We denote by d_n the thin diagonal in M^n corresponding to n points collapsing over one point. From the previous axioms, we prove that $T_n|_{M^n\setminus d_n}$ is a linear combination of products of $T_I, I \subsetneq \{1, \cdots, n\}$ with coefficients in $C^\infty(M^n \setminus d_n)$. So we encounter two problems:
1) Since the elements T_I are $\mathcal{H}$-valued distributions, we must justify that these products of distributions make sense in $M^n \setminus d_n$.
2) Even if the product makes sense T_n is still not defined over M^n, thus we must extend T_n on M^n.

Contents of the Thesis. In Chapter 1, we address the second of the previous questions of defining T_n on M^n, which amounts to extend a distribution t defined on $M \setminus I$ where M is a smooth manifold and I is a closed embedded submanifold. We give a geometric definition of scaling transversally to the submanifold I and of a weak homogeneity which are completely intrinsic (i.e. they do not depend on the choice of local charts). Our definition of weak homogeneity follows [54] and [53] and slightly differs from the definition of [26] which uses the Steinman scaling degree. We prove that if a distribution t is in $\mathcal{D}'(M \setminus I)$ and is weakly homogeneous of degree s then it has an extension $\bar{t} \in \mathcal{D}'(M)$ which is weakly homogeneous of degree s' for all $s' < s$. The extension sometimes requires a renormalization which is a subtraction of distributions supported on I i.e. local counterterms. The main difference with the work [26] is that we only have one definition of weak homogeneity and we use a continuous partition of unity. This chapter does not rely on microlocal analysis.

In Chapter 2, in order to solve the first problem of defining T_n, we must explain why the product of the T_I's in the formula which gives T_n makes sense and this is possible under some specific conditions on the wave front sets of the coefficients of the T_I's. So we are led to study the wave front sets of the extended distributions defined in Chapter 1. We find a geometric condition on $WF(t)$ named *soft landing condition* which ensures that the wave front of the extension is controlled. However this geometric condition is not sufficient and we explain this by a counterexample. We also give a geometric definition of local counterterms associated to a distribution t, which generalizes the counterterms of QFT textbooks in the context of curved space times. We show that the soft landing condition is equivalent to the fact that the local counterterms of t are smooth functions multiplied by distributions localized on the diagonal, i.e. they have a specific structure of finitely generated module over the ring $C^\infty(I)$. The new features of this Chapter are the soft landing condition which does not exist in the literature (only implicit in [26]), the definition of local counterterms associated to t and our theorem which proves that under certain conditions local counterterms are conormal distributions. Finally, our counterexample explains why in [26], the authors impose certain microlocal conditions on the unextended distribution t in order to control the wave front set of the extension.

In chapter 3, we prove that if we add one supplementary boundedness condition on t i.e. if t is weakly homogeneous in some topological space of distributions with prescribed wave front set, then the wave front $WF(\bar{t})$ of the extension is contained in the smallest possible set which is the union of the closure of the wave front of the unextended distribution $\overline{WF(t)}$ with the conormal C of I. Chapter 3 differs from [26] by the fact that we estimate $\overline{WF(t)}$ also in the case of renormalization with counterterms and our proof is much more detailed.

In chapter 4, we manage to prove that the conditions of Chapter 3 can be made completely geometric and coordinate invariant. We also prove the boundednes of the product and the pull-back operations on distributions in suitable microlocal topologies. Then we conclude Chapter 4 with the following theorem: if t is microlocally weakly homogeneous of degree $s \in \mathbb{R}$ then a "microlocal extension" $\bar{t}$ exists with minimal wave front set in $\overline{WF(t)} \cup C$ and $\bar{t}$ is microlocally weakly homogeneous of degree s' for all $s' < s$. Chapter 4 improves the results of Hörmander on products and pull-back of distributions since we prove that these operators are bounded maps for the suitable microlocal topologies. This seems to be a new result since in the literature only the sequential continuity of products and pull-back are proved.

In Chapter 5, we construct the two point function Δ_+ which is a distributional solution of the wave equation on M. We prove that $WF(\Delta_+)$ satisfies the microlocal spectrum condition of Radzikowski and finally we establish that Δ_+ is "microlocally weakly homogeneous" of degree -2. Chapter 5 contains a complete mathematical justification of the Wick rotation for which an explicit reference is missing although the idea of its proof is sketched in [74]. We also explicitly compute the wave front set of the holomorphic family $Q^s(\cdot + i0\theta)$ which cannot be found in [40], (we only found a computation of the **analytic** wave front set –in the sense of Sato-Kawai-Kashiwara– of $Q^s(\cdot + i0\theta)$ in [46] p. 90 example 2.4.3). Finally, our proof that the wave front set of Δ_+ (constructed as a perturbative series à la Hadamard) satisfies the microlocal spectrum condition seems to be missing in the literature. The construction appearing in [29] is not sufficient to prove that Δ_+ is microlocally weakly homogeneous of degree -2.

Chapter 6 is the final piece of this building. Inspired by the work of Borcherds, we quickly give our definition of a quantum field theory using the convenient language of Hopf algebras then we state the problem of defining a quantum field theory as equivalent to the problem of solving the equation (1) in T recursively in n on all configuration spaces M^n. We prove this recursively using all tools developed in the previous chapters, a careful partition of the configuration space generalizing ideas of R. Stora to the case of curved space times and an idea of polarization of wave front sets which translates microlocally the idea of positivity of energy.

Chapter 7 solves a conjecture of Bennequin and gives a nice geometric interpretation of the wave front set of any Feynman amplitude:

- it is parametrized by a Morse family,
- it is a union of smooth Lagrangian submanifolds of the cotangent space of configuration space.

In Chapter 8, which can be read independently of the rest except Chapter 1, using the language of currents, we treat the problem of preservation of symmetries by the extension procedure. Indeed, renormalization can break the symmetries of the unrenormalized objects and the fact that renormalization does not commute with the action of vector fields from some Lie algebra of symmetries is called anomaly and is measured by the appearance of local counterterms, which are far reaching generalizations of the notion of residues coming from algebraic geometry, (but generalized here to the current theoretic setting).

Finally, in chapter 9 we revisit the extension problem from the point of view of meromorphic regularization. We prove that under certain conditions on distributions, they can be meromorphically regularized then the extension consists in a subtraction of poles which are also local counterterms. To conclude this last Chapter, we introduce a lenght scale ℓ in the meromorphic renormalization and we prove that scaling in ℓ only gives polynomial divergences in $\log \ell$.

Chapter 1

The extension of distributions.

1.1 Introduction.

In the Stueckelberg ([72]) approach to quantum field theory, renormalization was formulated as a problem of division of distributions. For Epstein–Glaser ([21], [22]) , Stora ([57],[71]), and implicitly in Bogoliubov ([7]), it was formulated as a problem of extension of distributions, the latter approach is more general since the ambiguity of the extension is described by the renormalization group. This procedure was implemented on arbitrary manifolds (hence for curved Lorentzian spacetimes) by Brunetti and Fredenhagen in their groundbreaking paper of 2000 [26]. However, in the mathematical literature, the problem of extension of distributions goes back at least to the work of Hadamard and Riesz on hyperbolic equations ([63],[35]). It became a central argument for the proof of a conjecture of Laurent Schwartz ([65] p. 126,[49]): the problem was to find a fundamental solution E for a linear PDE with constant coefficients in $\mathbb{R}^n$, which means solving the equation $PE = \delta$ in the distributional sense. By Fourier transform, this is equivalent to the problem of extending $\widehat{P}^{-1}$ which is a honest smooth function on $\mathbb{R}^n \setminus \{\widehat{P} = 0\}$ as a distribution on $\mathbb{R}^n$, in such a way that $\widehat{P}\widehat{P}^{-1} = 1$ which makes the division a particular case of an extension. This problem set by Schwartz was solved positively by Lojasiewicz and Hörmander ([40],[68]). Recently, the more general extension problem was revisited in mathematics by Yves Meyer in his wonderful book [53]. In [53], Yves Meyer also explored some deep relations between the extension problem and Harmonic analysis (Littlewood–Paley and Wavelet decomposition). The extension problem was solved in [53] on $(\mathbb{R}^n \setminus \{0\})$. For the need of quantum field theory, we will extend his method to manifolds. In order to renormalize, one should find some way of measuring the wildness of the singularities of distributions. Indeed, we need to impose some growth condition on distributions because distributions cannot be extended in general! We estimate the wildness of the singularity by first defining an adequate notion of scaling with respect to a closed embedded submanifold I of a given manifold M, as done by Brunetti–Fredenhagen [26]. On $\mathbb{R}^{n+d}$ viewed as the cartesian product $\mathbb{R}^n \times \mathbb{R}^d$, the scaling is clearly defined by homotheties in the variables corresponding to the second factor $\mathbb{R}^d$. We adapt the definition of Meyer [53] in these variables and define the space of weakly homogeneous distributions of degree s which we call E_s.

We are able to represent all elements of E_s which are defined on $M \setminus I$ through a decomposition formula by a family $\left(u^\lambda\right)_{\lambda\in(0,1]}$ satisfying some specific hypothesis. The distributions $\left(u^\lambda\right)_{\lambda\in(0,1]}$ are the building blocks of the E_s and are the key for the renormalization. We establish the following correspondence

$$\left(u^\lambda\right)_{\lambda\in(0,1]} \longmapsto \int_0^1 \frac{d\lambda}{\lambda}\lambda^s (u^\lambda)_{\lambda^{-1}} + \text{nice terms}, \tag{1.1}$$

$$t \in E_s \longmapsto \left(u^\lambda\right)_{\lambda\in(0,1]} \text{ where } u^\lambda = \lambda^{-s} t_\lambda \psi, \tag{1.2}$$

the nice terms are distributions supported on the complement of I.

However this scaling is only defined in local charts and a scaling around a submanifold I in a manifold M depends on the choice of an Euler vector field. Thus we propose a geometrical definition of a class of Euler vector fields: to any closed embedded submanifold $I \subset M$, we associate the **ideal** $\mathcal{I}$ of smooth functions vanishing on I. A vector field ρ is called Euler vector field if

$$\forall f \in \mathcal{I}, \rho f - f \in \mathcal{I}^2. \tag{1.3}$$

This definition is clearly intrinsic. We prove that all scalings are equivalent hence all spaces of weakly homogeneous distributions are equivalent and that our definitions are in fact independent of the choice of Euler vector fields. Actually, we prove that all Euler vector fields are locally conjugate by a local diffeomorphism which fixes the submanifold I. So it is enough to study both E_s and the extension problem in a local chart. Meyer and Brunetti–Fredenhagen make use of a dyadic decomposition. We use instead a **continuous partition of unity** which is a continuous analog of the Littlewood–Paley decomposition. The continuous partition of unity has many advantages over the discrete approaches: 1) it provides a direct connection with the theory of Mellin transform, which allows to easily define meromorphic regularizations; 2) it gives elegant formulas especially for the poles and residues appearing in the meromorphic regularization (see Chapter 7); 3) it is well suited to the study of anomalies (see Chapter 6).

Relationship with other work. In Brunetti–Fredenhagen [26], the scaling around manifolds was also defined but they used two different definitions of scalings, then they showed that these actually coincide, whereas we only give one definition which is geometric. In mathematics, we also found some interesting work by Kashiwara–Kawai, where the concept of weak homogeneity was also defined ([54] Definition (1.1) p. 22).

1.2 Extension and renormalization.

1.2.1 Notation, definitions.

We work in $\mathbb{R}^{n+d}$ with coordinates (x,h), $I=\mathbb{R}^n\times\{0\}$ is the linear subspace $\{h=0\}$. For any open set $U\subset\mathbb{R}^{n+d}$, we denote by $\mathcal{D}(U)$ the space of test functions supported on U and for all compact $K\subset U$, we denote by $\mathcal{D}_K(U)$ the subset of all test functions in $\mathcal{D}(U)$ supported on K. We also use the seminorms:

$$\forall\varphi\in\mathcal{D}(\mathbb{R}^{n+d}),\pi_k(\varphi):=\sup_{|\alpha|\leqslant k}\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^{n+d})},$$

$$\forall\varphi\in C^\infty(\mathbb{R}^{n+d}),\forall K\subset\mathbb{R}^d,\pi_{k,K}(\varphi):=\sup_{|\alpha|\leqslant k}\sup_{x\in K}|\partial^\alpha\varphi(x)|.$$

We denote by $\mathcal{D}'(U)$ the space of distributions defined on U. The duality pairing between a distribution t and a test function φ is denoted by $\langle t,\varphi\rangle$. For a function, we define $\varphi_\lambda(x,h)=\varphi(x,\lambda h)$. For the vector field $\rho=h^j\frac{\partial}{\partial h^j}$, the following formula

$$\varphi_\lambda=e^{(\log\lambda)\rho\star}\varphi,$$

shows the relation between ρ and the scaling. Once we have defined the scaling for test functions, for any distribution f, we define the scaled distribution f_λ:

$$\forall\varphi\in\mathcal{D}(\mathbb{R}^{n+d}),\langle f_\lambda,\varphi\rangle=\lambda^{-d}\langle f,\varphi_{\lambda^{-1}}\rangle.$$

If f were a function, this definition would coincides with the naive scaling $f_\lambda(x,h)=f(x,\lambda h)$.

We give a definition of weakly homogeneous distributions in flat space following [53]. We call a subset $U\subset\mathbb{R}^{n+d}$ ρ-convex if $(x,h)\in U\implies\forall\lambda\in(0,1],(x,\lambda h)\in U$. We insist on the fact that since we pick $\lambda>0$, a ρ-convex domain may have *empty intersection* with I.

Definition 1.2.1 *Let U be an arbitrary ρ-convex open subset of $\mathbb{R}^{n+d}$. $E_s(U)$ is defined as the space of distributions t such that $t\in\mathcal{D}'(U)$ and*

$$\forall\varphi\in\mathcal{D}(U),\exists C(\varphi),\sup_{\lambda\in(0,1]}|\lambda^{-s}\langle t_\lambda,\varphi\rangle|\leqslant C(\varphi).$$

In the quantum field theory litterature, the wildness of distributions is measured by the Steinman scaling degree. We prefer the definition of Meyer,

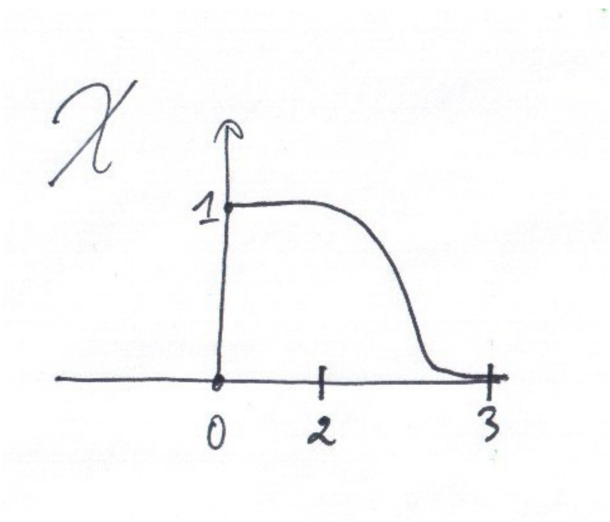

Figure 1.1: The function χ of Littlewood–Paley theory.

which exploits the properties of bounded sets in the space of distributions (this is related to bornological properties of $\mathcal{D}'(U)$).

We denote by $\frac{d\lambda}{\lambda}$ the multiplicative measure on $[0,1]$. We shall now give a definition of a class of maps $\lambda \mapsto u^\lambda$ with value in the space of distributions.

Definition 1.2.2 *For all* $1 \leqslant p \leqslant \infty$*, we define* $L^p_{\frac{d\lambda}{\lambda}}([0,1],\mathcal{D}'(U))$ *as the space of families* $(u^\lambda)_{\lambda\in(0,1]}$ *of distributions such that*

$$\forall\varphi\in\mathcal{D}(U), \lambda\mapsto\left\langle u^\lambda,\varphi\right\rangle\in L^p_{\frac{d\lambda}{\lambda}}([0,1],\mathbb{C}). \tag{1.4}$$

The Hörmander trick. We recall here the basic idea of Littlewood–Paley analysis ([53] p. 14). Pick a function χ which depends only on h such that $\chi = 1$ when $|h| \leqslant 2$ and $\chi = 0$ for $|h| \geqslant 3$. The Littlewood–Paley function $\psi(\cdot) = \chi(\cdot) - \chi(2\cdot)$ is supported on the annulus $1 \leqslant |h| \leqslant 3$. Then the idea is to rewrite the plateau function χ using the trick of the telescopic series

$$\chi = \chi(\cdot) - \chi(2\cdot) + \cdots + \chi(2^j\cdot) - \chi(2^{j+1}\cdot) + \cdots$$

and deduce a dyadic partition of unity

$$1 = (1-\chi) + \sum_{j=0}^{\infty}\psi(2^j.)$$

Our goal in this paragraph is to derive a continuous analog of the dyadic partition of unity. Let $\chi \in C^\infty(\mathbb{R}^{n+d})$ such that $\chi = 1$ in a neighborhood N_1 of I and χ vanishes outside a neighborhood N_2 of N_1. This implies χ satisfies

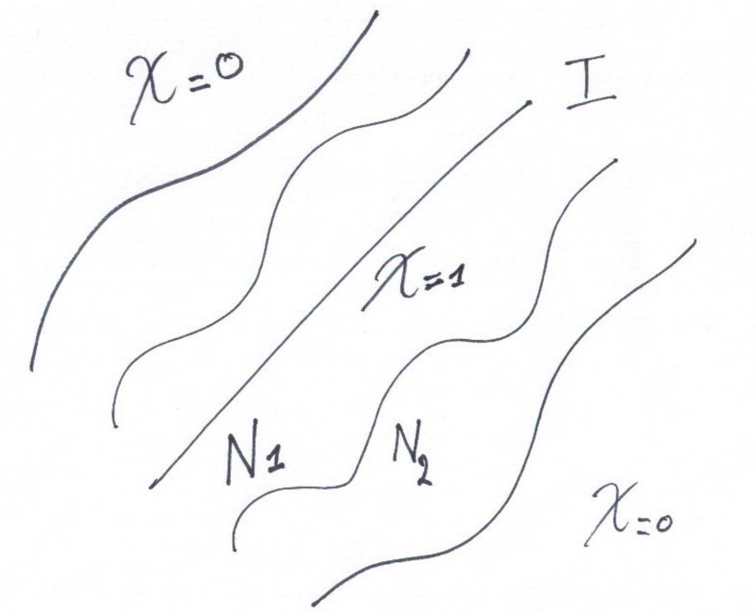

Figure 1.2: The neighborhoods N_1 and N_2.

the following constraint: for all compact set $K \subset \mathbb{R}^n, \exists (a,b) \in \mathbb{R}^2$ such that $b > a > 0$ and $\chi|_{(K\times\mathbb{R}^d)\cap\{|h|\leqslant a\}} = 1, \chi|_{(K\times\mathbb{R}^d)\cap\{|h|\geqslant b\}} = 0$. We find a convenient formula (inspired by [41] equation (8.5.1) p. 200 and [53] Formula (5.6) p. 28) for χ as an integral over a scale space indexed by $\lambda \in (0,1]$. First notice that $\chi(x, \frac{h}{\lambda}) \rightarrow_{\lambda\rightarrow 0} 0$ in L^1_{loc}. We repeat the Littlewood Paley trick in the continuous setting:

$$\chi(x,h) = \chi(x,h) - 0 = \int_0^1 \frac{d\lambda}{\lambda}\lambda\frac{d}{d\lambda}\left[\chi(x,\lambda^{-1}h)\right] = \int_0^1 \frac{d\lambda}{\lambda}\left(-\rho\chi\right)(x,\lambda^{-1}h)$$

Set

$$\psi = -\rho\chi. \tag{1.5}$$

Notice an important property of ψ: on each compact set $K \subset \mathbb{R}^n$, $\exists (a,b) \in \mathbb{R}^2$ such that $\psi|_{(K\times\mathbb{R}^d)}$ is supported on the annulus $(K\times\mathbb{R}^d)\cap\{a \leqslant |h| \leqslant b\}$. We obtain the formula

$$1 = (1-\chi) + \int_0^1 \frac{d\lambda}{\lambda}\psi_{\lambda^{-1}}, \tag{1.6}$$

which for the moment only has a heuristic meaning. The next proposition gives a precise meaning to the heuristic formula and gives a candidate formula for the extension problem.

Proposition 1.2.1 *Let $\chi \in C^\infty(\mathbb{R}^{n+d})$ such that $\chi = 1$ in a neighborhood N_1 of I and χ vanishes outside a neighborhood N_2 of N_1 and let $\psi = -\rho\chi$.*

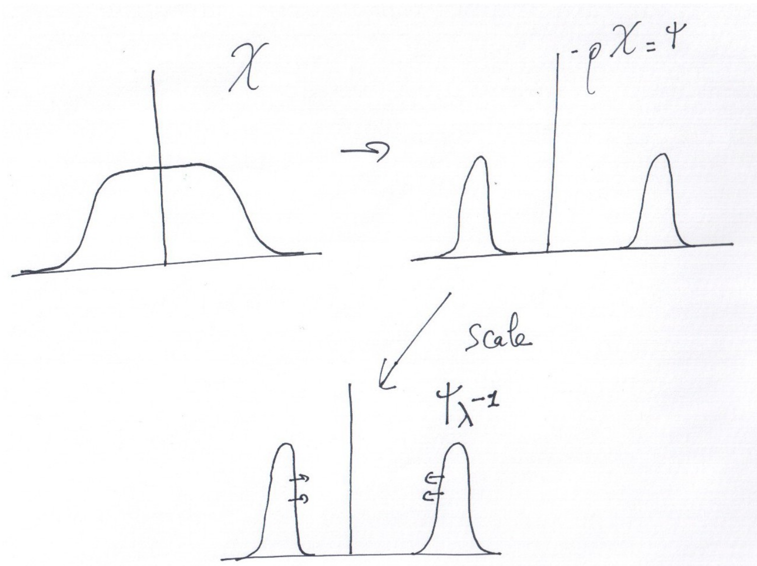

Figure 1.3: The function χ, the function ψ and the scaled $\psi_{\lambda^{-1}}$.

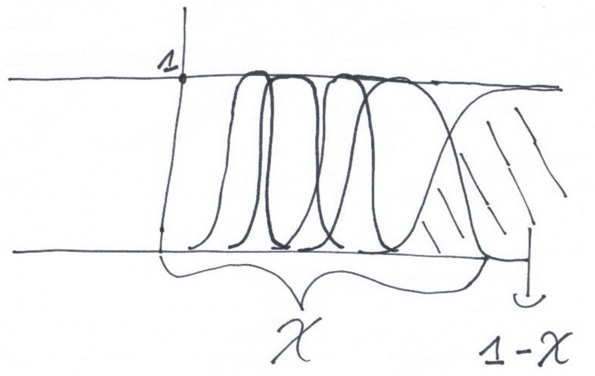

Figure 1.4: Partition of unity.

Then for all $\varphi \in \mathcal{D}(\mathbb{R}^{n+d})$ *such that* $\varphi = 0$ *in a neighborhood of* $I = \{h = 0\}$*, we find*

$$\langle t, \varphi \rangle = \int_0^1 \frac{d\lambda}{\lambda} \langle t\psi_{\lambda^{-1}}, \varphi \rangle + \langle t, (1-\chi)\varphi \rangle \,. \tag{1.7}$$

The formula $t = \int_0^1 \frac{d\lambda}{\lambda} \langle t\psi_{\lambda^{-1}}, \varphi \rangle + \langle t, (1-\chi)\varphi \rangle$ was inspired by Formula (5.8), (5.9) in [53] p. 28.

Proof — Let $\delta > 0$ such that $\varphi = 0$ when $|h| \leqslant \delta$. We can find $0 < a < b$ such that $[|h| > b \implies \chi = 0]$ and $[|h| > b \implies -\rho\chi = \psi = 0]$. Hence $\text{supp } \psi(x, \frac{h}{\lambda}) \subset \{|h| \leqslant \lambda b\}$ which implies $\forall \lambda \leqslant \frac{\delta}{b}, \varphi(x,h)\psi(x,\frac{h}{\lambda}) = 0$. We have the relation $\varphi = \varphi(1-\chi) + \varphi\chi = \int_{\frac{\delta}{b}}^1 \frac{d\lambda}{\lambda}\psi_{\lambda^{-1}}\varphi + \varphi(1-\chi)$ where the integral is well defined, we thus deduce $\forall \varepsilon \in [0, \frac{\delta}{b}]$

$$\varphi\chi = \int_\varepsilon^1 \frac{d\lambda}{\lambda} \underbrace{\psi_{\lambda^{-1}}\varphi}_{=0 \text{ for } \lambda \in [\varepsilon, \frac{\delta}{b}]} = \int_{\frac{\delta}{b}}^1 \frac{d\lambda}{\lambda}\psi_{\lambda^{-1}}\varphi$$

where the product makes perfect sense as a product of smooth functions, hence

$$\langle t\chi, \varphi \rangle = \langle t, \chi\varphi \rangle = \left\langle t, \int_\varepsilon^1 \frac{d\lambda}{\lambda}\psi_{\lambda^{-1}}\varphi \right\rangle = \int_\varepsilon^1 \frac{d\lambda}{\lambda} \left\langle t\psi(\frac{h}{\lambda}), \varphi \right\rangle$$

$$= \int_{\frac{\delta}{b}}^1 \frac{d\lambda}{\lambda} \left\langle t\psi(\frac{h}{\lambda}), \varphi \right\rangle = \int_0^1 \frac{d\lambda}{\lambda} \left\langle t\psi(\frac{h}{\lambda}), \varphi \right\rangle$$

where we can safely interchange the integral and the duality bracket. ■

Another interpretation of the Hörmander formula. The Hörmander formula gives a convenient way to write $\chi - \chi_{\varepsilon^{-1}}$.

$$\chi - \chi_{\varepsilon^{-1}} = \int_\varepsilon^1 \frac{d\lambda}{\lambda}\psi_{\lambda^{-1}}$$

then noticing that when $\varepsilon > 0$, for all $\lambda \in [\varepsilon, 1]$, $\psi_{\lambda^{-1}}$ is supported on the complement of a neighborhood of I, this implies that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^{n+d})$, for all $\varepsilon > 0$, we have the nice identity:

$$\int_\varepsilon^1 \frac{d\lambda}{\lambda} \langle t\psi_{\lambda^{-1}}, \varphi \rangle = \langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi \rangle \,.$$

Now if the function $\langle t\psi_{\lambda^{-1}}, \varphi \rangle$ is integrable on $[0,1]$ w.r.t. the measure $\frac{d\lambda}{\lambda}$, the **existence** of the integral $\int_0^1 \frac{d\lambda}{\lambda} \langle t\psi_{\lambda^{-1}}, \varphi \rangle$ will **imply** that the **limit**

$$\lim_{\varepsilon \to 0} \langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi \rangle \tag{1.8}$$

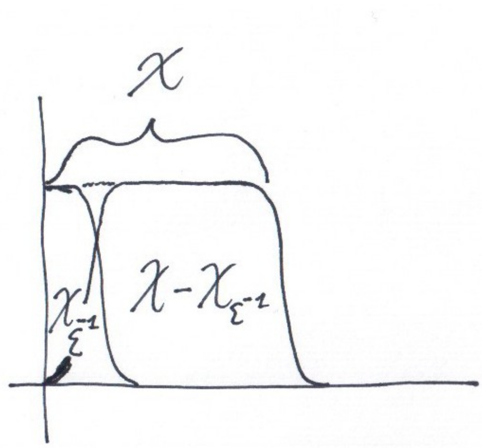

Figure 1.5: $\chi - \chi_{\varepsilon^{-1}}$.

exists. In the next sections, we prove that when the distribution t is in E_s for $s+d>0$, the integral formula $\int_\varepsilon^1 \frac{d\lambda}{\lambda}\langle t\psi_{\lambda^{-1}},\varphi\rangle$ converges when $\varepsilon\rightarrow 0$. Thus the limit (1.8) exists. However, when $t\in E_s$ when $s+d<0$, we must modify the formula $\int_\varepsilon^1 \frac{d\lambda}{\lambda}\langle t\psi_{\lambda^{-1}},\varphi\rangle$, which is divergent when $\varepsilon\rightarrow 0$, by subtracting a **local counterterm** $\langle c_\varepsilon,\varphi\rangle$ where $(c_\varepsilon)_\varepsilon$ is a family of distribution **supported** on I such that the limit

$$\lim_{\varepsilon\rightarrow 0}\left(\langle t\left(\chi-\chi_{\varepsilon^{-1}}\right),\varphi\rangle - \langle c_\varepsilon,\varphi\rangle\right), \tag{1.9}$$

makes sense. Notice that the renormalization does not affect the original distribution t on $M\setminus I$ since c_ε is supported on I.

1.2.2 From bounded families to weakly homogeneous distributions.

We construct an algorithm which starts from an arbitrary family of bounded distributions $(u^\lambda)_{\lambda\in(0,1]}$ supported on some annular domain, and builds a weakly homogeneous distribution of degree s. Actually, any distribution which is weakly homogeneous of degree s can be reconstructed from our algorithm as we will see in the next section. This is the key remark which allows us to solve the problem of extension of distributions. In this part, we make essential use of the Banach Steinhaus theorem on the dual of a Fréchet space recalled in appendix. We use the notation $t_\lambda(x,h)=t(x,\lambda h)$ and U is a ρ-convex open subset in $\mathbb{R}^{n+d}$.

Definition 1.2.3 *A family of distributions $(u^\lambda)_{\lambda\in(0,1]}$ is called uniformly supported on an annulus domain of U if for all compact set $K\subset\mathbb{R}^n$, there* *exists* $0 < a < b$ *such that* $\forall\lambda, u^\lambda|_{(K\times\mathbb{R}^d)\cap U}$ *is supported in a fixed annulus* $\{(x,h)|x\in K, a\leqslant |h|\leqslant b\}\cap U$.

The structure theorem gives us an algorithm to construct distributions in $E_s(U)$ given any family of distributions $(u^\lambda)_{\lambda\in(0,1]}$ bounded in $\mathcal{D}'(U\setminus I)$ and uniformly supported on an annulus domain of U.

Lemma 1.2.1 *Let* $(u^\lambda)_{\lambda\in(0,1]}$ *be a bounded family in* $\mathcal{D}'(U\setminus I)$ *which is uniformly supported on an annulus domain of* U*. Then the family* $\left(\lambda^{-d}u^\lambda_{\lambda^{-1}}\right)_{\lambda\in(0,1]}$ *is bounded in* $\mathcal{D}'(U)$.

Proof — If the family $(u^\lambda)_{\lambda\in(0,1]}$ is uniformly supported on an annulus domain of U, then for all compact set $K\subset\mathbb{R}^n$, there exists $0<a<b$ such that $\forall\lambda, u^\lambda|_{(K\times\mathbb{R}^d)\cap U}$ is supported in a fixed annulus $\mathcal{A}=\{a\leqslant|h|\leqslant b\}\cap((K\times\mathbb{R}^d)\cap U)$. If $u^\lambda|_{(K\times\mathbb{R}^d)\cap U}$ is a bounded family of distributions supported on the **fixed annulus** $\mathcal{A}=\underbrace{\{a\leqslant|h|\leqslant b\}\cap(K\times\mathbb{R}^d)\cap U}_{\text{compact in }\mathbb{R}^{n+d}}$, then the family u^λ satisfies the following estimate by Banach Steinhaus:

$$\forall K'\subset\mathbb{R}^{n+d}\text{compact}, \exists(k,C), \forall\varphi\in\mathcal{D}_{K'}(U), \sup_{\lambda\in(0,1]}|\left\langle u^\lambda,\varphi\right\rangle|\leqslant C\pi_k(\varphi),$$

and we notice that the estimate is still valid for test functions in $C^\infty((K\times\mathbb{R}^d)\cap U)$ (by compactness of $\mathcal{A}$):

$$\exists(k,C),\forall\varphi\in C^\infty((K\times\mathbb{R}^d)\cap U), \sup_{\lambda\in(0,1]}|\left\langle u^\lambda,\varphi\right\rangle|\leqslant C\pi_{k\mathcal{A}}(\varphi), \quad (1.10)$$

because u^λ is compactly supported in the h variables and φ is compactly supported in the x variables. For any test function $\varphi\in\mathcal{D}(U)$:

$$\lambda^{-d}|\left\langle u^\lambda_{\lambda^{-1}},\varphi\right\rangle| = \lambda^{-d}\lambda^d|\left\langle u^\lambda,\varphi(.,\lambda.)\right\rangle|\leqslant C\pi_{k\mathcal{A}}(\varphi_\lambda)$$

thus

$$\lambda^{-d}|\left\langle u^\lambda_{\lambda^{-1}},\varphi\right\rangle|\leqslant C\pi_k(\varphi) \quad (1.11)$$

because of the estimate (1.10) on the family $(u^\lambda)_\lambda$. This proves that the family $\left(\lambda^{-d}u^\lambda_{\lambda^{-1}}\right)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'(U\setminus I)$. ■

Corollary 1.2.1 *Let* $(u^\lambda)_{\lambda\in(0,1]}$ *be a bounded family in* $\mathcal{D}'(U\setminus I)$ *which is uniformly supported on an annulus domain of* U*. If* $s+d>0$*, then the integral*

$$\int_0^1\frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}} \quad (1.12)$$

converges in $\mathcal{D}'(U)$.

Proof — When $s+d>0$, $\lambda\mapsto\lambda^s u^\lambda_{\lambda^{-1}} = \underbrace{\lambda^{s+d}}_{\text{integrable}}\underbrace{\lambda^{-d}u^\lambda_{\lambda^{-1}}}_{\text{bounded}} \in L^1_{\frac{d\lambda}{\lambda}}([0,1],\mathcal{D}'(U\setminus I))$ and the integral $t=\int_0^1\frac{d\lambda}{\lambda}\lambda^{s+d}\lambda^{-d}u^\lambda_{\lambda^{-1}}$ converges in $L^1_{\frac{d\lambda}{\lambda}}([0,1],\mathcal{D}'(U\setminus I))$! By the estimate (1.11) on the bounded family $\lambda^{-d}u^\lambda_{\lambda^{-1}}$, we also have the estimate:

$$|\left\langle t,\varphi\right\rangle| = |\int_0^1\frac{d\lambda}{\lambda}\lambda^s\left\langle u^\lambda_{\lambda^{-1}},\varphi\right\rangle|$$

$$\leqslant\int_0^1\frac{d\lambda}{\lambda}\lambda^{s+d}\underbrace{|\lambda^{-d}\left\langle u^\lambda_{\lambda^{-1}},\varphi\right\rangle|}_{\leqslant C\pi_k(\varphi)\text{ by 1.11}}\leqslant C\pi_k(\varphi)\int_0^1\frac{d\lambda}{\lambda}\lambda^{s+d}=\frac{C}{s+d}\pi_k(\varphi).$$

■

Proposition 1.2.2 *Under the assumptions of Corollary (1.2.1),* $\int_0^1\frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}}\in E_s(U)$.

Proof — Recall that we proved that the integral $t=\int_0^1\frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}}$ converges in $\mathcal{D}'(U)$ and we would like to prove that $t\in E_s(U)$. We try to bound the quantity $\mu^{-s}t_\mu$:

$$\forall 0<\mu\leqslant 1, \mu^{-s}\left\langle t_\mu,\varphi\right\rangle=\mu^{-s-d}\left\langle t,\varphi_{\mu^{-1}}\right\rangle=\int_0^1\frac{d\lambda}{\lambda}\mu^{-s-d}\lambda^s\left\langle u^\lambda_{\lambda^{-1}},\varphi_{\mu^{-1}}\right\rangle$$

$$=\int_0^1\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda,\varphi_{\frac{\lambda}{\mu}}\right\rangle=\int_0^{\frac{1}{\mu}}\frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle u^{\lambda\mu},\varphi_\lambda\right\rangle.$$

We use the fact that there exists $R>0$ such that $\varphi\in\mathcal{D}(U)$ is supported inside the domain $\{|h|\leqslant R\}$. Then $\varphi_\lambda=\varphi(.,\lambda.)$ is supported in $\{|h|\leqslant\lambda^{-1}R\}$. We denote by π_1 the projection $\pi_1:=(x,h)\in\mathbb{R}^{n+d}\mapsto(x,0)\in\mathbb{R}^n\times\{0\}$ and we make the notation abuse $\pi_1(x,h)=(x)$. Then $K=\pi_1(\text{supp }\varphi)$ is compact in $\mathbb{R}^n$ thus, by assumption on the family u, $u^{\lambda\mu}|_{(K\times\mathbb{R}^d)\cap U}$ is supported in $\{a\leqslant|h|\leqslant b\}$ for some $0<a<b$ and $\left\langle u^{\lambda\mu},\varphi_\lambda\right\rangle$ must vanish when $\lambda^{-1}R\leqslant a\Leftrightarrow\lambda\geqslant\frac{R}{a}$. Finally:

$$\mu^{-s}\left\langle t_\mu,\varphi\right\rangle=\int_0^{\frac{R}{a}}\frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle u^{\lambda\mu},\varphi_\lambda\right\rangle.$$

Since $\varphi_\lambda\in C^\infty((K\times\mathbb{R}^d)\cap U)$, by estimate (1.10), we have $|\left\langle u^{\lambda\mu},\varphi_\lambda\right\rangle|\leqslant C\pi_{k,\mathcal{A}}(\varphi)\leqslant C\pi_k(\varphi)$ and

$$|\mu^{-s}\left\langle t_\mu,\varphi\right\rangle|\leqslant\left(\frac{R}{a}\right)^{s+d}\frac{C}{s+d}\pi_k(\varphi).$$

■

Proposition 1.2.3 *Let $(u^\lambda)_{\lambda\in(0,1]}$ be a bounded family in $\mathcal{D}'(U\setminus I)$ which is uniformly supported on an annulus domain of U. If $-m-1 < s+d \leqslant -m, m\in\mathbb{N}$, then the integral $\int_0^1 \frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}}$ needs a renormalization. There is a family $(\tau^\lambda)_{\lambda\in(0,1]}$ of distributions supported on I such that the renormalized integral*

$$\int_0^1 \frac{d\lambda}{\lambda}\lambda^s\left(u^\lambda_{\lambda^{-1}}-\tau^\lambda\right) \tag{1.13}$$

converges in $\mathcal{D}'(U)$.

Proof — If $-m-1 < s+d \leqslant -m$, then we repeat the previous proof except we have to subtract to φ its Taylor polynomial P_m of order m in h. We call I_m the Taylor remainder. Then $\varphi - P_m = I_m$. In coordinates, we get

$$\varphi(x,h) - \underbrace{\sum_{|i|\leqslant m}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)}_{P_m} = I_m(x,h) = \sum_{|i|=m+1} h^i H_i(x,h)$$

where $(H_i)_i$ are smooth functions. $R_\lambda(x,h) = R(x,\lambda h) = \lambda^{m+1}\sum_{|i|=m+1} h^i H_i(x,\lambda h)$. We define a distribution supported on I, which we call "counterterm":

$$\left\langle\tau^\lambda,\varphi\right\rangle = \left\langle u^\lambda_{\lambda^{-1}}, \sum_{|i|\leqslant m}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(\cdot,0)\right\rangle \tag{1.14}$$

where we abusively denoted the expression $\frac{\partial^i\varphi}{\partial h^i}\circ\pi_1$ by $\frac{\partial^i\varphi}{\partial h^i}(\cdot,0)$. We take into account the counterterm

$$\lambda^s\left\langle u^\lambda_{\lambda^{-1}}-\tau^\lambda,\varphi\right\rangle = \lambda^s\left\langle u^\lambda_{\lambda^{-1}},\varphi(x,h) - \sum_{|i|\leqslant m}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(\cdot,0)\right\rangle$$

$$= \lambda^s\left\langle u^\lambda_{\lambda^{-1}}, \sum_{|i|=m+1} h^i H_i(x,h)\right\rangle = \lambda^{s+d}\left\langle u^\lambda, \lambda^{(m+1)}\sum_{|i|=m+1} h^i H_i(x,\lambda h)\right\rangle$$

$$= \lambda^{(d+s+m+1)}\left\langle u^\lambda, \sum_{|i|=m+1} h^i H_i(x,\lambda h)\right\rangle$$

Hence

$$\int_0^1\frac{d\lambda}{\lambda}\lambda^s\left\langle u^\lambda_{\lambda^{-1}}-\tau^\lambda,\varphi\right\rangle = \int_0^1\frac{d\lambda}{\lambda}\underbrace{\lambda^{(d+s+m+1)}}_{\text{integrable}}\underbrace{\left\langle u^\lambda, \sum_{|i|=m+1} h^i H_i(x,\lambda h)\right\rangle}_{\text{bounded}}$$

since $\forall\lambda \in (0,1], h^i H_i(x,\lambda h) \in C^\infty((K\times\mathbb{R}^d)\cap U)$, we can use estimate (1.10)

$$\left|\int_0^1 \frac{d\lambda}{\lambda}\lambda^s \left\langle u^\lambda_{\lambda^{-1}}-\tau^\lambda,\varphi\right\rangle\right| \leqslant \frac{C}{d+s+m+1}\sup_{\lambda\in(0,1]}\pi_{k,\mathcal{A}}(\underbrace{\sum_{|i|=m+1} h^i H_i(x,\lambda h)}_{\text{derivatives of }\varphi\text{ order }m+1})$$

$$\int_0^1 \frac{d\lambda}{\lambda}|\lambda^s \left\langle u^\lambda_{\lambda^{-1}}-\tau^\lambda,\varphi\right\rangle| \leqslant \frac{\tilde{C}}{d+s+m+1}\pi_{k+m+1}(\varphi)$$

where the constant $\tilde{C}$ does not depend on φ and can be estimated by the integral remainder formula. ■

Proposition 1.2.4 *Under the assumptions of proposition (1.2.3), if s is not an integer then* $\int_0^1 \frac{d\lambda}{\lambda}\lambda^s\left(u^\lambda_{\lambda^{-1}}-\tau^\lambda\right)\in E_s(U)$.

Proposition 1.2.5 *Under the assumptions of proposition (1.2.3), if $s+d$ is a* ***non positive integer*** *then* $\int_0^1 \frac{d\lambda}{\lambda}\lambda^s\left(u^\lambda_{\lambda^{-1}}-\tau^\lambda\right)\in E_{s'}(U),\forall s'<s$, *and* $t=\int_0^1 \frac{d\lambda}{\lambda}\lambda^s\left(u^\lambda_{\lambda^{-1}}-\tau^\lambda\right)$ *satisfies the estimate*

$$\forall\varphi\in\mathcal{D}(U),\exists C, |\mu^{-s}\langle t_\mu,\varphi\rangle| \leqslant C\left(1+|\log\mu|\right). \quad (1.15)$$

Proof — To check the homogeneity of the renormalized integral is a little tricky since we have to take the scaling of counterterms into account. When we scale the smooth function then we should scale simultaneously the Taylor polynomial and the remainder

$$\varphi_\lambda = P_\lambda + R_\lambda$$

We want to know to which scale space $E_{s'}$ the distribution t belongs:

$$\mu^{-s'}\langle t_\mu,\varphi\rangle = \mu^{s-s'}\mu^{-s-d}\left\langle t,\varphi_{\mu^{-1}}\right\rangle = \mu^{s-s'}\int_0^1 \frac{d\lambda}{\lambda}\lambda^s\left\langle u^\lambda_{\lambda^{-1}}-\tau^\lambda,\mu^{-d-s}\varphi_{\mu^{-1}}\right\rangle$$

$$=\mu^{s-s'}\int_0^1\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^s\mu^{-d}\left\langle u^\lambda_{\lambda^{-1}},\varphi(x,\frac{h}{\mu})-\sum_{|i|\leqslant m}\frac{h^i}{\mu^i i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle$$

$$=\mu^{s-s'}\int_0^1\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda,\varphi(x,\frac{\lambda}{\mu}h)-\sum_{|i|\leqslant m}\frac{h^i}{\mu^i i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle.$$

$\varphi_{\frac{\lambda}{\mu}}$ is supported on $|h|\leqslant\frac{\mu R}{\lambda}$ thus when $\frac{R\mu}{\lambda}\leqslant a \Leftrightarrow \frac{R\mu}{a}\leqslant\lambda$, the support of $\varphi_{\frac{\lambda}{\mu}}$ does not meet the support of u^λ because u^λ is supported on $a\geqslant|h|$, whereas $\sum_{|i|\leqslant m} \frac{(\lambda h)^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)$ is supported everywhere because it is a Taylor polynomial. Consequently, we must split the integral in two parts

$$\mu^{-s}\left\langle t_\mu,\varphi\right\rangle = I_1 + I_2$$

$$I_1 = \int_0^{\frac{R\mu}{a}} \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda, I_{m,\frac{\lambda}{\mu}}\right\rangle$$

$$= \int_0^{\frac{R\mu}{a}} \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{(d+s+m+1)}\left\langle u^\lambda, \sum_{|i|=m+1} h^i H_i(x,\frac{\lambda}{\mu}h)\right\rangle$$

$$I_2 = \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda, I_{m,\frac{\lambda}{\mu}}\right\rangle$$

$$= \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda, \varphi(x,\frac{\lambda}{\mu}h) - \sum_{|i|\leqslant m}\frac{(\lambda h)^i}{\mu^i i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle$$

no contribution of $\varphi_{\frac{\lambda}{\mu}}$ since $\frac{R\mu}{a}\leqslant\lambda$

and we apply a variable change for I_1:

$$I_1 = \int_0^{\frac{R}{a}} \frac{d\lambda}{\lambda}\lambda^{(d+s+m+1)}\left\langle u_\mu^\lambda, \sum_{|i|=m+1} h^i H_i(x,\lambda h)\right\rangle$$

again by estimate (1.10)

$$\leqslant \left(\frac{R}{a}\right)^{-(d+s+m+1)} \frac{C}{s+d+m+1}\sup_{\lambda\in(0,1]}\pi_{k,\mathcal{A}}\left(\sum_{|i|=m+1} h^i H_i(x,\lambda h)\right)$$

and each H^i is a term in the Taylor remainder I_m of φ,

$$I_1 \leqslant C_1\pi_{k+m+1}(\varphi).$$

Notice that in the second term only the counterterm contributes

$$I_2 = \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle u^\lambda, -\sum_{|i|\leqslant m}\frac{(\lambda h)^i}{\mu^i i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle$$

$$= \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left\langle u^\lambda, -\sum_{|i|\leqslant m}\left(\frac{\lambda}{\mu}\right)^{s+d+i}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle.$$

Then notice that by assumption $s+d\leqslant -m$ and $|i|$ ranges from 0 to m which implies $s+d+|i|\leqslant 0$. When $s+d+|i|<0$:

$$\int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left|\left\langle u^\lambda, \left(\frac{\lambda}{\mu}\right)^{s+d+i}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle\right| \leqslant \underbrace{C_2\left|\left(\frac{1}{\mu}\right)^{s+d+i} - \left(\frac{R}{a}\right)^{s+d+i}\right|}_{\text{no blow up when }\mu\to 0}\pi_k(\varphi).$$

If $s+d<-m$ then $s+d+|i|$ is always strictly negative and there is no blow up when $\mu\to 0$, thus $t\in E_s$. If $s+d+m=0$ and for $|i|=m$:

$$\int_{\frac{R\mu}{a}}^{1}\frac{d\lambda}{\lambda}\left|\left\langle u^\lambda,\left(\frac{\lambda}{\mu}\right)^{s+d+i}\frac{h^i}{i!}\frac{\partial^i\varphi}{\partial h^i}(x,0)\right\rangle\right| \leqslant C_2|\log(\frac{R\mu}{a})|\pi_k(\varphi)$$

and the only term which blows up when $\mu\to 0$ is the logarithmic term. If $s+d=-m$ then $t\in E_{s'}$ for all $s'<s$ and $|\mu^{-s}\langle t_\mu,\varphi\rangle|$ has at most **logarithmic blow up**:

$$\exists(C_1,C_2)\ \ |\mu^{-s'}\langle t_\mu,\varphi\rangle| \leqslant \underbrace{\mu^{s-s'}\left(C_1\pi_{k+m+1}(\varphi)+C_2|\log(\frac{R\mu}{a})|\pi_k(\varphi)\right)}_{\text{bounded when } s'<s}.$$

■

1.3 Extension of distributions.

Conversely, if we start from any distribution t in $E_s(U\setminus I)$, then we can associate to it a bounded family $\left(u^\lambda\right)_{\lambda\in(0,1]}$. Then application of the previous results on the family $(u^\lambda)_\lambda$ allows to construct a distribution $\overline{t\chi}$ in $E_s(U)$. But the resulting distribution given by formulas (1.12) (1.13) coincides exactly with the extension formula $\int_0^1\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}$ on $U\setminus I$. Hence $\overline{t\chi}$ is an extension of $t\chi$. Moreover, if we started from a distribution $t\in E_s(U)$ then the reconstruction theorem provides us with a distribution which is equal to $t\chi$ up to a distribution supported on I, except for the case $s+d>0$ where the extension is unique if we do not want to increase the degree of divergence.

Proposition 1.3.1 *Let $t\in E_s(U\setminus I)$ and let $\psi=-\rho\chi$ where $\chi\in C^\infty(\mathbb{R}^{n+d})$, $\chi=1$ in a neighborhood N_1 of I and $\chi=0$ outside N_2 a neighborhood of N_1, then*

$$u^\lambda=\lambda^{-s}t_\lambda\psi \tag{1.16}$$

is a bounded family in $\mathcal{D}'(U\setminus I)$ which is uniformly supported on an annulus domain of U.

Proof — Consider the function $\psi=-\rho\chi$ used in our construction of the partition of unity of Hörmander. By construction, it is supported on an annulus domain of U. By definition, $t\in E_s(U\setminus I)$ implies $\lambda^{-s}t_\lambda$ is a bounded family of distributions in $\mathcal{D}'(U\setminus I)$, hence $u^\lambda=\lambda^{-s}t_\lambda\psi$ is a bounded family of distributions uniformly supported in supp ψ. ■

Once we notice

$$\int_0^1 \frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}} = \int_0^1 \frac{d\lambda}{\lambda}\lambda^s \left(\lambda^{-s} t_\lambda \psi\right)_{\lambda^{-1}} = \int_0^1 \frac{d\lambda}{\lambda} t\psi_{\lambda^{-1}},$$

the formula of the construction algorithm exactly coincides with the extension formula of Hörmander. Then we can deduce all the results listed below from simple applications of results derived for the family u^λ:

Theorem 1.3.1 *Let $t \in E_s(U \setminus I)$, if $s + d > 0$ then*

$$\forall \varphi \in \mathcal{D}(U), \bar{t}(\varphi) = \lim_{\varepsilon \to 0} \langle t(1 - \chi_{\varepsilon^{-1}}), \varphi\rangle \tag{1.17}$$

exists and defines an extension $\bar{t} \in \mathcal{D}'(U)$ and $\bar{t}$ is in $E_s(U)$.

The proof relies on the first identification

$$\int_0^1 \frac{d\lambda}{\lambda}\lambda^s u^\lambda_{\lambda^{-1}} = \int_0^1 \frac{d\lambda}{\lambda} t\psi(\frac{h}{\lambda}) = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{d\lambda}{\lambda} t\psi_{\lambda^{-1}} = \lim_{\varepsilon\to 0}\left\langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi\right\rangle,$$

where $\psi = -\rho\chi$. Then by definition of $\bar{t}$:

$$\bar{t} = \int_0^1 \frac{d\lambda}{\lambda} t\psi(\frac{h}{\lambda}) + \langle t(1-\chi), \varphi\rangle$$

$$= \lim_{\varepsilon\to 0}\left\langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi\right\rangle + \langle t(1-\chi), \varphi\rangle = \lim_{\varepsilon\to 0}\left\langle t(1-\chi_{\varepsilon^{-1}}), \varphi\right\rangle.$$

In the case $s + d > 0$, the last formula $\lim_{\varepsilon\to 0}\langle t(1-\chi_{\varepsilon^{-1}}), \varphi\rangle$ also appears in the very nice recent work [4] (but with different hypothesis and interpretation) and in fact goes back to Meyer [53] Definition 1.7 p. 15 and formula (3.16) p. 15.

Theorem 1.3.2 *Let $t \in E_s(U \setminus I)$, if $-m - 1 < s + d \leqslant -m \leqslant 0$ then*

$$\bar{t} = \lim_{\varepsilon\to 0}\left(\left\langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi\right\rangle - \langle c_\varepsilon, \varphi\rangle\right) + \langle t(1-\chi), \varphi\rangle \tag{1.18}$$

exists and defines an extension $\bar{t} \in \mathcal{D}'(U)$ where the ***local counterterms*** *c_ε is defined by*

$$\langle c_\varepsilon, \varphi\rangle = \left\langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \sum_{|i|\leqslant m} \frac{h^i}{i!}\varphi^i(x, 0)\right\rangle. \tag{1.19}$$

If s is not an integer then the extension $\bar{t}$ is in $E_s(U)$, otherwise $\bar{t} \in E_{s'}(U), \forall s' < s$.

The last case is treated by [4] and [26] in a slightly different way, they introduce a projection P from the space of C^∞ functions to the m-th power $\mathcal{I}^m$ of the ideal of smooth functions (of course by definition the restriction of this projection to $\mathcal{I}^m$ is the identity), and to construct this projection one has to subtract local counterterms as Meyer does.

A converse result.

Before we move on, let us prove a general converse theorem, namely that given any distribution $t \in \mathcal{D}'(U)$, we can find $s_0 \in \mathbb{R}$ such that for all $s \leqslant s_0$, $t \in E_s(U)$ (we believe such sort of theorems were first proved by Lojasiewicz and Alberto Calderon, [79]), this means morally that any distribution has "finite scaling degree" along an arbitrary vector subspace. We also have the property that $\forall s_1 \leqslant s_2, t \in E_{s_2} \implies t \in E_{s_1}$. This means that the spaces E_s are **filtered**. We work in $\mathbb{R}^{n+d}$ where $I = \mathbb{R}^n \times \{0\}$ and $\rho = h^j \frac{\partial}{\partial h^j}$:

Theorem 1.3.3 *Let U be a ρ-convex open set and $t \in \mathcal{D}'(U)$. If t is of order k, then $t \in E_s(U)$ for all $s \leqslant d + k$, where d is the* ***codimension*** *of $I \subset \mathbb{R}^{n+d}$. In particular any compactly supported distribution is in $E_s(\mathbb{R}^{n+d})$ for some s.*

Proof — First notice if a function $\varphi \in \mathcal{D}(U)$, then the family of scaled functions $(\varphi_{\lambda^{-1}})_{\lambda \in (0,1]}$ has support contained in a compact set $K = \{(x, \lambda h) | (x, h) \in \operatorname{supp} \varphi, \lambda \in (0, 1]\}$. We recall that for any distribution t, there exists k, C_K such that

$$\forall \varphi \in \mathcal{D}_K(U), |\langle t, \varphi\rangle| \leqslant C_K \pi_{K,k}(\varphi).$$

$$|\langle t_\lambda, \varphi\rangle| = |\lambda^{-d} \langle t, \varphi_{\lambda^{-1}}\rangle| \leqslant C_K \lambda^{-d} \pi_{K,k}(\varphi_{\lambda^{-1}}) \leqslant C_K \lambda^{-d-k} \pi_{K,k}(\varphi).$$

So we find that $\lambda^{d+k} \langle t_\lambda, \varphi\rangle$ is bounded which yields the conclusion. ∎

1.3.1 Removable singularity theorems.

Finally, we would like to conclude this section by a simple removable singularity theorem in the spirit of Riemann, (compare with Harvey-Polking [62] theorems (5.2) and (6.1)). In a renormalization procedure there is always an ambiguity which is the ambiguity of the extension of the distribution. Indeed, two extensions always differ by a distribution supported on I. The removable singularity theorem states that if $s + d > 0$ and if we demand that $t \in E_s(U \setminus I)$ should extend to $\bar{t} \in E_s(U)$, then the extension is **unique**. Otherwise, if $-m - 1 < s + d \leqslant -m$, then we bound the transversal order of the ambiguity. We fix the coordinate system (x^i, h^j) in $\mathbb{R}^{n+d}$ and $I = \{h = 0\}$. The collection of coordinate functions $(h^j)_{1 \leqslant j \leqslant d}$ defines a canonical collection of transverse vector fields $(\partial_{h^j})_j$. We denote by δ_I the unique distribution such that $\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+d})$,

$$\langle \delta_I, \varphi\rangle = \int_{\mathbb{R}^n} \varphi(x, 0) d^n x.$$

If $t \in \mathcal{D}'(\mathbb{R}^{n+d})$ with $\operatorname{supp} t \subset I$, then there exist unique distributions (once the system of transverse vector fields ∂_{h^j} is fixed) $t_\alpha \in \mathcal{D}'(\mathbb{R}^n)$, where each compact intersects $\operatorname{supp} t_\alpha$ for a finite number of multiindices α, such that $t(x, h) = \sum_\alpha t_\alpha(x) \partial_h^\alpha \delta_I(h)$ (see [65] theorem (36) and (37) p. 101–102 or [40] theorem (2.3.5)) where the ∂_h^α are derivatives in the **transverse** directions.

Theorem 1.3.4 *Let $t \in E_s(U \setminus I)$ and $\overline{t} \in E_{s'}(U \setminus I)$ its extension given by Theorem (1.3.1) and Theorem (1.3.2) $s' = s$ when $-s-d \notin \mathbb{N}$ or $\forall s' < s$ otherwise. Then $\tilde{t}$ is an extension in $E_{s'}(U)$ if and only if*

$$\tilde{t}(x,h) = \overline{t}(x,h) + \sum_{\alpha \leqslant m} t_\alpha(x)\partial_h^\alpha \delta_I(h),$$

where m is the integer part of $-s-d$. In particular when $s+d>0$ the extension is unique.

Remark: when $-s-d$ is a nonnegative integer, the counterterm is in E_s whereas the extension is in $E_{s'}, \forall s' < s$.
Proof — We scale an elementary distribution $\partial_h^\alpha \delta_I$:

$$\langle (\partial_h^\alpha \delta_I)_\lambda, \varphi \rangle = \lambda^{-d} \langle \partial_h^\alpha \delta_I, \varphi_{\lambda^{-1}} \rangle = (-1)^{|\alpha|} \lambda^{-d-|\alpha|} \langle \partial_h^\alpha \delta_I, \varphi \rangle$$

hence $\lambda^{-s}(\partial^\alpha \delta_I)_\lambda = \lambda^{-d-|\alpha|-s}\partial_h^\alpha \delta_I$ is bounded iff $d+s+|\alpha| \leqslant 0 \implies d+s \leqslant -|\alpha|$. When $s+d>0$, $\forall \alpha, \partial_h^\alpha \delta_I \notin E_s$ hence any two extensions in $E_s(U)$ cannot differ by a local counterterm of the form $\sum_\alpha t_\alpha \partial_h^\alpha \delta_I$. When $-m-1 < d+s \leqslant -m$ then $\lambda^{-s}(\partial_h^\alpha \delta_I)_\lambda$ is bounded iff $s+d+|\alpha| \leqslant 0 \Leftrightarrow -m \leqslant -|\alpha| \Leftrightarrow |\alpha| \leqslant m$. We deduce that $\partial_h^\alpha \delta_I \in E_s$ for all $\alpha \leqslant m$ which means that the scaling degree **bounds** the order $|\alpha|$ of the derivatives in the transverse directions. Assume there are two extensions in E_s, their difference is of the form $u = \sum_\alpha u_\alpha \partial_h^\alpha \delta_I$ by the structure theorem (36) p. 101 in [65] and is also in E_s which means their difference equals $u = \sum_{|\alpha| \leqslant m} u_\alpha \partial_h^\alpha \delta_I$. ∎

1.4 Euler vector fields.

We want to solve the extension problem for distributions on manifolds, in order to do so we must give a geometric definition of scaling transversally to a submanifold I closely embedded in a given manifold M. We will define a class of Euler vector fields which scale transversally to a given fixed submanifold $I \subset M$. Let M be a smooth manifold and $I \subset M$ an embedded submanifold without boundary. For the moment, all discussions are purely local. A classical result in differential geometry associates to each submanifold $I \subset M$ the **sheaf of ideal** $\mathcal{I}$ of functions vanishing on I.

Definition 1.4.1 *Let U be an open subset of M and I a submanifold of M, then we define the ideal $\mathcal{I}(U)$ as the collection of functions $f \in C^\infty(U)$ such that $f|_{I\cap U} = 0$. We also define the ideal $\mathcal{I}^2(U)$ which consists of functions $f \in C^\infty(U)$ such that $f = f_1 f_2$ where $(f_1, f_2) \in \mathcal{I}(U) \times \mathcal{I}(U)$.*

Definition 1.4.2 *A vector field ρ is locally defined on an open set U is called Euler if*

$$\forall f \in \mathcal{I}(U), \rho f - f \in \mathcal{I}^2(U). \tag{1.20}$$

Example 1.4.1 *$h^i\partial_{h^i}$ is Euler by application of Hadamard lemma, if f in $\mathcal{I}$ then $f = h^i H_i$ where the H_i are smooth functions, which implies $\rho f = f + h^i h^j \partial_{h^j} H_i \implies \rho f - f = h^i h^j \partial_{h^j} H_i$.*

In this definition, ρ is defined by testing against arbitrary restrictions of smooth functions $f|_U$ vanishing on I. Let G be the pseudogroup of local diffeomorphisms of M (i.e. an element Φ in G is defined over an open set $U \subset M$ and maps it diffeomorphically to an open set $\Phi(U) \subset M$) such that $\forall p \in I \cap U, \forall \Phi \in G, \Phi(p) \in I$.

Proposition 1.4.1 *Let ρ be **Euler**, then $\forall \Phi \in G$, $\Phi_*\rho$ is **Euler**.*

Proof — For this part, see [47] p. 92 for the definition and properties of the pushforward of a vector field: if $Y = \Phi_* X$ then $L_Y f = L_X(f \circ \Phi) \circ \Phi^{-1}$. We may write the last expression in terms of pull-back

$$L_{\Phi_* X} f = L_X(f \circ \Phi) \circ \Phi^{-1} = \Phi^{-1*}\left(L_X\left(\Phi^* f\right)\right). \qquad (1.21)$$

Then we apply the identity to $X = \rho, Y = \Phi_*\rho$, setting $L_{\Phi_*\rho} f = \Phi_*\rho f$ and $L_\rho f = \rho f$ for shortness:

$$\left(\left(\Phi_*\rho\right) f\right) = \Phi^{-1*}\left(\rho\left(\Phi^* f\right)\right).$$

Now since $\Phi \in G$, ρ is Euler and f an arbitrary function in $\mathcal{I}$.

$$\forall \Phi \in G, \forall f \in \mathcal{I}, \left(\Phi_*\rho\right) f - f = \Phi^{-1*}\left(\rho\left(\Phi^* f\right)\right) - \Phi^{-1*}\left(\Phi^* f\right) = \Phi^{-1*}\left(\rho\left(\Phi^* f\right) - \left(\Phi^* f\right)\right).$$

Since $\Phi(I) \subset I$, we have actually $\Phi^* f \in \mathcal{I}$ hence $\left(\rho\left(\Phi^* f\right) - \left(\Phi^* f\right)\right) \in \mathcal{I}^2$ and we deduce that $\Phi^{-1*}\left(\rho\left(\Phi^* f\right) - \left(\Phi^* f\right)\right) \in \Phi^{-1*}\mathcal{I}^2$. We will prove that $\Phi^*\mathcal{I}(U) = \mathcal{I}(\Phi(U))$.

$f \in \mathcal{I} \Leftrightarrow f|_I = 0 \Leftrightarrow f|_{\Phi(I)} = 0$ since $\Phi(I) \subset I \Leftrightarrow (f \circ \Phi)|_I = 0$ thus $\Phi^* f \in \mathcal{I}$.

Hence $\rho\left(\Phi^* f\right) - \left(\Phi^* f\right) \in \mathcal{I}^2$ by definition of ρ, finally we use the fact

$$\Phi^*\left(\mathcal{I}^2\right) = \{(fg)\circ\Phi; (f,g) \in \mathcal{I}^2\} = \{(f\circ\Phi)(g\circ\Phi); (f,g) \in \mathcal{I}^2\} = \left(\Phi^*\mathcal{I}\right)^2 = \mathcal{I}^2$$

since $\Phi^*\mathcal{I} = \mathcal{I}$ to deduce:

$$\Phi^{-1*}\left(\rho\left(\Phi^* f\right) - \left(\Phi^* f\right)\right) \in \mathcal{I}^2$$

which completes the proof. ■

Euler vector fields form a **sheaf** (check the definitions p. 289 in [47]) with the following nice additional properties:

- Given I, the set of *global* Euler vector fields defined on some open neighborhood of I is **nonempty**.

- For any local Euler vector field $\rho|_U$, for any open set $V \subset U$ there is a Euler vector field ρ' defined on a **global neighborhood** of I such that $\rho'|_V = \rho|_V$.

Proof — These two properties result from the fact that one can glue together Euler vector fields by a partition of unity subordinated to some cover of some neighborhood N of I. By paracompactness of M, we can pick an arbitrary locally finite open cover $\cup_{i\in I} V_i$ of I by open sets V_i, such that for each i, there is a local chart $(x,h) : V_i \mapsto \mathbb{R}^{n+d}$ where the image of I by the local chart is the subspace $\{h^j = 0\}$. We can define a Euler vector field $\rho|_{V_i}$, it suffices to pullback the vector field $\rho = h^j\partial_{h^j}$ in each local chart for V_i and by the example 1.4.1 this is a Euler vector field. The vector fields $\rho_i = \rho|_{V_i}$ do not necessarily coincide on the overlaps $V_i \cap V_j$. For any partition of unity $(\alpha_i)_i$ subordinated to this subcover, $\alpha_i \geqslant 0$, $\sum_i \alpha_i = 1$, consider the vector field ρ defined by the formula

$$\rho = \sum \alpha_i \rho_i \tag{1.22}$$

then $\forall f \in \mathcal{I}(U), \rho f - f = \sum \alpha_i \rho_i f - \sum \alpha_i f = \sum \alpha_i (\rho_i f - f) \in \mathcal{I}^2(U)$. ■

We can find the general form for all possible Euler vector fields ρ in arbitrary coordinate system (x,h) where $I = \{h = 0\}$.

Lemma 1.4.1 *$\rho|_U$ is **Euler** if and only if for all $p \in I \cap U$, in **any arbitrary** local chart (x,h) centered at p where $I = \{h = 0\}$, ρ has the standard form*

$$\rho = h^j \frac{\partial}{\partial h^j} + h^i A_i^j(x,h)\frac{\partial}{\partial x^j} + h^i h^j B_{ij}^k(x,h)\frac{\partial}{\partial h^k} \tag{1.23}$$

where A, B are smooth functions of (x,h).

Proof — We use the sum over repeated index convention. Let us start with an arbitrary $f \in \mathcal{I}(U)$. Set $\rho = B^i(x,h)\partial_{h^i} + L^i(x,h)\partial_{x^i}$ and we use

$$f \in \mathcal{I} \implies f = h^j \frac{\partial f}{\partial h^j}(0,0) + x^i h^j \frac{\partial^2 f}{\partial x^i \partial h^j}(0,0) + O(|h|^2)$$

First compute ρf up to order two in h:

$$\rho f = B^j(x,h)\partial_{h^j} f + L^i(x,h)\partial_{x^i} f$$

$$= B^j(x,h)\frac{\partial f}{\partial h^j}(0,0) + B^j(x,h)x^i\frac{\partial^2 f}{\partial h^j \partial x^i}(0,0) + h^j L^i(x,h)\frac{\partial^2 f}{\partial h^j \partial x^i}(0,0) + O(|h|^2)$$

then the condition $\rho f - f \in \mathcal{I}^2$ reads $\forall f \in \mathcal{I}$,

$$B^j(x,h)\frac{\partial f}{\partial h^j}(0,0) + \left(B^j(x,h)x^i + h^j L^i(x,h)\right)\frac{\partial^2 f}{\partial h^j \partial x^i}(0,0)$$

$$= h^j \frac{\partial f}{\partial h^j}(0,0) + x^i h^j \frac{\partial^2 f}{\partial x^i \partial h^j}(0,0) + O(|h|^2)$$

Now we set $f(x,h) = h^j$ which is an element of $\mathcal{I}$, and substitute it in the previous equation, by uniqueness of the Taylor expansion

$$B^j(x,h) = h^j + O(|h|^2)$$

but this implies

$$h^j \frac{\partial f}{\partial h^j}(0,0) + h^j x^i \frac{\partial^2 f}{\partial h^j \partial x^i}(0,0) + h^j L^i(x,h) \frac{\partial^2 f}{\partial h^j \partial x^i}(0,0)$$

$$= h^j \frac{\partial f}{\partial h^j}(0,0) + x^i h^j \frac{\partial^2 f}{\partial x^i \partial h^j}(0,0) + O(|h|^2)$$

$$\implies h^j L^i(x,h) \frac{\partial^2 f}{\partial h^j \partial x^i}(0,0) = O(|h|^2) \implies L^i \in \mathcal{I}$$

finally $\rho = B^i(x,h)\partial_{h^i} + L^i(x,h)\partial_{x^i}$ where $B^j(x,h) = h^j + \mathcal{I}^2$ and $L^i \in \mathcal{I}$ which gives the final generic form. ■

Fix N an open neighborhood of I with smooth boundary ∂N, the boundary ∂N forms a tube around I. If the Euler ρ restricted to ∂N points outward, this means that the Euler ρ can be exponentiated to generate a one-parameter group of local diffeomorphism: $t \mapsto e^{-t\rho} : N \mapsto N$, N is thus ρ-**convex**. I is the fixed point set of this dynamical system. The one parameter family acts on any section of a natural bundle functorially defined over M, hence on smooth compactly supported sections of the tensor bundles over M particularly on $\Omega_c^d(M)$.

Example 1.4.2 *Choose a local chart $(x,h) : U \mapsto \mathbb{R}^{n+d}$ where I is given by $\{h = 0\}$, the scaling $\left(e^{\log\lambda\rho *} f\right)$ satisfies the differential identity*

$$\lambda \frac{d}{d\lambda}\left(e^{\log\lambda\rho *} f\right) = \rho\left(e^{\log\lambda\rho *} f\right). \tag{1.24}$$

In the case of the canonical Euler $\rho = h^j \frac{\partial}{\partial h^j}$, we also have identity:

$$\lambda \frac{d}{d\lambda} f(x,\lambda h) = \left(h^j \frac{df}{dh^j}\right)(x,\lambda h) = (\rho f)(x,\lambda h),$$

from which we deduce that $\left(e^{\log\lambda\rho *} f\right)(x,h) = f(x,\lambda h)$ which is true because both the l.h.s. and r.h.s. satisfy the differential equation $(\lambda\frac{d}{d\lambda} - \rho)f = 0$ and coincide at $\lambda = 1$.

We generalize the definition of weakly homogeneous distributions to the case of manifolds but this definition is ρ dependent:

Definition 1.4.3 *Let U be ρ-convex open set. The set $E_s^\rho(U)$ is defined as the set of distributions $t \in \mathcal{D}'(U)$ such that*

$$\forall\varphi \in \mathcal{D}(U), \exists C(\varphi), \sup_{\lambda\in(0,1]} |\left\langle \lambda^{-s} t_\lambda, \varphi \right\rangle| \leqslant C(\varphi).$$

1.4.1 Invariances

We gave a global definition of the space E_s^ρ but this definition depends on the Euler ρ. Recall that G is the group of local diffeomorphisms preserving I. On the one hand, we saw that the class of Euler vector fields is invariant by the action of G on the other hand it is not obvious that for any two Euler vector fields ρ_1, ρ_2, there is an element $\Phi \in G$ such that $\Phi_*\rho_1 = \rho_2$.

Denote by $S(\lambda) = e^{\log\lambda\rho}$ the scaling operator defined by the Euler ρ. $S(\lambda)$ is a multiplicative group homomorphism, it satisfies the identity $S(\lambda_1)S(\lambda_2) = S(\lambda_1\lambda_2)$.

Proposition 1.4.2 *Let p in I, let U be an open set containing p and let ρ_1, ρ_2 be two Euler vector fields defined on U and $S_a(\lambda) = e^{\log\lambda\rho_a}, a = 1, 2$ the corresponding scalings. Then there exists a neighborhood $V \subset U$ of p and a one-parameter family of diffeomorphisms $\Phi \in C^\infty([0,1] \times V, V)$ such that, if for all $\lambda \in [0,1]$, $\Phi(\lambda) = \Phi(\lambda, .) : V \mapsto V$, then $\Phi(\lambda)$ satisfies the equation:*

$$S_2(\lambda) = S_1(\lambda) \circ \Phi(\lambda).$$

Proof — We use a local chart $(x, h) : U \mapsto \mathbb{R}^{n+d}$ centered at p, where $I = \{h = 0\}$. We set $\rho = h^j\partial_{h^j}$ and $S(\lambda) = e^{\log\lambda\rho}$ and we try to solve the two problems $S(\lambda)^*t = \Phi_a(\lambda)^*\left(S_a(\lambda)^*t\right)$ for $a = 1$ or 2. We must have the following equation

$$\Phi_a(\lambda)^* = S(\lambda)^*S_a^{-1}(\lambda)^* \implies \Phi_a(\lambda) = S_a^{-1}(\lambda) \circ S(\lambda).$$

If so, the map $\Phi_a(\lambda)$ satisfies the differential equation

$$\lambda\frac{\partial}{\partial\lambda}\Phi_a(\lambda)^* = \lambda\frac{\partial}{\partial\lambda}S(\lambda)^*S_a^{-1}(\lambda)^*$$

$$= \rho S(\lambda)^*S_a^{-1}(\lambda)^* + S(\lambda)^*(-\rho_a)S_a^{-1}(\lambda)^*$$

$$= \rho S(\lambda)^*S_a^{-1}(\lambda)^* + S(\lambda)^*(-\rho_a)S^{-1}(\lambda)^*S(\lambda)^*S_a^{-1}(\lambda)^*$$

$$= \left(\rho - Ad_{S^{-1}(\lambda)}\rho_a\right)\Phi_a(\lambda)^*$$

$$\implies \lambda\frac{\partial}{\partial\lambda}\Phi_a(\lambda) = \left(\rho - S^{-1}(\lambda)_\star\rho_a\right)(\Phi_a(\lambda)) \text{ with } \Phi_a(1) = Id$$

where we used the Lie algebraic formula (1.21): $\left(\left(\Phi_*\rho\right)f\right) = \Phi^{-1*}\left(\rho\left(\Phi^*f\right)\right) = \left(Ad_\Phi\rho\right)f$. Let f be a smooth function and X a vector field. We use formula (1.21) to compute the pushforward of fX by a diffeomorphism Φ:

$$L_{\Phi_*(fX)}\varphi = (\Phi^{-1*}f)L_{\Phi_*X}\varphi. \tag{1.25}$$

We use the general form (1.23) for a Euler vector field:

$$\rho_a = h^j\frac{\partial}{\partial h^j} + h^iA_i^j(x,h)\frac{\partial}{\partial x^j} + h^ih^jB_{ij}^k(x,h)\frac{\partial}{\partial h^k}$$

hence we apply formula 1.25:

$$S^{-1}(\lambda)_*\rho_a = S^{-1}(\lambda)_*\left(h^j\frac{\partial}{\partial h^j}\right)+S^{-1}(\lambda)_*\left(h^iA_i^j\frac{\partial}{\partial x^j}\right)+S^{-1}(\lambda)_*\left(h^ih^jB_{ij}^k\frac{\partial}{\partial h^k}\right)$$

$$= (S(\lambda)^*h^j)S^{-1}(\lambda)_*\frac{\partial}{\partial h^j}+S(\lambda)^*(h^iA_i^j)S^{-1}(\lambda)_*\frac{\partial}{\partial x^j}+S(\lambda)^*(h^ih^jB_{ij}^k)S^{-1}(\lambda)_*\frac{\partial}{\partial h^k}$$

$$= \lambda h^j\lambda^{-1}\partial_{h^j} + \lambda h^iA_i^j(x,\lambda h)\frac{\partial}{\partial x^j} + \lambda^2h^ih^jB_{ij}^k(x,\lambda h)\lambda^{-1}\frac{\partial}{\partial h^k}$$

$$= h^j\partial_{h^j} + \lambda h^iA_i^j(x,\lambda h)\frac{\partial}{\partial x^j} + \lambda h^ih^jB_{ij}^k(x,\lambda h)\frac{\partial}{\partial h^k}$$

$$\Longrightarrow \rho - S_*^{-1}(\lambda)\rho_a = -\lambda\left(h^iA_i^j(x,\lambda h)\frac{\partial}{\partial x^j} + h^ih^jB_{ij}^k(x,\lambda h)\frac{\partial}{\partial h^k}\right).$$

If we define the vector field $X(\lambda) = -\left(h^iA_i^j(x,\lambda h)\frac{\partial}{\partial x^j} + h^ih^jB_{ij}^k(x,\lambda h)\frac{\partial}{\partial h^k}\right)$ then

$$\frac{\partial \Phi_a}{\partial\lambda}(\lambda) = X(\lambda,\Phi_a(\lambda)) \text{ with } \Phi_a(1) = Id. \tag{1.26}$$

$\Phi_a(\lambda)$ satisfies a non autonomous ODE, the vector field

$$X(\lambda) = -\left(h^iA_i^j(x,\lambda h)\frac{\partial}{\partial x^j} + h^ih^jB_{ij}^k(x,\lambda h)\frac{\partial}{\partial h^k}\right)$$

depends smoothly on (λ,x,h). We have to prove that by choosing a suitable neighborhood of $p\in I$, there is always a solution of (1.26) on the interval $[0,1]$ in the sense that there is no blow up at $\lambda=0$. For any compact $K\subset\{|h|\leqslant\varepsilon_1\}$, we have the estimates $\forall(x,h)\in K,\forall\lambda\in[0,1], |h^ih^jB_{ij}(x,\lambda h)|\leqslant b|h|^2$ and $|h^iA_i(x,\lambda h)|\leqslant a|h|$. Hence as long as $|h|\leqslant\varepsilon_1$, we have $|\frac{dh}{d\lambda}|\leqslant b|h|^2\leqslant b\varepsilon_1|h|$. Then for any Cauchy data $(x(1),h(1))\in K$ such that $|h(1)|\leqslant\varepsilon_2$, we compute the maximal interval $I=(\lambda_0,1]$ such that for all $\lambda\in[\lambda_0,1]$ we have $|h(\lambda)|\leqslant\varepsilon_1$. An application of Gronwall lemma ([73] Theorem 1.17 p. 14) to the differential inequality $|\frac{dh}{d\lambda}|\leqslant b\varepsilon_1|h|$ yields $\forall\lambda\in I, |h(\lambda)|\leqslant e^{(1-\lambda)\varepsilon_1 b}|h(1)|$. Hence, if we choose λ in such a way that $e^{(1-\lambda)\varepsilon_1 b}\varepsilon_2\leqslant\varepsilon_1$ then $|h(\lambda)|\leqslant e^{(1-\lambda)\varepsilon_1 b}|h(1)|\leqslant e^{(1-\lambda)\varepsilon_1 b}\varepsilon_2\leqslant\varepsilon_1$ thus $\lambda\in I$ by definition. Hence, we conclude that if we choose $\varepsilon_2\leqslant\frac{\varepsilon_1}{e^{\varepsilon_1 b}}$ then

$$[0,1] = \{\lambda|e^{(1-\lambda)\varepsilon_1 b}\frac{\varepsilon_1}{e^{\varepsilon_1 b}}\leqslant\varepsilon_1\}\subset\{\lambda|e^{(1-\lambda)\varepsilon_1 b}\varepsilon_2\leqslant\varepsilon_1\}\subset I$$

and by classical ODE theory the equation (1.26) always has a smooth solution $\lambda\mapsto\Phi_a(\lambda)$ on the interval $[0,1]$, the open set V on which this existence result holds is the restriction of the chart $U\cap\{|h|\leqslant\varepsilon_2\}$. Now, to conclude properly in the case both ρ_1,ρ_2 are not equal to $\rho=h^j\frac{\partial}{\partial h^j}$ then we apply the previous result

$$S(\lambda) = S_1(\lambda)\circ\Phi_1(\lambda) = S_2(\lambda)\circ\Phi_2(\lambda) \Longrightarrow S_2(\lambda) = S_1(\lambda)\circ\Phi_1(\lambda)\circ\Phi_2^{-1}(\lambda)$$

hence $S_2(\lambda)^*t = \left(\Phi_1(\lambda)\circ\Phi_2^{-1}(\lambda)\right)^* S_1(\lambda)^*t$ ∎

We keep the notations and assumptions of the above proposition and proof, we give an elementary proof of the conjugation without the use of Sternberg Chen theorem:

Corollary 1.4.1 *Let $\rho_a, a = (1,2)$ be two Euler vector fields and $S_a(\lambda) = e^{\log\lambda\rho_a}, a = (1,2)$ the two corresponding scalings. In the chart $(x,h), I = \{h = 0\}$ around p, let $\rho = h^j\partial_{h^j}$ be the canonical Euler vector field and $S(\lambda) = e^{\log\lambda\rho}$ the corresponding scaling and $\Phi_a(\lambda)$ be the family of diffeomorphisms $\Phi_a(\lambda) = S_a^{-1}(\lambda)\circ S(\lambda)$ which has a* ***smooth limit*** *$\Psi_a = \Phi_a(0)$ when $\lambda \rightarrow 0$. Then $\Psi_a \in G$ locally conjugates the hyperbolic dynamics:*

$$\forall\mu, \Psi_a \circ S(\mu) \circ \Psi_a^{-1} = S_a(\mu) \quad (1.27)$$

$$\Phi_a(\mu) = \Psi_a \circ S(\mu^{-1}) \circ \Psi_a^{-1} \circ S(\mu) \quad (1.28)$$

$$\rho_a = \Psi_{a\star}\rho. \quad (1.29)$$

Hence in any coordinate chart, in a neighborhood of any point $(x,0) \in I$, all Euler are locally conjugate by an element of G to the standard Euler $\rho = h^j\partial_{h^j}$. *Proof* — The map $\lambda \mapsto S(\lambda)$ is a group homomorphism from $(\mathbb{R}^*, \times) \mapsto (G, \circ)$:

$$\Phi_a(\lambda)\circ S(\mu) = \left(S_a^{-1}(\lambda)\circ S(\lambda)\right)\circ S(\mu) = S_a^{-1}(\lambda)\circ S(\lambda\mu)$$

$$= S_a(\mu)\circ S_a^{-1}(\mu)\circ S_a^{-1}(\lambda)\circ S(\lambda\mu) = S_a(\mu)\circ S_a^{-1}(\lambda\mu)\circ S(\lambda\mu) = S_a(\mu)\circ\Phi_a(\lambda\mu)$$

finally $\forall(\lambda,\mu)$, we find $\Phi_a(\lambda)\circ S(\mu) = S_a(\mu)\circ\Phi_a(\lambda\mu) \implies \Phi_a(0)\circ S(\mu) = S_a(\mu)\circ\Phi_a(0)$ at the limit when $\lambda \rightarrow 0$ where the limit makes sense because Φ_a is smooth in λ at 0. To obtain the pushforward equation $\rho_a = \Psi_{a\star}\rho$, just differentiate the last identity w.r.t. μ. ■

Beware that the conjugation theorem is only true in a neighborhood V_p of some given point $p \in I$. ρ_1, ρ_2 are *not necessarily conjugate globally* in a neighborhood of I. There might be topological obstructions for a global conjuguation. The local diffeomorphism $\Psi = \Phi_a(0)$ makes the following diagram

$$\begin{array}{ccccc} & V & \overset{S(\lambda)}{\rightarrow} & V & \\ \Psi & \downarrow & & \downarrow & \Psi \\ & V & \underset{S_a(\lambda)}{\rightarrow} & V & \end{array}$$

commute. We keep the notational conventions of the above corollary:

Lemma 1.4.2 *Let p in I and U be an open set containing p, let ρ_1, ρ_2 be two Euler vector fields defined on U then there exists an open neighborhood V of p on which $\forall s, E_s^{\rho_1}(V) = E_s^{\rho_2}(V)$.*

Proof — Set $\Phi(\lambda) = S_1^{-1}(\lambda)\circ S_2(\lambda)$, Φ depends smoothly in λ by Proposition 1.4.2 and $V = \bigcap_{\lambda\in[0,1]}\Phi^{-1}(\lambda)(U)$.

$$\forall\varphi \in \mathcal{D}(V), \lambda^{-s}\left\langle S_2(\lambda)^*t, \varphi\right\rangle = \lambda^{-s}\left\langle \Phi^*(\lambda)\left(S_1(\lambda)^*t\right), \varphi\right\rangle$$

$$= \lambda^{-s} \left\langle S_1(\lambda)^* t, \underbrace{\left(\Phi(\lambda)^{-1*}\varphi\right) |\det(D\Phi(\lambda)^{-1})|}_{\text{bounded in } \mathcal{D}(U)} \right\rangle$$

which is bounded by the hypothesis $t \in E_s^{\rho_1}$ which means by definition that $\lambda^{-s} S_1(\lambda)^* t$ is bounded in $\mathcal{D}'(U)$. ■

We illustrate the previous method in the following example:

Example 1.4.3 *We work in $\mathbb{R}^2, n = d = 1$ with coordinates (x, h), let $\rho_1 = h\partial_h, \rho_2 = h\partial_h + h\partial_x$. Let $t(x, h) = f(x)g(h)$ where f is an arbitrary distribution and g is homogeneous of degree s:*

$$\lambda^{-s} g(\lambda h) = g(h).$$

Then t is homogeneous of degree s with respect to ρ_1 thus $t \in E_s^{\rho_1}$. We will study the scaling behaviour when we scale with ρ_2, $S_2(\lambda)(x, h) = e^{\log \lambda \rho_2}(x, h) = (x + (\lambda - 1)h, \lambda h)$:

$$\int_{\mathbb{R}^2} S_2(\lambda)^* \left(f(x)g(h)\right) \varphi(x, h) dxdh = \int_{\mathbb{R}^2} f\left(x + (\lambda - 1)h\right) g(\lambda h) \varphi(x, h) dxdh$$

Use Proposition (1.4.2) and first determine $\Phi(\lambda)$ in such a way that the equation $\forall \lambda, S_2(\lambda) = S_1(\lambda) \circ \Phi(\lambda)$ is satisfied. We find $\Phi(\lambda)(x, h) = S_1^{-1}(\lambda) \circ S_2(\lambda) = S_1^{-1}(\lambda)(x + (\lambda - 1)h, \lambda h) = (x + (\lambda - 1)h, h)$. Applying the previous result to our example reduces to a simple change of variables in the integral:

$$\int_{\mathbb{R}^2} S_2(\lambda)^* \left(f(x)g(h)\right) \varphi(x, h) dxdh = \int_{\mathbb{R}^2} f(x) g(\lambda h) \varphi(x + (1 - \lambda)h, h) dxdh$$

$$= \lambda^s \int_{\mathbb{R}^2} f(x)g(h) \underbrace{\varphi(x + (1 - \lambda)h, h)}_{\textit{bounded family of test functions}} dxdh.$$

Then the result is straightforward and we can conclude $t \in E_s^{\rho_2}$.

Local invariance

Definition 1.4.4 *A distribution t is said to be locally E_s^ρ at p if there exists an open ρ-convex set $U \subset M$ such that $\overline{U}$ is a neighborhood of p and such that $t \in E_s^\rho(U)$.*

Corollary (1.4.1) and lemma (1.4.2) imply the following local statement:

Theorem 1.4.1 *Let $p \in I$, if t is locally E_s^ρ at p for some Euler vetor field ρ, then it is so for any other Euler vector field.*

A comment on the statement of the theorem, first the definition of ρ-convexity allows U to have *empty intersection* with I, because the definition of ρ-convexity is $\forall p \in U, \forall \lambda \in (0,1], S(\lambda)[p] \in U$, the fact that $\lambda > 0$ allows the case of empty intersection with I. The previous theorem allows to give a definition of the space of distributions $E_s(U)$ that are weakly homogeneous of degree s which makes no mention of the choice of Euler vector field:

Definition 1.4.5 *A distribution t is weakly homogeneous of degree s at p if t is locally E_s^ρ at p for some ρ. $E_s(U)$ is the space of all distributions $t \in \mathcal{D}'(U)$ such that $\forall p \in (I \cap \overline{U})$, t is weakly homogeneous of degree s at p.*

If we look at the definition 1.4.5, and we take into account that the space of distributions on open sets of M forms a sheaf, we deduce the following gluing property: if there is a collection U_i and a collection $t_i \in \mathcal{D}'(U_i)$ such that $\forall i$, $t_i \in E_s(U_i)$ and $t_i = t_j$ on every intersection $U_i \cap U_j$, then for $U = \cup_i U_i$ there is a unique $t \in \mathcal{D}'(U)$ which lives in $E_s(U)$ and coincides with t_i on U_i for all i. From this gluing property, and since the property of being weakly homogeneous of degree s at p is *open*, we can deduce that it is sufficient to check the property on a cover $(U_i)_i$ of U by local charts $(x,h)_i : U_i \subset N \mapsto \Omega_i \subset \mathbb{R}^{n+d}$ where $t|_{U_i}$ is in $E_s^{\rho_i}(U_i)$ for the canonical Euler ρ_i given by the chart.

Theorem 1.4.2 *Let U be an open neighborhood of $I \subset M$, if $t \in E_s(U \setminus I)$ then there exists an extension $\overline{t}$ in $E_{s'}(U)$ where $s' = s$ if $-s-d \notin \mathbb{N}$ and $s' < s$ otherwise.*

Apply Theorem 1.4.1, restrict to local charts $(x,h)_i : (U_i \setminus I) \mapsto (\Omega_i \setminus I)$ where $t|_{U_i \setminus I} = t_i \circ (x,h)_i$ where $t_i \in E_s(\Omega_i \setminus I)$, then extend each t_i on Ω_i, $\overline{t_i} \in E_s(\Omega_i)$, pullback the extension denoted by $\overline{t|_{U_i}} \in E_s(U_i)$ on U_i, then glue together all $\overline{t|_{U_i}}$ (they coincide on $(U_i \cap U_j) \setminus I$ but might not coincide on I but this does not matter !) by a partition of unity $(\varphi_i)_i$ subordinated to the cover $(U_i)_i$. The extension reads $\overline{t} = \sum_i \varphi_i \overline{t|_{U_i}}$.

The extension depends only on ρ, χ. Instead of using the Taylor expansion in local coordinates, we can use the identity

$$\sum_{|\alpha|=n} \frac{h^\alpha}{\alpha!} \partial_h^\alpha f(x,0) = \frac{1}{n!} \left(\left(\frac{d}{dt} \right)^n e^{\log t \rho *} f \right) |_{t=0}(x,h)$$

We can define the counterterms and the renormalized distribution by the equations:

$$\left\langle \tau^\lambda, \varphi \right\rangle = \lim_{t\rightarrow 0} \left\langle te^{-\log \lambda\rho *} (-\rho\chi), \sum_0^m \frac{1}{n!}\left(\frac{d}{dt}\right)^n e^{\log t\rho *}\varphi \right\rangle \qquad (1.30)$$

$$\left\langle \overline{t}, \varphi \right\rangle = \left\langle te^{-\log \lambda\rho *} (-\rho\chi), I_m(\varphi) \right\rangle + \langle t(1-\chi), \varphi\rangle \qquad (1.31)$$

$$I_m(\varphi) = \int_0^1 \frac{d\lambda}{\lambda}\frac{1}{m!}\int_0^1 ds(1-s)^m \left(\frac{\partial}{\partial s}\right)^{m+1} e^{\log s\rho\star}\varphi \qquad (1.32)$$

where we made an effort to convince the reader that the formulas only depend on ρ and χ.

1.5 Appendix.

The Banach–Steinhaus theorem. We will frequently use the Banach–Steinhaus theorem in more general spaces than Banach spaces. We recall here basic results about Fréchet spaces using Gelfand–Shilov [28] as our main reference. Let E be a locally convex topological vector space where the topology is given by a countable family of norms, ie E is a Fréchet space in modern terminology and "countably normed space" in Gelfand–Shilov terminology. Hence it is a **complete metric space** (the topology induced by the metric is exactly the same as the topology induced by the family of norms) (section 3.4 in [28]). Following [28], we assume the family of norms defining the topology are ordered $\|.\|_p \leqslant \|.\|_{p+1}$, where we denote by E_p the completion of E with respect to the norm $\|.\|_p$ which makes E_p a **Banach space**. Then we have the sequence of continuous inclusions $E = ... \subset E_{p+1} \subset E_p \subset ...$ and $E = \bigcap_p E_p$.

A **complete metric space** satisfies the Baire property: any countable union of closed sets with empty interior has empty interior. A consequence of the Baire property is that if a set $U \subset E$ is closed, convex, centrally symmetric ($U = -U$) and absorbant, then it must contain a neighborhood of the origin for the Fréchet topology of E. In 4.1 of [28], starting from the definition of the continuity of a linear map ℓ on E, the authors deduced the existence of p and the corresponding seminorm $\|.\|_p$ such that $\forall x \in E, \ell(x) \leqslant C\|x\|_p$. Following the interpretation of 4.3 in [28], if we denote by E_p the completion of E relative to the norm $\|.\|_p$ then ℓ defines by **Hahn–Banach** a non unique element of E'_p, the topological dual of E_p. Then the main theorem of 5.3 in [28] characterizes strongly bounded sets in the topological dual E' of E. A set $B \subset E'$ is strongly bounded iff there is p such that $B \subset E'_p$ and elements of B are bounded in the norm of E'_p.

$$\exists C, \forall f \in B, \sup_{\|\varphi\|_p \leqslant 1} |\langle f, \varphi\rangle| \leqslant C.$$

The weak topology in E' is generated by the collection of open sets

$$\{f; |\langle f, \varphi\rangle| < \varepsilon\}$$

By definition, if A is a weakly bounded set, then:

$$\forall\varphi, \sup_{f\in A} |\langle f, \varphi\rangle| < \infty.$$

In 5.5 it is proved that weakly bounded sets of E' are in fact strongly bounded in E'. Let A be a weakly bounded set in E'. Then the set $B = \{\varphi; \forall f \in A, |\langle f, \varphi\rangle| < 1\}$ is closed, convex, centrally symmetric ($U = -U$) and absorbant therefore it must contain a neighborhood of the origin by lemma of section 3.4.

$$\{\|\varphi\|_p \leqslant C\} \subset B$$

for a certain seminorm $\|.\|_p$ by definition of a neighborhood of the origin in a Fréchet space. By definition elements of A are bounded on this neighborhood of the origin

$$\begin{aligned} &\forall f \in A, \varphi \in B, |\langle f, \varphi\rangle| < 1 \\ \implies &\forall f \in A, \|\varphi\|_p \leqslant C, |\langle f, \varphi\rangle| < 1 \\ \implies &\forall f \in A, |\langle f, \varphi\rangle| \leqslant C^{-1}\|\varphi\|_p. \end{aligned}$$

Now we will apply these abstract results in the case of bounded families of distributions:

Theorem 1.5.1 *Let $U \subset \mathbb{R}^d$ be an open subset. If A is a weakly bounded family of distributions in $\mathcal{D}'(U)$:*

$$\forall\varphi \in \mathcal{D}(U), \sup_{t\in A} \langle t, \varphi\rangle < \infty$$

then for all compact subset $K \subset U$:

$$\exists p, \exists C_K, \forall t \in A, \forall\varphi \in \mathcal{D}_K(U), |\langle t, \varphi\rangle| \leqslant C_K \pi_p(\varphi).$$

Proof — Set $\|\varphi\|_p = \pi_p(\varphi)$, it is well known this is a norm. The family A is **weakly bounded** in the dual $\mathcal{D}'(K)$ of the Fréchet space $\mathcal{D}(K) = \bigcap_k C_0^k(K)$ ie the intersection of all spaces of C^k functions supported in K. It is thus strongly bounded in the dual space $\mathcal{D}'(K)$ and translating the strong boundedness into estimates yields the result. ■

Theorem 1.5.2 *Let K be a fixed compact subset of $\mathbb{R}^d$. If A is a family of distributions in $\mathcal{D}'_K(U)$ **supported** on $K \subset U$ and*

$$\forall\varphi \in C^\infty(U), \sup_{t\in A} \langle t, \varphi\rangle < \infty,$$

then $\forall K_2$ which is a compact neighborhood of K, $\exists p, \exists C$,

$$\forall t \in A, \forall\varphi \in C^\infty(U), |\langle t, \varphi\rangle| \leqslant C\pi_{p,K_2}(\varphi).$$

Proof — In the second case, first we find a compact set K_2 such that K_2 is a neighborhood of K. We set the Fréchet $E = \bigcap_k C_0^k(K_2)$ which is the intersection of all C^k functions supported in K_2. These functions should not necessarily vanish on the complement of K. Then we pick any plateau function χ such that $\chi|_K = 1$ and $\chi = 0$ on the complement of K_2. $t \in A$ is supported on K thus $\forall t \in A, \forall \varphi \in C^\infty(U), |\langle t, \varphi\rangle| = |\langle t, \chi\varphi\rangle|$ then we reduce to the previous theorem: $\forall t \in A, \forall \varphi \in C^\infty(U), |\langle t, \varphi\rangle| = |\langle t, \chi\varphi\rangle| \leqslant C_{K_2} \sup_{|\alpha|\leqslant p} |\partial^\alpha \chi\varphi|_{L^\infty} \leqslant C \sup_{|\alpha|\leqslant p} |\partial^\alpha \varphi|_{L^\infty(K_2)}$. ■

Corollary 1.5.1 *Let U be an arbitrary open domain, $t \in E_s(U)$ iff $t \in \mathcal{D}'(U)$ is a distribution on U*

$$\forall \varphi \in \mathcal{D}(U), \exists C(\varphi), \sup_{\lambda\in[0,1]} |\lambda^{-s} t_\lambda, \varphi| \leqslant C(\varphi)$$

$$\Leftrightarrow \forall K \subset U, \exists (p, C_K), \forall \varphi \in \mathcal{D}_K(U), \sup_{\lambda\in[0,1]} |\lambda^{-s} t_\lambda, \varphi| \leqslant C_K \pi_p(\varphi).$$

Chapter 2

A prelude to the microlocal extension.

2.0.1 Introduction.

First, let us recall the problem which was solved in Chapter 1. We started from a smooth manifold M and a closed embedded submanifold $I \subset M$. We defined a general setting in which we could scale transversally to I using the flow generated by a class of vector fields called **Euler** vector fields. Then for each distribution $t \in \mathcal{D}'(M \setminus I)$ which was weakly homogeneous of degree s in some precise sense (we called $E_s(M \setminus I)$ the space of such distributions):
- the notion of weak homogeneity was made independent of the choice of Euler ρ,
- we proved that t has an extension $\overline{t} \in E_{s'}(M)$ for some s'. We also understood that the problem of extension is essentially a local problem and that everything can be reduced to the extension problem in $\mathbb{R}^{n+d}$ with coordinates (x, h), $I = \mathbb{R}^n \times \{0\} = \{h = 0\}$ and where the scaling is defined by $\rho = h^j \frac{\partial}{\partial h^j}$. All the "geometry" is somehow contained in the possibility of choosing another Euler vector field. In fact, the pseudogroup G of local diffeomorphisms of $\mathbb{R}^{n+d}$ preserving I acts on the space of Euler vector fields.

However this gives no a priori information on the wave front set of the extension $\overline{t}$. But in QFT, we need conditions on $WF(\overline{t})$ in order to define products of distributions. By the pull-back theorem of Hörmander ([40] thm 8.2.4), there is no reason for $WF(t_\lambda)$ to be equal to $WF(t)$. Hence in order to control the wave front set of $\overline{t}$, the first step is to build some cone Γ which bounds the wave front set of all scaled distributions t_λ and a natural candidate for Γ is $\Gamma = \bigcup_{\lambda \in (0,1]} WF(t_\lambda)$. We denote by $(x, h; k, \xi)$ the coordinates in $T^\star\mathbb{R}^{n+d}, (x; k) \in T^\star\mathbb{R}^n, (h; \xi) \in T^\star\mathbb{R}^d$. We use the notation $T^\bullet M$ for the cotangent bundle $T^\star M$ with the zero section removed. Denote by C_ρ the set $\{(x, h; k, 0) | k \neq 0\} \subset T^\bullet\mathbb{R}^{n+d}$. We call $C = \{(x, 0; 0, \xi) | \xi \neq 0\}$

the intersection of the conormal bundle of I with $T^\bullet\mathbb{R}^{n+d}$. In the first part of this Chapter, we will explain the geometric interpretation of the set C_ρ and how it depends on the choice of Euler ρ. C_ρ plays an important role for the determination of the analytical structure of local counterterms: if $WF(t)$ does not meet $C_\rho = \{(x,h;k,0)|k\neq 0\}$, then the **local counterterms** constructed from t (1.30) are distributions with wave front set in the **conormal** (we meet them again in Chapter 6 under the form of anomaly counterterms). Whereas the condition $WF(t)\cap C_\rho = \emptyset$ depends on the choice of ρ, the stronger condition $\overline{WF(t)}|_I \subset C$ does not depend on ρ and implies that for any choice of Euler ρ, $WF(t)\cap C_\rho = \emptyset$ in some neighborhood of I.

The problem of the closure of Γ over I. So we are led to study under which conditions on $WF(t)$ the cone Γ defined by $\Gamma = \bigcup_{\lambda\in(0,1]} WF(t_\lambda)$ satisfies the constraint $\overline{\Gamma}|_I \subset C$, where $\overline{\Gamma}$ is the closure of $\Gamma\subset T^\bullet(M\setminus I)$ in $T^\bullet M$. Then we find a necessary and sufficient condition on $WF(t)$ which we call **soft landing condition** for $\overline{\Gamma}|_I$ to lie in C. The fact that $WF(t)$ satisfies the soft landing condition guarantees that whatever generalized Euler vector field ρ we choose, the counterterms are conormal distributions supported on I. Furthermore, it is a condition which allows to control the wave front set of the extension as we will see in Chapter 3.

The soft landing condition is not sufficient in order to control the wave front set. Assume that $t\in E_s(M\setminus I)$ and $WF(t)$ satisfies the soft landing condition. Under these assumptions, we address the question: in which sense $\lim_\varepsilon \int_\varepsilon^1 \frac{d\lambda}{\lambda} t\psi(\frac{h}{\lambda})$ converges to $\overline{t}$? More precisely for what topology on $\mathcal{D}'(M)$ do we have convergence ? We already know from Theorems 1.3.1 and 1.3.2 in Chapter 1 that the integral converges in the **weak topology** of $\mathcal{D}'$ but this is not sufficient since it does not imply the convergence in stronger topologies which control wave front sets as the following examples show: indeed in (2.4.1), we construct a distribution t such that $t(1-\chi_{\varepsilon^{-1}}) \underset{\varepsilon\rightarrow 0}{\rightarrow} t$ in $\mathcal{D}'$, whereas $\forall\varepsilon\in(0,1], t(1-\chi_{\varepsilon^{-1}})$ is smooth in $M\setminus I$, the wave front of t can contain any ray $p\in T^\bullet M|_I$ in the cotangent cone over I. Our example shows that generically, we cannot control the wave front set of $\lim_{\varepsilon\rightarrow 0} t(1-\chi_{\varepsilon^{-1}})$ even if the limit exists in $\mathcal{D}'$ and each $t(1-\chi_{\varepsilon^{-1}})\in\mathcal{D}'_\Gamma$ has wave front set in a given cone Γ. Thus our assumptions that $t\in E_s(M\setminus I)$ and $WF(t)$ satisfies the soft landing condition are **not sufficient to control the wave front set of the extension** $\overline{t}$. We will later prove in Chapter 3, that the supplementary condition that $\lambda^{-s}t_\lambda$ be **bounded in** $\mathcal{D}'_\Gamma(M\setminus I)$ (see Definition 2.0.2) is sufficient to have the estimate $WF(\overline{t})\subset\overline{WF(t)}\cup C$.

Notation and preliminary definitions. In this paragraph, we recall results on distribution spaces that we will use in the proof of our main

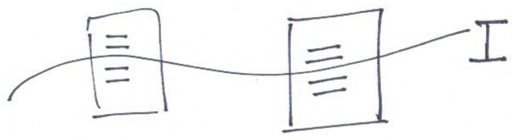

Figure 2.1: The conormal bundle to I.

theorem which controls the wave front set of the extension. Furthermore the seminorms that we define here allow to write proper estimates. For any cone $\Gamma\subset T^\bullet\mathbb{R}^d$, we let $\mathcal{D}'_\Gamma$ be the set of distributions with wave front set in Γ. We define the set of seminorms $\|.\|_{N,V,\chi}$ on $\mathcal{D}'_\Gamma$.

Definition 2.0.1 *For all $\chi\in\mathcal{D}(\mathbb{R}^d)$, for all closed cone $V\subset(\mathbb{R}^d\setminus\{0\})$ such that $(supp\ \chi\times V)\cap\Gamma=\emptyset$, $\|t\|_{N,V,\chi}=\sup_{\xi\in V}|(1+|\xi|)^N\widehat{t\chi}(\xi)|$.*

We recall the definition of the topology $\mathcal{D}'_\Gamma$ (see [1] p. 14 and [33] Chapter 6 p. 333),

Definition 2.0.2 *The topology of $\mathcal{D}'_\Gamma$ is the weakest topology that makes all seminorms $\|.\|_{N,V,\chi}$ continuous and which is stronger than the weak topology of $\mathcal{D}'(\mathbb{R}^d)$. Or it can be formulated as the topology which makes all seminorms $\|.\|_{N,V,\chi}$ and the seminorms of the weak topology:*

$$\forall\varphi\in\mathcal{D}\left(\mathbb{R}^d\right),|\left\langle t,\varphi\right\rangle|=P_\varphi\left(t\right)\qquad(2.1)$$

continuous.

We say that B is bounded in $\mathcal{D}'_\Gamma$ if B is bounded in $\mathcal{D}'$ and if for all seminorms $\|.\|_{N,V,\chi}$ defining the topology of $\mathcal{D}'_\Gamma$,

$$\sup_{t\in B}\|t\|_{N,V,\chi}<\infty.$$

2.1 Geometry in cotangent space.

We will denote by $C=(TI)^\perp\cap T^\bullet M$ the intersection of the conormal bundle $(TI)^\perp$ with the cotangent cone $T^\bullet M$. For any subset Γ of $T^\bullet M$ and for any subset U of M we denote by $\Gamma|_U$ the set $\Gamma\cap T^\bullet U$ where $T^\bullet U$ is the restriction of the cotangent cone over U.

Associating a fiber bundle to a generalized Euler ρ. We work with Euler vector fields ρ defined on a neighborhood $\mathcal{V}$ of I then $\mathcal{V}$ fibers over I in such a way that the leaves of these fibrations are the set of all flow lines ending at a given of point of I, these leaves are invariant by the flow of ρ.

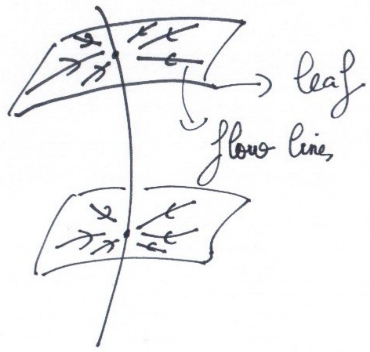

Figure 2.2: The foliation, endpoints of flow lines and leaves.

Definition 2.1.1 *Define the map* $\pi^\rho : p \in \mathcal{V} \mapsto \lim_{t\rightarrow\infty} e^{-t\rho}(p) \in I$.

Proposition 2.1.1 *Let* ρ *be a generalized Euler vector field defined on a neighborhood* $\mathcal{V}$ *of* I*, then* $\mathcal{V}$ *fibers over* I*,* $\pi^\rho : \mathcal{V} \mapsto I$.

Proof — It is sufficient to check that the fibration is trivial over an open neighborhood of any $p \in I$ ([47] Definition 6.1 p. 257). We proved that for any $p \in I$, there is a local chart (x, h) of M around p where $I = \{h = 0\}$ and the vector field ρ writes $h^j\partial_{h^j}$. In this chart, the fibration takes the trivial form

$$(x, h) \in \mathbb{R}^{n+d} \mapsto (x) \in \mathbb{R}^n.$$

■

Definition 2.1.2 *We define a subset* C_ρ *as the union of the conormals of the leaves of the fibration* $\pi^\rho : \mathcal{V} \mapsto I$. C_ρ *is a coisotropic set of* $T^\star M$.

C, C_ρ **in local coordinates.** In the sequel, we always work in local charts $(x, h) \in \mathbb{R}^{n+d}$ where $I = \{h = 0\}$. We denote by $(x, h; k, \xi)$ the coordinates in cotangent space $T^\star\mathbb{R}^{n+d}$, where k (resp ξ) is dual to x (resp h). The scaling is defined by the Euler vector field $\rho = h^j\partial_{h^j}$. There is no loss of generality in reducing to this case because we proved that locally we can always reduce to this canonical situation (cf Chapter 1). In local coordinates $C = \{(x, 0; 0, \xi)|\xi \neq 0\}$ and $C_\rho = \{(x, h; k, 0)|k \neq 0\}$.

Lemma 2.1.1 *Let* $t \in \mathcal{D}'(M \setminus I)$*. If* $\overline{WF(t)}|_I \subset C$ *then for any Euler* ρ*, there exists a neighborhood* $\mathcal{V}$ *of* I *for which* $WF(t)|_\mathcal{V} \cap C_\rho = \emptyset$.

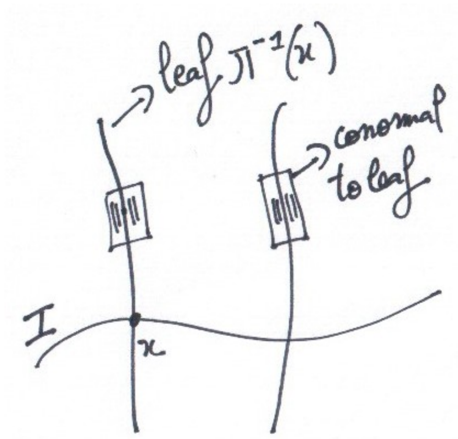

Figure 2.3: The representation of C^ρ as a union of conormal bundles of the leaves of the foliation.

Proof — Since the property we want to prove is open, it is sufficient to establish it on some open neighborhood of any $p \in I$. So consider a local chart $(x, h) : \Omega \mapsto \mathbb{R}^{n+d}$ where $p = (0,0), I = \{h = 0\}$, $\rho = h^j\partial_{h^j}$ and Ω is a compact set. By a simple contradiction argument, if for all $|h| \leqslant \varepsilon$, $WF(t)|_{\Omega\cap\{0<|h|\leqslant\varepsilon\}} \cap C_\rho \neq \emptyset$, we can find a sequence $(x_n, h_n; \frac{k_n}{|k_n|}, 0)$ in $WF(t)$ such that $(x_n, h_n) \in \Omega, h_n \to 0$, then extracting a convergent subsequence yields a contradiction with the assumption $\overline{WF(t)|_I} \subset C$. ■

Lifted flows on cotangent space. It will be crucial in the proof of Theorem 3.2.1 to control the wave front of the extension to understand the dynamics of the lift of the Euler flow on cotangent space. When we scale a distribution t by the one-parameter family $\Phi_\lambda = e^{\log\lambda\rho\star}$, we need to compute the wave front of $\Phi_\lambda^* t$. This is described by the pull-back theorem of Hörmander (see [40] Theorem 8.2.4) as the image of $WF(t)$ by the flow $T^*\Phi_\lambda^{-1}$.

Two interpretations of the lifted flow in cotangent space. We give here two points of view on this lifting. In the first one, the sections of the cotangent bundle are viewed as sections of the bundle of one forms $\Omega^1(M)$. The second interpretation is more in the spirit of symplectic geometry and will be useful for the microlocal interpretation of the flow (see Chapter 5).

1. ρ defines a flow on M and, as any diffeomorphism, this flow can be lifted to the cotangent space T^*M. Actually any diffeomorphism $\Phi : M \mapsto M$ lifts by the formula

$$T^\star\Phi : (x, \eta) \mapsto \left(\Phi(x), \eta \circ d\Phi^{-1}|_{\Phi(x)}\right) \tag{2.2}$$

which in coordinates representation $(x,h)\mapsto(x,\lambda h)$ in $\mathbb{R}^{n+d}$ reads:

$$(x,h;k,\xi)\in T^\star\mathbb{R}^{n+d}\mapsto(x,\lambda h;k,\lambda^{-1}\xi)\in T^\star\mathbb{R}^{n+d}.$$

2. The symbol of the differential operator ρ is $\sigma(\rho)=-ih^j\xi_j$. We compute its symplectic gradient $\sigma(\rho)\in C^\infty(T^\star M)$ for the symplectic form $i(dk\wedge dx+d\xi\wedge dh)$
$$h^j\partial_{h^j}-\xi_j\partial_{\xi_j},$$
and we take the flow of this vector field (for more on the symbol map see [23] p. 198) .

Experts in microlocal analysis use this lifted flow in the "Change-of-variables formula" for pseudodifferential operators, see the formula at the bottom of p. 222 in [23] and Formula 61.20 p. 334 in [23].

2.2 Geometric and metric topological properties of Γ.

We work in $\mathbb{R}^{n+d}$ with coordinates (x,h), $I=\mathbb{R}^n\times\{0\}$ is the linear subspace $\{h=0\}$, the scaling is given by the vector field $\rho=h^j\frac{\partial}{\partial h^j}$ and we use the notation $f_\lambda(x,h)=f(x,\lambda h)$. We restrict to a compact set K which is ρ-convex. The goal of the first part is to find conditions on Γ so that $\forall\lambda\in(0,1],WF(t_\lambda)\subset\Gamma$. We first use the pull-back theorem of Hörmander to describe $WF(t_\lambda)$.

The pull back theorem of Hörmander. Recall the definition of $\Phi^*\Gamma$ for $\Phi:X\mapsto Y$ a smooth diffeomorphism beetween two smooth manifolds (X,Y) and $\Gamma\in T^\bullet Y$,

$$\Phi^*\Gamma=\{(x;\xi\circ d\Phi_x)|(\Phi(x);\xi)\in\Gamma\}.$$

In the case Φ is a diffeomorphism, Φ is invertible and we have the simpler formula:
$$\Phi^*\Gamma=\{\Phi^{-1}(y);\xi\circ D\Phi_{\Phi^{-1}(y)}|\ (y;\xi)\in\Gamma\}.$$

For $\Phi(\lambda):(x,h)\mapsto(x,\lambda h)$, we thus have

$$\Phi(\lambda)^*\Gamma=\{(x,\lambda^{-1}h,k,\lambda\xi)|(x,h;k,\xi)\in\Gamma\}$$

and also $\Phi(\lambda)^*\Gamma|_K=\{(x,h;k,\xi)|(x,\lambda h,k,\lambda^{-1}\xi)\in\Gamma,(x,h)\in K\}=\Phi(\lambda)^*\Gamma\cap\left(K\times(\mathbb{R}^{n+d})^\star\right)$. If $t\in\mathcal{D}'_\Gamma$ then $\Phi^\star t\in\mathcal{D}'_{\Phi^\star\Gamma}$ by application of the pull-back theorem of Hörmander (8.2.4 in [40] or [23] theorem 63.1) where Hörmander uses the notation ${}^td\Phi_x\xi$ for $\xi\circ d\Phi_x$.

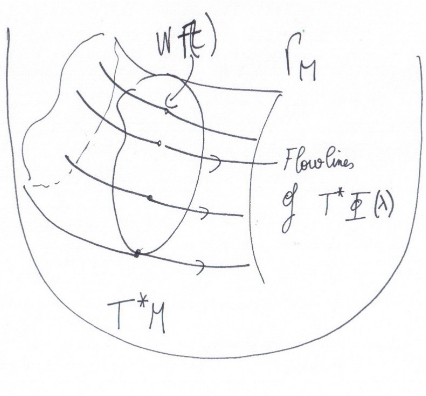

Figure 2.4: Γ_M as the union of all flowlines intersecting $WF(t)$.

The fundamental equation. We wish actually to compute $\bigcup_{\lambda\in(0,1]} WF(t_\lambda)$. Let U be any ρ-convex subset of M. We construct a geometric upper bound $\Gamma_M(WF(t))$ such that $\bigcup_{\lambda\in(0,1]} WF(t_\lambda)|_U \subset \Gamma_M(WF(t))$, where $\Gamma_M(WF(t))$ has a transparent geometrical meaning.

Definition 2.2.1 *Let ρ be a Euler vector field and U a ρ-convex subset of M. Let $WF(t)$ be given, then the set $\Gamma_M(WF(t))|_U$ is defined as the union of all curves of the flow $\lambda \mapsto T^\star(e^{\log\lambda\rho})$ which intersect $WF(t)$ and the projection on the base space of which lie in U. Let T be the maximal time of existence of the flow $e^{\log\lambda\rho}$*

$$\Gamma_M(WF(t))|_U = \{T^\star e^{\log\lambda\rho}(p)|p \in WF(t), \lambda\in(0,T)\} \cap T^\bullet U. \qquad (2.3)$$

$\Gamma_M(WF(t))|_U$ is also defined as the smallest subset of $T^\star_U M$ which contains $WF(t)\cap T^\star_U M$ and which is stable by $T^\star e^{\log\lambda\rho}$ for $\lambda\in(0,1]$. It is **entirely determined** by ρ and $WF(t)$.

Proposition 2.2.1 *For all $\lambda\in(0,1]$, $WF(t_\lambda)|_U \subset \Gamma_M(WF(t))|_U$.*

This is immediate from the definition of $\Gamma_M(WF(t))$ and the pullback theorem. In the sequel, we use a local chart to identify a neighborhood of $p\in I$ with the $\left(h^j\frac{\partial}{\partial h^j}\right)$-convex set $U=\{0<|h|\leqslant\varepsilon, x\in K\}$ for some ε and where K is a compact set of $\mathbb{R}^n$. We want to describe geometrically the set $\Gamma_M(WF(t))$. The intuitive idea is that it is enough to know $\Gamma_M(WF(t))$ on a vertical slice $\{|h|=\varepsilon\}$ just by following the integral curves of the flow intersecting $\Gamma_M(WF(t))|_{|h|=\varepsilon}$. We solve a Cauchy problem for the set Γ_M, in the sense that we fix some geometric Cauchy data $\Gamma_M|_{|h|=\varepsilon}$ on the boundary $\{|h|=\varepsilon\}$ of the domain then we use the geometric characterization of $\Gamma_M|_U$ given by equation (2.3). It is a geometric version of *the method of characteristics* in PDE.

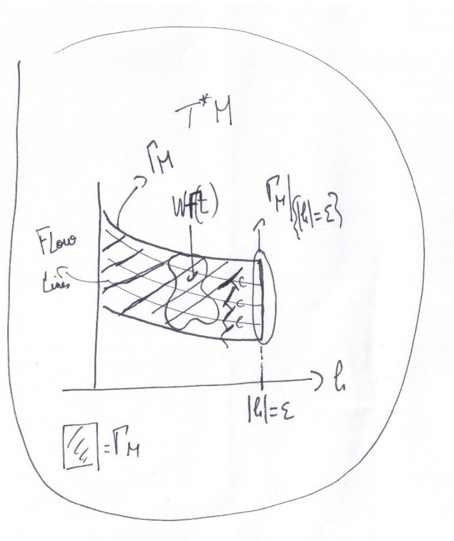

Figure 2.5: The WF(t), the foliation of Γ_M by flowlines and the restriction over $|h| = \varepsilon$.

Proposition 2.2.2 *Let* $U = \{(x,h)|0 < |h| \leqslant \varepsilon, x \in K\} \subset \mathbb{R}^{n+d}$ *where* K *is a compact subset of* $\mathbb{R}^n$ *and for some* $\varepsilon > 0$. *Let* $\Gamma_M(WF(t))|_U$ *be defined by Definition (2.2.1). Then* $\Gamma_M(WF(t))|_{U\cap\{|h|=\varepsilon\}}$ *entirely determines* $\Gamma_M(WF(t))|_U$ *by the equation:*

$$\Gamma_M(WF(t))|_U = \{T^\star\Phi_\lambda(p)|p \in \Gamma_M(WF(t))|_{U\cap\{|h|=\varepsilon\}}, 0 < \lambda \leqslant 1\}. \quad (2.4)$$

Proof — By definition, $\Gamma_M(WF(t))|_U$ is fibered by curves $\Gamma_M(WF(t))|_U = \{\Phi_\lambda(p)|p \in \Gamma_M(WF(t))|_U, \lambda \in (0,1]\} \cap T^\bullet\{0 < |h| \leqslant \varepsilon\}$. Each of these curves must intersect the boundary $|h| = \varepsilon$ in $T^\bullet U$ hence $\Gamma_M(WF(t))|_U$ is the set of all curves $(T^\star\Phi_\lambda(p))_{0<\lambda\leqslant 1}$ for $p \in \Gamma_M(WF(t))|_{U\cap\{|h|=\varepsilon\}}$. ■

For a given cone $WF(t)$ and $\Gamma_M(WF(t))$ defined by the equation (2.3), we believe it is natural to demand that $\overline{\Gamma_M}|_I$ is contained in the conormal C because this ensures that $\Gamma_M(WF(t))$ never meets C_ρ for arbitrary choices of generalized Euler vector fields ρ. This condition is crucial for QFT because it ensures that counterterms are conormal distributions supported on I, we will discuss this in Theorem (2.3.1). We introduce a local condition on $WF(t)$ named local **soft landing condition** at p which ensures that for some neighborhood V_p of p, $\overline{\Gamma_M(WF(t))}|_{I\cap V_p} \subset C$:

Definition 2.2.2 *$WF(t)$ satisfies the **soft landing condition** at p if there exists ρ and a local chart $(x,h) \in C^\infty(U, \mathbb{R}^{n+d}), I = \{h = 0\}$ at $p \in U$ for which $\rho = h^j\frac{\partial}{\partial h^j}$ and such that*

$$\exists\varepsilon > 0, \exists\delta > 0, WF(t)|_{U\cap\{|h|\leqslant\varepsilon\}} \subset \{|k| \leqslant \delta|h||\xi|\}. \quad (2.5)$$

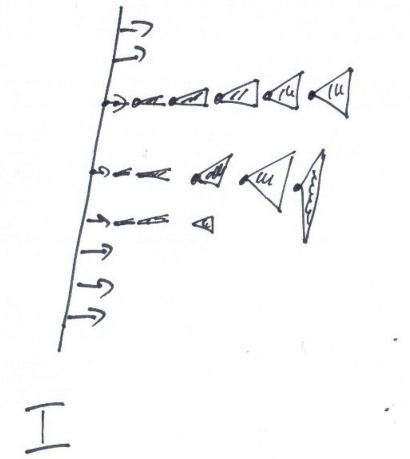

Figure 2.6: The soft landing condition forces the elements of WF to converge to the conormal of I.

Notice that the scale invariance of estimate $|k| \leqslant \delta |h||\xi|$ implies the stability of the soft landing condition by scaling with $\rho = h^j \frac{\partial}{\partial h^j}$. The above definition depends on the choice of ρ, however since by 1.4.1, two Eulers ρ_1, ρ_2 are always locally conjugated by an element Ψ of the pseudogroup G, Ψ transforms the Euler by pushforward, $\Psi_\star \rho_1 = \rho_2$, and the local chart by pullback. To prove that the local soft landing condition does not depend on the choice of Euler vector field, it suffices to prove Γ satisfies the local soft landing condition at p implies $\Psi(\Gamma)$ satisfies the soft landing condition at $\Psi(p)$ for all $\Psi \in G$.

The soft landing condition is stable by action of G.

We prove in Propositions 2.2.3 that the soft landing condition is locally stable by the action of the pseudogroup G of local diffeomorphisms fixing I.

The geometric reformulation of the soft landing condition. We are led to reformulate the local soft landing condition in a more geometric flavor which, once established, makes the claim of stability rather trivial. We denote by $\mathbb{U}^\star\mathbb{R}^{n+d}$ the unit cosphere bundle. Let $\pi_1 : (x, h; k, \xi) \in \mathbb{U}^\star\mathbb{R}^{n+d} \mapsto (x, h) \in \mathbb{R}^{n+d}$ and $\pi_2 : (x, h; k, \xi) \in \mathbb{U}^\star\mathbb{R}^{n+d} \mapsto (k, \xi) \in \mathbb{U}^{n+d-1}$. We introduce the following distance on the cosphere bundle $d_{\mathbb{U}^\star\mathbb{R}^{n+d}}(p, q) = d_{\mathbb{R}^{n+d}}(\pi_1(p), \pi_1(q)) + d_{\mathbb{U}^{n+d-1}}(\pi_2(p), \pi_2(q))$. Let us consider $\mathbb{U}\Gamma$ the trace of Γ on $\mathbb{U}^\star\mathbb{R}^{n+d}$ and also $\mathbb{U}C$ the trace of the conormal bundle of I in $\mathbb{U}^\star\mathbb{R}^{n+d}$.

Definition 2.2.3 *The set* Γ *satisfies the local soft landing condition on* U *if and only if for any element* $p \in \mathbb{U}\Gamma$ *such that* $\pi_1(p) \in U$*, the distance of* p *with the conormal trace* $\mathbb{U}C$ *is controlled by the distance beetween* $\pi_1(p)$ *and* I*:*

$$\forall K \subset U, \exists \delta, \forall p \in \Gamma_S, \pi_1(p) \in K, d_{\mathbb{S}^\star\mathbb{R}^{n+d}}(p, C_S) \leqslant \delta d_{\mathbb{R}^{n+d}}(\pi_1(p), I).$$

We will quickly explain the equivalence of this definition with the definition (2.2.2),

$$lt\ \ k| \leqslant \delta|h||\xi|$$
$$\longleftrightarrow$$
$$\frac{|k|}{|\xi|} \leqslant \delta|h|$$
$$\Leftrightarrow |\tan\theta((k,\xi);(0,\xi))| \leqslant \delta|h| \implies |\theta((k,\xi);(0,\xi))| \leqslant \delta'|h|$$
$$\implies d_{\mathbb{S}^{n+d-1}}(\pi_2(p),\pi_2(C)) \leqslant \delta' d_{\mathbb{R}^{n+d}}(\pi_1(p), I)$$
$$\implies d_{\mathbb{S}^\star\mathbb{R}^{n+d}}(p, C_S) = d_{\mathbb{S}^{n+d-1}}(\pi_2(p),\pi_2(C)) + d_{\mathbb{R}^{n+d}}(\pi_1(p), I)$$
$$\leqslant (1+\delta')d_{\mathbb{R}^{n+d}}(\pi_1(p), I).$$

Conversely,

$$d_{\mathbb{S}^\star\mathbb{R}^{n+d}}(p, C_S) \leqslant \delta d_{\mathbb{R}^{n+d}}(\pi_1(p), I)$$
$$\implies d_{\mathbb{S}^{n+d-1}}(\pi_2(p),\pi_2(C)) \leqslant \delta d_{\mathbb{R}^{n+d}}(\pi_1(p), I)$$
$$\implies |\theta((k,\xi);(0,\xi))| \leqslant \delta|h|$$
$$\implies |\tan\theta((k,\xi);(0,\xi))| \leqslant \delta'|h| \implies \frac{|k|}{|\xi|} \leqslant \delta'|h|.$$

The invariance by G. The geometrical reformulation in terms of distance combined with Proposition 4.3.1 makes obvious the following claim:

Proposition 2.2.3 *Let* $\Psi : U \mapsto U$ *be a local diffeomorphism in* G*,* $\sigma = T^\star\Psi$ *be the corresponding lift on* $T^\star U$ *and* Γ *be a closed conic set in* $T^\bullet M$*. Then if* Γ *satisfies the local* ***soft landing condition*** *at* $\pi_1 \circ \sigma(p) \in U$*, then* $\sigma \circ \Gamma$ *satisfies the local* ***soft landing condition*** *at* $\pi_1 \circ \sigma(p)$*.*

By 1.4.1, this implies:

Proposition 2.2.4 *If* $WF(t)$ *satisfies the soft landing condition locally at* p *for some* ρ *and some associated chart, then for any local chart* $(x, h) \in C^\infty(U, \mathbb{R}^{n+d}), I = \{h = 0\}$ *and associated Euler* $\rho = h^j\frac{\partial}{\partial h^j}$*,* $WF(t)$ *satisfies the soft landing condition locally at* p*.*

Definition 2.2.4 $WF(t)$ *satisfies the soft landing condition if for all* $p \in I$*, it satisfies the soft landing condition locally at* p*.*

Consequences of the soft landing condition.

Lemma 2.2.1 *Let $t \in \mathcal{D}'(M \setminus I)$. If $WF(t)$ satisfies the soft landing condition, then $\overline{WF(t)}|_I \subset C$. In particular, this implies for all Euler ρ, there exists a neighborhood $\mathcal{V}$ of I such that $WF(t) \cap C^\rho = \emptyset$.*

Proof — By definition of the soft landing condition, it suffices to work locally at each $p \in I$. For each p, there exists some open set U s.t. $\exists \delta > 0, WF(t)|_{U\cap\{|h|\leqslant\varepsilon\}} \subset \{|k| \leqslant \delta|h||\xi|\}$ which implies $\overline{WF(t)}|_{U\cap\{h=0\}} \subset \{k = 0\} \implies \overline{WF(t)}|_{I\cap U} \subset C$. Actually $\overline{WF(t)}|_{I\cap U} \subset C \implies WF(t)|_{U\cap\{|h|\leqslant\varepsilon\}} \cap C_\rho = \emptyset$ for ε small enough by Lemma 2.1.1. For each p, we were able to find an open set U_p and $\varepsilon_p > 0$ such that $WF(t)|_{U_p\cap\{|h|\leqslant\varepsilon_p\}} \cap C_\rho = \emptyset$ then $\cup_{p\in I} U_p \cap \{|h| \leqslant \varepsilon_p\}$ forms an open cover of I and extracting a subcover $\mathcal{V} = \cup_{n\in\mathbb{N}} U_{p_n} \cap \{|h| \leqslant \varepsilon_{p_n}\}$ gives a neighborhood $\mathcal{V}$ of I such that $WF(t) \cap C^\rho = \emptyset$. ■

Theorem 2.2.1 *Let $t \in \mathcal{D}'(M \setminus I)$. $WF(t)$ satisfies the soft landing condition if and only if*

$$\overline{\Gamma_M(WF(t))}|_I \subset C = (TI)^\perp, \tag{2.6}$$

where $\Gamma_M(WF(t))$ is defined by Equation (2.3).

Proof — It suffices to work locally at each $p \in I$. The sense $\Rightarrow$ is simple. The set $\{|k| \leqslant \delta|h||\xi|\}$ is clearly invariant by the flow $(x, h; k, \xi) \to (x, \lambda h; k, \lambda^{-1}\xi)$. If $p \in WF(t)$ then by hypothesis $p \in \{|k| \leqslant \delta|h||\xi|\}$, hence the whole curve $\lambda \mapsto \Phi_\lambda(p)$ lies in $\{|k| \leqslant \delta|h||\xi|\}$ thus by definition $\Gamma_M = \{\Phi_\lambda(p)|p \in WF(t), \lambda \in (0,\infty), \Phi_\lambda(p) \in T^\bullet (0 < |h| \leqslant \varepsilon)\} \subset \{|k| \leqslant \delta|h||\xi|\}$. Since $\{|k| \leqslant \delta|h||\xi|\}$ is closed then $\overline{\Gamma_M} \subset \{|k| \leqslant \delta|h||\xi|\}$ and on $I = \{h = 0\}$ we must have $k = 0$ thus $\overline{\Gamma_M}|_I \subset C$. Hence $\Gamma_M|_I \subset C$. To establish the converse sense $\Leftarrow$, we use the proposition (2.2.2). If $\overline{\Gamma_M(WF(t))}|_I \subset C$ then by Lemma 2.1.1, $\Gamma_M(WF(t))|_{|h|=\varepsilon} \cap \{(x, h; k, 0)|k \neq 0\} = \emptyset$ for ε small enough. This implies that $\exists\delta > 0$ s.t. $\Gamma|_{|h|=\varepsilon} \subset \{|k| \leqslant \delta\varepsilon|\xi|\}$. Indeed let us proceed by contradiction. Assume the contrary, then for any $n \in \mathbb{N}^*$, there exist $(x_n, h_n; k_n\xi_n) \in \Gamma|_{|h|=\varepsilon}$ s.t. $k_n > n|\xi_n|$ and w.l.g. $|k_n| = 1$. By compactness, we can extract a subsequence which converges to $(x, h; k, 0)$. This hypothesis translates in an estimate $\Gamma_M|_{|h|=\varepsilon} \subset \{|k| \leqslant \delta\varepsilon|\xi|\}$ for a certain $\delta > 0$. Now the idea is to scale this estimate in order to have a general estimate for all h.

$$p \in \Gamma_M|_{|h|=\varepsilon} \implies p = (x, h; k, \xi) \in \{|k| \leqslant \delta\varepsilon|\xi|\} \subset \{|k| \leqslant \delta|h||\xi|\}$$

by the estimate $\Gamma_M|_{|h|=\varepsilon} \subset \{|k| \leqslant \delta\varepsilon|\xi|\}$ and because $|h| = \varepsilon$,

$$\implies \forall\lambda \in (0, 1], \Phi_\lambda(p) = (x, \lambda h; k, \lambda^{-1}\xi) \in \{|k| \leqslant \delta\lambda|h|\lambda^{-1}|\xi|\} = \{|k| \leqslant \delta|h||\xi|\}$$

Hence by proposition (2.2.2) we find

$$\Gamma_M|_{0<|h|\leqslant\varepsilon} = \{\Phi_\lambda(p)|p\in\Gamma_M|_{|h|=\varepsilon}, 0<\lambda\leqslant 1\}\subset\{|k|\leqslant\delta|h||\xi|\} \quad (2.7)$$

$$\overline{\Gamma_M|_{0<|h|\leqslant\varepsilon}} = \{\Phi_\lambda(p)|p\in\Gamma_M|_{|h|=\varepsilon}, \lambda\in[0,1]\}\subset\{|k|\leqslant\delta|h||\xi|\} \quad (2.8)$$

and we proved the claim because $WF(t)|_{0<|h|\leqslant\varepsilon}\subset\overline{\Gamma_M|_{0<|h|\leqslant\varepsilon}}$. ■

A counterexample which shows the optimality of the soft landing condition.

We give a counterexample which proves $\overline{WF(t)}|_I\subset C$ does not imply $\overline{\Gamma_M(WF(t))}|_I\subset C$ ie the soft landing condition (2.2.2) is in fact optimal. We work in $\mathbb{R}^2$ with coordinates (x,h). The Euler vector field writes $\rho = h\partial_h$. If $WF(t) = \{(x,h;\lambda 1,\lambda h^{-\frac{1}{2}})|\lambda\in\mathbb{R}_+\}$ then it is immediate that $\overline{WF(t)}|_I\subset C = \{(x,0;0,\xi)\}$. However $WF(t)$ does not satisfy the soft landing condition since we find that the sequence of points $(x,\frac{1}{n^2};1,n)$ belongs to $WF(t)$. By definition of $\Gamma=\bigcup_{\lambda\in(0,1]}WF(t_\lambda)$, we find that

$$\Gamma = \{(x,\lambda^{-1}h,k,\lambda\xi)|(x,h,k,\xi)\in WF(t),\lambda\in(0,1]\}$$

thus setting $\lambda_n=\frac{1}{n}$, we find that the sequence $(x,n\frac{1}{n^2};1,\frac{n}{n}) = (x,\frac{1}{n};1,1)$ belongs to Γ thus $\lim_{n\to\infty}(x,\frac{1}{n};1,1) = (x,0;1,1)\in\overline{\Gamma}|_I$ which does not live in the conormal.

2.3 The counterterms are conormal distributions.

We fix the coordinate system (x^i,h^j) in $\mathbb{R}^{n+d}$ and $I=\{h=0\}$. We first recall a deep theorem of Schwartz (see [65] Theorems 36 p. 101) about the structure of distributions supported on $I\subset\mathbb{R}^{n+d}$. We denote by δ_I the unique distribution such that $\forall\varphi\in\mathcal{D}(\mathbb{R}^{n+d})$,

$$\langle\delta_I,\varphi\rangle = \int_{\mathbb{R}^n}\varphi(x,0)d^nx.$$

The collection of coordinate functions $(h^j)_{1\leqslant j\leqslant d}$ defines a canonical collection of transverse vector fields $(\partial_{h^j})_j$. If $t\in\mathcal{D}'(\mathbb{R}^{n+d})$ with supp $t\subset I$, then there exist a unique family of distributions (once the system of transverse vector fields ∂_{h^j} is fixed) $t_\alpha\in\mathcal{D}'(\mathbb{R}^n)$, with $\{\text{supp } t_\alpha\}$ locally finite, such that $t(x,h)=\sum_\alpha t_\alpha(x)\partial_h^\alpha\delta_I(h)$ (see [65] Theorem 36 p. 101-102 or [40] theorem 2.3.5)) where the ∂_h^α are derivatives in the **transverse** directions.

What happens in the case of manifolds ? From the point of view of L. Schwartz, the only thing to keep in mind is that a distribution supported on a submanifold I is always well defined locally and the representation of this distribution is unique once we fix a system of coordinate functions $(h^j)_j$ which are transverse to I ([65] Theorem 37 p. 102). For any distribution $t_\alpha \in \mathcal{D}'(I)$, if we denote by $i : I \hookrightarrow M$ the canonical embedding of I in M then $i_\star t_\alpha$ is the push-forward of t_α in M:

$$\forall \varphi \in \mathcal{D}(M), \langle i_\star t_\alpha, \varphi \rangle = \langle t_\alpha, \varphi \circ i \rangle .$$

The next lemma completes Theorem 1.3.4 proved in Chapter 1. Here the idea is that we add a constraint on the **local counterterm** t, namely that $WF(t)$ is contained in the conormal of I. Then we prove that the coefficients t_α appearing in the Schwartz representation are in fact **smooth** functions.

Lemma 2.3.1 *Let $t \in \mathcal{D}'(M)$ such that t is supported on I, then*
1) t has a unique decomposition as locally finite linear combinations of transversal derivatives of push-forward to M of distributions t_α in $\mathcal{D}'(I)$: $t = \sum_\alpha \partial_h^\alpha (i_\star t_\alpha)$,
and 2) $WF(t)$ is contained in the conormal of I if and only if $\forall \alpha$, t_α is smooth.

Proof — In local coordinates, let

$$t(x,h) = \sum_\alpha \partial_h^\alpha (t_\alpha(x)\delta_I(h)) = \sum_\alpha t_\alpha(x)\partial_h^\alpha \delta_I(h).$$

Assume t_α is not smooth then $WF(t_\alpha)$ would be **non empty**. Then $WF(t_\alpha)$ contains an element $(x_0; k_0)$. Pick $\chi \in \mathcal{D}(R^n)$ such that $\chi(x_0) \neq 0$ then

$$\mathcal{F}(t_\alpha \chi \partial_h^\alpha \delta_I)(k,\xi) = \widehat{t_\alpha \chi}(k)(-i\xi)^\alpha,$$

hence we find a codirection $(\lambda k_0, \lambda \xi), k_0 \neq 0$ in which the product $\widehat{t_\alpha \chi}\widehat{\partial_h^\alpha \delta_I}$ is not rapidly decreasing, hence there is a point $(x,0)$ such that $(x,0;k_0,\xi_0) \in WF(t)$ (by lemma 8.2.1 in [40]) which is in contradiction with the fact that $WF(t) \subset C = \{(x,0,0,\xi) | \xi \neq 0\}$. The reader can use Theorem 8.1.5 in [40] for the converse. ■

Combining with Theorem 1.3.4, we obtain:

Corollary 2.3.1 *Let $t \in \mathcal{D}'(\mathbb{R}^{n+d})$ and supp $t \subset I$. If $WF(t) \subset C$ and $t \in E_s(\mathbb{R}^{n+d}), -m-1 < s+d \leqslant -m$, then $t(x,h) = \sum_\alpha t_\alpha(x)\partial_h^\alpha \delta_I(h)$, where $\forall \alpha$, $t_\alpha \in C^\infty(\mathbb{R}^n)$ and $|\alpha| \leqslant m$.*

Corollary 2.3.2 *Let M be a smooth manifold and I a closed embedded submanifold. For $-m-1 < s+d \leqslant -m$, the space of distributions $t \in E_s(M)$ such that supp $t \in I$ and $WF(t)$ is contained in the conormal of I is a finitely generated module of **rank** $\frac{m+d!}{m!d!}$ over the ring $C^\infty(I)$.*

Proof — In each local chart (x,h) where $I=\{h=0\}$, $t=\sum_\alpha t_\alpha(x)\partial_h^\alpha\delta_I(h)$ where the lenght $|\alpha|$ is bounded by m by the above corollary and $\forall\alpha$, $t_\alpha\in C^\infty(I)$. This improves on the result given by the structure theorem of Laurent Schwartz since we now know that the t_α are smooth. ■

Recall π is the fibration which in local coordinates where $\rho=h^j\frac{\partial}{\partial h^j}$ writes $\pi:(x,h)\mapsto x$ and i is the embedding of I in M. Recall the formula 1.30 for the counterterms which are used to renormalize the Hörmander extension formula:

$$\langle\tau_\lambda,\varphi\rangle=\left\langle t\psi(\frac{h}{\lambda}),\sum_{|\alpha|\leqslant m}\frac{h^\alpha}{\alpha!}\pi^{\rho\star}i^\star\left(\partial_h^\alpha\varphi\right)\right\rangle. \tag{2.9}$$

We give here a general definition of local counterterms of t that covers the counterterms of Chapter 1, the anomaly counterterms of Chapter 6 and the poles of the meromorphic regularization of Chapter 7:

Definition 2.3.1 *Let us fix a system $(h^j)_{1\leqslant j\leqslant d}$ of coordinate functions transverse to I. The vector space of local counterterms of $t\in\mathcal{D}'(M\setminus I)$ is defined as the vector space generated by all distribution τ supported on I which can be represented by the formula:*

$$\forall\varphi\in\mathcal{D}(M),\langle\tau,\varphi\rangle=\langle t\psi,\pi^{\rho\star}i^\star\left(\partial_h^\alpha\varphi\right)\rangle, \tag{2.10}$$

where ψ vanishes in a neighborhood of I and π : supp $\psi\mapsto I$ is a proper mapping.

The next theorem we will prove is very simple yet extremely important conceptually for QFT in curved space times. In classical QFT textbooks, one should subtract polynomials of momenta to renormalize divergent integrals. By inverse Fourier transform these counterterms become sums of derivatives of delta functions supported on vector subspaces of configuration space. In curved space times, there is no concept of polynomials of momenta but the notion of conormal distribution supported on a submanifold still makes sense and replaces the concept of polynomials of momenta. We start by a simple lemma:

Lemma 2.3.2 *Let $t\in\mathcal{D}'(M\setminus I)$ and τ be a distribution defined by the formula*

$$\forall\varphi\in\mathcal{D}(M),\langle\tau,\varphi\rangle=\langle t\psi,(\partial_h^\alpha\varphi)\circ i\circ\pi\rangle, \tag{2.11}$$

where ψ vanishes in a neighborhood of I and π : supp $\psi\mapsto I$ is a proper mapping. If $WF(t\psi)\cap C_\rho=\emptyset$ then $WF(\tau)$ is contained in the conormal C.

Proof — We can prove our claim in local charts and reduce to the flat case $\mathbb{R}^{n+d}$. τ can be reformulated as a product of the pushforward of $t\psi$ by the **fibration** $\pi:(x,h)\in\mathbb{R}^{n+d}\mapsto x\in\mathbb{R}^n$ with a derivative of delta distribution. The idea of the proof is to use the Fubini theorem where integration is performed in a specific order. To clearly understand the strategy, let us write $\langle t\psi, \partial^\alpha\varphi(x,0)\rangle$ in integral form

$$\int_{\mathbb{R}^{n+d}} d^n x d^d h t(x,h)\psi(x,h)\partial^\alpha\varphi(x,0)$$

$$= \int_{\mathbb{R}^n} d^n x \left(\int_{\mathbb{R}^d} d^d h t(x,h)\psi(x,h)\right)\partial^\alpha\varphi(x,0)$$

$$= \int_{\mathbb{R}^n} d^n x \underbrace{\left(\int_{\pi^{-1}(x)} d^d h t(x,h)\psi(x,h)\right)}_{\text{integrated along fibers}}\partial^\alpha\varphi(x,0).$$

This formula suggests the coefficient $t_\alpha(x)$ in the Schwartz representation formula is just equal to the integral $\left(\int_{\pi^{-1}(x)} d^d h t(x,h)\psi(x,h)\right)$. Then the distribution $x \mapsto t_\alpha(x) = \int_{\pi^{-1}(x)} d^d h t(x,h)\psi(h)$ is the **pushforward** $\pi_*(t\psi)$ where we **integrated** $t\psi$ **along the fibers of the fibration** π. The wave front set of $\pi_*(t\psi)$ can be computed by proposition (1.3.4) page 20 of [17]. $WF(\pi_*(t\psi)) = \{(x;k) | \exists h, (x,h;k,0) \in WF(t\psi)\}$, since $WF(t\psi) \cap C_\rho = \emptyset$ then $WF(\pi_*(t\psi))$ is empty hence $\pi_*(t\psi) \in C^\infty(I)$. Finally, if we set $t_\alpha = \pi_*(t\psi)$ then the counterterm τ writes $\tau(x,h) = t_\alpha(x)\partial_h^\alpha\delta_I(h)$ where $t_\alpha \in C^\infty(I)$ and is a conormal distribution in the terminology of Hörmander (see [40] 8.1.5). ■

Combining Lemmas 2.3.2, 2.3.1, 2.1.1 and fixing a system of coordinates functions $(h^j)_j$ transversal to I yields the theorem:

Theorem 2.3.1 *Let $t \in \mathcal{D}'(M \setminus I)$. If $\overline{WF(t)}|_I \subset C$, then there exists a neighborhood $\mathcal{V}$ of I such that for all τ defined by the formula*

$$\forall\varphi \in \mathcal{D}(M), \langle \tau, \varphi\rangle = \langle t\psi, \pi^{\rho\star} i^\star (\partial_h^\alpha\varphi)\rangle, \tag{2.12}$$

where ψ vanishes in a neighborhood of I and π : supp $\psi \mapsto I$ is a proper mapping and supp $\psi \subset \mathcal{V}$, $WF(\tau) \subset C$. In particular, τ is represented in a unique way by $\tau = \sum_\alpha \partial_h^\alpha (i_\star\tau_\alpha)$ where $\forall\alpha, \tau_\alpha \in C^\infty(I)$.

2.4 Counterexample.

We work in $T^\star\mathbb{R}^{n+d}$ with coordinates $(x,h;k,\xi)$ and $I = \{h = 0\}$. In this section, we prove that for any $p \in T^\bullet\mathbb{R}^{n+d}|_I$, we can construct $t \in C^\infty(\mathbb{R}^{n+d} \setminus \{h = 0\}) \cap L^\infty(\mathbb{R}^{n+d})$ in such a way that $p \in WF(t)$. t is a bounded function hence defines a unique element $t \in \mathcal{D}'(\mathbb{R}^{n+d})$.

Lemma 2.4.1 *For all* $p = (x_0, 0; k, \xi) \in T^\bullet\mathbb{R}^{n+d}|_I$*, there exists* $t \in C^\infty(\mathbb{R}^{n+d} \setminus \{h = 0\}) \cap L^\infty(\mathbb{R}^{n+d})$ *such that* $p \in WF(t)$*. In particular, when* $p = (0, 0; \epsilon, 0)$ *then we can choose*

$$t(x, h) = \int_{\mathbb{R}^{n+d}} d\xi dk e^{i(x.k+h.\xi)} a(k, \xi) \left(1 + |k| + |\xi|\right)^{-n-d-1},$$

where $a(k, \xi) = e^{-\frac{|k|^2+|\xi|^2-(k.\epsilon)^2}{(k.\epsilon)}} (1 - \alpha(k, \xi))$ *when* $k.\epsilon > 0$ *and* 0 *otherwise, where* $\alpha = 1$ *in a neighborhood of* 0.

The contruction of t was inspired by [41] Example 8.2.4 p. 188 and the lecture notes of Louis Boutet de Monvel [15] (8.7) p. 80.
Proof — Without loss of generality, we can reduce to the specific case where $\varepsilon = (1, 0, \dots, 0)$ and $\xi = 0$ by coordinate change. Notice $t \in L^\infty(\mathbb{R}^{n+d})$,

$$|t| \leqslant \int_{\mathbb{R}^{n+d}} d\xi dk \left(1 + |k| + |\xi|\right)^{-n-d-1}$$

and

$$\widehat{t}(k, \xi) = e^{-\frac{\sum_{i=2}^n k_i^2+|\xi|^2}{k_1^2}} \left(1 + |k| + |\xi|\right)^{-n-d-1} (1 - \alpha)$$

does not decrease faster than any polynomial inverse when $k_2 = \dots = k_n = \xi_1 = \dots = \xi_d = 0, k_1 > 0$ which implies by Proposition 8.1.3 p. 254 in [40] that $WF(t)$ is **nonempty**. $\widehat{t}$ is a smooth symbol on $T^\bullet\mathbb{R}^{n+d}$ ([67] p. 98–99) which does not depend on (x, h) and the Fourier phase $(x.k+h.\xi)$ has critical points only at $x = h = 0$ thus by Theorem 9.47 p. 102–103 in [67], we find that the singular support of t reduces to $(0, 0)$ thus $WF(t) \subset T^\bullet_{(0,0)}\mathbb{R}^{n+d}$ and $t \in C^\infty(\mathbb{R}^{n+d} \setminus \{h = 0\}) \cap L^\infty(\mathbb{R}^{n+d})$. But $WF(t)$ should be non empty and the projection on the second factor $(x, h; k, \xi) \in T^\bullet\mathbb{R}^{n+d} \mapsto (k, \xi) \in \mathbb{R}^{n+d}$ should be contained in $\{k_2 = \dots = k_n = \xi_1 = \dots = \xi_d = 0, k_1 > 0\}$ so $WF(t) = (0, 0; \lambda\varepsilon, 0), \lambda > 0$. ■

The distribution t is bounded hence weakly homogeneous of degree 0, thus the extension $\lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{d\lambda}{\lambda} t\psi_{\lambda^{-1}} = \lim_{\varepsilon\to 0} t(1-\chi_{\varepsilon^{-1}})$ exists in $\mathcal{D}'(\mathbb{R}^{n+d})$ by Theorem 1.3.1, is unique in $E_0(\mathbb{R}^{n+d})$ by Theorem 1.3.4 and just corresponds to the extension of t in $\mathcal{D}'$ by integration against test functions. However, $\forall\varepsilon, \int_\varepsilon^1 \frac{d\lambda}{\lambda} t\psi_{\lambda^{-1}} = t(1 - \chi_{\varepsilon^{-1}}) \in C^\infty(\mathbb{R}^{n+d})$:

Theorem 2.4.1 *For all* $p = (x_0, 0; k, \xi) \in T^\bullet\mathbb{R}^{n+d}|_I$*, there exists a smooth function* $t \in E_0(\mathbb{R}^{n+d} \setminus I)$ *(thus* $WF(t) = \emptyset$*) which has a unique extension* $\overline{t}$ *in* $E_0(\mathbb{R}^{n+d})$ *such that* $p \in WF(\overline{t})$.

2.5 Appendix.

The module structure of distributions supported on I. The concept of delta distribution δ_I of a submanifold I is *not intrinsically* defined but a certain sheaf associated to I is canonically defined: let U be an open set of M and $(h^j)_{j=1,\cdots,d} \in \mathcal{I}(U)^d$ a collection of sections of the sheaf $\mathcal{I}$ of functions vanishing on $I \cap U$ such that the differentials $dh^j, j = 1, \cdots, d$ are linearly independent ($(h^j)_{1\leqslant j\leqslant d}$ are transversal coordinates of a local chart). The map $h : U \mapsto \mathbb{R}^d$ allows to pullback $\delta_0^{\mathbb{R}^d} \in \mathcal{D}'(\mathbb{R}^d)$ on U, and we denote this pullback $h^\star\delta_0^{\mathbb{R}^d}$ by $\delta_{\{h=0\}}$. If we chose another system of defining functions h' for I, then $\delta_{\{h'=0\}} = \underbrace{|\frac{dh}{dh'}|}_{\in C^\infty(I)} \delta_{\{h=0\}}$, where $|\frac{dh}{dh'}| = \det(\frac{dh^j}{dh'^i})_{ij}$. Thus the left module $C^\infty(I)\delta_{\{h=0\}}$ defined over U has intrinsic meaning (analoguous to the space of sections of a vector bundle). Patching by a partition of unity gives a sheaf of modules of rank 1 over $C^\infty(I)$. Acting on the sections of this sheaf by differential operators of order k defines a module of rank $\frac{d+k!}{d!k!}$ over $C^\infty(I)$.

Chapter 3

The microlocal extension.

Introduction. Let M be a smooth manifold and $I \subset M$ be a closed embedded submanifold of M. In Chapter 2, we gave a necessary and sufficient condition on $WF(t), t \in \mathcal{D}'(M \setminus I)$ that ensured that the union $\Gamma = \bigcup_{\lambda\in(0,1]} WF(t_\lambda)$ of the wave front sets of all scaled distribution t_λ has the property $\overline{\Gamma}|_I \subset C$ where C is the conormal of I. We saw this condition named **soft landing condition** (Definitions 2.2.2 and 2.2.3) was not sufficient to control the wave front set of the extension $\overline{t}$. Our goal in this chapter is to add a boundedness condition which ensures the control of the wave front set of the extension. Our plan starts with a geometric investigation of the dynamical properties of the scaling flow $e^{\log\lambda\rho}$ in cotangent space and show certain asymptotic behaviour of this flow.

3.1 Dynamics in cotangent space.

In this section, we use the terminology and notation of section 1 of Chapter 2. We investigate the **asymptotic behaviour** of the lifted flow $T^\star\Phi_\lambda$.

Decomposition in stable and unstable sets. We interpret C, C_ρ as stable and unstable sets for the lifted flow $T^\star e^{t\rho}$ in cotangent space. We work locally, let $p \in I$ and V_p a neighborhood of p in M, we fix a chart $(x, h) : V_p \mapsto \mathbb{R}^{n+d}$ in which $\rho = h^j \frac{\partial}{\partial h^j}$.

Proposition 3.1.1 *The flow $T^*e^{t\rho}$ lifted to the cotangent cone $T^\bullet V_p$ has the following property:*

$$\lim_{t\to+\infty} T^*e^{t\rho}(p) \in (C_\rho \cap T^\bullet V_p) \tag{3.1}$$

$$\lim_{t\to-\infty} T^*e^{t\rho}(p) \in (C \cap T^\bullet V_p) \tag{3.2}$$

in an ***open dense*** *subset $T^\bullet V_p$.*

Proof — In coordinates (x,h) in which $I=\{h=0\}$ and the flow has simple form $(x,h)\mapsto(x,e^t h)$, the action lifts to $(x,h;k,\xi)\in T^\star\mathbb{R}^{n+d}\mapsto(x,e^t h;k,e^{-t}\xi)\in T^\star\mathbb{R}^{n+d}$. We study the limit $t\to-\infty$, two cases arise:

- **generically** $\xi\neq 0$, then $(x,e^t h;k,e^{-t}\xi)\sim(x,e^t h;e^t k,\xi)$ (because it is a cotangent cone) converges to $(x,0;0,\xi)$, it is immediate to deduce $\{(x,0;0,\xi)|\xi\neq 0\}=(TI)^\perp=C$ is the stable set of the flow. Notice the conormal bundle is an **intrinsic geometric object** and **does not depend on the choice** of vector field ρ.
- Otherwise $\xi=0$, $(x,\lambda h;k,0)\to(x,0;k,0)$, the limit must lie in $\{(x,0;k,0)|k\neq 0\}\subset C^\rho$ which we will later see belongs to the unstable set.

Conversely if $t\to\infty$:

- **generically** $k\neq 0$, then $(x,e^t h;k,e^{-t}\xi)$ converges to $(x,0;k,0)$, it is immediate to deduce $\{(x,h;k,0)|k\neq 0\}=C_\rho$ is the unstable cone.

The flow $\lim_{t\to\infty}Te^{t\rho}$ sends all **conic sets in the complement** of C to the coisotropic set C_ρ. ■

Beware that the wave front set $WF(\Phi^*u)$ is the image of $WF(u)$ by the map $T^\star\Phi^{-1}$. If $\Phi=e^{\log\lambda\rho}$ then the interesting flow for the pull back will be $T^\star e^{-\log\lambda\rho}$ when $\lambda\to 0$. This is why the properties established in the proposition 3.1.1 are crucial in the proof of the main theorem. Especially, we will use the fact that the flow $Te^{-\log\lambda\rho}$, when $\lambda\to 0$ sends all **conic sets in the complement** of C to the coisotropic set C_ρ.

3.1.1 Definitions.

In this subsection, we recall results on distribution spaces that we will use in our proof of the main theorem which controls the wave front set of the extension. Furthermore the seminorms that we define here allow to write proper estimates. We denote by θ the weight function $\xi\mapsto(1+|\xi|)$. For any cone $\Gamma\subset T^\star\mathbb{R}^d$, let $\mathcal{D}'_\Gamma$ be the set of distributions with wave front set in Γ. We define the set of seminorms $\|.\|_{N,V,\chi}$ on $\mathcal{D}'_\Gamma$.

Definition 3.1.1 *For all $\chi\in\mathcal{D}(\mathbb{R}^d)$, for all closed cone $V\subset\mathbb{R}^d\setminus\{0\}$ such that (supp $\chi\times V)\cap\Gamma=\emptyset$, $\|t\|_{N,V,\chi}=\sup_{\xi\in V}|(1+|\xi|)^N\widehat{t\chi}(\xi)|$.*

We recall the definition of the topology $\mathcal{D}'_\Gamma$ (see [1] p. 14),

Definition 3.1.2 *The topology of $\mathcal{D}'_\Gamma$ is the weakest topology that makes all seminorms $\|.\|_{N,V,\chi}$ continuous and which is stronger than the weak topology of $\mathcal{D}'(\mathbb{R}^d)$. Or it can be formulated as the topology defined by all seminorms $\|.\|_{N,V,\chi}$ and the seminorms of the weak topology:*

$$\forall\varphi\in\mathcal{D}\left(\mathbb{R}^d\right),|\langle t,\varphi\rangle|=P_\varphi(t). \tag{3.3}$$

We say that B is bounded in $\mathcal{D}'_\Gamma$, if B is bounded in $\mathcal{D}'$ and if for all seminorms $\|.\|_{N,V,\chi}$ defining the topology of $\mathcal{D}'_\Gamma$,

$$\sup_{t\in B}\|t\|_{N,V,\chi}<\infty.$$

We also use the seminorms:

$$\forall\varphi\in\mathcal{D}(\mathbb{R}^d),\pi_m(\varphi)=\sup_{|\alpha|\leqslant m}\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)},$$

$$\forall\varphi\in\mathcal{E}(\mathbb{R}^d),\forall K\subset\mathbb{R}^d,\pi_{m,K}(\varphi)=\sup_{|\alpha|\leqslant m}\|\partial^\alpha\varphi\|_{L^\infty(K)}.$$

3.2 Main theorem.

In this section, we prove the main theorem of this chapter which gives a sufficient condition to control the wave front set of the extension $\overline{t}$. The condition is as follows: Let $t\in E_s(M\setminus I)$ and assume $WF(t)$ satisfies the **soft landing condition**, and assume that $\lambda^{-s}t_\lambda$ is **bounded** in $\mathcal{D}'_\Gamma$ where $\Gamma=\bigcup_{\lambda\in(0,1]}WF(t_\lambda)$. Then our theorem claims that $WF(\overline{t})\subset\overline{WF(t)}\cup C$ for the extension $\overline{t}$.

Theorem 3.2.1 *Let $s\in\mathbb{R}$ such that $s+d>0$, $\mathcal{V}$ be a ρ-convex neighborhood of I and $t\in\mathcal{D}'(\mathcal{V}\setminus I)$. Assume that $WF(t)$ satisfies the soft landing condition and that $\lambda^{-s}t_\lambda$ is* ***bounded*** *in $\mathcal{D}'_\Gamma(\mathcal{V}\setminus I)$ where $\Gamma=\bigcup_{\lambda\in(0,1]}WF(t_\lambda)\subset T^\bullet(M\setminus I)$. Then the wave front set of the extension $\overline{t}$ of t given by Theorem 1.3.1 is such that $WF(\overline{t})\subset WF(t)\cup C$.*

We saw in Chapter 2 that the hypothesis that $WF(t)$ satisfies the **soft landing condition** is *equivalent* to the requirement that $\overline{\Gamma}|_I\subset C$ in particular, this implies that $\Gamma\cap C_\rho=\emptyset$ in a sufficiently small neighborhood of I and $\overline{WF(t)}|_I\subset\overline{\Gamma}|_I\subset C$. Hence we have the relation $WF(\overline{t})\subset WF(t)\cup C=\overline{WF(t)}\cup C$.

3.2.1 Proof of the main theorem.

For the proof, it suffices to work in flat space $\mathbb{R}^{n+d}$ with coordinates $(x,h)\in\mathbb{R}^n\times\mathbb{R}^d$ where $I=\{h=0\}$ and $\rho=h^j\frac{\partial}{\partial h^j}$, since the hypothesis of the theorem and the result are local and open properties.

Proof — We denote by Ξ the set $\overline{WF(t)}\cup C$. The weight function $(1+|k|+|\xi|)$ is denoted by θ. In order to establish the inclusion $WF(\overline{t})\subset\Xi$, it suffices to prove that for all $p=(x_0,h_0;k_0,\xi_0)\notin\Xi$, there exists χ s.t. $\chi(x_0,h_0)\neq0$, V a closed conic neighborhood of (k_0,ξ_0) such that $\|\overline{t}\|_{N,V,\chi}<+\infty$ for all N. Let $p=(x_0,h_0;k_0,\xi_0)\notin\Xi$, then:

Either $h_0\neq0$, and we choose χ in such a way that $\chi=0$ on I thus $t\chi=\overline{t}\chi$ and we are done since $\|\overline{t}\|_{N,V,\chi}=\|t\|_{N,V,\chi}<+\infty$.

Either $h_0 = 0$ thus $k_0 \neq 0$ since $p \notin C$. Since $|k_0| > 0$, there exists $\delta' > 0$ s.t.

$$|k_0| \geqslant 2\delta'|\xi_0|.$$

We set $V = \{(k,\xi) | |k| \geqslant \delta'|\xi|\}$. By the soft landing condition,

$$\exists \varepsilon_1 > 0, \exists \delta > 0,\ WF(t)|_{|h|\leqslant \varepsilon_1} \subset \{|k| \leqslant \delta |h||\xi|\},$$

$$\text{and } \Gamma|_{|h|\leqslant\varepsilon_1} \subset \{|k| \leqslant \delta|h||\xi|\}.$$

If we choose $\varepsilon > 0$ in such a way that $\delta\varepsilon < \delta'$ and $\varepsilon < \varepsilon_1$, then for any function χ s.t. supp $\chi \subset \{|h| \leqslant \varepsilon\}$, by the previous steps, we obtain that (supp $\chi \times V) \cap \Gamma = \emptyset$. From now on, χ and V are given.

1. Recall $\psi = -\rho\chi'$ is the Littlewood–Paley function on $\mathbb{R}^{n+d}$, and supp $\psi = \{a \leqslant |h| \leqslant 1\}, 0 < a < 1$ does not meet $I = \{h = 0\}$. ψ is defined on $\mathbb{R}^{n+d}$ but is not compactly supported in the x variable. We start from the definition of scaling given in Meyer ([53]) Definition 2.1 p. 45 Definition 2.2 p. 46:

$$\langle t_\lambda \psi, g\rangle = \lambda^{-d} \langle t\psi_{\lambda^{-1}}, g_{\lambda^{-1}}\rangle .$$

We pick the test functions g defined by:

$$g(x,h) = e^{-i(kx+\xi h)}\chi(x,h),$$

then application of the identity which defines the scaling gives:

$$\widehat{t\psi_{\lambda^{-1}}\chi} = \lambda^d \widehat{t_\lambda \chi_\lambda \psi}(k, \lambda\xi)$$

The trick is to notice that $\psi\chi_\lambda$ has a compact support which does not meet $I = \{h = 0\}$, because supp $\psi \subset \{a \leqslant |h| \leqslant b\}$ and $\chi(x, \lambda h)$ is compactly supported in x uniformly in λ. Thus we can find a compact subset $K \subset \mathbb{R}^{n+d}$ such that $\forall \lambda$, supp $\chi_\lambda\psi \subset K$ and $K \cap I = \emptyset$ hence the above Fourier transforms are well defined. Set the family of cones $V_\lambda = \{(k, \lambda\xi) | (x,\xi) \in V\}$. By definition of the seminorms $\|.\|_{N,V,\chi}$, we get

$$\|t\psi_{\lambda^{-1}}\|_{N,V,\chi} = \sup_{(k,\xi)\in V} (1 + |k| + |\xi|)^N |\widehat{t\psi_{\lambda^{-1}}\chi}|$$

$$= \sup_{(k,\xi)\in V} (1 + |k| + |\xi|)^N \lambda^d |\widehat{t_\lambda\chi_\lambda\psi}|(k, \lambda\xi),$$

we isolate the interesting term

$$(1+|k|+|\xi|)^N \lambda^d |\widehat{t_\lambda\chi_\lambda\psi}|(k,\lambda\xi) = \frac{(1+|k|+|\xi|)^N}{(1+|k|+\lambda|\xi|)^N}(1+|k|+\lambda|\xi|)^N \lambda^d |\widehat{t_\lambda\chi_\lambda\psi}|(k,\lambda\xi).$$

We also have

$$\sup_{(k,\xi)\in V} (1 + |k| + \lambda|\xi|)^N \lambda^d |\widehat{t_\lambda\chi_\lambda\psi}|(k, \lambda\xi) \leqslant \|\lambda^d t_\lambda\psi\|_{N,V_\lambda,\chi_\lambda},$$

by definition of $V_\lambda = \{(k, \lambda\xi) | (k,\xi) \in V\}$.

2. Hence, we are reduced to prove that the quantity $\frac{(1+|k|+|\xi|)^N}{(1+|k|+\lambda|\xi|)^N}$ remains bounded for $(k,\xi)\in V$. If so, we are able to apply estimates in Step 2 to bound $\|t\psi_{\lambda^{-1}}\|_{N,V,\chi}$ in function of $\|\lambda^d t_\lambda\psi\|_{N,V_\lambda,\chi_\lambda}$. The difficulty comes from the values of λ close to $\lambda=0$. But we find the following condition
$$\sup_{\lambda\in(0,1],(k,\xi)\in V}\frac{(1+|k|+|\xi|)^N}{(1+|k|+\lambda|\xi|)^N}<(1+\delta'^{-1})^N, \tag{3.4}$$
this follows from:
$$(k,\xi)\in V \implies \delta'|\xi|\leqslant |k|$$
$$\implies 1\leqslant\frac{1+|k|+|\xi|}{1+|k|+\lambda|\xi|}\leqslant\frac{1+(1+\delta'^{-1})|k|}{1+|k|}\leqslant(1+\delta'^{-1}),$$
and implies the estimate
$$\|t\psi_{\lambda^{-1}}\|_{N,V,\chi}\leqslant\lambda^d C\|t_\lambda\psi\|_{N,V_\lambda,\chi_\lambda},$$
where $C=(1+\delta'^{-1})^N$. By rescaling, we also have
$$\forall\varepsilon>0,\ \|t\psi_{\lambda^{-1}}\|_{N,V,\chi}\leqslant\left(\frac{\lambda}{\varepsilon}\right)^d C\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\|_{N,V_{\frac{\lambda}{\varepsilon}},\chi_{\frac{\lambda}{\varepsilon}}}. \tag{3.5}$$

3. We return to $V\subset\{|k|\geqslant\delta'|\xi|\}$ thus
$$\text{supp }\chi\times V\subset\{|k|\geqslant\delta'|h||\xi|\}$$
since supp $\chi\subset\{|h|\leqslant\varepsilon\}$ and ε can always be chosen $\leqslant 1$. For all $\lambda\leqslant\varepsilon$, we have the sequence of inclusions:
$$\text{supp }(\chi_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}})\times V_{\frac{\lambda}{\varepsilon}}\subset\text{supp }\chi\psi_{\varepsilon^{-1}}\times V\subset\{|k|\geqslant\delta'|h||\xi|\},$$
from which we deduce an improvement of the rescaled estimate (3.5):
$$\forall\lambda\leqslant\varepsilon,\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\|_{N,V_{\frac{\lambda}{\varepsilon}},\chi_{\frac{\lambda}{\varepsilon}}}\leqslant\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}\|_{N,V,\varphi'}$$
for some function $\varphi'\in\mathcal{D}(\mathbb{R}^{n+d})$ s.t. $\varphi'=1$ on supp $\chi\psi_{\varepsilon^{-1}}$, $\varphi'=0$ in a neighborhood of I and (supp $\varphi'\times V)\cap\Gamma=\emptyset$ (such φ' always exists by choosing ε small enough in the first step of the proof and by choosing supp φ' slightly larger than supp $\chi\psi_{\varepsilon^{-1}}$). We have gained the fact that the term $\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}\|_{N,V,\varphi'}$ on the r.h.s. is expressed in terms of a seminorm $\|.\|_{N,V,\varphi'}$ where the cone V does not depend on λ. We still have to get rid of the dependance of the function $\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}$ in λ. We use our estimates for the product of a smooth function and

a distribution (see Estimate 3.9), for any **arbitrary** cone W which is a **neighborhood** of V:

$$\begin{aligned} &\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}\|_{N,V,\varphi'} \\ \leqslant\ & C\pi_{2N}\left(\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}\right)\left(\|t_{\frac{\lambda}{\varepsilon}}\|_{N,W,\varphi'}+\|\theta^{-m}t_{\frac{\lambda}{\varepsilon}}\varphi'\|_{L^\infty}\right), \end{aligned} \tag{3.6}$$

where $\|.\|_{N,W,\varphi'}$ is a seminorm of $\mathcal{D}'_\Gamma$. By using the hypothesis of the theorem that $\lambda^{-s}t_\lambda$ is bounded in $\mathcal{D}'_\Gamma$, we deduce that

$$\sup_{\lambda\in(0,\varepsilon]}\left(\frac{\lambda}{\varepsilon}\right)^{-s}\|t_{\frac{\lambda}{\varepsilon}}\|_{N,W,\varphi'}<+\infty.$$

The above inequality combined with the estimate (3.6), the estimate 3.5 and Theorem 4.1.2 applied to the bounded family $(\lambda^{-s}t_\lambda)_{\lambda\in(0,1]}$ gives us:

$$\forall\lambda\leqslant\varepsilon,\exists C',\|t\psi_{\lambda^{-1}}\|_{N,V,\chi}\leqslant C'\left(\frac{\lambda}{\varepsilon}\right)^{s+d}.$$

4. This suggests we should decompose the integral $\int_0^1\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}$ in two parts:

$$\|\bar{t}\|_{N,V,\chi}=\|\int_0^1\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}\|_{N,V,\chi}$$

$$\leqslant\|\int_0^\varepsilon\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}\|_{N,V,\chi}+\|\int_\varepsilon^1\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}\|_{N,V,\chi}$$

$$\leqslant\int_0^\varepsilon\frac{d\lambda}{\lambda}\|t\psi_{\lambda^{-1}}\|_{N,V,\chi}+\underbrace{\|t(\chi-\chi_{\varepsilon^{-1}})\|_{N,V,\chi}}_{<+\infty},$$

because $t(\chi-\chi_{\varepsilon^{-1}})$ is supported away from $\{h=0\}$. This reduces the study to $\int_0^\varepsilon\frac{d\lambda}{\lambda}\|t\psi_{\lambda^{-1}}\|_{N,V,\chi}$ which is bounded by $C'\int_0^\varepsilon\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\varepsilon}\right)^{s+d}<+\infty$.

5. We try to give an explicit bound which "summarizes" all our previous arguments:

$$\begin{aligned} &\int_0^\varepsilon\frac{d\lambda}{\lambda}\|t\psi_{\lambda^{-1}}\|_{N,V,\chi} \\ \leqslant\ & \frac{C\varepsilon^{s+d}}{2^{s+d}(s+d)}\sup_{\lambda\in(0,\varepsilon]}\left(\frac{\lambda}{\varepsilon}\right)^{-s}\pi_{2N}(\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}})\left(\|t_{\frac{\lambda}{\varepsilon}}\|_{N,W,\varphi'}+\|\theta^{-m}\widehat{t_{\frac{\lambda}{\varepsilon}}}\varphi'\|_{L^\infty}\right). \end{aligned} \tag{3.7}$$

■

What do we need to reproduce the estimate (3.7) for families? We keep the same notation as in the proof and statement of theorem (3.2.1). The previous proof works for a fixed distribution t. We would like to reconsider the proof of the main theorem for a family $(t_\mu)_\mu$ of distributions bounded in $\mathcal{D}'_\Gamma$. The validity of the previous theorem relied on the final estimate (3.7):

$$\int_0^\varepsilon \frac{d\lambda}{\lambda}\|t\psi_{\lambda^{-1}}\|_{N,V,\chi} \leqslant \frac{C\varepsilon^{s+d}}{2^{s+d}(s+d)} \sup_{\lambda\in(0,\varepsilon]} \left(\frac{\lambda}{\varepsilon}\right)^{-s} \pi_{2N}(\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}) \left(\|t_{\frac{\lambda}{\varepsilon}}\|_{N,W,\varphi'} + \|\theta^{-m}\widehat{t_{\frac{\lambda}{\varepsilon}}\varphi'}\|_{L^\infty}\right). \tag{3.8}$$

where the constants of the inequality are **independent** of t. Hence the proof and the final estimate still works for the family of distributions $\mu^{-s}t_\mu$ since the family $\lambda^{-s}(\mu^{-s}t_\mu)_\lambda = (\lambda\mu)^{-s}t_{\lambda\mu}$ is bounded in $\mathcal{D}'_\Gamma(\mathcal{V}\setminus I)$ uniformly in (λ,μ). Thus we have the proposition:

Proposition 3.2.1 *If t satisfies the assumptions of theorem (3.2.1), then the family $(\mu^{-s}\overline{t}_\mu)_{\mu\in(0,1]}$ is bounded in $\mathcal{D}'_{\Gamma\cup C}(\mathcal{V})$.*

3.2.2 The renormalized version of the main theorem.

What do we need to extend the proof of the main theorem to the case with counterterms ? In the course of the proof of 3.2.1, we used that $\lambda^{-s}t_\lambda$ is bounded in $\mathcal{D}'_\Gamma$. When $-m-1 < s+d \leqslant m$, we need to introduce counterterms in the Hörmander formula. We outline the proof of the renormalized case following the main steps of the proof of Theorem 3.2.1. We will sometimes denote by $\mathcal{F}[f]$, the Fourier transform $\widehat{f}$ of a Schwartz distribution f and we denote by $e_{k,\xi}$ the Fourier character $e_{k,\xi} : (x,h) \mapsto e^{i(kx+\xi h)}$.

- The first step is identical, for $p=(x_0,0;k_0,\xi_0)\notin \overline{WF(t)}\cup C$, $k_0\neq 0$ we find a neighborhood supp $\chi\times V$ of p such that supp $\chi\times V\cap\Gamma=\emptyset$ where $V\subset\{|k|\geqslant \delta'|\xi|\}$ and supp $\chi\subset\{|h|\leqslant\varepsilon\}$ for some $\varepsilon,\delta'>0$.

- For the computational step, we must use the Taylor formula with integral remainder to take into account the subtraction of counterterms:

$$\mathcal{F}\left[(t\psi_{\lambda^{-1}}-\tau_\lambda)\chi\right](k,\xi) = \left\langle t\psi_{\lambda^{-1}}, \underbrace{\left(1-\sum_{|\alpha|\leqslant m}\frac{h^\alpha}{\alpha!}(-\partial)^\alpha\delta_{h=0}\right)}_{\text{subtraction of local counterterm}} e_{k,\xi}\chi\right\rangle$$

$$= \left\langle t\psi_{\lambda^{-1}}, \underbrace{\frac{1}{m!}\int_0^1 du(1-u)^m\left(\frac{\partial}{\partial u}\right)^{m+1} e_{k,u\xi}\chi_u}_{\text{Taylor remainder}}\right\rangle$$

$$= \frac{1}{m!}\int_0^1 du(1-u)^m\left(\frac{\partial}{\partial u}\right)^{m+1} \widehat{t\psi_{\lambda^{-1}}\chi_u}(k,u\xi)$$

$$= \lambda^d \frac{1}{m!} \int_0^1 du (1-u)^m \left(\frac{\partial}{\partial u}\right)^{m+1} \widehat{t_\lambda \psi \chi_{\lambda u}}(k, u\lambda\xi)$$

$$= \lambda^{d+m+1} \frac{1}{m!} \int_0^\lambda \frac{du}{\lambda} (1-\frac{u}{\lambda})^m \left(\frac{\partial}{\partial u}\right)^{m+1} \widehat{t_\lambda \psi \chi_u}(k, u\xi).$$

by variable change. We also introduce a rescaled version of the previous identity with a variable parameter $\varepsilon > 0$ in such a way that the cut-off function $\psi_{\varepsilon^{-1}}$ on the r.h.s. restrict the expression under the Fourier symbol to the domain $|h| \leqslant \varepsilon$:

$$\forall \varepsilon > 0, \mathcal{F}\left[\left(t\psi_{\lambda^{-1}} - \tau_\lambda\right)\chi\right](k,\xi)$$

$$= \left(\frac{\lambda}{\varepsilon}\right)^{d+m} \frac{1}{m!} \int_0^{\frac{\lambda}{\varepsilon}} du (1-\frac{\varepsilon u}{\lambda})^m \left(\frac{\partial}{\partial u}\right)^{m+1} \mathcal{F}\left(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \chi_u\right)(k, u\xi).$$

Since $\psi_{-1} \subset \{|h| \leqslant \varepsilon\}$, we have the estimate

$$\partial_u^{m+1} \mathcal{F}\left(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \chi_u\right)(k, u\xi) \leqslant (1+\varepsilon|\xi|)^{m+1} \sup_{0\leqslant j\leqslant m+1} \left|\mathcal{F}(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \partial_u^j \chi_u)(k, u\xi)\right|,$$

by Leibniz rule.

$$|(1+|k|+|\xi|)^N \left(\frac{\partial}{\partial u}\right)^{m+1} \mathcal{F}\left(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \chi_u\right)(k, u\xi)|$$

$$\leqslant (1+|k|+|\xi|)^{N+m+1} \sup_{0\leqslant j\leqslant m+1} \left|\mathcal{F}(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \partial_u^j \chi_u)(k, u\xi)\right|$$

$$\leqslant \frac{(1+|k|+|\xi|)^{N+m+1}}{(1+|k|+u|\xi|)^{N+m+1}} (1+|k|+u|\xi|)^{N+m+1} \sup_{0\leqslant j\leqslant m+1} \left|\mathcal{F}(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \partial_u^j \chi_u)(k, u\xi)\right|.$$

- Following the proof of Theorem 3.2.1, we find that the hypothesis (3.4) $V \subset \{\delta'|\xi| \leqslant |k|\}$ implies the estimate

$$\sup_{(k,\xi)\in V} \frac{(1+|k|+|\xi|)^{N+m+1}}{(1+|k|+u|\xi|)^{N+m+1}} \leqslant (1+\delta'^{-1})^{N+m+1}$$

from which we deduce:

$$\forall (k,\xi) \in V, \exists C,\ |(1+|k|+|\xi|)^N \left(\frac{\partial}{\partial u}\right)^{m+1} \mathcal{F}\left(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \chi_u\right)(k, u\xi)|$$

$$\leqslant C(1+|k|+u|\xi|)^{N+m+1} \sup_{0\leqslant j\leqslant m+1} \left|\mathcal{F}(t_{\frac{\lambda}{\varepsilon}} \psi_{\varepsilon^{-1}} \partial_u^j \chi_u)(k, u\xi)\right|.$$

- Thus $\forall u \leqslant \frac{\lambda}{\varepsilon}$:

$$\|\theta^N \left(\frac{\partial}{\partial u}\right)^{m+1} \mathcal{F}\left(t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_u\right)(k,u\xi)\|_{L^\infty(V)}$$
$$\leqslant C \sup_{0\leqslant j\leqslant m+1} \|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\|_{N+m+1,V_u,\partial_u^j\chi_u}$$

where $V_u = \{(k,u\xi)|(k,\xi)\in V\}$. If we denote by $\chi_u^{(j)} = \partial_u^j\chi_u$, by the same argument as in the proof of Theorem 3.2.1, for all $u \leqslant \frac{\lambda}{\varepsilon}, \lambda \leqslant \varepsilon$, we have the inclusion $\text{supp}\left(\psi_{\varepsilon^{-1}}\chi_u^{(j)}\right) \times V_u \subset \text{supp}\left(\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}^{(j)}\right) \times V_{\frac{\lambda}{\varepsilon}}$ where $\text{supp}\left(\psi_{\varepsilon^{-1}}\chi_{\frac{\lambda}{\varepsilon}}^{(j)}\right) \times V \cap \Gamma = \emptyset$, which implies the estimate

$$\|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\|_{N+m+1,V_u,\chi_u^{(j)}} \leqslant \|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_u^{(j)}\|_{N+m+1,V,\varphi'}$$

where φ' is any function in $\mathcal{D}(\mathbb{R}^{n+d})$ such that $\varphi' = 1$ on $\text{supp}\,(\psi_{\varepsilon^{-1}}\chi)$ and $\text{supp}\,\varphi' \times V \cap \Gamma = \emptyset$. Finally, we find that

$$\|\,(t\psi_{\lambda^{-1}} - \tau_\lambda)\,\|_{N,V,\chi}$$
$$\leqslant C\left(\frac{\lambda}{\varepsilon}\right)^{d+m} \frac{1}{m!}\int_0^{\frac{\lambda}{\varepsilon}} du(1-\frac{\varepsilon u}{\lambda})^m \sup_{u\in(0,1],0\leqslant j\leqslant m+1} \|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_u^{(j)}\|_{N+m+1,V,\varphi'}$$
$$\leqslant C\left(\frac{\lambda}{\varepsilon}\right)^{d+m+1} \frac{1}{m+1!}\sup_{u\in(0,1],0\leqslant j\leqslant m+1} \|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_u^{(j)}\|_{N+m+1,V,\varphi'}$$

where we use the simple identity $\frac{1}{m+1} = \int_0^1 du(1-u)^m$. Then we use the estimates (3.9) for the product of the bounded family of smooth functions $\psi_{\varepsilon^{-1}}\chi_u^{(j)}$ and the family of distributions $t_{\frac{\lambda}{\varepsilon}}$ and the assumption that $\lambda^{-s}t_\lambda$ is bounded in $\mathcal{D}'_\Gamma$ to establish the estimate

$$\sup_{u\leqslant 1} \|t_{\frac{\lambda}{\varepsilon}}\psi_{\varepsilon^{-1}}\chi_u^{(j)}\|_{N+m+1,V,\varphi'} \leqslant C'\left(\frac{\lambda}{\varepsilon}\right)^s$$

for all $0 \leqslant j \leqslant m+1$. Then we can conclude in the same way as in the proof of Theorem 3.2.1:

$$\|\int_0^1 \frac{d\lambda}{\lambda}\,(t\psi_{\lambda^{-1}} - \tau_\lambda)\,\|_{N,V,\chi}$$
$$\leqslant \|\underbrace{t(\chi-\chi_{\varepsilon^{-1}})}_{\in\mathcal{D}'_{WF(t)}}\|_{N,V,\chi} + \|\underbrace{\int_\varepsilon^1 \tau_\lambda}_{\in\mathcal{D}'_C}\|_{N,V,\chi} + \underbrace{\int_0^\varepsilon \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\varepsilon}\right)^{s+d+m+1}}_{\text{integrable}} \frac{C}{m+1!}C',$$

where the last term is finite.

Theorem 3.2.2 *Theorem 3.2.1 holds under the weaker assumption $s \in \mathbb{R}$. Moreover if $-s-d \in \mathbb{N}$ then $\lambda^{-s'}\overline{t}_\lambda$ is bounded in $\mathcal{D}'_{\Gamma\cup C}(\mathcal{V})$ for all $s' < s$, if $-s-d \notin \mathbb{N}$ then $\lambda^{-s}\overline{t}_\lambda$ is bounded in $\mathcal{D}'_{\Gamma\cup C}(\mathcal{V})$.*

3.3 Appendix

3.3.1 Estimates for the product of a distribution and a smooth function.

Theorem 3.3.1 *Let $m \in \mathbb{N}$ and $\Gamma \subset T^\bullet(\mathbb{R}^d)$. Let V be a closed cone in $\mathbb{R}^d \setminus 0$ and $\chi \in \mathcal{D}(\mathbb{R}^d)$. Then for every N and every closed conical neighborhood W of V such that $(\text{supp } \chi \times W) \cap \Gamma = \emptyset$, there exists a constant C such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and for all $t \in \mathcal{D}'_\Gamma(\mathbb{R}^d)$ such that $\|\theta^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty$:*

$$\|t\varphi\|_{N,V,\chi} \leqslant C\pi_{2N,K}(\varphi)(\|t\|_{N,W,\chi} + \|\theta^{-m}\widehat{t\chi}\|_{L^\infty}). \tag{3.9}$$

Proof — We denote by θ the weight function $\xi \mapsto (1+|\xi|)$ and $e_\xi := x \mapsto e^{-ix.\xi}$ the Fourier character. If the cone V is given, we can always define a thickening W of the cone V such that W is a closed conic neighborhood of V:

$$W = \{\eta \in \mathbb{R}^d \setminus \{0\} | \exists \xi \in V, |\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \leqslant \delta\},$$

intuitively this means that small angular perturbations of covectors in V will lie in the neighborhood W. If $(\text{supp } \chi \times V) \cap \Gamma = \emptyset$ then δ can be chosen **arbitrarily small** in such a way that $(\text{supp } \chi \times W) \cap \Gamma = \emptyset$. We compute the Fourier transform of the product:

$$|\widehat{t\varphi\chi}(\xi)| = |\left\langle t\varphi, e_\xi \chi \right\rangle| = |\widehat{t\chi} \star \widehat{\varphi}|(\xi)$$
$$\leqslant \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)| d\eta.$$

We reduce to the estimate

$$\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)| d\eta$$

$$\leqslant \underbrace{\int_{|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \leqslant \delta} |\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)| d\eta}_{I_1(\xi)} + \underbrace{\int_{|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \geqslant \delta} |\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)| d\eta}_{I_2(\xi)},$$

we will estimate separately the two terms $I_1(\xi), I_2(\xi)$. Start with $I_1(\xi)$, if $\xi \in V$ then $|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \leqslant \delta \implies \eta \in W$ and by definition of the seminorms, we have the estimate

$$\forall N, |\widehat{t\chi}(\eta)| \leqslant \|t\|_{N,W,\chi}(1+|\eta|)^{-N}$$

then we use a trick due to Eskin, since $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we also have $|\widehat{\varphi}(\xi - \eta)| \leqslant \|\theta^{2N}\widehat{\varphi}\|_{L^\infty}(1+|\xi-\eta|)^{-2N} \leqslant C\pi_{2N}(\varphi)(1+|\xi-\eta|)^{-2N}$ where $C = d^N\text{Vol}\,(\text{supp } \varphi)$ depends on N and on the volume of supp φ. Hence

$$\int_{|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \leqslant \delta} |\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)| d\eta$$

$$\leqslant C\pi_{2N}(\varphi)\|t\|_{N,W,\chi}(1+|\xi|)^{-N}\int_{\mathbb{R}^d}\frac{(1+|\xi|)^N}{(1+|\eta|)^N(1+|\xi-\eta|)^{2N}}d\eta$$
$$\leqslant C\pi_{2N}(\varphi)\|t\|_{N,W,\chi}(1+|\xi|)^{-N}C_1$$

where $C_1 = \sup_{|\xi|}\int_{\mathbb{R}^d}\frac{(1+|\xi|)^N}{(1+|\eta|)^N(1+|\xi-\eta|)^{2N}}d\eta$ is finite when $N \geqslant d+1$. To estimate the second term $I_2(\xi)$, we use the inequality $|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\geqslant\delta$ which implies the angle beetween covectors is bounded below by an angle $\alpha = 2\arcsin\frac{\delta}{2} > 0$. By definition $\frac{\eta}{|\eta|}$ is in $\mathbb{R}^d\setminus(W\cup\{0\})$, and $\frac{\xi}{|\xi|}\in V\subset W$ hence the angle between $\frac{\xi}{|\xi|},\frac{\eta}{|\eta|}$ must be larger than $\alpha = 2\arcsin\frac{\delta}{2}$. Then the trick is to deduce lower bounds from the identity $a^2+b^2-2ab\cos c = (a-b\cos c)^2+b^2\sin^2 c = (b-a\cos c)^2+a^2\sin^2 c$, thus

$$\forall(\xi,\eta)\in(V\times^c W)\,,|(\sin\alpha)\eta|\leqslant|\xi-\eta|,|(\sin\alpha)\xi|\leqslant|\xi-\eta|.$$

We start again from the estimate on the Fourier tranform of φ, $\forall N$:

$$|\widehat{\varphi}(\xi-\eta)|\leqslant C\pi_{2N}(\varphi)(1+|\xi-\eta|)^{-2N}\leqslant C\pi_{2N}(\varphi)(1+|(\sin\alpha)\eta|)^{-N}(1+|(\sin\alpha)\xi|)^{-N}$$
$$\leqslant C\pi_{2N}(\varphi)|\sin\alpha|^{-2N}(1+|\eta|)^{-N}(1+|\xi|)^{-N}$$
$$\int_{|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}|\geqslant\delta}|\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)|d\eta$$
$$\leqslant C\pi_{2N}(\varphi)|\sin\alpha|^{-2N}(1+|\xi|)^{-N}\int_{\mathbb{R}^d}(1+|\eta|)^{-N}|\widehat{t\chi}(\eta)|d\eta$$
$$\leqslant C\pi_{2N}(\varphi)|\sin\alpha|^{-2N}(1+|\xi|)^{-N}\int_{\mathbb{R}^d}(1+|\eta|)^{-N}\|\theta^{-m}\widehat{t\chi}\|_{L^\infty}(1+|\eta|)^m d\eta$$

where m is the order of the distribution, finally

$$I_2(\xi)\leqslant C_2\pi_{2N}(\varphi)(1+|\xi|)^{-N}\|\theta^{-m}\widehat{t\chi}\|_{L^\infty}$$

where $C_2 = C|\sin\alpha|^{-2N}\int_{\mathbb{R}^d}(1+|\eta|)^{-N}(1+|\eta|)^m d\eta$ is finite when $N\geqslant m+d+1$. Gathering the two estimates, we have

$$\int_{\mathbb{R}^d}|\widehat{\varphi}(\xi-\eta)\widehat{t\chi}(\eta)|d\eta$$
$$\leqslant C\pi_{2N}(\varphi)(1+|\xi|)^{-N}\left(C_1\|t\|_{N,W,\chi}+C_2\|\theta^{-m}\widehat{t\chi}\|_{L^\infty}\right)$$

but recall the estimate on the right hand side is relevant provided $\delta > 0$ which implies $\alpha > 0$, δ depends on the choice of the cone W, the estimate is true for any cone W such that dist $({}^cW\cap\mathbb{S}^{d-1}, V\cap\mathbb{S}^{d-1})\geqslant\delta$. We have a final estimate

$$\|t\varphi\|_{N,V,\chi}\leqslant C\pi_{2N}(\varphi)(\|t\|_{N,W,\chi}+\|\theta^{-m}\widehat{t\chi}\|_{L^\infty})$$

where C is a constant which depends on N,V,W and the volume of supp φ. ■

Chapter 4

Stability of the microlocal extension.

Introduction. In Chapter 3, we saw that there is a subspace of distributions of $\mathcal{D}'(M \setminus I)$ for which we could control the wave front set of the extension $\bar{t} \in \mathcal{D}'(M)$. In fact, we proved that if $WF(t)$ satisfies the soft landing condition and $\lambda^{-s}t_\lambda$ is bounded in $\mathcal{D}'_\Gamma$, then $WF(\bar{t}) \subset \overline{WF(t)} \bigcup C$. Our assumptions obviously depend on the choice of some Euler vector field ρ. Actually, our objective in this technical part is to investigate the dependence of these conditions on the choice of ρ, their stability when we pull-back by diffeomorphisms and when we multiply distributions both satisfying these hypotheses. This is absolutely necessary in order to prove by recursion that all vacuum expectation values $\langle 0|T(a_1(x_1)...a_n(x_n))|0\rangle$ are well defined in the distributional sense.

4.1 Notation, definitions.

We denote by θ the weight function $\xi \mapsto (1 + |\xi|)$. We recall a theorem of Laurent Schwartz (see [65] p. 86 Theorem (22)) which gives a concrete representation of bounded families of distributions.

Theorem 4.1.1 *For a subset $B \subset \mathcal{D}'(\mathbb{R}^d)$ to be bounded it is neccessary and sufficient that for any domain Ω with compact closure, there is a multi-index α such that $\forall t \in B, \exists f_t \in C^0(\Omega)$ where $t|_\Omega = \partial^\alpha f_t$ and $\sup_{t\in B} \|f_t\|_{L^\infty(\Omega)} < \infty$.*

We give an equivalent formulation of the theorem of Laurent Schwartz in terms of Fourier transforms:

Theorem 4.1.2 *Let $B \subset \mathcal{D}'(\mathbb{R}^d)$.*

$$\forall \chi \in \mathcal{D}(\mathbb{R}^d), \exists m \in \mathbb{N}, \quad \sup_{t\in B} \|\theta^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty$$

$\Leftrightarrow B$ *weakly bounded in* $\mathcal{D}'(\mathbb{R}^d) \Leftrightarrow B$ *strongly bounded in* $\mathcal{D}'(\mathbb{R}^d)$.

We refer the reader to the appendix of this chapter for a proof of the above theorem. For any cone $\Gamma \subset T^\star\mathbb{R}^d$, let $\mathcal{D}'_\Gamma$ be the set of distributions with wave front set in Γ. We define the set of seminorms $\|.\|_{N,V,\chi}$ on $\mathcal{D}'_\Gamma$.

Definition 4.1.1 *For all* $\chi \in \mathcal{D}(\mathbb{R}^d)$*, for all closed cone* $V \subset (\mathbb{R}^d \setminus \{0\})$ *such that* $(supp\ \chi \times V) \cap \Gamma = \emptyset$*,* $\|t\|_{N,V,\chi} = \sup_{\xi\in V} |(1+|\xi|)^N \widehat{t\chi}(\xi)|$.

We recall the definition of the topology $\mathcal{D}'_\Gamma$ (see [1] p14),

Definition 4.1.2 *The topology of* $\mathcal{D}'_\Gamma$ *is the weakest topology that makes all seminorms* $\|.\|_{N,V,\chi}$ *continuous and which is stronger than the weak topology of* $\mathcal{D}'(\mathbb{R}^d)$*. Or it can be formulated as the topology which makes all seminorms* $\|.\|_{N,V,\chi}$ *and the seminorms of the weak topology:*

$$\forall \varphi \in \mathcal{D}\left(\mathbb{R}^d\right), |\left\langle t,\varphi\right\rangle| = P_\varphi\left(t\right) \tag{4.1}$$

continuous.

We say that B is bounded in $\mathcal{D}'_\Gamma$, if B is bounded in $\mathcal{D}'$ and if for all seminorms $\|.\|_{N,V,\chi}$ defining the topology of $\mathcal{D}'_\Gamma$,

$$\sup_{t\in B} \|t\|_{N,V,\chi} < \infty.$$

We also use the seminorms:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), \pi_m(\varphi) = \sup_{|\alpha|\leqslant m} \|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^d)},$$

$$\forall \varphi \in \mathcal{E}(\mathbb{R}^d), \forall K \subset \mathbb{R}^d, \pi_{m,K}(\varphi) = \sup_{|\alpha|\leqslant m} \|\partial^\alpha \varphi\|_{L^\infty(K)}.$$

Warning! In this chapter, we will prove that if Γ_1, Γ_2 are two closed conic sets in $T^\bullet\mathbb{R}^d$ such that $\Gamma_1 \cap -\Gamma_2 = \emptyset$, if we set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$, then the product $(t_1, t_2) \in \mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2} \mapsto t_1 t_2 \in \mathcal{D}'_\Gamma$ is **jointly and separately sequentially continuous** and **bounded** for the topology of $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$. In fact, Professor Alesker informed us that he found a counterexample which proves that the product is **not topologically bilinear continuous**. This comes from the fact that the space $\mathcal{D}'_\Gamma$ is not bornological (see [11]), for instance a bounded linear map from $\mathcal{D}'_\Gamma$ to $\mathbb{C}$ **may not be continuous**. We also prove that the pull-back by a smooth diffeomorphism $t \in \mathcal{D}'_\Gamma \mapsto t \circ \Phi \in \mathcal{D}'_{\Phi^\star\Gamma}$ is sequentially continuous and **bounded** from $\mathcal{D}'_\Gamma$ to $\mathcal{D}'_{\Phi^\star\Gamma}$.

4.2 The product of distributions.

4.2.1 Approximation and coverings.

In order to prove various theorems on the product of distributions and to discuss the action of Fourier integral operators on distributions, we should be able to approximate any conic set of $T^\bullet\mathbb{R}^d$ by some union of simple cartesian products of the form $K \times V \subset T^\bullet\mathbb{R}^d$ where K is a compact set in space and V is a closed cone in $\mathbb{R}^{d\bullet}$. We denote by $\mathbb{R}^d \overset{\pi_1}{\leftarrow} T^\star\mathbb{R}^d \overset{\pi_2}{\rightarrow} \mathbb{R}^{d*}$ the two projections on the base space $\mathbb{R}^d$ and the momentum space $\mathbb{R}^{d*}$ respectively.

Lemma 4.2.1 *Let Γ_1, Γ_2 be two* ***non intersecting*** *closed conic sets in $T^\bullet\mathbb{R}^d$. Then there is a family of closed cones $(V_{j1}, V_{j2})_{j\in J}$ and a cover $(U_j)_{j\in J}$ of $\mathbb{R}^d$ such that*

$$\Gamma_k \subset \bigcup_{j\in J} U_j \times V_{jk}$$

and $\forall j \in J, V_{j1} \cap V_{j2} = \emptyset$.

Proof — For all $x \in \mathbb{R}^d$, let $U_x(\varepsilon)$ be an open ball of radius ε around x and $\Gamma_k|_x = \Gamma_k \cap T^\bullet_x\mathbb{R}^d$. Let $V_{kx}(\varepsilon) = \pi_2\left(\Gamma_k|_{\overline{U_x(\varepsilon)}}\right)$ be a closed cone which contains $\Gamma_k|_x$. We first establish that since $\Gamma_1|_x \cap \Gamma_2|_x = \emptyset$ and $\bigcap_{\varepsilon>0} \pi_2\left(\Gamma_k|_{U_x(\varepsilon)}\right) = \Gamma_k|_x$ we may assume that we can choose ε small enough in such a way that $V_{1x} \cap V_{2x} = \emptyset$: assume that there exists a decreasing sequence $\varepsilon_n \to 0$ such that

$$\forall n, V_{1x}(\varepsilon_n) \cap V_{2x}(\varepsilon_n) = \emptyset,$$

then let $\eta_n \in V_{1x}(\varepsilon_n) \cap V_{2x}(\varepsilon_n)$ for all n where we may assume that $|\eta_n| = 1$. Using the definition of $V_{kx}(\varepsilon_n)$, there is a sequence x_{kn} s.t. $(x_{kn}; \eta_n) \in \Gamma_k|_{\overline{U_x(\varepsilon_n)}}$. $(x_{kn}; \eta_n)$ lives in the compact set $\overline{U_x(\varepsilon_0)} \times \mathbb{S}^{d-1}$ and we can therefore extract a convergent subsequence which converges to $(x_k; \eta_k) \in \Gamma_k$ since Γ_k is closed. Furthermore $\eta_1 = \eta_2 = \eta$ and $x_{kn} \in \overline{U_x(\varepsilon_n)}$ implies $\lim_{n\to\infty} x_{kn} = x$ thus $(x; \eta) \in \Gamma_1 \cap \Gamma_2$, contradiction ! For all x, we thus have $\Gamma_k|_{U_x} \subset U_x \times V_{kx}$. Since $(U_x)_{x\in\mathbb{R}^d}$ forms an open cover of $\mathbb{R}^d$, we can extract a locally finite subcover $(U_j)_{j\in J}$ and $\Gamma_k \subset \bigcup_{j\in J} U_j \times V_{jk}$. ■

Lemma 4.2.2 *Let Γ be a closed conic set in $T^\bullet\mathbb{R}^d$. For every partition of unity $(\varphi_j^2)_{j\in J}$ of $\mathbb{R}^d$ and family of functions $(\alpha_j)_{j\in J}$ in $C^\infty(\mathbb{R}^d \setminus 0)$, homogeneous of degree 0, $0 \leqslant \alpha_j \leqslant 1$ such that $\Gamma \bigcap \left(\bigcup_{j\in J} supp\ \varphi_j \times supp\ (1-\alpha_j)\right) = \emptyset$, we have*

$$\forall t \in \mathcal{D}'_\Gamma, t = \sum_{j\in J} \underbrace{\varphi_j\mathcal{F}^{-1}\left(\alpha_j\widehat{t\varphi_j}\right)}_{\textit{singular part}} + \underbrace{\varphi_j\mathcal{F}^{-1}\left((1-\alpha_j)\widehat{t\varphi_j}\right)}_{\textit{smooth part}}.$$

Proof — Let $\mathcal{D}'_\Gamma$ denote the set of all distributions with wave front set in Γ. We use the highly non trivial lemma 8.2.1 of [40]: Let $t \in \mathcal{D}'_\Gamma$, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, for any V such that $(\text{supp } \varphi \times V) \cap \Gamma = \emptyset$, we have $\forall N, \|t\|_{N,V,\varphi} < \infty$. Set the family of functions $V_j = \text{supp } (1-\alpha_j)$ then $(\text{supp } \varphi_j \times \text{supp } (1-\alpha_j)) \cap \Gamma = \emptyset$ hence $(1-\alpha_j)\widehat{t\varphi_j}$ has fast decrease at infinity and its inverse Fourier transform is a smooth function which yields the result. ∎

4.2.2 The product is bounded.

A relevant example of products of distributions first appeared in the work of Alberto Calderon in 1965. A nice exposition of this work can be found in the article [52] by Yves Meyer. Actually, Meyer defines Γ-*holomorphic* distributions as Schwartz distributions in $S'\left(\mathbb{R}^d\right)$ the Fourier transform of which is supported on a cone $\overline{\Gamma} \subset \mathbb{R}^d$ where $\Gamma \subset \mathbb{R}^d \setminus 0$ is defined by the inequality $0 < |\xi| \leqslant \delta\xi_d$ where $\delta > 1$. Notice that ξ_d must be positive and that $0 \notin \Gamma + \Gamma$. Then Meyer defines the functional spaces L^p_α which are analogs of the classical Sobolev spaces $W^{\alpha,p}$ for positive α, and proves that for any pair $(t_1, t_2) \in L^p_\alpha \times L^q_\beta$ the product t_1t_2 makes sense, t_1t_2 is Γ-*holomorphic* and belongs to the functional space $L^r_{\alpha+\beta}$ where $r^{-1} = p^{-1} + q^{-1}$. Most importantly, Meyer proves there is a *bilinear continuous mapping* P_Γ which satisfies a Hölder like estimate and coincides with the product when t_1, t_2 are Γ-*holomorphic.*

In the same spirit, we will prove bilinear estimates for the product of distributions. The bilinear estimates are formulated in terms of the seminorms $\|.\|_{N,V,\chi}$ defining the topology of $\mathcal{D}'_\Gamma$ and the seminorms:

$$\|\theta^{-m}\widehat{t\chi}\|_{L^\infty}. \tag{4.2}$$

which control boundedness in $\mathcal{D}'$ (but they do not define the weak topology of $\mathcal{D}'$). We closely follow the exposition of [23] thm (14.3).

Lemma 4.2.3 *Let Γ_1, Γ_2 be two conic sets in $T^\bullet\mathbb{R}^d$. If $\Gamma_1 \cap -\Gamma_2 = \emptyset$, then there exists a partition of unity $(\varphi_j^2)_{j\in J}$ and a family of closed cones $(W_{j1}, W_{j2})_{j\in J}$ in $\mathbb{R}^d \setminus 0$ such that $\forall j \in J, W_{j1} \cap -W_{j2} = \emptyset$ and $\Gamma_k \subset \left(\bigcup_{j\in J} supp(\varphi_j) \times W_{jk}\right), (k = 1,2)$.*

Proof — We use our approximation lemma for Γ_1 and $-\Gamma_2$. The approximation lemma gives us a pair of covers

$$\Gamma_k \subset \bigcup_{j\in J} U_j \times W_{jk}, k \in \{1,2\},$$

then pick a partition of unity $(\varphi_j^2)_{j\in J}$ subordinated to the cover $\bigcup_{j\in J} U_j$ and we are done. ∎

Lemma 4.2.4 *Let Γ_1, Γ_2 be two cones in $T^\bullet\mathbb{R}^d$ and let m_1, m_2 be given non negative integers. Assume $\Gamma_1 \cap -\Gamma_2 = \emptyset$ then for all $\chi \in \mathcal{D}(\mathbb{R}^d)$, for all $N_2 \geqslant N_1 + d + 1$ there exists C such that for all $(t_1, t_2) \in \mathcal{D}'_{\Gamma_1}(\mathbb{R}^d) \times \mathcal{D}'_{\Gamma_2}(\mathbb{R}^d)$ satisfying $\|\theta^{-m_1}\widehat{t_1\chi\varphi_j}\|_{L^\infty} < +\infty$ and $\|\theta^{-m_2}\widehat{t_2\chi\varphi_j}\|_{L^\infty} < +\infty$, we have the bilinear estimate:*

$$\|\theta^{-(m_1+m_2+d)}\widehat{t_1t_2\chi^2}(\xi)\|_{L^\infty}$$

$$\leqslant C\sum_{j\in J}\left(\|\theta^{-m_1}\widehat{t_1\chi\varphi_j}\|_{L^\infty} + \|t_1\chi\|_{N_1,V_{j1},\varphi_j}\right)\left(\|\theta^{-m_2}\widehat{t_2\chi\varphi_j}\|_{L^\infty} + \|t_2\chi\|_{N_2,V_{j2},\varphi_j}\right)$$

for some seminorms $\|.\|_{N_k,V_{jk},\varphi_j}$ of $\mathcal{D}'_{\Gamma_k}, k = 1, 2$.

Before we prove the lemma, let us explain the crucial consequence of this lemma for the product of distributions. Let $B_k, k \in \{1, 2\}$ be bounded subsets of $\mathcal{D}'_{\Gamma_k}(\mathbb{R}^d), k \in \{1, 2\}$. Then for each fixed χ, there exists a pair m_1, m_2 such that the r.h.s. of the bilinear estimate is bounded for all t_1, t_2 describing $B_1 \times B_2$ by theorem (4.4.2). Thus for each fixed $\chi^2 \in \mathcal{D}(\mathbb{R}^d)$, there exists an integer $m_1 + m_2 + d$ such that $\|\theta^{-(m_1+m_2+d)}\widehat{t_1t_2\chi^2}(\xi)\|_{L^\infty}$ is bounded for all t_1, t_2 describing $B_1 \times B_2$. Then this implies again by (4.4.2) that t_1t_2 is bounded in $\mathcal{D}'(\mathbb{R}^d)$. So the consequence of this lemma can be summarized as follows

Corollary 4.2.1 *Let Γ_1, Γ_2 be two cones in $T^\bullet\mathbb{R}^d$. Assume $\Gamma_1 \cap -\Gamma_2 = \emptyset$. Then the product $(t_1, t_2) \in \mathcal{D}'_{\Gamma_1}(\mathbb{R}^d) \times \mathcal{D}'_{\Gamma_2}(\mathbb{R}^d) \mapsto t_1t_2 \in \mathcal{D}'(\mathbb{R}^d)$ is well defined and bounded.*

Now let us return to the proof of lemma (6.4.1).
Proof — By Lemma 4.2.3 $\Gamma_k \subset \bigcup_{j\in J} \text{supp } \varphi_j \times W_{jk}, k \in \{1, 2\}$ for a partition of unity $(\varphi_j^2)_{j\in J}$ and for a family of closed cones $(W_{j1}, W_{j2})_{j\in J}$ in $\mathbb{R}^d \setminus 0$ such that $\forall j \in J, W_{j1} \cap -W_{j2} = \emptyset$. In a similar way to the construction of the approximation lemma, we have

$$t_1t_2\chi^2 = \sum_{j\in J}(\chi\varphi_jt_1)(\chi\varphi_jt_2) = \sum_{j\in J} t_{j1}t_{j2}.$$

where we set $t_{jk} = (\chi\varphi_jt_k)$. Set $\alpha_{jk}, k \in \{1, 2\}$ a smooth function on $\mathbb{R}^d \setminus \{0\}$, $\alpha_{jk} = 1$ on W_{jk}, homogeneous of degree 0 such that supp $(\alpha_{j1}) \cap$ $-$supp $(\alpha_{j2}) = \emptyset$. We decompose the convolution product $I(\xi) = \int_{\mathbb{R}^d} d\eta t_{j1}(\xi - \eta)t_{j2}(\eta)$ into four parts:

$$I_1 = \int_{\mathbb{R}^d} d\eta\alpha_{j1}\widehat{t_{j1}}(\xi - \eta)\alpha_{j2}\widehat{t_{j2}}(\eta) \tag{4.3}$$

$$I_2 = \int_{\mathbb{R}^d} d\eta(1 - \alpha_{j1})\widehat{t_{j1}}(\xi - \eta)\alpha_{j2}\widehat{t_{j2}}(\eta) \tag{4.4}$$

$$I_3 = \int_{\mathbb{R}^d} d\eta\alpha_{j1}\widehat{t_{j1}}(\xi - \eta)(1 - \alpha_{j2})\widehat{t_{j2}}(\eta) \tag{4.5}$$

$$I_4 = \int_{\mathbb{R}^d} d\eta(1 - \alpha_{j1})\widehat{t_{j1}}(\xi - \eta)(1 - \alpha_{j2})\widehat{t_{j2}}(\eta) \tag{4.6}$$

We would like to estimate $I(\xi)$ for **arbitrary** ξ. Let us first discuss the more singular term I_1. The key point is that its integrand vanishes outside the domain $|\eta| \leqslant \frac{|\xi|}{\sin\delta}$ for some δ. Indeed, we observe that $\text{supp }\alpha_{j1} \cap -\text{supp }\alpha_{j2} = \emptyset$ means that for any $(\zeta_1,\zeta_2) \in \text{supp }\alpha_{j1} \times \text{supp }\alpha_{j2}$, the angle θ between ζ_1 and ζ_2 is less than $\pi - \delta$ for a given $\delta > 0$.

Hence if $\zeta_1 = \xi - \eta \in \text{supp }\alpha_{j1}$ and $\zeta_2 = \eta \in \text{supp }\alpha_{j2}$ the angle between ζ_1 and ζ_2 is bounded from below:

$$|\zeta_1+\zeta_2|^2 = \langle \zeta_1+\zeta_2, \zeta_1+\zeta_2\rangle = |\zeta_1|^2 + |\zeta_2|^2 + 2\cos\theta|\zeta_1||\zeta_2|$$

$$= (|\zeta_1| + \cos\theta|\zeta_2|)^2 + \sin^2\theta|\zeta_2|^2 \geqslant \sin^2\theta|\zeta_2|^2 \geqslant \sin^2\delta|\zeta_2|^2,$$

hence $|\sin\delta||\eta| \leqslant |\xi|$ and $|\sin\delta||\xi-\eta| \leqslant |\xi$ by symmetry between ζ_1,ζ_2. Thus

$$|I_1| \leqslant \int_{|\xi|\geqslant|\sin\delta||\eta|} d\eta \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} (1+|\xi-\eta|)^{m_1}(1+|\eta|)^{m_2}$$

if $|\xi|$ is fixed we integrate a rational function over a ball

$$|I_1| \leqslant |\sin\delta|^{-m_1-m_2} \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} (1+|\xi|)^{m_1+m_2} \int_0^{\frac{|\xi|}{|\sin\delta|}} r^{d-1}dr$$

$$\leqslant C_1 \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} (1+|\xi|)^{m_1+m_2+d}$$

where $C_1 = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}|(\sin\delta)^{-d-m_1-m_2}|$ does not depend on t_1, t_2. We have estimated the more singular term, set $\text{supp }(1-\alpha_{jk}) = V_{jk}$, we choose α_{jk} in such a way that $V_{jk} = \overline{{}^cW_{jk}}$. The estimation of others terms is simple and relies on the key inequalities $\frac{(1+|\eta|)}{(1+|\xi|)(1+|\xi-\eta|)} \leqslant 1$ and $\frac{(1+|\xi-\eta|)}{(1+|\xi|)(1+|\eta|)} \leqslant 1$. We gather all results:

$$I_1 \leqslant \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{n}{2})} \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} (1+|\xi|)^{m_1+m_2+d}$$

$$I_2 \leqslant \|t_1\chi\|_{m_2+d+1,V_{j1},\varphi_j} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} \int_{\mathbb{R}^d} d\eta (1+|\xi-\eta|)^{-(m_2+d+1)}(1+|\eta|)^{m_2}$$

$$\leqslant \|t_1\chi\|_{m_2+d+1,V_{j1},\varphi_j} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} (1+|\xi|)^{m_2} \int_{\mathbb{R}^d} d\eta \frac{(1+|\eta|)^{m_2}}{(1+|\xi|)^{m_2}(1+|\xi-\eta|)^{(m_2+d+1)}}$$

$$I_3 \leqslant \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|t_2\chi\|_{m_1+d+1,V_{j2},\varphi_j} \int_{\mathbb{R}^d} d\eta (1+|\xi-\eta|)^{m_1}(1+|\eta|)^{-(m_1+d+1)}$$

$$\leqslant \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty} \|t_2\chi\|_{m_1+d+1,V_{j2},\varphi_j} (1+|\xi|)^{m_1} \int_{\mathbb{R}^d} d\eta \frac{(1+|\xi-\eta|)^{m_1}}{(1+|\xi|)^{m_1}(1+|\eta|)^{m_1+d+1}}$$

$$I_4 \leqslant \|t_1\chi\|_{N_1,V_{j1},\varphi_j} \|t_2\chi\|_{N_2,V_{j2},\varphi_j} (1+|\xi|)^{-N_1} \int_{\mathbb{R}^d} d\eta \frac{(1+|\xi|)^{N_1}}{(1+|\xi-\eta|)^{N_1}(1+|\eta|)^{N_2}}.$$

We write the estimates in a more compact form where we replaced the integrals by constants $(C_i)_{1\leqslant i\leqslant 4}$:

$$I_1 \leqslant C_1\|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty}\|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}(1+|\xi|)^{m_1+m_2+d} \tag{4.7}$$

$$I_2 \leqslant C_2\|t_1\chi\|_{m_2+d+1,V_{j1},\varphi_j}\|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}(1+|\xi|)^{m_2} \tag{4.8}$$

$$I_3 \leqslant C_3\|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty}\|t_2\chi\|_{m_1+d+1,V_{j2},\varphi_j}(1+|\xi|)^{m_1} \tag{4.9}$$

$$I_4 \leqslant C_4\|t_1\chi\|_{N_1,V_{j1},\varphi_j}\|t_2\chi\|_{N_2,V_{j2},\varphi_j}(1+|\xi|)^{-N_1} \tag{4.10}$$

then we summarize the whole estimate, if $N_2 \geqslant N_1+d+1$:

$$(1+|\xi|)^{-m_1-m_2-d}|I|$$

$$\leqslant C\left(\|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty}+\|t_1\chi\|_{N_1,V_{j1},\varphi_j}\right)\left(\|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}+\|t_2\chi\|_{N_2,V_{j2},\varphi_j}\right).$$

■

Lemma 4.2.5 *Let Γ_1,Γ_2 be two cones in $T^\bullet\mathbb{R}^d$ and m_1,m_2 some non negative integers. Assume $\Gamma_1\cap-\Gamma_2=\emptyset$. Set $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_1+\Gamma_2$. Then for all seminorm $\|.\|_{N,V,\chi^2}$ of $\mathcal{D}'_\Gamma$ where $N\geqslant\sup_{k=1,2}m_k+d+1$, there exists C such that for all $(t_1,t_2)\in\mathcal{D}'_{\Gamma_1}(\mathbb{R}^d)\times\mathcal{D}'_{\Gamma_2}(\mathbb{R}^d)$ satisfying $\|\theta^{-m_1}\widehat{t_1\chi}\|_{L^\infty}<\infty,\|\theta^{-m_2}\widehat{t_2\chi}\|_{L^\infty}<\infty$, we have the bilinear estimate:*

$$\|t_1t_2\|_{N,V,\chi^2}\leqslant C\sum_{j\in J}\|t_2\chi\|_{2N,V_{j2},\varphi_j}\|\theta^{-m_1}\widehat{t_1\varphi_j\chi}\|_{L^\infty}$$

$$+\|t_1\chi\|_{2N,V_{j1},\varphi_j}\|\theta^{-m_2}\widehat{t_2\varphi_j\chi}\|_{L^\infty}+\|t_1\|_{2N,V_{j1},\varphi_j}\|t_2\|_{N,V_{j2},\varphi_j}$$

for some seminorms $\|.\|_{N,V_{jk},\varphi_j}$ of $\mathcal{D}'_{\Gamma_k},k=1,2$.

Proof — Let V be a closed cone of $\mathbb{R}^d$ such that supp $\chi\times V$ does not meet $\Gamma_1\cup\Gamma_2\cup\Gamma_1+\Gamma_2$. Now, it is always possible to use the cover given by the approximation lemma *fine enough* so that for all $j\in J$, V will not meet $W_{j1}\cup W_{j2}\cup(W_{j1}+W_{j2})$. We would like to estimate $I(\xi)$ for $\xi\notin W_{j1}\cup W_{j2}\cup(W_{j1}+W_{j2})$. But $\alpha_{j2}(\eta)\alpha_{j1}(\xi-\eta)\neq0 \implies (\eta,\xi-\eta)\in W_{j2}\times W_{j1} \implies \xi=(\xi-\eta)+\eta\in W_{j1}+W_{j2}$. Thus if $\xi\notin W_{j1}+W_{j2}$ then $\alpha_{j2}(\eta)\alpha_{j1}(\xi-\eta)=0$ for all η, hence $I_1(\xi)=0$ when $\xi\in V$. We set supp $(1-\alpha_{jk})=V_{jk}$ which is a cone in which t_{jk} decreases faster than any inverse of polynomial function. By definition:

$$|(1-\alpha_{jk})\widehat{t}_{jk}|(\xi)\leqslant\|t_k\chi\|_{N,V_{jk},\varphi_j}(1+|\xi|)^{-N}$$

also for $\alpha_{jk}\widehat{t}_{jk}$ where $t_{jk}=(t_k\chi)\varphi_j$, we have:

$$|\alpha_{jk}\widehat{t}_{jk}|(\xi)\leqslant\|(1+|\xi|)^{-m_k}\widehat{t_{jk}}\|_{L^\infty}(1+|\xi|)^{m_k}$$

where m_k is the order of the compactly supported distribution $t_k\chi$. We can estimate I_4 in a simple way:

$$|I_4|(\xi) \leqslant \|t_1\chi\|_{2N,V_{j1},\varphi_j}\|t_2\chi\|_{N,V_{j2},\varphi_j}(1+|\xi|)^{-N}\int_{\mathbb{R}^d} d\eta \frac{(1+|\xi|)^N}{(1+|\xi-\eta|)^{2N}(1+|\eta|)^N}$$

$$|I_4|(\xi) \leqslant C_N\|t_1\chi\|_{2N,V_{j1},\varphi_j}\|t_2\chi\|_{N,V_{j2},\varphi_j}(1+|\xi|)^{-N},$$

where $C_N = \int_{\mathbb{R}^d} d\eta \frac{(1+|\xi|)^N}{(1+|\xi-\eta|)^{2N}(1+|\eta|)^N} \leqslant \int_{\mathbb{R}^d} d\eta\,(1+|\eta|)^{-N}$.

To estimate I_2, let us first notice that if α_{jk} were smooth at 0 then we could identify the "good function" $(1-\alpha_{j1})\widehat{t}_{j1}(\eta)$ with the Fourier transform of a Schwartz function and "the bad function" $\alpha_{j2}\widehat{t}_{j2}(\eta)$ with the Fourier transform of a distribution. Denoting by $\theta(\xi,\eta)$ the angle between ξ and η, we cut I_2 into two parts:

$$I_2(\xi) = \int_{\theta(\xi,\eta)\leqslant\delta}(1-\alpha_{j1})\widehat{t}_{j1}(\xi-\eta)\alpha_{j2}\widehat{t}_{j2}(\eta)+\int_{\theta(\xi,\eta)\geqslant\delta}(1-\alpha_{j1})\widehat{t}_{j1}(\xi-\eta)\alpha_{j2}\widehat{t}_{j2}(\eta)$$

We set the cone $W'_{kj} = \{\xi|\text{dist}\,(\frac{\xi}{|\xi|}, W_{kj}) \leqslant \delta\}$ for some $\delta > 0$ in such a way that the following sequence of inclusions holds:

$$W_{kj} \subset \text{supp}\ \alpha_{jk} \subset W'_{kj}.$$

The restrictions $\xi \in V, \eta \in \text{supp}\ \alpha_{j2}$ impose the angle $\theta(\xi,\eta)$ between them satisfies the bound $\theta \geqslant dist(V\cap\mathbb{S}^{d-1}, \text{supp}\ \alpha_{j2}\cap\mathbb{S}^{d-1}) > 0$, hence if $\delta < dist(V\cap\mathbb{S}^{d-1}, W_{j2}\cap\mathbb{S}^{d-1})$ then

$$\forall\xi\in V, I_2(\xi) = \int_{\theta(\xi,\eta)\geqslant\delta}(1-\alpha_{j1})\widehat{t}_{j1}(\xi-\eta)\alpha_{j2}\widehat{t}_{j2}(\eta),$$

but the estimate $\theta(\xi,\eta) \geqslant \delta$ exactly means that the angle between ξ,η is bounded from below hence we use the bounds

$$|\xi-\eta| \geqslant \sin\delta|\xi|, |\xi-\eta| \geqslant \sin\delta|\eta|$$

which implies

$$(1+|\xi-\eta|)^{-2N} \leqslant (1+\sin\delta|\xi|)^{-N}(1+\sin\delta|\eta|)^{-N} \leqslant (\sin\delta)^{-2N}(1+|\xi|)^{-N}(1+|\eta|)^{-N}$$

which implies the following bounds for I_2:

$$\forall\xi\in V, |I_2|(\xi)$$

$$\leqslant \int_{\theta(\xi,\eta)\geqslant\delta} d\eta\|t_1\chi\|_{2N,V_{j1},\varphi_j}(1+|\xi-\eta|)^{-2N}\|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}(1+|\eta|)^{m_2}$$

$$\leqslant \|t_1\chi\|_{2N,V_{j1},\varphi_j}\|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}(1+|\xi|)^{-N}|\sin\delta|^{-2N}\int_{\mathbb{R}^d} d\eta(1+|\eta|)^{-N}(1+|\eta|)^{m_2}.$$

Provided that $dist(V \cap \mathbb{S}^{d-1}, W_{j2} \cap \mathbb{S}^{d-1}) > \delta > 0$ and $N \geqslant m_2 + d + 1$, the integral on the right hand side absolutely converges. Setting $C_2 = |\sin\delta|^{-2N} \int_{\mathbb{R}^d} d\eta (1+|\eta|)^{-N}(1+|\eta|)^{m_2}$ yields the estimate

$$\forall \xi \in V, |I_2|(\xi) \leqslant C_2 \|t_1\chi\|_{2N,V_{j1},\varphi_j} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty}(1+|\xi|)^{-N}.$$

Now for $I_3(\xi)$, after the variable change

$$\int_{\mathbb{R}^d} d\eta |\alpha_{j1} t_{j1}(\xi-\eta)(1-\alpha_{j2})t_{j2}(\eta)| = \int_{\mathbb{R}^d} d\eta |\alpha_{j1} t_{j1}(\eta)(1-\alpha_{j2})t_{j2}(\xi-\eta)|,$$

we repeat the exact same proof as above with the roles of the indices $1, 2$ exchanged.

$$\forall \xi \in V, |I_3|(\xi) \leqslant C_3 \|t_2\chi\|_{2N,V_{j2},\varphi_j} \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty}(1+|\xi|)^{-N}$$

where $C_3 = |\sin\delta|^{-2N} \int_{\mathbb{R}^d} d\eta (1+|\eta|)^{-N}(1+|\eta|)^{m_1}$. Gathering the three terms, we obtain:

$$\forall \xi \in V, |I|(\xi) \leqslant C(\|t_2\chi\|_{2N,V_{j2},\varphi_j} \|\theta^{-m_1}\widehat{t_{j1}}\|_{L^\infty}$$

$$+\|t_1\chi\|_{2N,V_{j1},\varphi_j} \|\theta^{-m_2}\widehat{t_{j2}}\|_{L^\infty} + \|t_1\chi\|_{2N,V_{j1},\varphi_j}\|t_2\chi\|_{N,V_{j2},\varphi_j})(1+|\xi|)^{-N}.$$

■

Let us explain the boundedness properties of the product. Let $B_k, k \in \{1,2\}$ be bounded subsets of $\mathcal{D}'_{\Gamma_k}(\mathbb{R}^d), k \in \{1,2\}$. Then for each V satisfying the hypothesis of the lemma for each χ, there exists a pair (m_1, m_2) such that the r.h.s. of the bilinear estimate is bounded for all t_1, t_2 describing $B_1 \times B_2$ by theorem (4.4.2). Thus the seminorm $\|t_1 t_2\|_{N,V,\chi^2}$ is bounded for all $t_1, t_2 \in B_1 \times B_2$. The joint and partial sequential continuity of the product simply follows from the above arguments. As a corollary of the previous lemmas, we deduce the following important

Theorem 4.2.1 *Let Γ_1, Γ_2 be two cones in $T^\bullet\mathbb{R}^d$. Assume $\Gamma_1 \cap -\Gamma_2 = \emptyset$. Set $\Gamma = (\Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2))$, where $x, \xi \in \Gamma_1 + \Gamma_2$ means that $\xi = \xi_1 + \xi_2$ for some $(x, \xi_1) \in \Gamma_1, (x, \xi_2) \in \Gamma_2$. Then the product*

$$(t_1, t_2) \in D'_{\Gamma_1} \times D'_{\Gamma_2} \mapsto t_1 t_2 \in D'_\Gamma$$

is well defined and ***bounded****.*

4.2.3 The soft landing condition is stable by sum.

We have studied the boundedness properties of the product. The main theorem of Chapter 3 singled out an essential property of the wave front set of distributions which was the **soft landing condition**. Our goal in this subsection will be to check that this condition on wave front sets is *stable* by products. If $WF(t_i)_{\in\{1,2\}}$ satisfies the soft landing condition and $WF(t_1) \cap (-WF(t_2)) = \emptyset$ on $M \setminus I$, then what happens to $WF(t_1 t_2)$?

Proposition 4.2.1 *Let Γ_1, Γ_2 be two closed conic sets which both satisfy the soft landing condition and Γ_1, Γ_2 are such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Then the cone $\Gamma_1 \cup \Gamma_2 \cup \Gamma_1 + \Gamma_2$ satisfies the* ***soft landing condition****.*

Proof — We just have to prove that $\Gamma_1 + \Gamma_2$ satisfies the soft landing condition because taken individually, $\Gamma_i, \in \{1,2\}$ already satisfy the soft landing condition. We denote $(x_i, h_i; k_k, \xi_i)$ a point in $\Gamma_i, \in \{1,2\}$. We also denote $\eta_i = (k_i, \xi_i)$. In the course of the proof, we use the norm $|\eta| = |k| + |\xi|$ and the result does not depend on the choice of this norm since all norms are equivalent.

1. We start from the hypothesis that $\Gamma_i, \in \{1,2\}$ both satisfy the soft landing condition

   $$\forall i \in \{1,2\}, \exists \varepsilon_i > 0, \exists \delta_i > 0, \Gamma_i|_{K \cap |h| \leqslant \varepsilon} \subset \{|k| \leqslant \delta |h| |\xi|\}$$

   but this implies that for the points of the form $(x, h; \eta_1) + (x, h; \eta_2) = (x, h; \eta_1 + \eta_2) \in (\Gamma_1 + \Gamma_2)|_{(x,h)}$, we have the inequality

   $$|k_1 + k_2| \leqslant \sup_{\in\{1,2\}} \delta_i |h| \left(|\xi_1| + |\xi_2|\right),$$

   from now on, we set $\sup_{\in\{1,2\}} \delta_i = \delta$.

2. In order to estimate the sum $(|\xi_1| + |\xi_2|)$, we will use the fact that $\Gamma_1 \cap -\Gamma_2 = \emptyset$. This can be translated in the estimate

   $$\forall (x, h; \eta_i) \in \Gamma_i|_K, \exists \delta' > 0, \delta' \left(|\eta_1| + |\eta_2|\right) \leqslant |\eta_1 + \eta_2|$$

   $$\Longrightarrow \delta' \left(|k_1| + |k_2| + |\xi_1| + |\xi_2|\right) \leqslant |k_1 + k_2| + |\xi_1 + \xi_2|$$

   $$\Longrightarrow |\xi_1| + |\xi_2| \leqslant \frac{1 - \delta'}{\delta'} |k_1 + k_2| + \frac{1}{\delta'} |\xi_1 + \xi_2|,$$

   where we can always assume we chose $\delta' < 1$.

3. Combining the two previous estimates, we obtain

   $$|k_1| + |k_2| \leqslant \delta |h| \left(|\xi_1| + |\xi_2|\right) \leqslant \delta |h| \left(\frac{1 - \delta'}{\delta'} |k_1 + k_2| + \frac{1}{\delta'} |\xi_1 + \xi_2|\right).$$

   Now we choose ε' small enough in such a way that $\forall |h| \leqslant \varepsilon'$ $0 < \delta \varepsilon' \frac{1-\delta'}{\delta'} < 1$. Then this implies the final estimate

   $$\forall |h| \leqslant \varepsilon', |k_1 + k_2| \leqslant \frac{\delta |h|}{\delta'} (1 - \delta \varepsilon' \frac{1 - \delta'}{\delta'})^{-1} |\xi_1 + \xi_2|$$

which means $\Gamma_1 + \Gamma_2$ satisfies the **soft landing condition**. ■

4.3 The pull-back by diffeomorphisms.

Our goal in this part consists in studying the lift to $T^\star M$ of diffeomorphisms of M fixing I since the symplectomorphisms of $T^\star M$ will determine the action on wave front sets. In this section, we will work in a local chart of M in $\mathbb{R}^{n+d}$ with coordinates (x,h) where I is given by the equation $\{h=0\}$.

4.3.1 The symplectic geometry of the vector fields tangent to I and of the diffeomorphisms leaving I invariant.

We will work at the infinitesimal level within the class $\mathfrak{g}$ of vector fields tangent to I defined by Hörmander ([40] vol 3 Lemma (18.2.5)). First recall their definition in coordinates (x,h) where $I=\{h=0\}$: the vector fields X tangent to I are of the form

$$h^j a_j^i(x,h)\partial_{h^i} + b^i(x,h)\partial_{x^i}$$

and they form an infinite dimensional Lie algebra denoted by $\mathfrak{g}$ which is a Lie subalgebra of $Vect(M)$. Actually, these vector fields form a module over the ring $C^\infty(M)$ finitely generated by the vector fields $h^i\partial_{h^j}, \partial_{x^i}$. This module was defined by Melrose and is associated to a vector bundle called the Tangent Lie algebroid of I. This module is naturally filtered by the vanishing order of the vector field on I.

Definition 4.3.1 *Let $\mathcal{I}$ be the ideal of functions vanishing on I. For $k\in\mathbb{N}$, let F_k be the submodule of vector fields tangent to I defined as follows, $X\in F_k$ if $X\mathcal{I}\subset\mathcal{I}^{k+1}$.*

This definition of the filtration is completely coordinate invariant. We also immediately have $F_{k+1}\subset F_k$. Note that $F_0=\mathfrak{g}$.

Cotangent lift of vector fields.

We recall the following fact, any vector field $X\in Vect(M)$ lifts functorially to a *Hamiltonian vector field* $X^\star\in Vect(T^\star M)$ (for more on Hamiltonian vector fields, see [2] 3.5 page 14) by the following procedure

$$X = X^i\frac{\partial}{\partial z^i}\in Vect(M)\overset{\sigma}{\mapsto}\sigma(X)=X^i\xi_i\in C^\infty(T^\star M)$$

$$\mapsto X^\star=\{\sigma(X),.\}=X^i\frac{\partial}{\partial z^i}-\xi_i\frac{\partial X^i}{\partial z^i}\frac{\partial}{\partial\xi_i},$$

where $\{.,.\}$ is the Poisson bracket of $T^\star M$. Notice the projection on M of $X^\star$ is X and $X^\star$ is **linear** in the cotangent fibers. This means the action of vector fields is lifted to an action by Hamiltonian symplectomorphisms of $T^\star M$. The map $X\in\mathfrak{g}\mapsto\sigma(X)\in C^\infty(T^\star M)$ from the Lie algebra $\mathfrak{g}$ to the *Poisson ideal* $\mathcal{I}_{(TI)^\perp} \subset C^\infty(T^\star M)$ can be interpreted as a "universal" **moment map** in Poisson geometry since to each element X of the Lie algebra $\mathfrak{g}$ which acts symplectically as a vector field $X^\star \in Vect(T^\star M)$, we associate a function which is the Hamiltonian of $X^\star$ (as explained to us by Mathieu Stiénon).

Lemma 4.3.1 *Let X be a vector field in $\mathfrak{g}$. Then $X^\star$ is tangent to the conormal $(TI)^\perp$ of I and the symplectomorphism $e^{X^\star}$ leaves the conormal* ***globally invariant****. In particular, if $X \in F_2$, then $X^\star$ vanishes on the conormal $(TI)^\perp$ of I and $(TI)^\perp$ is contained in the set of fixed points of the symplectomorphism $e^{X^\star}$.*

Proof — Any vector in $\mathfrak{g}$ admits the decomposition $h^j a^i_j(x,h)\partial_{h^i} + b^i(x,h)\partial_{x^i}$. Thus the symbol map $\sigma(X) \in C^\infty(T^\star M)$ equals $h^j a^i_j(x,h)\xi_i + b^i(x,h)k_i$. This function vanishes on the conormal bundle $(TI)^\perp$ which is a Lagrangian submanifold. Now we are reduced to the following problem: given a function f in a symplectic manifold which vanishes along a Lagrangian submanifold C, what can be said about the symplectic gradient $\nabla_\omega f$ along C ? Since $f|_L = 0$, for all $v \in TL$, $df(v) = 0$. But $\forall v \in TL, 0 = df(v) = \omega(\nabla_\omega f, v)$ which means that $\nabla_\omega f$ is in the orthogonal of TL for the symplectic form ω. Since L is a **Lagrangian** submanifold of $T^\star M$, this orthogonal is equal to TL, finally $\nabla_\omega f \in TL$. If $X \in F_1$, then $\sigma(X) = h^j h^i a^l_{ji}(x,h)\xi_l + h^i b^l_i(x,h)k_l$ by the Hadamard lemma. The symplectic gradient $X^\star$ is given by the formula

$$X^\star = \frac{\partial\sigma(X)}{\partial k_i}\partial_{x^i} - \frac{\partial\sigma(X)}{\partial x^i}\partial_{k_i} + \frac{\partial\sigma(X)}{\partial \xi_i}\partial_{h^i} - \frac{\partial\sigma(X)}{\partial h^i}\partial_{\xi_i},$$

thus $X^\star = 0$ when $k = 0, h = 0$ which means $X^\star = 0$ on the conormal $(TI)^\perp$ of I. ■

Proposition 4.3.1 *Let ρ_1, ρ_2 be two Euler vector fields and $\Phi(\lambda) = e^{-\log\lambda\rho_1} \circ e^{\log\lambda\rho_2}$. Then the cotangent lift $T^\star\Phi(\lambda)$ restricted to $(TI)^\perp$ is the identity map:*

$$T^\star\Phi(\lambda)|_{(TI)^\perp} = Id|_{(TI)^\perp}.$$

In particular, the diffeomorphism $\Psi = \Phi(0)$ (Corollary 1.4.1) which conjugates ρ_1 with ρ_2 satisfies the same property.

Proof — Let us set

$$\Phi(\lambda) = e^{-\log\lambda\rho_1} \circ e^{\log\lambda\rho_2} \tag{4.11}$$

which is a family of diffeomorphisms which depends smoothly in $\lambda \in [0,1]$ according to 1.4.2, then $\Phi(0)$ is the diffeomorphism which locally conjugates ρ_1 and ρ_2 (Corollary 1.4.1). The proof is similar to the proof of proposition 1.4.2, $\Phi(\lambda)$ satisfies the differential equation:

$$\lambda\frac{d\Phi(\lambda)}{d\lambda} = e^{-\log\lambda\rho_1}(\rho_2 - \rho_1)\, e^{\log\lambda\rho_1}\Phi(\lambda) \text{ where } \Phi(1) = Id \tag{4.12}$$

we reformulated this differential equation as

$$\frac{d\Phi(\lambda)}{d\lambda} = X(\lambda)\Phi(\lambda), \Phi(1) = Id \tag{4.13}$$

where the vector field $X(\lambda) = \frac{1}{\lambda}e^{-\log\lambda\rho_1}(\rho_2 - \rho_1)e^{\log\lambda\rho_1}$ depends smoothly in $\lambda \in [0,1]$. The cotangent lift $T^\star\Phi_\lambda$ satisfies the differential equation

$$\frac{dT^\star\Phi(\lambda)}{d\lambda} = X^\star(\lambda)T^\star\Phi(\lambda), T^\star\Phi(1) = Id \tag{4.14}$$

Notice that $\forall\lambda \in [0,1], X(\lambda) \in F_1$ which implies that for all λ the lifted Hamiltonian vector field $X^\star(\lambda)$ will vanish on $(TI)^\perp$ by the lemma (4.3.1). Since $T^\star\Phi(1) = Id$ obviously fixes the conormal, this immediately implies that $\forall\lambda, T^\star\Phi(\lambda)|_{(TI)^\perp} = Id|_{(TI)^\perp}$. ■

4.3.2 The pull-back is bounded.

The problem we solve. We start from a distribution $t \in \mathcal{D}'(M \setminus I)$ such that $WF(t)$ satisfies the soft landing condition. We assumed that there exists a generalized Euler ρ_1 and a small neighborhood $\mathcal{V}$ of I such that $\lambda^{-s}e^{-\log\lambda\rho_1 *}t$ is bounded in $\mathcal{D}'_\Gamma(\mathcal{V} \setminus I)$ where $\Gamma = \bigcup_{\lambda\in(0,1]} WF(e^{\log\lambda\rho_1\star}t)$. Under these conditions, by the main theorem of Chapter 3, we know that the extension $\bar{t}$ is well defined, $WF(\bar{t}) \subset \overline{WF(t)} \cup C$ and for every $s' < s$, $\lambda^{-s'}e^{\log\lambda\rho_1}\bar{t}$ is bounded in $\mathcal{D}'_{\overline{\Gamma}\cup C}(\mathcal{V})$. We proved (Proposition 1.4.2 Chapter 1) that when we change the Euler vector field from ρ_1 to ρ_2, we have:

$$\lambda^{-s}e^{\log\lambda\rho_2 *}t = \Phi(\lambda)^\star \underbrace{\left(\lambda^{-s}e^{\log\lambda\rho_1 *}t\right)}_{\text{bounded in } \mathcal{D}'_{\Gamma_1}}.$$

The above equation motivates us to study a more general question, is the image of a bounded set in $\mathcal{D}'_\Gamma$ by a diffeomorphism Φ still a bounded family in $\mathcal{D}'_{\Phi^\star\Gamma}$?

4.3.3 The action of Fourier integral operators.

Fourier integral operators are abbreviated FIO. In this section, we will work exclusively in $\mathbb{R}^d$ since our problem is local. To solve our problem, we will have to revisit a deep theorem of Hörmander (see [40] theorem 8.2.4) which describes the wave front set of distributions under pull back. However, we will reprove a variant of this theorem which is tailored for applications in QFT. First, we prove the theorem for a specific subclass of FIO (as discussed in [23]) which contains the space of diffeomorphisms and we also give explicit bounds for the seminorms of $\mathcal{D}'_\Gamma$. We deliberately choose to discuss everything in the language of canonical relations and symplectomorphisms since these are at the core of the geometric ideas involved in the proof.

A quick reminder about the formalism of FIO.

We recall the definition of a specific class of FIO following [23]. And we will frequently use several notions that can be found in [23].

The definition of Eskin's FIO. We adapt the definition of [23] to our context, we consider operators of the form:

$$U : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}'(\mathbb{R}^d) \mapsto \mathcal{D}'(\mathbb{R}^d)$$

$$(\varphi, t) \mapsto U_\varphi t = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\eta e^{iS(x,\eta)} a(x,\eta) \widehat{t\varphi}(\eta) \tag{4.15}$$

where S is smooth, homogeneous of degree 1 in η and $\det \frac{\partial^2 S}{\partial x \partial \eta} \neq 0$, we do not assume $a = 0$ if $|\eta| < 1$ since for diffeomorphisms $a = 1$, and this does only change the FIO modulo smoothing operator (see [23] p. 330). The Schwartz kernel of U_φ is the Fourier distribution which by a slight abuse of notation reads:

$$U_\varphi(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\eta e^{iS(x,\eta) - iy.\eta} a(x, \eta) \varphi(y).$$

See [23] p. 341.

Lemma 4.3.2 *Let Φ be a diffeomorphism of $\mathbb{R}^d$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then there exists an operator U_φ as in 4.15 such that $\forall t \in \mathcal{D}'(\mathbb{R}^d)$, $U_\varphi(t) = \Phi^\star(t\varphi)$.*

We will later choose φ as an element of an ad hoc partition of unity defined by the approximation lemmas (4.2.1,4.2.2). *Proof* — Our proof follows the strategy outlined in [17] proposition (1.3.3). The idea is to write down $t\varphi$ as the inverse Fourier transform of $\widehat{t\varphi}$.

$$t\varphi = \mathcal{F}^{-1}\left(\widehat{t\varphi}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\eta e^{ix.\eta} \widehat{t\varphi}(\eta)$$

Now, we pull-back $t\varphi$ by the diffeomorphism Φ :

$$\Phi^*(t\varphi)(x) = \Phi^* \mathcal{F}^{-1}\left(\widehat{t\varphi}\right)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\eta e^{i\Phi(x).\eta} \widehat{t\varphi}(\eta)$$

Now setting $S(x;\eta) = \Phi(x).\eta$, we recognize the phase function S appearing in (4.15). ■

In the following, given a generating function S, we denote by σ the canonical transformation defined by:

$$\sigma : (y;\eta) \mapsto (x;\xi), \xi = \frac{\partial S}{\partial x}(x,\eta), y = \frac{\partial S}{\partial \eta}(x,\eta), \tag{4.16}$$

see Equation (61.2) p. 330 in [23].

Theorem 4.3.1 *Let $(t_\mu)_\mu$ be bounded in $\mathcal{D}'_\Gamma(\Omega), \Omega \subset \mathbb{R}^d$. Let U be a* ***proper*** *operator as defined in (4.15) with amplitude $a = 1$ and generating function S and σ the corresponding canonical relation. Then $\left(Ut_\mu\right)_\mu$ is bounded in $D'_{\sigma\circ\Gamma}(\Omega)$.*

We will decompose the proof of the theorem in many different lemmas. Our strategy goes as follows, we have some bounds on $\widehat{t\varphi}$ where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ because we know that $t \in \mathcal{D}'_\Gamma$ by the hypothesis of the theorem and we want to deduce from these bounds some estimates on the Fourier transform $\mathcal{F}\left(\chi U\left(t\varphi\right)\right)$. We first prove a lemma which gives an estimate of $WF(U\left(t\varphi\right))$.

Lemma 4.3.3 *Let U be a* ***proper*** *operator as defined in (4.15) with amplitude $a = 1$ and generating function S, σ the corresponding canonical transformation and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Then for all $t \in \mathcal{D}'_\Gamma$, $WF(U_\varphi t) \subset \sigma \circ \Gamma$.*

Proof — We denote by $(y;\eta)$ and $(x;\xi)$ the coordinates in $T^\star\mathbb{R}^d$. Let t be a distribution and U a FIO of the form (4.15) with phase function $S(x;\eta) - \langle y, \eta\rangle$. Then Theorem 63.1 in Eskin (see [23] p. 340) expresses $WF(U_\varphi t)$ in terms of the image $\sigma \circ WF(t\varphi)$ of $WF(t\varphi)$ by the canonical relation σ generated by S. To apply the theorem of Eskin, we use the fact that $t\varphi$ compactly supported

$$\implies \|\theta^{-m}\widehat{t\varphi}\|_{L^\infty} < +\infty \implies \theta^{-m-\frac{d+1}{2}}\widehat{t\varphi} \in L^2(\mathbb{R}^d) \Leftrightarrow \widehat{t\varphi} \in H^{-m-\frac{d+1}{2}}.$$

$$U_\varphi t(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} dy d\eta e^{i[S(x;\eta)-y.\eta]} t\varphi(y) \tag{4.17}$$

$$\sigma : (y;\eta) \mapsto (x;\xi), \xi = \frac{\partial S}{\partial x}(x,\eta), y = \frac{\partial S}{\partial \eta}(x,\eta). \tag{4.18}$$

The canonical transformation is the same as equation 61.2 p. 330 in [23]. For convenience, we will write in local coordinates $\sigma(y,\eta) = (x(y,\eta), \xi(y,\eta))$. In the particular case of a diffeomorphism $x \mapsto \Phi(x)$,

$$\frac{\partial S}{\partial \eta}(x,\eta) = \Phi(x), \frac{\partial S}{\partial x}(x,\eta) = \eta \circ d\Phi$$

and the corresponding family of canonical relations is

$$\sigma : (y,\eta) \mapsto (\Phi^{-1}(y), \eta \circ d\Phi). \tag{4.19}$$

■

Motivated by this result, we will test $\Phi^*(t\varphi)$ on seminorms $\|.\|_{N,V,\chi}$, for a cone V and test function χ such that supp $\chi \times V$ does not meet $\sigma \circ \Gamma$.

Lemma 4.3.4 *Let U be given by 4.15, σ the corresponding canonical relation, m a nonnegative integer, $\alpha \in C^\infty(\mathbb{R}^d \setminus 0)$, homogeneous of degree 0,*

$\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $V \subset (\mathbb{R}^d \backslash 0)$ a closed cone. If $(supp\ \chi \times V) \bigcap \sigma \circ (supp\ \varphi \times supp\ \alpha) = \emptyset$ and $(supp\ \varphi \times supp\ (1-\alpha)) \bigcap \Gamma = \emptyset$ then for all N, there exists C_N s.t. for all $t \in \mathcal{D}'_\Gamma$ satisfying $\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty} < +\infty$:

$$\|U\left(t\varphi\right)\|_{N,V,\chi} \leqslant C_N(1+|\xi|)^{-N}\left(\|\theta^{-m}\widehat{t\varphi}\|_{L^\infty} + \|t\|_{N+d+1,W,\varphi}\right) \tag{4.20}$$

where $W = supp(1-\alpha)$.

Proof — Our method of proof is based on the method of stationary phase and a geometric interpretation. In the course of our proof, we will explain why constants appearing in all our estimates do not depend on t but only on U and Γ. This is the only way to obtain an estimate which is valid for families $(t_\mu)_\mu$ bounded in $\mathcal{D}'_\Gamma$. In order to bound $\|U(t\varphi)\|_{N,V,\chi}$, we must first compute the Fourier transform of $\chi U(t\varphi)$:

$$\mathcal{F}\left(\chi U(t\varphi)\right)(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} dx d\eta \chi(x) e^{i[S(x;\eta)-x.\xi]}\widehat{t\varphi}(\eta) \tag{4.21}$$

We then extract the *oscillatory integral* on which we will apply the method of stationary phase:

$$I(\xi,\eta) = \int_{\mathbb{R}^d} dx e^{i[S(x;\eta)-x.\xi]}\chi(x) = \int_{\mathbb{R}^d} dx e^{i\psi(x,\xi,\eta)}\chi(x),$$

where the phase $\psi(x,\xi,\eta) = [S(x;\eta) - x.\xi]$. We reformulate the expression giving $\mathcal{F}\left(\chi U\left(t\varphi\right)\right)(\xi)$ in terms of the oscillatory integral $I(\xi,\eta)$:

$$\mathcal{F}\left(\chi U\left(t\varphi\right)\right)(\xi) = \int_{\mathbb{R}^d} d\eta I(\xi,\eta)\widehat{t\varphi}(\eta).$$

Then the idea is to split the integral in two parts, in one part the oscillatory integral $I(\xi,\eta)$ behaves nicely and decreases fastly at infinity, ie $\forall N, (1+|\xi|+|\eta|)^N I(\xi,\eta)$ is bounded. In the second part, the oscillatory integral is bounded but this domain corresponds to the codirections in which $\widehat{t\varphi}$ has fast decrease at infinity. The method of stationary phase states (see [70] p. 330,341) that the integral I is rapidly decreasing in the codirections (ξ,η) for which ψ is **noncritical**, i.e. $d_x\psi(x;\xi,\eta) \neq 0$. We compute the critical set of the phase

$$d_x\psi(x;\xi,\eta) = d_xS(x,\eta) - \xi.$$

Hence the critical set $d_x\psi = 0$ is given by the equations

$$\{(\eta,\xi)|d_xS(x,\eta) - \xi = 0, x \in \text{supp } \chi\}, \tag{4.22}$$

we thus naively set

$$\forall \xi,\ \Sigma(\xi) := \{(y,\eta)|\exists x \in \text{supp } \chi, d_xS(x,\eta) - \xi = 0, y = \frac{\partial S}{\partial \eta}(x,\eta)\}. \tag{4.23}$$

Motivated by the geometric relation between the generating function S and the canonical relation σ (by Equation (4.16)), we interpret $\Sigma(\xi)$ in terms of the canonical transformation σ:

$$\Sigma(\xi) = \{(y,\eta)|\exists x \in \text{supp } \chi, \sigma(y,\eta) = (x,\xi)\} \tag{4.24}$$

$$\text{or} \quad \Sigma(\xi) = \sigma^{-1} \circ (\text{supp } \chi \times \{\xi\}) \,. \tag{4.25}$$

Hence $\Sigma(\xi)$ is the inverse image of supp $\chi \times \{\xi\}$ by the canonical relation σ. Let us recall that π_2 projects $T^\bullet\mathbb{R}^d$ on the second factor $\mathbb{R}^{d\star}$. We define

$$R(\xi) = \pi_2 (\Sigma(\xi)) = \{\eta|\exists x \in \text{supp } \chi, d_x S(x,\eta) - \xi = 0\}$$

which has the following analytic interpretation, for fixed ξ, $R(\xi)$ contains the critical set ("bad η's") of $I(\xi,\eta)$. We **admit temporarily** that

$$\sigma \circ (\text{supp } \varphi \times \text{supp } \alpha) \bigcap (\text{supp } \chi \times V) = \emptyset$$

implies supp α does not meet $\bigcup_{\xi\in V} R(\xi)$ (we will prove this claim in Lemma (4.3.5)). We are led to define a neighborhood $R_\varepsilon(\xi)$ of $R(\xi)$ for which $\forall \xi \in V, R_\varepsilon(\xi) \cap \text{supp } \alpha = \emptyset$:

$$R_\varepsilon(\xi) = \{\eta|\exists x \in \text{supp } \chi, |d_x S(x,\eta) - \xi| \leqslant \varepsilon\}.$$

Denote by $R^c_\varepsilon(\xi)$ the complement of $R_\varepsilon(\xi)$.

$$R^c_\varepsilon(\xi) = \{\eta|\forall (x,\xi) \in \text{supp } \chi \times V, |d_x S(x;\eta) - \xi| > \varepsilon\}$$

$$R^c_\varepsilon(\xi) = \{\eta|\forall (x,\xi) \in \text{supp } \chi \times V, |d_x \psi(\xi,\eta)| > \varepsilon\}.$$

We use the following result in Duistermaat, $\forall N, \exists C_N$ s.t.

$$\forall (\xi,\eta) \in V \times R^c_\varepsilon(\xi), \, |I(\xi,\eta)| \leqslant C_N \left(1 + |\eta| + |\xi|\right)^{-N}. \tag{4.26}$$

The proof of this result is based on the fact that we are away from the critical set $R(\xi)$ and from application of the stationary phase ([17] Proposition 2.1.1 p. 11). The constant C_N depends only on N, χ, S, ε.

Recall we made the assumption there is a function $\alpha \in C^\infty(\mathbb{R}^n \setminus 0)$, homogeneous of degree 0 such that $\forall \xi \in V, R_\varepsilon(\xi)$ does not meet supp α, and supp $\varphi \times$ supp $(1-\alpha)$ does not meet Γ. We cut the Fourier transform in two pieces:

$$I(\xi) = \mathcal{F}\left(\chi U \left(t_\mu \varphi_j\right)\right)(\xi) = I_1 + I_2$$

where

$$I_1(\xi) = \int_{R_\varepsilon(\xi)} d\eta I(\xi,\eta) \widehat{t\varphi}(\eta) \tag{4.27}$$

$$I_2(\xi) = \int_{R^c_\varepsilon(\xi)} d\eta I(\xi,\eta) \widehat{t\varphi}(\eta). \tag{4.28}$$

Observe $I_1(\xi)=\int_{R_\varepsilon(\xi)} d\eta I(\xi,\eta)\alpha\widehat{t\varphi}(\eta)+\int_{R_\varepsilon(\xi)} d\eta I(\xi,\eta)(1-\alpha)\widehat{t\varphi}(\eta)=\int_{R_\varepsilon(\xi)} d\eta I(\xi,\eta)(1-\alpha)\widehat{t\varphi}(\eta)$ since we assumed $\forall\xi\in V,\text{supp }\alpha\cap R_\varepsilon(\xi)=\emptyset$. By Paley–Wiener theorem, we know that $\exists m, \|\theta^{-m}\widehat{t\varphi}\|_{L^\infty}<\infty$. We use this inequality and stationary phase estimate (4.26)

$$|I_2|(\xi)=|\int_{R^c_\varepsilon(\xi)} d\eta I(\xi,\eta)\widehat{t\varphi}(\eta)|\leqslant C_{N+m+d+1}\int_{R^c_\varepsilon(\xi)} d\eta(1+|\eta|+|\xi|)^{-N-m-d-1}|\widehat{t\varphi}(\eta)|$$

$$\leqslant C_{N+m+d+1}\int_{R^c_\varepsilon(\xi)} d\eta(1+|\eta|+|\xi|)^{-N-m-d-1}(1+|\eta|)^m\|\theta^{-m}\widehat{t\varphi}\|_{L^\infty}$$

$$\leqslant C_{N+m+d+1}(1+|\xi|)^{-N}\|\theta^{-m}\widehat{t\varphi}\|_{L^\infty}\int_{\mathbb{R}^d} d\eta(1+|\eta|)^{-d-1}$$

hence $I_2(\xi)\leqslant C'_{N+m+d+1}(1+|\xi|)^{-N}\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty}$ where $C'_{N+m+d+1}$ is a constant which depends only on N, χ, S, ε. Now to estimate I_1, set $W:=\text{supp }(1-\alpha)$:

$$I_1(\xi)=\int_{R_\varepsilon(\xi)} d\eta I(\xi,\eta)(1-\alpha)\widehat{t\varphi}(\eta)$$

by a change of variable in (4.27) so that η does appear on the right hand side,

$$|I_1(\xi)|\leqslant\int_{\mathbb{R}^d} dx|\chi(x)|\int_{R_\varepsilon(\xi)} d\eta|(1-\alpha)\widehat{t\varphi}(\eta)|$$

because $|I(\xi,\eta)|\leqslant\int_{\mathbb{R}^d} dx|\chi(x)|$,

$$|I_1(\xi)|\leqslant\int_{\mathbb{R}^d} dx|\chi(x)|\int_{R_\varepsilon(\xi)} d\eta\|t\|_{N,W,\varphi}\,(1+|\eta|)^{-N}.$$

Recall the definition of $R_\varepsilon(\xi)=\{\eta|\exists x\in\text{supp }\chi,|d_xS(x,\eta)-\xi|\leqslant\varepsilon\}$. The defining inequality $|d_xS(x,\eta)-\xi|\leqslant\varepsilon$ implies that on $R_\varepsilon(\xi)$:

$$|d_xS(x;\eta)-\xi|\leqslant\varepsilon\implies|\xi|-\varepsilon\leqslant|d_xS(x;\eta)|\leqslant|\xi|+\varepsilon.$$

This estimate is relevant if $|\xi|>\varepsilon$. Then we use the fact that $\eta\mapsto d_xS(x,\eta)$ does not meet the zero section when $\eta\neq 0$ and depends smoothly on $x\in\text{supp }\chi$ (in the case of a diffeomorphism, we find $d_xS(x,\eta)=\eta\circ d\Phi(x)$), so there is a constant $c>0$ such that

$$\forall(x,\eta)\in\text{supp }\chi\times\mathbb{R}^d, c^{-1}|\eta|\leqslant|d_xS(x,\eta)|\leqslant c|\eta|. \tag{4.29}$$

Combining with the previous estimate gives $\forall\xi\in V,\forall\eta\in R_\varepsilon(\xi),|\xi|-\varepsilon\leqslant c|\eta|$ which can be translated as the inclusion of sets

$$R_\varepsilon(\xi)\subset\{c^{-1}(|\xi|-\varepsilon)\leqslant|\eta|\}=\mathbb{R}^d\setminus B\left(0,\frac{|\xi|-\varepsilon}{c}\right) \tag{4.30}$$

$$I_1(\xi) \leqslant \int_{\mathbb{R}^d} dx|\chi(x)| \int_{c^{-1}(|\xi|-\varepsilon)\leqslant|\eta|} d\eta \|t\|_{N+d+1,W,\varphi} (1+|\eta|)^{-N-d-1}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\int_{\mathbb{R}^d} dx|\chi(x)|\right) \|t\|_{N+d+1,W,\varphi} \int_{c^{-1}(|\xi|-\varepsilon)}^{\infty} (1+r)^{-N-d-1} r^{d-1} dr$$

$$\leqslant \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\int_{\mathbb{R}^d} dx|\chi(x)|\right) \|t\|_{N+d+1,W,\varphi} \int_{c^{-1}(|\xi|-\varepsilon)}^{\infty} r^{-N-2} dr$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\int_{\mathbb{R}^d} dx|\chi(x)|\right) \|t\|_{N+d+1,W,\varphi} \frac{\left(c^{-1}(|\xi|-\varepsilon)\right)^{-N-1}}{N+1}$$

$$\leqslant C_{N+1} \|t\|_{N+d+1,W,\varphi} (1+|\xi|)^{-N-1}.$$

where C_{N+1} does not depend on t but only on Γ. ■

In the previous lemma, we made two assumptions that we are going to prove, we recall some useful definitions:

$$\forall \xi \in V, \Sigma(\xi) = \sigma^{-1} \circ (\text{supp } \chi \times \{\xi\}), R(\xi) = \pi_2(\Sigma(\xi))$$

and $R_\varepsilon(\xi)$ is a family of neighborhoods of $R(\xi)$ which tends to $R(\xi)$ as $\varepsilon \to 0$.

Lemma 4.3.5 *For any closed conic set V and $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $(supp\ \chi \times V) \cap (\sigma \circ \Gamma) = \emptyset$, there exists a pseudodifferential partition of unity $(\alpha_j, \varphi_j)_j$ such that*

$$\forall \xi \in V, R_\varepsilon(\xi) \cap supp\ \alpha_j = \emptyset \tag{4.31}$$

$$\Gamma \subset \bigcup_{j \in J} supp\ \varphi_j \times supp\ \alpha_j. \tag{4.32}$$

Proof — χ and V are given in such a way that

$$(\text{supp } \chi \times V) \cap (\sigma \circ \Gamma) = \emptyset \underset{\sigma \text{ diffeo}}{\Leftrightarrow} \sigma^{-1} (\text{supp } \chi \times V) \cap \Gamma = \emptyset.$$

We then use Lemma 4.2.1, 4.2.2 to cover Γ by $\left(\bigcup_j \text{supp } \varphi_j \times \text{supp } \alpha_j\right)$ where $\alpha_j \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is homogeneous of degree 0 and we choose the cover fine enough in such a way that

$$\left(\sigma^{-1} \circ (\text{supp } \chi \times V)\right) \cap \left(\bigcup_j \text{supp } \varphi_j \times \text{supp } \alpha_j\right) = \emptyset.$$

But this implies

$$\forall j, \left(\bigcup_{\xi \in V} \sigma^{-1} (\text{supp } \chi \times \{\xi\})\right) \cap (\text{supp } \varphi_j \times \text{supp } \alpha_j) = \emptyset$$

$$\Leftrightarrow \left(\bigcup_{\xi\in V}\Sigma(\xi)\right)\cap(\text{supp }\varphi_j\times\text{supp }\alpha_j)=\emptyset \implies \left(\bigcup_{\xi\in V}R(\xi)\right)\cap\text{supp }\alpha_j=\emptyset,$$

the last line follows by projecting with π_2. Finally by choosing ε small enough, we can always assume $\forall\xi\in V, R_\varepsilon(\xi)\cap\text{supp }\alpha_j=\emptyset$: assume the converse holds, i.e. $\forall n$, $\exists\xi_n\in V$, $\exists x_n\in\text{supp }\chi$, $\exists\eta_n\in R_{\frac{1}{n}}(\xi_n)\cap\text{supp }\alpha_j$ w.l.g. assume $|\eta_n|=1$ then by definition of $R_{\frac{1}{n}}(\xi_n)$, we find that

$$|\xi_n-\frac{\partial S}{\partial x}(x_n,\eta_n)|<\frac{1}{n}$$

and estimate (4.29) $\implies$ $|d_xS(x_n,\eta_n)|\leqslant c|\eta_n|=c \implies |\xi_n|<c+\frac{1}{n}$. This means the sequence (x_n,ξ_n,η_n) lives in a compact set, thus we can extract a subsequence which converges to $(x,\xi,\eta)\in\text{supp }\chi\times V\times\text{supp }\alpha_j$ and $\eta\in R(\xi)\cap\text{supp }\alpha_j$, contradiction ! ∎

Then we give the final lemma which concludes the proof of theorem (4.3.1).

Lemma 4.3.6 *Let U be an operator given in (4.15) with symbol $a=1$ and σ the corresponding canonical transformation. For any closed conic set V and $\chi\in\mathcal{D}(\mathbb{R}^d)$ such that (supp $\chi\times V)\cap(\sigma\circ\Gamma)=\emptyset$, there exists a finite family of seminorms $(\|.\|_{N,W_j,\varphi_j})_{j\in J'}$ for $\mathcal{D}'_\Gamma$ such that $\forall N,\exists C_N,\forall t\in\mathcal{D}'_\Gamma$ s.t. $\forall j\in J',\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty}<+\infty$:*

$$\|Ut\|_{N,V,\chi}\leqslant\sum_{j\in J'}C_N\left(\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty}+\|t\|_{N+2d+1,W_j,\varphi_j}\right).$$

Proof — There is still a problem due to the noncompactness of the support of t, there is no reason the sum $\sum_{j\in J}t\varphi_j$ ($(\varphi_j)_{j\in J}$ is a partition of unity of $\mathbb{R}^d$ given by Lemma 4.3.5) should be finite thus we do not necessarily have one fixed m for which $\forall j\in J,\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty}<+\infty$. However, $\chi Ut=\sum_{j\in J'}\chi Ut\varphi_j$ where J' is any subset of J such that $\sum_{j\in J'}\varphi_j=1$ on the **compact** set $\pi_1\left(\sigma^{-1}(\text{supp }\chi\times V)\right)$, thus J' can be chosen finite. Now we use the pseudodifferential partition of unity indexed by J' to patch everything together:

$$\forall\xi\in V,|\mathcal{F}(\chi Ut)|(\xi)\leqslant\sum_{j\in J'}|\int_{\mathbb{R}^{2d}}dxd\eta e^{i[S(x;\eta)-x.\xi]}\chi(x)\widehat{t\varphi_j}(\eta)|$$

$$\leqslant\sum_{j\in J'}C_N(1+|\xi|)^{-N}\left(\|\theta^{-m}\widehat{t\varphi_j}\|_{L^\infty}+\|t\|_{N+2d+1,W_j,\varphi_j}\right)$$

by estimate (4.20) where $W_j=\text{supp }(1-\alpha_j)$. And this final estimate generalizes directly to families of distributions $(t_\mu)_\mu$:

$$\|Ut_\mu\|_{N,V,\chi}\leqslant\sum_{j\in J'}C_N\left(\|\theta^{-m}\widehat{t_\mu\varphi_j}\|_{L^\infty}+\|t_\mu\|_{N+2d+1,W_j,\varphi_j}\right).$$

∎

For t_μ in a bounded family of distributions, there is a finite integer m (which depends on the finite partition of unity φ_j) such that the r.h.s. of the above inequality is bounded thus all seminorms $\|.\|_{N,V,\chi}$ for $\mathcal{D}'_{\sigma\circ\Gamma}$ are bounded. Finally, it remains to check that the pull-back by a diffeomorphism of a weakly bounded family of distributions is weakly bounded, the proof is a simple application of the variable change formula for distributions ([23] formula (3.7) p. 10).

Consequences for the scaling with different Eulers.

Definition 4.3.2 *t is microlocally weakly homogeneous of degree s at $p \in I$ for ρ if $WF(t)$ satisfies the local soft landing condition at p, there exists a ρ-convex open set V_p such that $(\lambda^{-s} e^{\log\lambda\rho *} t)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'_\Gamma(V_p \setminus I)$ for some $\Gamma \subset T^\bullet V_p$ which satisfies the soft landing condition.*

In particular, if $(\lambda^{-s} e^{\log\lambda\rho *} t)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'_\Gamma(V_p \setminus I)$ for $\Gamma = \bigcup_{\lambda\in(0,1]} WF(t_\lambda)$ then t is microlocally weakly homogeneous of degree s since $WF(t)$ satisfies the soft landing condition implies $\Gamma = \bigcup_{\lambda\in(0,1]} WF(t_\lambda)$ also does.

Theorem 4.3.2 *Let $t \in \mathcal{D}'(M \setminus I)$. If t is microlocally weakly homogeneous of degree s at $p \in I$ for some ρ then it is so for any ρ.*

Proof — Let ρ_1, ρ_2 be two Euler vector fields and t is microlocally weakly homogeneous of degree s at $p \in I$ for ρ_1. We use Proposition 1.4.2 which states that locally there exists a smooth family of diffeomorphisms $\Phi(\lambda) : V_p \mapsto V_p$ such that $\forall\lambda \in [0,1], \Phi(\lambda)(p) = p$ and $\Phi(\lambda)$ relates the two scalings:

$$e^{\log\lambda\rho_2 *} = \Phi(\lambda)^* e^{\log\lambda\rho_1 *}.$$

Then $\Phi(\lambda)^\star$ is a Fourier integral operator which depends smoothly on a parameter $\lambda \in [0,1]$. $\lambda^{-s} e^{\log\lambda\rho_1 *} t$ is bounded in $\mathcal{D}'_{\Gamma_1}(V \setminus I)$, then we apply Theorem (4.3.1) to deduce that the family

$$\Phi(\lambda)^* \left(\lambda^{-s} e^{\log\lambda\rho_1 *} t\right)_\lambda = \left(\lambda^{-s} e^{\log\lambda\rho_2 *} t\right)_\lambda$$

is in fact bounded in $\mathcal{D}'_{\Gamma_2}(V_p)$, with Γ_2 given by the equation

$$\Gamma_2 = \bigcup_{\lambda\in[0,1]} \sigma_\lambda \circ \Gamma_1$$

where $\sigma_\lambda = T^\star \Phi^{-1}(\lambda)$. ■

The previous theorem allows us to define a space of distributions $E_s(U)$ that are microlocally weakly homogeneous of degree s, the definition being independent of the choice of Euler vector field ρ:

Definition 4.3.3 *t is microlocally weakly homogeneous of degree s at p if t is microlocally weakly homogeneous of degree s at p for some ρ. $E_s^\mu(U)$ is the space of all distributions $t \in \mathcal{D}'(U)$ such that $\forall p \in (I \cap \overline{U})$, t is microlocally weakly homogeneous of degree s at p.*

We now state a general theorem which summarizes all our investigations in the first four chapters of this thesis and is a microlocal analog of Theorem 1.4.2,

Theorem 4.3.3 *Let U be an open neighborhood of $I \subset M$, if $t \in E_s^\mu(U \setminus I)$ then there exists an extension $\overline{t}$ in $E_{s'}^\mu(U) \cap \mathcal{D}'_{WF(t) \cup C}(U)$ where $s' = s$ if $-s-d \notin \mathbb{N}$ and $s' < s$ otherwise.*

4.4 Appendix.

We recall a deep theorem of Laurent Schwartz (see [65] p. 86 theorem (22)) which gives a concrete representation of bounded families of distributions.

Theorem 4.4.1 *For a subset $B \subset \mathcal{D}'(\mathbb{R}^d)$ to be bounded it is neccessary and sufficient that for any domain Ω with compact closure, there is an multiindex α such that $\forall t \in B, \exists f_t \in C^0(\Omega)$ where $t|_\Omega = \partial^\alpha f_t$ and $\sup_{t\in B} \|f_t\|_{L^\infty(\Omega)} < \infty$.*

We give an equivalent formulation of the theorem of Laurent Schwartz in terms of Fourier transforms:

Theorem 4.4.2 *Let $B \subset \mathcal{D}'(\mathbb{R}^d)$.*

$$\forall \chi \in \mathcal{D}(\mathbb{R}^d), \exists m \in \mathbb{N}, \sup_{t\in B} \|(1+|\xi|)^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty$$

$$\Leftrightarrow B \text{ weakly bounded in } \mathcal{D}'(\mathbb{R}^d) \Leftrightarrow B \text{ strongly bounded in } \mathcal{D}'(\mathbb{R}^d).$$

Proof — We will not recall here the proof that B is weakly bounded is equivalent to B is strongly bounded (by Banach Steinhaus see the appendix of Chapter 1). Assume $\forall \chi \in \mathcal{D}'(\mathbb{R}^d), \exists m \in \mathbb{N}, \sup_{t\in B} \|(1+|\xi|)^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty$. We fix an arbitrary test function φ. There is a function $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\chi = 1$ on the support of φ. Then

$$|\langle t, \varphi\rangle| = |\langle t\chi, \varphi\rangle| = |\langle \widehat{t\chi}, \widehat{\varphi}\rangle|$$

$$= |\int_{\mathbb{R}^d} d^d\xi (1+|\xi|)^{-d-1}(1+|\xi|)^{-m}\widehat{t\chi}(\xi)(1+|\xi|)^{m+d+1}\widehat{\varphi}(\xi)|$$

$$\leqslant \underbrace{\int_{\mathbb{R}^d} d^d\xi (1+|\xi|)^{-d-1}}_{\text{integrable}} |(1+|\xi|)^{-m}\widehat{t\chi}(\xi)||(1+|\xi|)^{m+d+1}\widehat{\varphi}(\xi)|$$

$$\leqslant C\|(1+|\xi|)^{-m}\widehat{t\chi}\|_{L^\infty}\pi_{m+d+1}(\varphi),$$

finally

$$\sup_{t\in B}|\langle t,\varphi\rangle| \leqslant C\pi_{m+d+1}(\varphi)\sup_{t\in B}\|(1+|\xi|)^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty.$$

Conversely, we can always assume B to be strongly bounded, then for all $\chi \in \mathcal{D}_K(\mathbb{R}^d)$, the family $(\chi e_\xi)_{\xi\in\mathbb{R}^d}$ where $e_\xi(x) = e^{-ix.\xi}$ has fixed compact support K. Then there exists m and a universal constant C such that

$$\forall t \in B, \forall\varphi \in \mathcal{D}(K), |\langle t,\varphi\rangle| \leqslant C\pi_m(\varphi)$$

thus

$$\forall t \in B, |\widehat{t\chi}|(\xi) = |\langle t,\chi e_\xi\rangle| \leqslant C\pi_m(\chi e_\xi),$$

now notice that $\pi_m(\chi e_\xi)$ is polynomial in ξ of degree m thus $\sup_\xi |(1+|\xi|)^{-m}\pi_m(\chi e_\xi)$ is bounded. But then $(1+|\xi|)^{-m}|\widehat{t\chi}(\xi)| \leqslant C\underbrace{|(1+|\xi|)^{-m}\pi_m(\chi e_\xi)|}_{\text{bounded in }\xi}$ and thus $\sup_{t\in B}\|\theta^{-m}\widehat{t\chi}\|_{L^\infty} < +\infty$. ■

Chapter 5

The two point function $\langle 0|\phi(x)\phi(y)|0\rangle$.

Introduction. Hadamard states are nowadays widely accepted as possible physical states of the free quantum field theory on a curved space-time. The Hadamard condition plays an essential role in the perturbative construction of interacting quantum field theory [26]. Since the work of Radzikowski [60], the "Hadamard condition" (renamed microlocal spectrum condition) is formulated as a requirement on the wave front set of the associated two-point function Δ_+ which is necessarily a bisolution of the wave equation in the globally hyperbolic space-time. The construction of solutions of the wave equation in a globally hyperbolic space-time by the parametrix method, following Hadamard [35] and Riesz [63], is by now classical in the mathematical literature. For space-times of the form $\mathbb{R}\times M$ where M is a compact Riemannian manifold, it is well known that $\Delta_+ = \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}$ where $e^{it\sqrt{-\Delta}}$ is a Fourier integral operator constructed in [19] theorem (1.1) p. 43 with the wave front set satisfying the Hadamard condition (see also [76] théorème 1 p. 2). However, to our knowledge, only the recent work of C. Gérard and M. Wrochna [29] treats the non static space-times case (although [45] constructed Hadamard states on space-times with compact Cauchy surfaces). Furthermore, for the purpose of renormalizing interacting quantum field theory, we need to establish that Δ_+ has finite "**microlocal scaling degree**" (following the terminology of [26]), which is a **stronger assumption** than establishing that $WF(\Delta_+)$ satisfies the Hadamard condition.

The goal of this chapter is to prove that $\Gamma = WF(\Delta_+)$ satisfies the microlocal spectrum condition and that Δ_+ is microlocally weakly homogeneous of degree -2 in the sense of Chapter 4 (means in the notation of Chapter 4 that $\Delta_+ \in E^{\mu}_{-2}$). Although our goal is not to construct Δ_+ on flat space, as preliminary, we spend some time to present various different mathematical interpretations of the Wightman function Δ_+ in the flat case and give many formulas that are scattered in the mathematical literature.

We provide proofs (or give precise references whenever we do not give all the details) of so called "well known facts", as for instance the *Wick rotation*, which cannot be easily found in the mathematical literature. In fact, our work done in the flat case will be useful when we pass to the curved case.

Our plan and some historical comments. The first section deals with the Wightman function Δ_+ in $\mathbb{R}^{n+1}$. We start with the expression of the Wightman function given by Reed and Simon [67]: Δ_+ is the inverse Fourier transform $\mathcal{F}^{-1}(\mu)$ of a Lorentz invariant measure μ supported by the positive mass hyperboloid in momentum space. This beautiful interpretation also appears in the book of Laurent Schwartz [64]. This gives a first proof that Δ_+ is a tempered distribution. The formalism of functional calculus immediately allows us to relate $\mathcal{F}^{-1}(\mu)$ with the function $\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}$ of the Laplace operator $-\Delta$, $\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}$ is a solution in the space of operators of the wave equation. From the inverse Fourier transform formula, Δ_+ is interpreted as an oscillatory integral ([67]), hence by a theorem of Hörmander, this gives us a first possible way to compute the WF of Δ_+.

Then we give a second approach to the Wightman function: we notice the striking similarity of $\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}$ with the Poisson kernel $\frac{e^{-\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}$, and the fact that they should be the same formula if we could treat the time variable t as a complex variable. To carry out this program, we first compute the inverse Fourier transform w.r.t. to the variables ξ of the Poisson kernel $\frac{e^{-\tau|\xi|}}{|\xi|}$, we obtain the function $\frac{C}{\tau^2+\sum_{i=1}^n (x^i)^2}$ which can be viewed as the Schwartz kernel of the operator $e^{-t\sqrt{-\Delta}}\sqrt{-\Delta}^{-1}$. This computation relies on the beautiful **subordination identity** connecting the Poisson operator and the Heat kernel. Then we show how to make sense of the analytic continuation in time of the Poisson kernel $\frac{C}{\tau^2+\sum_{i=1}^n x_i^2}$, called the wave Poisson kernel and which corresponds to the operator $e^{i(t+i\tau)\sqrt{-\Delta}}\sqrt{-\Delta}^{-1}$. This allows to recover Δ_+ when the complexified time $(\tau - it)$ becomes purely imaginary, justifying the famous **Wick rotation** and giving a third proof that Δ_+ is a distribution. In fact, to generalize this idea to static space-times of the form $\mathbb{R}\times M$ where M is a noncompact Riemannian manifolds, we can use the machinery of functional calculus defined in the monograph [74] (see also [75]), from the relation

$$f(\sqrt{-\Delta_g}) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty} \widehat{f}(t)e^{it\sqrt{-\Delta_g}},$$

one can easily define the analytic continuation in time of the Poisson kernel $e^{-\tau\sqrt{-\Delta_g}}\sqrt{-\Delta_g}^{-1}$, hence define the Wick rotation of $e^{-\tau\sqrt{-\Delta_g}}\sqrt{-\Delta_g}^{-1}$ where Δ_g denotes the Laplace–Beltrami operator on the noncompact Riemannian manifold, this will be the object of future investigations. Finally, we arrive at the formula which expresses the kernel of the Wightman function as a distribution defined as the boundary value of a holomorphic function

$$\frac{C}{Q(.+i0\theta)} = \lim_{\varepsilon\to 0^+} \frac{C}{(x^0 \pm i\varepsilon)^2 - \sum_{i=1}^n (x^i)^2},$$

where $Q(x) = (x^0)^2 - \sum_{i=1}^n (x^i)^2$ and $\theta = (1,0,0,0)$. Applying general theorems of Hörmander, this gives a fourth proof of the fact that Δ_+ is a distribution and a second way to estimate the wave front set of Δ_+. Along the way, we prove that $\log\left((x^0+i0)^2 - \sum_1^n (x^i)^2\right)$ and the family $\left((x^0+i0)^2 - \sum_{i=1}^n (x^i)^2\right)^s$ are distributions with wave front set satisfying the microlocal condition condition.

Going to the curved case. There are two conceptual difficulties when we pass to the curved case, the first is to intrinsically define objects on M^2 which generalize the singularity $Q^{-1}(\cdot + i0\theta)$ of Δ_+ and the powers of Q in general. The starting point is to pull back distributions and functions defined on $\mathbb{R}^{n+1}$ by a map $F : \mathcal{V} \subset M^2 \mapsto \mathbb{R}^{n+1}$ constructed by inverting the exponential geodesic map. This well-known technique was already used in [35] and [63] and is expounded in many recent works ([5], [78]), however none of these works present a computation of the wave front set of the pulled back singular term $F^\star Q^{-1}(\cdot + i0\theta)$. Here we prove that the wave front set of the singular term $F^\star Q^{-1}(\cdot + i0\theta)$ satisfies the Hadamard condition as stated in [60].

The second step consists in pulling back certain distributions in $\mathcal{D}'(M)$ on $\mathbb{R}^{n+1}$ in order to set and solve the system of transport equations. For all $p \in M$, we define a map $E_p : \mathbb{R}^{n+1} \mapsto M$ which allows to pull-back functions, differential operators and the metric on $\mathbb{R}^{n+1}$ ($\mathbb{R}^{n+1}$ is identified with the exponential chart centered at p).

Once these two difficulties are solved, and all proper geometric objects are defined, it is simple to follow the classical construction of Hadamard [35] to obtain a parametrix with suitable wave front set.

5.1 The flat case.

Fix the Lorentz invariant quadratic form $Q(x^0, x^1, \dots, x^n) = (x^0)^2 - \sum_1^n (x^i)^2$ in $\mathbb{R}^{n+1}$. In the book of Laurent Schwartz [64], the study of particles is related to the problem of finding Lorentz invariant tempered distributions of positive type on $\mathbb{R}^{n+1}$. By Fourier transform and application of the Bochner theorem (p. 60,66 in [64]), it is equivalent to the problem of finding positive Lorentz invariant measures $\mu \in \left(C^0(\mathbb{R}^{n+1})\right)'$ in momentum space. Then μ is called a scalar particle. If the particle is *elementary*, it is required that μ is extremal which means that $\mu = \sum \alpha_i \mu_i$ holds iff μ_i are proportional to μ. This notion of extremal measure is the analogue in functional analysis of the notion of irreducible representations of a group in representation theory.

We also require that μ has *positive energy* i.e. μ is supported on $\{x^0 \geqslant 0\}$. Before we discuss Lorentz invariant measures, we would like to give a simple formula which is a reinterpretation of the usual Lebesgue integration in $\mathbb{R}^{n+1}$ in terms of *slicing* by the orbits of the Lorentz group:

$$\int_{\mathbb{R}^{n+1}} f \wedge_{\mu=0}^{n} dx^{\mu} = \int_{-\infty}^{\infty} dm \int_{Q=m} f \frac{\wedge_{\mu=0}^{n} dx^{\mu}}{dQ} \tag{5.1}$$

as a consequence of the coarea formula of Gelfand Leray ([43], [83]). Notice that we can produce natural Lorentz invariant measures by modifying this integral, instead of integrating over the Lebesgue measure dm over the real line, we integrate against an arbitrary measure $\rho(m)$:

Proposition 5.1.1 *Any Lorentz invariant measure of positive energy μ can be represented by the formula*

$$\mu(f) = \int_{-\infty}^{\infty} \rho(m) \int_{Q=m} f \frac{\wedge_{\nu=0}^{n} dx^{\nu}}{dQ} + cf(0) \tag{5.2}$$

where the measure ρ is in fact the push-forward of μ:

$$\rho = Q_*(\mu).$$

In particular, by Bochner theorem, any tempered positive distribution μ invariant by $O(n,1)_+^{\uparrow}$ can be represented by

$$\mu(f) = \int_{-\infty}^{\infty} \rho(m) \int_{Q=m} \widehat{f} \frac{\wedge_{\nu=0}^{n} dx^{\nu}}{dQ} + c \int_{\mathbb{R}^{n+1}} d^{n+1}x f(x). \tag{5.3}$$

Proof — The proof is given in full detail in [67] Theorem 9.33 p. 75 and also the classification of all Lorentz invariant distributions was given by Méthée. ■

From now on, we assume μ has positive energy. Inspired by the previous proposition, we claim

Proposition 5.1.2 *Any extremal measure of positive energy μ in $\mathbb{R}^{n+1}$ which is invariant by the group $O(n,1)_+^{\uparrow}$ of time and orientation preserving Lorentz transformations is supported on one orbit of $O(n,1)_+^{\uparrow}$.*

Proof — It was proved in a very general setting in [64] p. 72. The orbits of $O(n,1)_+^{\uparrow}$ in the positive energy region $\{x^0 \geqslant 0\}$ are connected components of constant mass hyperboloids for $m > 0$, the half null cone $(x^0)^2 - |x|^2 = 0, x^0 > 0$ and the fixed point $\{0\}$ of the group action:

$$\bigcup_{m>0} \underbrace{\{(x^0)^2 - |x|^2 = m^2, x^0 > 0\}}_{\text{positive mass hyperboloids}} \bigcup \underbrace{\{(x^0)^2 - |x|^2 = 0, x^0 > 0\}}_{\text{halfcone}} \bigcup \underbrace{\{0\}}_{\text{origin}}.$$

Let μ be an $O(n,1)^{\uparrow}_{+}$ invariant measure on $\mathbb{R}^{n+1}$. Let Q be the canonical $O(n,1)$ invariant quadratic form. Then the push-forward $Q_*\mu$ is a well defined measure on $\mathbb{R}^+$ (since μ has positive energy) because Q is smooth and the support of $Q_*\mu$ contains the masses of the particles. Assume the support of μ contains two points which are in disjoint orbits of $O(n,1)^{\uparrow}_{+}$, then the push-forward $Q_*\mu$ is supported at two different points m_1, m_2. Then pick a smooth function $0 \leqslant \chi \leqslant 1$ such that $\chi(m_1) = 1$ and $\chi(m_2) = 0$ and consider the pair of push pull measures

$$Q^* \left(\chi Q_*\mu\right), Q^* \left((1-\chi)Q_*\mu\right).$$

These are measures with different supports, hence linearly independent, and

$$\mu = H(x^0)Q^* \left(\chi Q_*\mu\right) + H(x^0)Q^* \left((1-\chi)Q_*\mu\right)$$

which contradicts the extremality of μ. ■

Now, let μ be an extremal measure of positive energy. We already saw the support of μ is one orbit of $O(n,1)^{\uparrow}_{+}$, a hyperboloid of mass $m > 0$. Here we give an interpretation of the $O(n,1)^{\uparrow}_{+}$ invariant measure μ supported by the mass shell m of positive energy (which is unique by theorem 9.37 in [67]) in terms of the Gelfand–Leray distributions (see [43]). We introduce the following notations:

$$\xi = (\xi^\mu)_{0\leqslant\mu\leqslant n} = (\xi^0, \xi^i_{1\leqslant i\leqslant n}) = (\xi^0, \overrightarrow{\xi}).$$

Proposition 5.1.3 *Let $\Omega = d\xi^0 \wedge d^n\overrightarrow{\xi}$ be the canonical measure in $\mathbb{R}^{n+1}$ and $Q = (\xi^0)^2 - \sum_{i=1}^n (\xi^i)^2$. Then we can construct an $O(n,1)^{\uparrow}_{+}$ invariant measure μ supported by the component of positive energy of $Q = m$ given by the formulas:*

$$\mu(f) = \left\langle \delta_m, \left(\int_{Q=m} f\frac{\Omega}{dQ}\right)\right\rangle = \int_{\mathbb{R}^n} \frac{d^n\overrightarrow{\xi}}{2\sqrt{m^2+|\overrightarrow{\xi}|^2}} f((m^2+|\overrightarrow{\xi}|^2)^{\frac{1}{2}}, \overrightarrow{\xi}). \tag{5.4}$$

Proof — Let us remark that the Lebesgue measure in momentum space $\Omega = d\xi^0 \wedge d^n\overrightarrow{\xi}$ is $O(n,1)$ invariant because the determinant of any element in $O(n,1)$ equals 1. Let us compute the δ function $\delta_{\{(\xi^0)^2-|\overrightarrow{\xi}|^2=m,\xi^0\geqslant 0\}}(\Omega)$ as defined in Gelfand–Shilov [43] :

$$\delta_{\{(\xi^0)^2-|\overrightarrow{\xi}|^2=m,\xi^0\geqslant 0\}}(d\xi^0\wedge d^n\overrightarrow{\xi}) = \int_{\{\xi^0=\sqrt{m^2+|\overrightarrow{\xi}|^2}\}} \frac{d\xi^0\wedge d^n\overrightarrow{\xi}}{d((\xi^0)^2-(m^2+|\overrightarrow{\xi}|^2))}$$

The Gelfand-Leray form $\frac{d\xi^0\wedge d^n\overrightarrow{\xi}}{d((\xi^0)^2-(m^2+|\overrightarrow{\xi}|^2))}$ is the ratio of two Lorentz invariant forms. More explicitely, we compute this ratio in the parametrization

$\overrightarrow{\xi} \in \mathbb{R}^n \mapsto ((m^2 + |\overrightarrow{\xi}|^2)^{\frac{1}{2}}, \overrightarrow{\xi}) \in \mathbb{R}^{n+1}$ of the mass hyperboloid:

$$\frac{d\xi^0 \wedge d^n\overrightarrow{\xi}}{d((\xi^0)^2 - (m^2 + |\overrightarrow{\xi}|^2))}|_{\xi^0=\sqrt{m^2+|\overrightarrow{\xi}|^2}} = \frac{d\xi^0 \wedge d^n\overrightarrow{\xi}}{2(\xi^0 d\xi^0 - \left\langle \overrightarrow{\xi}, d\overrightarrow{\xi}\right\rangle)}|_{\xi^0=\sqrt{m^2+|\overrightarrow{\xi}|^2}}$$

$$= \frac{d^n\overrightarrow{\xi}}{2\xi^0}|_{\xi^0=\sqrt{m^2+|\overrightarrow{\xi}|^2}}$$

because $\frac{d^n\overrightarrow{\xi}}{2\xi^0} \wedge 2(\xi^0 d\xi^0 - \left\langle \overrightarrow{\xi}, d\overrightarrow{\xi}\right\rangle) = d\xi^0 \wedge d^n\overrightarrow{\xi}$

$$= \frac{d^n\overrightarrow{\xi}}{2\sqrt{m^2 + |\overrightarrow{\xi}|^2}},$$

we thus connect with the formula found in [67] p. 70,74. ■

Once we have this measure μ in momentum space, we would like to recover the distribution it defines by computing the inverse Fourier transform $\mathcal{F}^{-1}(\mu)$ in $\mathbb{R}^{n+1}$.

Proposition 5.1.4 *Assume $\Delta_+ = \mathcal{F}^{-1}(\mu)$ where μ is an extremal measure of mass m, $O(n,1)_+^{\uparrow}$ invariant and of positive energy, then Δ_+ is given by the formula*

$$\Delta_+(x;m) = \frac{1}{2(2\pi)^{n+1}} \int_{\mathbb{R}^n} \frac{e^{-ix^0(m^2+|\overrightarrow{\xi}|^2)^{\frac{1}{2}} + i\overrightarrow{x}.\overrightarrow{\xi}}}{(m^2 + |\overrightarrow{\xi}|^2)^{\frac{1}{2}}} d^n\overrightarrow{\xi}. \tag{5.5}$$

Proof — To prove the claim, we use the Gelfand–Leray notation and the beautiful identity $e^{i\tau f}\omega = e^{i\tau t}dt \int_{t=f} \frac{\omega}{df}$ ([83] page 124 lemma (5.12)), which allows to rewrite the Reed Simon formula:

$$\delta_{\{(\xi^0)^2 - |\overrightarrow{\xi}|^2 = m, \xi^0 \geqslant 0\}}(e^{i(x^0\xi^0 + \overrightarrow{x}.\overrightarrow{\xi})}\Omega)$$

$$= \int e^{i(x^0\xi^0 + \overrightarrow{x}.\overrightarrow{\xi})} d\xi^0 \int_{\xi^0=\sqrt{m^2+|\overrightarrow{\xi}|^2}} \frac{d\xi^0 \wedge d^n\overrightarrow{\xi}}{d(\xi^{02} - (m^2 + |\overrightarrow{\xi}|^2))}$$

$$= \int_{\mathbb{R}^n} e^{i(x^0\sqrt{m^2+|\overrightarrow{\xi}|^2} + \overrightarrow{x}.\overrightarrow{\xi})} \frac{d^n\overrightarrow{\xi}}{2\sqrt{m^2 + |\overrightarrow{\xi}|^2}},$$

we recognize the inverse Fourier transform of a distribution supported by the positive sheet of the hyperboloid. ■

If we provisionnally call t the variable x^0 then the above proposition allows to interpret Δ_+ as the Schwartz kernel of the operator $\frac{e^{it\sqrt{-\Delta+m^2}}}{\sqrt{-\Delta+m^2}}$ where Δ is the Laplace operator acting on $\mathbb{R}^n$. Also notice that the evolution operator $t\mapsto U(t)=\frac{e^{it\sqrt{-\Delta+m^2}}}{\sqrt{-\Delta+m^2}}$ satisfies the square root Klein–Gordon equation: $\partial_t - i\sqrt{-\Delta+m^2}U=0$, thus $\Delta_+(x;m)$ is a solution of the Klein Gordon equation and for any $u\in H^s(\mathbb{R}^n)$, $u_+=\Delta_+(t;m)*u$ is a solution of the Klein Gordon equation which has **positive energy** i.e. its Fourier transform is supported in the positive hyperboloid.

5.1.1 The Poisson kernel, the Wick rotation and the subordination identity.

To define Δ_+ as the inverse Fourier transform of the measure μ is not very satisfactory since it does not give an explicit formula for Δ_+ in space variables. We will prove that $\Delta_+ = C((x^0+i0)^2-|x|^2)^{-1}$ where we explain how to make sense of the term on the right hand side as a tempered distribution by the process of **Wick rotation**.

Lemma 5.1.1 *The family of Schwartz distributions*

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix.\xi-y|\xi|}}{|\xi|}d^n\xi=\frac{e^{-y\sqrt{-\Delta}}}{\sqrt{-\Delta}}\delta(x)=\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})}\frac{1}{(y^2+|x|^2)^{\frac{n-1}{2}}}\tag{5.6}$$

is holomorphic in $y\in\{y|Re(y)>0\}$ and continuous in $y\in\{y|Re(y)\geqslant 0\}$ with values in $S'(\mathbb{R}^n)$.

Similar computations of Poisson integrals are presented in [69] p. 60, 130, [23] and [74] (3.5).
Proof — Our proof follows [74] (3.5). Everything relies on the following identity (see the identity β in [69] p. 61)

$$\frac{e^{-Ay}}{A}=\frac{1}{\pi^{1/2}}\int_0^\infty e^{\frac{-y^2}{4t}}e^{-A^2t}t^{-\frac{1}{2}}dt\tag{5.7}$$

which is derived from the **subordination identity** (5.22) in [74]

$$e^{-Ay}=\frac{y}{2\pi^{1/2}}\int_0^\infty e^{\frac{-y^2}{4t}}e^{-A^2t}t^{-\frac{3}{2}}dt\tag{5.8}$$

by integrating w.r.t. y and by noticing that when $y=0$ our formula (5.7) coincides with the Hadamard–Fock–Schwinger formula:

$$\int_0^\infty t^{-\frac{1}{2}}e^{-tA^2}dt=\int_0^\infty t^{\frac{1}{2}}e^{-tA^2}\frac{dt}{t}$$

$$=A^{-1}\int_0^\infty t^{\frac{1}{2}}e^{-t}\frac{dt}{t}=A^{-1}\Gamma(\frac{1}{2})=A^{-1}\pi^{\frac{1}{2}}$$

since $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$. Next we need functional calculus in our proof since we want to apply the subordination identity with $A = \sqrt{-\Delta}$. We then get an identity for functions of the **operator** $\sqrt{-\Delta}$. We apply these operators to the delta function supported at 0:

$$\frac{e^{-\sqrt{-\Delta}y}}{\sqrt{-\Delta}}\delta_0 = \left(\frac{1}{\pi^{1/2}}\int_0^\infty e^{\frac{-y^2}{4t}} e^{t\Delta} t^{-\frac{1}{2}} dt\right)\delta_0,$$

we recognize on the left hand side a distributional solution of the Poisson operator $\partial_y^2 + \Delta_x$ and on the right hand side, we recognize the Heat kernel $e^{t\Delta}\delta_0 = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$. Substituting in the previous formula,

$$\frac{e^{-\sqrt{-\Delta}y}}{\sqrt{-\Delta}}\delta_0 = \frac{1}{\pi^{1/2}}\int_0^\infty e^{\frac{-y^2}{4t}} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} t^{-\frac{1}{2}} dt = \frac{1}{(4\pi)^{\frac{n}{2}}\pi^{\frac{1}{2}}}\int_0^\infty dt e^{-\frac{y^2+|x|^2}{4t}} \frac{1}{t^{\frac{n+1}{2}}}$$

set $t = \frac{1}{4s}$ and we get

$$\frac{1}{(4\pi)^{\frac{n}{2}}\pi^{\frac{1}{2}}}\int_0^\infty \frac{ds}{4s^2} e^{-(y^2+|x|^2)s}(4s)^{\frac{n+1}{2}} = \frac{1}{2\pi^{\frac{n+1}{2}}}\int_0^\infty ds e^{-(y^2+|x|^2)s} s^{\frac{n-3}{2}}$$

finally by a variable change in the formula of the Gamma function

$$\frac{e^{-\sqrt{-\Delta}y}}{\sqrt{-\Delta}}\delta_0 = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \frac{1}{(y^2+|x|^2)^{\frac{n-1}{2}}}$$

■

We give an interpretation of $((t \pm i0)^2 - |x|^2)^{-\frac{n-1}{2}}$ as an oscillatory integral.

Theorem 5.1.1 *The limit* $\lim_{\varepsilon\to 0}((t\pm i\varepsilon)^2 - |x|^2)^{-\frac{n-1}{2}}$ *makes sense in* $S'(\mathbb{R}^n)$ *and satisfies the identity:*

$$((t \pm i0)^2 - |x|^2)^{-\frac{n-1}{2}} = \frac{(-1)^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})(4\pi)^{\frac{n-1}{2}}}\int_{\mathbb{R}^n} d^n\xi \frac{1}{|\xi|} e^{\pm it|\xi|} e^{ix.\xi} \tag{5.9}$$

Proof — The key argument of the proof is to justify the analytic continuation of the Poisson kernel, this is called Wick rotation in physics textbooks. Notice that $\frac{e^{-y\sqrt{-\Delta}}}{\sqrt{-\Delta}}\delta_0$ is the Schwartz kernel of a well defined operator $\frac{e^{-\sqrt{-\Delta}y}}{\sqrt{-\Delta}}$. Through the partial Fourier transform w.r.t. the variable x, the operator $\frac{e^{-y\sqrt{-\Delta}}}{\sqrt{-\Delta}}$ corresponds to the multiplication by $\frac{e^{-y|\xi|}}{|\xi|}$. Consider now the function $\frac{1}{|\xi|}e^{-y|\xi|}$, **when** $n \geqslant 2$ this function is analytic in $\{y, Re(y) > 0\}$ with value Schwartz distribution in ξ because

$$\forall y \in \{Re(y) \geqslant 0\}, |\frac{1}{|\xi|}e^{-y|\xi|}| \leqslant \frac{1}{|\xi|} \in L^1_{loc}(\mathbb{R}^n).$$

Notice that the above estimate is still true when $Re(y) \to 0^+$ hence $\frac{1}{|\xi|}e^{-y|\xi|}$ is a well defined Schwartz distribution in ξ for $Re(y) \geqslant 0$ (it is continuous in y with value tempered distribution). Finally, we can continue this operator in the y variable in the domain $Re(y) \geqslant 0$, set $y = \tau + it$ and let τ tends to zero in $\mathbb{R}^+$. Set $\frac{e^{-\sqrt{-\Delta}(\tau \pm it)}}{\sqrt{-\Delta}}\delta_0 = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\frac{1}{((\tau \pm it)^2+|x|^2)^{\frac{n-1}{2}}}$ then at the limit we find

$$\begin{array}{ccc} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix.\xi-(\tau\pm it)|\xi|}}{|\xi|}d^n\xi & = & \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\frac{1}{(-(t\mp i\tau)^2+|x|^2)^{\frac{n-1}{2}}} \\ \downarrow_{\tau\to 0^+} & & \downarrow_{\tau\to 0^+} \\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{ix.\xi\mp it|\xi|}}{|\xi|}d^n\xi & = & \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\frac{1}{(|x|^2-(t\mp i0)^2)^{\frac{n-1}{2}}} \end{array}$$

■

5.1.2 Oscillatory integral.

For QFT, we are interested in the formula (5.9) for $n = 3$.

$$((t \pm i0)^2 - |x|^2)^{-1} = C_n \int_{\mathbb{R}^n} d^n\overrightarrow{\xi}\frac{1}{|\overrightarrow{\xi}|}e^{\pm it|\overrightarrow{\xi}|}e^{-i\overrightarrow{x}.\overrightarrow{\xi}}, C_n = \frac{(-1)^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n-1}{2})(4\pi)^{\frac{n-1}{2}}}. \tag{5.10}$$

It provides a **definition** of $((t \pm i0)^2 - |\overrightarrow{\xi}|^2)^{-1}$ as an oscillatory integral or Lagrangian distribution in $\mathbb{R}^{n+1}$,

$$C_n \int_{\mathbb{R}^n} d^n\overrightarrow{\xi} e^{i\phi_\pm(t,\overrightarrow{x};\overrightarrow{\xi})}\frac{1}{|\overrightarrow{\xi}|} \tag{5.11}$$

with phase function $\phi_\pm(t,\overrightarrow{x};\overrightarrow{\xi}) = \sum_{i=1}^n -x^i\xi_i \pm t\sqrt{\sum_1^n(\xi_i)^2} = -\overrightarrow{x}.\overrightarrow{\xi} \pm t|\overrightarrow{\xi}|$. The idea is to use the interpretation of $((t \pm i0)^2 - |x|^2)^{-1}$ as an oscillatory integral to compute $WF((t \pm i0)^2 - |x|^2)^{-1}$.

Proposition 5.1.5 *We claim*

$$WF\left(C_n\int_{\mathbb{R}^n} d^n\xi e^{i(-x.\xi\pm t|\xi|)}\frac{1}{|\xi|}\right) = \{(0,0;\pm|\xi|,-\overrightarrow{\xi})\}\cup\{(|x|,x_i;\pm\lambda,-\frac{\lambda x_i}{|x|})|\lambda>0,|x|\neq 0\}.$$

Proof — This computation can be found in [67] example 7 p. 101. The function $\phi = t|\overrightarrow{\xi}| - x.\overrightarrow{\xi}$. satisfies the axioms of a phase function because it is homogeneous of degree 1 in ξ, smooth outside $\overrightarrow{\xi} = 0$ and $d_{x,t}\phi$ never vanishes as soon as $|\overrightarrow{\xi}| \neq 0$ which implies that it defines a **phase function** in the sense of Hörmander. We first compute the critical set of ϕ denoted by Σ_ϕ and defined by the equation $\{d_\xi\phi = 0\}$:

$$d_\xi(t|\overrightarrow{\xi}| - x.\overrightarrow{\xi}) = t\sum_{\mu=1}^n \frac{\xi_\mu}{|\xi|}d\xi_\mu - x^\mu d\xi_\mu = 0 \Leftrightarrow t = |x|, x^\mu = \frac{\xi_\mu}{|\xi|}|x|.$$

We will later see in Chapter 6 that this defines a Morse family

$$\big((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}, \phi\big)$$

and the wave front set is parametrized by the Lagrange immersion $\lambda_\phi \Sigma_\phi$ in $T^\star\mathbb{R}^{n+1}$ of the critical set defined by the Morse family:

$$\lambda_\phi \Sigma_\phi = \{(t, x; \partial_t\phi, \partial_x\phi) | \partial_\xi \phi = 0\}$$

$$= \{(x = 0, t = 0; |\xi|, -\xi)\} \cup \{(t, x; |\xi|, -\xi)| t = |x|, x^\mu = \frac{\xi_\mu}{|\xi|}|x|, \xi \neq 0\}$$

$$= \{(0, 0; |\xi|, -\xi)\} \cup \{(|x|, x^\mu; \lambda, -\frac{\lambda x^\mu}{|x|})|\lambda > 0, |x| \neq 0\}.$$

To conclude, we see that the sign in front of t in the phase $\phi_\pm(t, x; \xi) = \pm t|\xi| - x.\xi$ will decide of the positivity or negativity of the energy of $WF(\Delta_+)$. ■

5.2 The holomorphic family $\big((x^0 + i0)^2 - \sum_{i=1}^n (x^i)^2\big)^s$.

We give a detailed derivation of the main steps needed for the computation of the wave front set of the family $\big((x^0 + i0)^2 - \sum_{i=1}^n (x^i)^2\big)^s$, $s \in \mathbb{C}$ and $\log\big((x^0 + i0)^2 - \sum_{i=1}^n (x^i)^2\big)$ using the general theory of boundary values of holomorphic functions along convex sets developped by Hörmander [40]. The result is given in [40] p. 322 without any detail, also a similar treatment in the literature can be found in [77]. We carefully follow the exposition of [40] (8.7) but we specialize to the simpler case of the quadratic form $Q = (x^0)^2 - \sum_{i=1}^n (x^i)^2$ which makes the explanations much clearer and allows us to give direct arguments.

Let C^+ denote the set $\{y | Q(y) > 0, y^0 > 0\}$, C^+ is an open cone called the *future cone*. We denote by q the unique symmetric bilinear map associated to the quadratic form Q.

Microhyperbolicity. Given $\theta = (1, 0, 0, 0)$. We recall that Q is said to be microhyperbolic (see definition 8.7.1 in [40]) w.r.t. θ in an open set $\Omega \subset \mathbb{R}^n$ if $\forall x \in \mathbb{R}^n, \exists t(x) > 0$, such that $\forall t, 0 < t < t(x)$, $Q(x + it\theta) \neq 0$.

Proposition 5.2.1 *The quadratic form $Q(x) = (x^0)^2 - \sum_{i=1}^n (x^i)^2$ is microhyperbolic with respect to any vector $\theta \in C^+$.*

Proof — We are supposed first to fix a vector $\theta \in C^+$, and we must check Q is microhyperbolic with respect to θ. In fact, we prove a stronger result: $\forall x, \forall \varepsilon > 0$, $Q(x + i\varepsilon\theta) \neq 0$. If $Q(x + i\varepsilon\theta) = Q(x) - \varepsilon^2 Q(\theta) + 2i\varepsilon q(x, \theta) = 0$ then the imaginary part $\mathrm{Im}Q(x + i\varepsilon\theta) = 0$ must vanish hence $q(x, \theta) = 0$. Hence we would have $Q(x) \leqslant 0$ since $\theta \in C^+$ and finally $Q(x + i\varepsilon\theta) = Q(x) - \varepsilon^2 Q(\theta) < 0$. Contradiction ! ■

The domain $T^C=\mathbb{R}^{n+1}+iC^+$ is called a *tube cone*. We want to define the limits in the sense of distributions $\lim_{y\rightarrow 0, y\in C^+} Q^s(x+iy)$ of the holomorphic function Q^s.

The Vladimirov approach.

In the Vladimirov approach which is similar to Hörmander's, we have to prove Q^s is slowly increasing in the algebra $\mathcal{O}(T^C)$ of functions holomorphic in the tube cone T^C (see [77]). In fact, in our case, we would have to prove an estimate of the form

$$\forall z=x+iy, |Q^s(z)|\leqslant \left(1+d(y,\partial C^+)^{-2Re(s)}\right). \tag{5.12}$$

where $d(y,\partial C^+)$ is defined as the distance beetween $y\in C^+$ and the boundary ∂C^+ of the future cone. Then we know (see Theorem 4 p. 204 in [77]) that the Fourier Laplace transform $\mathcal{F}$ is an algebra isomorphism from $\left(\mathcal{O}(T^C),\times\right)$ to the algebra $(\mathcal{S}'(C^\circ),\star)$ of tempered distribution supported in the dual cone $C^\circ\subset\mathbb{C}^4$ endowed with the **convolution product**. However, both the Hörmander and Vladimirov approaches rely on an estimate which roughly says the holomorphic function $Q^s(z)$ has moderate growth when the imaginary part y of z tends to zero in the Tube cone T^C.

Stratification of the space of zeros. For a fixed point $x_0\in\mathbb{R}^{n+1}$, we study the Jets of the map $x\mapsto Q(x)$ at the point x_0. The Minkowski space $\mathbb{R}^{n+1}$ is partitioned by the **lowest order of homogeneity** of the Taylor expansion of Q. Lojasiewicz describes this construction as the **stratification** of the space $\mathbb{R}^{n+1}$ by the **orders of the zeros** of Q. We study the Taylor expansion of Q at x_0 by looking at the map $y\mapsto Q(x_0+y)$. We find three distinct situations:

- $Q(x_0)\neq 0$ thus $Q(x_0+y)=q(x_0,x_0)+O(|y|)$, the term of lowest homogeneity is $q(x_0,x_0)$ and is homogeneous of degree 0 in y
- $Q(x_0)=0, x_0\neq 0$ thus $Q(x_0+y)=2q(x_0,y)+O(|y|^2)$, the term of lowest homogeneity is $2q(x_0,y)$ and is homogeneous of degree 1 in y
- $x_0=0$ thus $Q(0+y)=q(y,y)+O(|y|^3)$, the term of lowest homogeneity is $q(y,y)$ and is quadratic hence homogeneous of degree 2 in y.

Following Hörmander, we denote by $Q_{x_0}(y)$ the term of lowest homogeneity in y. The term of lowest homogeneity allows to construct a geometric structure over $\mathbb{R}^{n+1}$ called the tuboid.

Construction of the tuboid. To every $x_0 \in \mathbb{R}^{n+1}$, we associate the cone Γ_{x_0} ([40] Lemma 8.7.3) defined as the connected component of

$$\{y|Q_{x_0}(y) \neq 0\} \tag{5.13}$$

which contains the vector $\theta = (1,0,0,0)$.

Lemma 5.2.1 *Let $Q = (x^0)^2 - \sum_{i=1}^n (x^i)^2$ and $\theta = (1,0,0,0)$. For every $x_0 \in \mathbb{R}^{n+1}$, let Γ_{x_0} be the cone defined as above.*

- *If $Q(x_0) \neq 0$ then $\Gamma_{x_0} = \{y|Q_{x_0}(y) \neq 0\} = \mathbb{R}^{n+1}$ since the term of lowest homogeneity $Q(x_0)$ does not depend on y.*
- *If $Q(x_0) = 0, x_0 \neq 0$ then $\{y|Q_{x_0}(y) \neq 0\} = \{y|q(x_0,y) \neq 0\} = \{y|q(x_0,y) > 0\} \bigcup \{y|q(x_0,y) < 0\}$ contains two connected components the upper and lower half spaces associated to $Q(x_0,.)$, $\Gamma_{x_0} = \{y|q(x_0,\theta)q(x_0,y) > 0\}$.*
- *If $x_0 = 0$ then $\Gamma_{x_0} = \{y|q(y,y) > 0, y_0 > 0\}$, it is the space of all **future oriented timelike** vectors.*

The domain $\Lambda = \{x_0 + i\Gamma_{x_0}|x_0 \in \mathbb{R}^{n+1}\} \subset \mathbb{C}^4$ is called a *tuboid* in the terminology of Vladimirov.

Choice of the branch of the log function. In order to define the complex powers $Q^s(x+iy) = e^{s\log Q(x+iy)}$ and $\log Q(x+iy)$, we must specify the branch of the log function that we use. We choose the branch of the log in the domain $0 < \arg Q(z) < 2\pi$, for $Q = (x^0)^2 - \sum_{i=1}^n (x^i)^2$. For this determination of the log (see [48] Proposition 4.1), by the proof of Proposition 5.2.1, we see that $Q(x+i\varepsilon\theta)$ avoids the positive reals .

Proposition 5.2.2 $\lim_{\varepsilon\to 0} \log Q(.+i\varepsilon\theta)$ *converges to a smooth function in the **nonconnected open set** $Q \neq 0$.*

Proof — We are going to prove that $\lim \log Q(.+i\varepsilon\theta) \in C^\infty(\{Q \neq 0\})$. We notice that the set $\{Q(x_0) \neq 0\}$ contains three open connected domains, and we classify the convergence of $\log Q(.+i\varepsilon\theta)$ on each of these connected domains:

$$Q(x_0) < 0 \implies \forall x \in U_{x_0}, \log Q(x+i\varepsilon\theta) \to \log|Q(x)| + i\pi \tag{5.14}$$
$$Q(x_0) > 0, x_0^0 > 0, \implies \forall x \in U_{x_0}, \log Q(x+i\varepsilon\theta) \to \log|Q(x)| \tag{5.15}$$
$$Q(x_0) > 0, x_0^0 < 0, \implies \forall x \in U_{x_0}, \log Q(x+i\varepsilon\theta) \to \log|Q(x)| + 2i\pi \tag{5.16}$$

Thus for every x_0 such that $Q(x_0) \neq 0$, there is a small neighborhood of x_0 in which the family of analytic functions $\log Q(.+i\varepsilon\theta)$ converges uniformly to a **smooth** function. ∎

We only have to study the case $Q(x_0) = 0$.

The moderate growth estimate along T^C.

Hörmander proved an important estimate in [40] lemma 8.7.4 which is a specific case of the celebrated Lojasiewicz inequality. We have to slightly modify his result, actually we prove the estimate of lemma 8.7.4, plus the property that $Q(x+iy)$ never meets the positive half line $\mathbb{R}_+$ for x, y in appropriate domains. Let $\theta = (1,0,0,0)$. For every $x_0 \in \mathbb{R}^{n+1}$ such that $Q(x_0)=0$, let Γ_{x_0} be the cone defined as the connected component of

$$\{y | Q_{x_0}(y) \neq 0\} \tag{5.17}$$

which contains the vector θ.

Proposition 5.2.3 *For any closed conic subset $V_{x_0} \subset \Gamma_{x_0}$, there exists $\delta, \delta' > 0$ and U_{x_0} is a neighborhood of x_0 such that for all $(x,y) \in U_{x_0} \times V_{x_0}, |y| \leqslant \delta$ the following estimate is satisfied:*

$$\exists m \in \mathbb{Z}, \delta' |y|^m \leqslant |Q(x+iy)| \tag{5.18}$$

and $Q(x+iy)$ does not meet $\mathbb{R}_+$.

Proof — We fix x_0. We also prove that we can choose U_{x_0} in such a way that $U_{x_0} \times V_{x_0}$ tends to $\{x_0\} \times \Gamma_{x_0}$ for any net of cones V_{x_0} which converges to Γ_{x_0}. We study the two usual cases:

- if $Q(x_0)=0, x_0 \neq 0$, any closed cone V_{x_0} contained in

  $$\Gamma_{x_0} = \{y | q(x_0,\theta) q(x_0,y) > 0\}$$

  should be contained in

  $$\{y | q(x_0,\theta) q(x_0,y) \geqslant 2\delta |y|\}$$

  for some $\delta > 0$ small enough (when $\delta \to 0$ we recover Γ_{x_0}). Let us consider the continuous map $f := x \mapsto \inf_{y \in V_{x_0}, |y|=1} q(x_0,\theta) q(x,y)$. By definition of V_{x_0}, $f(x_0) \geqslant 2\delta$ therefore the set $f^{-1}[\delta, +\infty)$ contains a neighborhood of x_0. We set $U_{x_0} = f^{-1}[\delta,+\infty) = \{x | \forall y \in V_{x_0}, q(x_0,\theta) q(x,y) \geqslant \delta |y|\}$, then U_{x_0} is a neighborhood of x_0. It is immediate by definition of U_{x_0} that for all $(x,y) \in U_{x_0} \times V_{x_0}$, we have $|q(x,y)| \geqslant \delta |q(x_0,\theta)|^{-1} |y|$ which is the moderate growth estimate and we also find that $\operatorname{Im} Q(x+iy) = 2q(x,y)$ never vanishes. Thus $Q(x+iy)$ avoids $\mathbb{R}_+$.

- if $x_0 = 0$ then $\Gamma_0 = \{y | q(y,y) > 0, y_0 > 0\}$ is the space of all **future oriented timelike** vectors. If we set $y = t\theta, \theta = (1,0,0,0)$, we find that

  $$\forall x, |Q(x+iy)| \geqslant |Q(y)| \tag{5.19}$$

in fact the unique **critical point** of the map $(x,t)\mapsto Q(x+it\theta)$ is the point $x=0$. But then this inequality is invariant by the group $O_+^{\uparrow}(n,1)$ of time and orientation preserving Lorentz transformations. Thus the previous estimate is true for any $y\in\Gamma_0$ and reads:

$$\forall x,\forall y\in\Gamma_0, |Q(x+iy)|\geqslant|Q(y)|. \tag{5.20}$$

To properly conclude, we use the fact that y is contained in a closed subcone V_0 of the interior future cone $q(y,y)>0$, thus there is a constant $\delta<1$ such that

$$(x,y)\in K \implies \sum_{i=1}^n (y^i)^2\leqslant \delta(y^0)^2$$

this implies the estimates

$$\sum_{\mu=0}^n (y^\mu)^2=(y^0)^2+\sum_{i=1}^n (y^i)^2\leqslant(1+\delta)(y^0)^2 \implies (y^0)^2\geqslant\frac{\sum_{\mu=0}^n (y^\mu)^2}{1+\delta}$$

and also the estimate $\forall(x,y)\in U_0\times V_0$, where $U_0=|x|\leqslant\delta$:

$$q(y,y)=(y^0)^2-\sum_{i=1}^n (y^i)^2\geqslant(y^0)^2-\delta(y^0)^2 \implies q(y,y)\geqslant(1-\delta)(y^0)^2$$

finally, combining the two previous estimates gives

$$\frac{(1-\delta)\sum_{\mu=0}^n (y^\mu)^2}{1+\delta}\leqslant q(y,y),$$

which yields the inequalities, $\forall(x,y)\in U_0\times V_0$:

$$\frac{(1-\delta)\sum_{\mu=0}^n (y^\mu)^2}{1+\delta}\leqslant q(y,y)\leqslant|Q(x+iy)|, \tag{5.21}$$

setting $\delta'=\frac{1-\delta}{1+\delta}$ proves the claim.

■

Corollary 5.2.1 *Thus for all $y\in\Gamma_x$, $\log Q(x+iy)$ and $Q^s(x+iy)$ are well defined analytic functions of the variable $z=x+iy$ for the **branch** of the* log*: $0<\arg Q(z)<2\pi$.*

The tube cone T^C is $O(n,1)_+^{\uparrow}$ **invariant** thus our arguments would be still valid for any vector θ in the orbit of $(1,0,0,0)$ by $O(n,1)_+^{\uparrow}$. Thus all results of proposition 5.2.3 are **independent** of the choice of θ in the open cone $Q(\theta)>0,\theta^0>0$. The key inequality (5.21) also appears in a less precise form in the proof of Proposition 4.1 p. 352 in [48].

Partial results by the Vladimirov approach. In the course of the proof of proposition (5.2.3), we rediscovered the Lorentz invariant inequality $\forall z=x+iy\in T^C, |Q(z)|\geqslant |Q(y)|$. We notice that $\forall y\in C, Q(y)=2\Delta^2(y)$ where $\Delta(y)=\left(\frac{(y^0)^2-|y|^2}{2}\right)^{\frac{1}{2}}$ is the Euclidean distance beetween y and the boundary of C. Immediately, we deduce that for $Re(s)\leqslant 0$:

$$|\,(Q(z))^s\,|\leqslant (2\Delta^2(y))^{Re(s)}\leqslant M(s)(1+\Delta^{2Re(s)}(y)),$$

this means Q^s is in the algebra $H(C)$ of slowly increasing functions in $O(T^C)$ (where $O(T^C)$ is the algebra of holomorphic functions in T^C). Application of theorems of Vladimirov proves the existence of a boundary value $\lim_{y\rightarrow 0, z=x+iy\in T^C} Q^s(z)$ in the space of tempered distributions when $y\rightarrow 0$ in C. The limit is understood as a tempered distribution and also the Fourier transform of Q^s is a tempered distribution in $\mathcal{S}'(C^\circ)$ which is the algebra for the convolution product of Schwartz distributions supported on the dual cone C° of C. In the terminology of Yves Meyer, the boundary value $Q^s(.+i0\theta)$ is C° *holomorphic.*

Existence of the boundary value as a distribution. The previous estimates allow us to prove a moderate growth property which is the requirement to apply Theorems 3.1.15 and 8.4.8 in [40] giving existence of Boundary values and control of the wave front set:

Proposition 5.2.4 *For any closed conic subset $V_{x_0}\subset\Gamma_{x_0}$, there exists a sufficiently small neighborhood U_{x_0} of x_0 such that for all $x+iy\in U_{x_0}+iV_{x_0}, |y|\leqslant\delta$,*

$$|\log(Q(x+iy))|\leqslant\frac{C}{|y|}\tag{5.22}$$

$$|Q^s(x+iy)|\leqslant C|y|^{2Re(s)}\tag{5.23}$$

Thus the hypothesis of theorem 3.1.15 of [40] are satisfied for $\log(Q(z)), Q^s(z)$.
Proof — Since $\forall(x,y)\in U_{x_0}\times V_{x_0}, 0<|y|\leqslant\delta$, we have $Q(x+iy)\notin\mathbb{R}_+$, then we must have $\log Q(x+iy)=\log|Q(x+iy)|+i\text{arg}\,(Q(x+iy))$ where $0<\arg(Q)<2\pi$ which implies $|\log Q(x+iy)|<\log|Q(x+iy)|+2\pi$. Recall that we have estimates of the form

$$\forall(x,y)\in U_{x_0}\times V_{x_0}, 0<|y|\leqslant\delta, \delta|y|^m\leqslant|Q(x+iy)|$$

We can assume without loss of generality that $0<C|y|^m<1$ and $|Q(x+iy)|\leqslant 1$. Then we have

$$\forall(x,y)\in U_{x_0}\times V_{x_0}, 0<|y|\leqslant\delta, \delta|y|^m\leqslant|Q(x+iy)|\implies |Q^s(x+iy)|\leqslant(\delta|y|^m)^{Re(s)}$$

for $Re(s) \leqslant 0$. And also $\forall (x,y) \in U_{x_0} \times V_{x_0}, 0 < |y| \leqslant \delta, \delta|y|^m \leqslant |Q(x+iy)|$

$$\implies \log \delta|y|^m \leqslant \log|Q(x+iy)| \implies |\log|Q(x+iy)|| \leqslant |\log(\delta|y|^m)|.$$

Thus we find

$$|\log Q(x+iy)| \leqslant 2\pi + |\log\delta| + m|\log(|y|)|.$$

■

Corollary 5.2.2 *Application of Theorem 3.1.15 in [40] implies $Q^s(.+i0y)$ and $\log Q(.+i0y)$ for $y \in \Gamma$ are both well defined on $\mathbb{R}^{n+1}$ as boundary values of holomorphic functions.*

The proof that $Q^s(.+i0y)$ defines a tempered distribution is only sketched in [48] Proposition 4.1 and it is proved in [46] in example 2.4.3 p. 90 that these are hyperfunctions in the sense of Sato but this is not enough to prove these are genuine distributions. Notice that the existence and definition of the boundary values $Q^s(.+i0y)$ and $\log Q(.+i0y)$ **does not depend** on the choice of y provided y lives in the open cone C^+, but since this cone is $O(n,1)^{\uparrow}_{+}$ invariant, the distributions $Q^s(.+i0y)$ and $\log Q(.+i0y)$ are $O(n,1)^{\uparrow}_{+}$ invariant.

The wave front set of the boundary value.

Theorem 5.2.1 *The wave front set of $Q^s(.+i0\theta)$ and $\log Q(.+i0\theta)$ is contained in the set:*

$$\{(x;\tau dQ)|\tau x^0 > 0, Q(x)=0\} \bigcup \{(0;\xi)|Q(\xi,\xi) \geqslant 0, \xi_0 > 0\}. \quad (5.24)$$

Proof — We want to apply Theorem 8.7.5 in [40] in order to obtain the result explained in [40] on p. 322. More precisely, we want to apply Theorem 8.4.8 of [40] which gives the wave front set of boundary values of holomorphic functions. Application of Theorem 8.4.8 of [40] claims that for each point x_0 such that $Q(x_0)=0$,

$$WF(\log Q(U_{x_0}+i0V_{x_0})) \subset U_{x_0} \times V_{x_0}^{\circ}$$

where $V_{x_0}^{\circ} = \{\eta|\forall y \in V_{x_0}, \eta(y) \geqslant 0\}$ is the dual cone of V_{x_0}. But since this upper bound is true for **any** closed subcone $V_{x_0} \subset \Gamma_{x_0}$ and corresponding neighborhood U_{x_0} containing x_0, by picking an *increasing* family $V_{\delta,x_0} = \{y|q(x_0,y) \geqslant 2\delta|y|\}$ and the corresponding *decreasing* family of neighborhoods $U_{x_0,\delta} = \{x|\forall y \in V_{\delta,x_0}, |q(x,y)| \geqslant \delta|y|, |x-x_0| \leqslant \delta\}$, when $\delta \to 0$, we find that the wave front set of the boundary value over each point x_0 should be contained in the **dual cone** $\Gamma_{x_0}^{\circ} = \{\eta|\forall y \in \Gamma_{x_0}, \eta(y) \geqslant 0\}$ of

Γ_{x_0}. Our job consists in determining this **dual cone** $\Gamma_{x_0}^\circ$ for all x_0 such that $Q(x_0)=0$ ie in the singular support of $Q^s(.+i0\theta)$. As usual there are two cases: $Q(x_0)=0, x_0\neq 0$ and $x_0=0$.

For $Q(x_0)=0, x_0\neq 0$, consider the cone

$$\{y|q(x_0,y)\neq 0\} \tag{5.25}$$

this cone contains *two connected components* separated by the hyperplane $H=\{y|q(x_0,y)=0\}$, we should set Γ_{x_0} equal to the connected component which contains θ,

$$\Gamma_{x_0}=\{y|q(x_0,y)q(x_0,\theta)>0\}.$$

However, since $q(x_0,\theta)=x_0^0$ and $dQ_{x_0}(y)=q(x_0,y)$, it is much more convenient to reformulate Γ_{x_0} as the half space

$$\Gamma_{x_0}=\{y|\eta(y)>0\},\eta=x_0^0 dQ_{x_0} \tag{5.26}$$

for the linear form $y\mapsto\eta=x_0^0 dQ_{x_0}(y)$. By definition, this half space is the convex enveloppe of the linear form η thus the dual cone $\Gamma_{x_0}^\circ$ of the half space Γ_{x_0} consists in the positive scalar multiples of the linear form η generating this half space, finally $\Gamma_{x_0}^\circ=\{\tau dQ_{x_0}|\tau x_0^0>0\}$.

When $x_0=0$, consider the cone

$$\{y|q(y,y)\neq 0\} \tag{5.27}$$

this cone contains *three connected components* depending on the sign of Q and y^0, we should set Γ_0 equals to the connected component which contains θ:

$$\Gamma_0=\{y|q(y,y)>0,y^0>0\}. \tag{5.28}$$

By a straightforward calculation

$$\Gamma_0^\circ=\{\eta|\forall y\in\Gamma_0,\eta(y)\geqslant 0\}=\{\eta|Q(\eta)\geqslant 0,\eta^0\geqslant 0\},$$

which is the future cone in dual space. Finally,

$$WF\log Q(.+i0\theta)\subset\left(\bigcup_{x_0\neq 0,Q(x_0)=0}\Gamma_{x_0}^\circ\right)\bigcup\Gamma_0^\circ$$

and we have the same upper bound for $WFQ^s(.+i0\theta)$. ∎

The proof of this theorem cannot be found in physics textbooks and is not even sketched in [40] (where it is only stated as an example of direct application of Theorem 8.7.5 in [40]). A nice consequence of theorems proved in this section is that it makes sense of **complex powers** of the Wightman function Δ_+. Our work differs from the work of Marcel Riesz because the Riesz family $\square^s$ does not have the right wave front set, for all s $\square^s\neq\Delta_+^s$, actually $\square^{-1}$ is a **fundamental solution** of the wave equation whereas the Wightman function Δ_+ is an actual **solution** of the wave equation.

5.3 Pull-backs and the exponential map.

The moving frame. Let (M, g) be a pseudo-Riemannian manifold and TM its tangent bundle. We denote by $(p; v)$ an element of TM, where $p \in M$ and $v \in T_pM$. Let $\mathcal{N}$ be a neighborhood of the zero section $\underline{0}$ in TM for which the map $(p; v) \in \mathcal{N} \mapsto (p, \exp_p(v)) \in M^2$ is a local diffeomorphism onto its image ($\exp_p : T_pM \mapsto M$ is the exponential geodesic map). Thus the subset $\mathcal{V} = \exp \mathcal{N} \subset M^2$ is a neighborhood of d_2 and the inverse map $(p_1, p_2) \in \mathcal{V} \mapsto (p_1; \exp^{-1}_{p_1}(p_2)) \in \mathcal{N}$ is a well defined diffeomorphism. Let Ω be an open subset of M and $(e_0, ..., e_n)$ be **an orthonormal moving frame** on Ω ($\forall p \in \Omega, g_p(e_\mu(p), e_\nu(p)) = \eta_{\mu\nu}$), and $(\alpha^\mu)_\mu$ the corresponding orthonormal moving coframe.

The pull-back. We denote by ϵ_μ the canonical basis of $\mathbb{R}^{n+1}$, then the data of the orthonormal moving coframe $(\alpha^\mu)_\mu$ allows to define the submersion

$$F := (p_1, p_2) \in \mathcal{V} \mapsto F^\mu(p_1, p_2)\epsilon_\mu = \underbrace{\alpha^\mu_{p_1}}_{\in T^\star_{p_1}M} \underbrace{(\exp^{-1}_{p_1}(p_2))}_{\in T_{p_1}M}\epsilon_\mu \in \mathbb{R}^{n+1}. \quad (5.29)$$

For any distribution f in $\mathcal{D}'(\mathbb{R}^{n+1})$, the composition

$$(p_1, p_2) \in \mathcal{V} \mapsto f \circ F(p_1, p_2) = f \circ \left(\alpha^\mu_{p_1}(\exp^{-1}_{p_1}(p_2))\epsilon_\mu\right)$$

defines the pull-back of f on $\mathcal{V} \subset M^2$. If f is $O(n,1)^\uparrow_+$ invariant, then the pull-back defined as above **does not depend on the choice of orthonormal moving frame** $(e_\mu)_\mu$ and is thus **intrinsic** (since all orthonormal moving frames are related by gauge transformations in $C^\infty(M, O(n,1)^\uparrow_+)$). We apply this construction to the family $Q^s(h + i0\theta) \in \mathcal{D}'\left(\mathbb{R}^{n+1}\right)$ constructed in Corollary (5.2.2) as boundary value of holomorphic functions, and we obtain the distribution $(p_1, p_2) \in \mathcal{V} \mapsto Q^s \circ \left(\alpha^\mu_{p_1}(\exp^{-1}_{p_1}(p_2))\epsilon_\mu\right)$. This allows to **canonically** pull-back $O(n,1)^\uparrow_+$ invariant distributions to distributions defined on a neighborhood of d_2.

Example 5.3.1 *The quadratic function $Q(h) = h^\mu\eta_{\mu\nu}h^\nu$ is $O(n,1)^\uparrow_+$ invariant in $\mathbb{R}^{n+1}$. The pull back of Q by F on $\mathcal{V}$ gives*

$$Q \circ F(p_1, p_2) = \alpha^\mu_{p_1}(\exp^{-1}_{p_1}(p_2))\eta_{\mu\nu}\alpha^\nu_{p_1}(\exp^{-1}_{p_1}(p_2))$$

which is the "square of the pseudodistance" between the two points (p_1, p_2) called Synge's world function in the physics literature. Following [35], we will denote this function by $\Gamma(p_1, p_2)$.

5.3.1 The wave front set of the pull-back.

We compute the wave front set of $Q^s \circ F$.

The expression of $WF(Q^s(.+i0\theta))$ in terms of $\eta_{\mu\nu}$. Notice that $WFQ^s(.+i0\theta)$ can be written in the form:

$$WFQ^s(.+i0\theta)=\{(h^\mu;\lambda\eta_{\mu\nu}h^\nu)|Q(h)=0,h^0\lambda>0\}\cup\{(0;k)|Q(k)\geqslant 0,k_0>0\},\tag{5.30}$$

where the condition $h^0\lambda>0$ plays an important role in ensuring that the momentum $\lambda\eta_{\mu\nu}h^\nu$ has **positive energy**.

The pull-back theorem of Hörmander in our case. Denote by t the distribution $Q^s(.+i0\theta)$. An application of the pull-back theorem ([40] Theorem 8.2.4) in our situation gives

$$WF(F^\star t)\subset\{(p_1,p_2;k\circ d_{p_1}F,k\circ d_{p_2}F)|(F(p_1,p_2),k)\in WF(t)\}\tag{5.31}$$

We denote by $(p_1,p_2;\eta_1,\eta_2)$ an element of $T^\star\mathcal{V}\subset T^\star M^2$ and $(h^\mu;k_\mu)$ the coordinates in $T^\star\mathbb{R}^{n+1}$. The pull-back with indices reads:

$$(p_1,p_2;k\circ d_{p_1}F,k\circ d_{p_2}F)=(p_1,p_2;k_\mu d_{p_1}F^\mu,k_\mu d_{p_2}F^\mu).$$

Step 1, we first compute $WF(F^\star t)$ outside the set $d_2=\{p_1=p_2\}$. The condition $(F(p_1,p_2),k)\in WF(t)$ in (5.31), reads by (5.30) $(F^\mu(p_1,p_2);k_\mu)=(F^\mu(p_1,p_2);\lambda\eta_{\mu\nu}F^\nu(p_1,p_2))$. We obtain

$$(p_1,p_2;\lambda k\circ d_{p_1}F,\lambda k\circ d_{p_2}F)=(p_1,p_2;\lambda F^\mu\eta_{\mu\nu_2}d_{p_1}F^{\nu_2},\lambda F^\mu\eta_{\mu\nu_2}d_{p_2}F^{\nu_2})$$

and also $F^\mu(p_1,p_2)\eta_{\mu\nu}F^\nu(p_1,p_2)=0$. Now set $\Gamma(p_1,p_2)=F^\mu(p_1,p_2)\eta_{\mu\nu}F^\nu(p_1,p_2)$. The key observation is that $d_{p_1}\Gamma=2F^\mu\eta_{\mu\nu}d_{p_1}F^\nu$ and $d_{p_2}\Gamma=2F^\mu\eta_{\mu\nu}d_{p_2}F^\nu$, hence:

$$WF(F^\star t)\subset\{(p_1,p_2;\lambda d_{p_1}\Gamma,\lambda d_{p_2}\Gamma)|\Gamma(p_1,p_2)=0,\lambda F^0(p_1,p_2)>0,\lambda\in\mathbb{R}\}$$
$$\cup\{(p_1,p_2;k\circ d_{p_1}F,k\circ d_{p_2}F)|p_1=p_2,Q(k)\geqslant 0,k_0>0\}.$$

The geometric interpretation of the last formula.

Definition 5.3.1 *A distribution $t\in\mathcal{D}'\left(M^2\right)$ satisfies the Hadamard condition, if and only if $WF(t)\subset\{(p_1,p_2;-\eta_1,\eta_2)|(x_1;\eta_1)\sim(x_2;\eta_2),\eta_2^0>0\}$.*

Our convention for the Hadamard condition is the opposite of the convention of Theorem 3.9 p. 33 in [45]. The Hadamard condition is a condition on the wave front set of a distributional bisolution of the wave equation which ensures it represents a quasi free state of the free quantum field theory in curved space time ([45]).

The function Γ is the pseudo Riemannian analogue of the square geodesic distance and will be discussed in paragraph (5.4.3). We first interpret the term

$$\{(p_1,p_2;\lambda d_{p_1}\Gamma,\lambda d_{p_2}\Gamma)|\Gamma(p_1,p_2)=0,\lambda F^0(p_1,p_2)>0\}$$

appearing in the last formula as the subset of all elements in $T^\star\mathcal{V}$ of the conormal bundle of the conoid $\{\Gamma = 0\}$ such that $(\eta_2)_0$ has **constant** sign: this is exactly the **Hadamard condition**. If we use the metric to lift the indices, $d_{p_1}\Gamma\left(e_\mu(p_1)\right)\eta^{\mu\nu}e_\nu(p_1)$ and $d_{p_2}\Gamma\left(e_\mu(p_2)\right)\eta^{\mu\nu}e_\nu(p_2)$ are the Euler vector fields $\nabla_1\Gamma, \nabla_2\Gamma$ defined by Hadamard. We will later prove in proposition (5.4.3) that the vectors $\nabla_1\Gamma, -\nabla_2\Gamma$ are parallel along the null geodesic connecting p_1 and p_2, proving $(d_{p_1}\Gamma, -d_{p_2}\Gamma)$ are in fact **coparallel** along this null geodesic.

Step 2, "Diagonal". For any function F on M^2, we uniquely decompose the total differential in two parts as follows

$$dF = d_{p_1}F + d_{p_2}F, \text{ where } d_{p_1}F|_{\{0\}\times T_{p_2}M} = 0, d_{p_2}F|_{T_{p_1}M\times\{0\}} = 0.$$

Let i be the inclusion map $i := p \in M \mapsto (p,p) \in d_2 \subset M$ then $\forall p \in M, F\circ i(p) = 0 \implies d_pF\circ i = 0 \Leftrightarrow d_{p_1}F\circ di + d_{p_2}F\circ di = 0$. Since

$$d_{p_2}F^\mu(p,p) = d_{p_2}\alpha^\mu_{p_1}\left(\exp^{-1}_{p_1}(p_2)\right)|_{p_1=p_2=p} = \alpha^\mu_{p_1}\left(d_{p_2}\exp^{-1}_{p_1}(p_2)\right)|_{p_1=p_2=p} = \alpha^\mu(p),$$

because $d_{p_2}\exp^{-1}_{p_1}(p_2)|_{p_1=p_2=p} = Id_{T_pM\mapsto T_pM} = e_\mu(p)\alpha^\mu(p)$. Thus $d_{p_1}F^\mu(p,p) = -\alpha^\mu(p)$ and

$$\{(p_1,p_2;k\circ d_{p_1}F, k\circ d_{p_2}F)|p_1 = p_2, Q(k)\geqslant 0, k_0 > 0\}$$

$$= \{(p,p;-k_\mu\alpha^\mu(p), k_\mu\alpha^\mu(p))|p\in M, Q(k)\geqslant 0, k_0>0\}.$$

Then summarizing step 1 and step 2, let us denote by $\Lambda \subset T^\bullet\left(M^2\setminus d_2\right)$ the conormal bundle of the set $\{\Gamma = 0\}$ with the zero section removed:

Theorem 5.3.1 *The wave front set of the distributions $Q^s(\cdot + i0\theta)\circ F$ and $\log Q(\cdot + i0\theta)\circ F$ is contained in*

$$\left(\Lambda\bigcup\{(p,p;-\eta,\eta)|g_p(\eta,\eta)\geqslant 0\}\right)\bigcap\{(p_1,p_2;\eta_1,\eta_2)|\eta_2^0>0\}, \qquad (5.32)$$

where $\Lambda \subset T^\bullet\left(M^2\setminus d_2\right)$ is the conormal of $\{\Gamma = 0\}$ with the zero section removed.

Remarks:
a)If we denote by $\overline{\Lambda}$ the closure of the conormal $\Lambda\subset T^\bullet\left(M^2\setminus d_2\right)$ in $T^\bullet M^2$, then $(\Lambda\bigcup\{(p,p;-\eta,\eta)|g_p(\eta,\eta)\geqslant 0\}) = \overline{\Lambda}+\overline{\Lambda}$.
b) $\{(p,p;-\eta,\eta)|g_p(\eta,\eta)\geqslant 0\}$ is contained in the conormal $(Td_2)^\perp$ of d_2.

Corollary 5.3.1 *The families $Q^s(.+i0\theta)\circ F$ and $\log Q(.+i0\theta)\circ F$ satisfy the **Hadamard condition**.*

Discussion of the sign convention for the energy. We want to discuss some sign conventions. Recall that if $(h;k)\in WF(Q(.+i0\theta)^s)$ (resp $WF(Q(.-i0\theta)^s)$) then k has positive (resp negative) energy. Denote $(p_1,p_2;\eta_1,\eta_2)$ an element of the wave front set of $F^\star Q^s(\cdot\pm i0\theta)$. If we want η to be a covector of **positive energy** (resp negative energy), then we must consider the distribution $F^\star Q^s(.+i0\theta)$ (resp $F^\star Q^s(.-i0\theta)$).

Notice that in the physics literature, the boundary value is determined using a Cauchy hypersurface determined by a function $T: M\mapsto\mathbb{R}$:

$$\left(\Gamma(p_1,p_2)+i\varepsilon(T(p_1)-T(p_2))+\varepsilon^2\right)^s .$$

The proof that it defines a well defined distribution is never given and the wave front set of this boundary value was never computed. Furthermore, the formula is not obviously covariant since it relies on the existence of a foliation of space-times by Cauchy hypersurfaces.

5.3.2 The pull back of the phase function.

In order to connect with the interpretation of the wave front set in terms of Lagrangian manifold, we imitate what we did for $((x^0\pm i0)^2-\sum_{i=1}^n(x^i)^2)^{-1}$, we pull-back the oscillatory integral representation on $\mathcal{V}\subset M^2$ by the smooth map F.

Theorem 5.3.2 *The distribution $F^*\left(Q(.+i0\theta)\right)^{-1}$ is the Lagrangian distribution given by the formula*

$$C_n\int_{\mathbb{R}^n} d^n\xi e^{i(\phi_\pm\circ F)(p_1,p_2;\xi)}\frac{1}{|\xi|},$$

this Lagrangian distribution with phase function $\phi_\pm\circ F$ has a wave front set which satisfies the Hadamard condition.

Proof — Let us only sketch the proof. First we use Proposition (5.1.5) to determine the wave front set of the oscillatory integral $C_n\int_{\mathbb{R}^n}d^n\xi e^{i\phi_\pm(h;\xi)}\frac{1}{|\xi|}$. It is the same wave front set as for $((h^0\pm i0)^2-\sum_1^n(h^i)^2)^{-1}$, then we apply the pull-back theorem of Hörmander in order to define the wave front set on the curved space and it exactly follows the same proof as for the pull back theorem (5.3.1). ∎

5.4 The construction of the parametrix.

Our parametrix construction is based on the work of Hadamard [35] (see also [18]). The construction is done in the neighborhood $\mathcal{V}$ of d_2. Recall by 5.29 that $F(p_1,p_2)=e^\mu_{p_1}\left(\exp^{-1}_{p_1}(p_2)\right)\epsilon_\mu$.

The Hadamard expansion. We construct the parametrix locally in $\mathcal{V}$ by successive approximations. Inspired by the flat case, we look for an expansion of the form

$$\Delta_+ = U(p_1,p_2)\left(Q^{-1}\circ F\right)(p_1,p_2)$$
$$+\sum_{k=0}^{\infty} V_k(p_1,p_2)\Gamma^k(p_1,p_2)\left(\log Q\circ F\right)(p_1,p_2)$$

where $\Gamma(p_1,p_2)=Q\circ F$ is the square of the pseudodistance and each term of the asymptotic expansion has an intrinsic meaning.

5.4.1 The meaning of the asymptotic expansions.

Our goal is to construct U, V_k in $C^\infty(\mathcal{V})$. First, we would like to make an important remark. The series $\sum_k V_k\Gamma^k$ does not usually converge. However, we can still make sense of the asymptotic expansion $\sum_k V_k\Gamma^k$ as the asymptotic expansion of the **composite function** $V(.,.;\Gamma)$ in $C^\infty(\mathcal{V}\times\mathbb{R})$ where only the germs of map $r\mapsto V(.,.;r)$ at $r=0$ are defined (V is not uniquely defined).

The Borel lemma.

Proposition 5.4.1 *For any sequence of smooth functions $(V_k)_k$ in $(C^\infty(\mathcal{V}))^{\mathbb{N}}$, there exists a smooth function $r\mapsto V(.,.;r)$ in $C^\infty(\mathcal{V}\times\mathbb{R})$ such that the coefficients of the Taylor series in the variable r of V is equal to the sequence V_k:*

$$V_k(p_1,p_2)=\frac{1}{k!}\frac{\partial^k V}{\partial r^k}(p_1,p_2;0). \tag{5.33}$$

Proof — The proof is an application of the idea of the proof of the Borel lemma which states that any sequence $(a_k)_k$ can be realized as the Taylor series of a smooth function at 0. The proof we give is due to Malgrange [50]. Let $\Omega\subset\mathcal{V}$ be an open subset with compact closure, then $\sup_\Omega |V_k| = a_k<\infty$. Let $\chi(r)$ be a cut-off function near $r=0$, $\chi=1$ in a neighborhood of zero and vanishes when $r\geqslant 1$. We fix any sequence b_k, s.t. $b_k>0$ growing sufficiently fast such that $\forall k, \sup_{r\in\mathbb{R}^+,\alpha\leqslant k-1}|\partial_r^\alpha a_k\chi(rb_k)r^k|\leqslant\frac{1}{2^k}$. Then $\sum V_k\chi(\frac{r}{b_k})r^k$ is a smooth function whose Taylor coefficients are the V_k. The series $\sum V_k\chi(\frac{r}{b_k})r^k$ is bounded and defines a smooth function only on the set Ω. Let $(\varphi_j)_{j\in J}$ be a collection of compactly supported functions in M^2 such that $\sum_{j=J}\varphi_j=1$ in a neighborhood of d_2 and vanishes outside $\mathcal{V}$. For each $j\in J$, since supp φ_j is compact the previous construction gives us a sequence $(b_{k_j})_{k_j}$. This gives us a final series $U=\sum_{j\in J,k\in\mathbb{N}}\varphi_j V_k\chi(\frac{r}{b_{kj}})r^k$ which is a smooth function supported in $\mathcal{V}$ such that

$$V(.,.;\Gamma)=\sum_{j\in J,k\in\mathbb{N}}\varphi_j V_k\chi(\frac{\Gamma}{b_{kj}})\Gamma^k\sim\sum_{k\in\mathbb{N}}V_k\Gamma^k.$$

■

This remark cannot be found in any physics textbook. It is given in [25] Lemma 4.3.2. Finally, if we know the sequence of coefficients V_k, we find a function V such that $V(p_1, p_2; r) = \sum V_k(p_1, p_2) r^k$, thus $V(p_1, p_2; \Gamma)$ is a well defined smooth function.

5.4.2 The invariance properties of the Beltrami operator $\Box^g$ and of gradient vector fields.

Let (M, g) be a pseudo Riemannian manifold and let us define the Dirichlet energy $\mathcal{E}(u; g)$ by the equation:

$$\mathcal{E}(u; g) = \int_M \frac{1}{2} \langle \nabla u, \nabla u \rangle_g \, d\text{vol}_g. \tag{5.34}$$

We will follow the exposition of Hélein (see [37]) and define the Beltrami operator $\Box^g$ for a general metric g by the first variation of the Dirichlet energy:

$$\delta\mathcal{E}(u, g)(\varphi) = \int_M \langle \nabla u, \nabla \varphi \rangle_g \, d\text{vol}_g = -\int_M (\Box^g u) \, \varphi d\text{vol}_g, \tag{5.35}$$

(see [37] Equation (1.5) p. 3).

The operator $\Box^g$. Let Φ be a diffeomorphism of M, and

$$\Phi : (M, \Phi^\star g) \mapsto (M, g)$$

the associated isometry, then the Dirichlet energy satisfies the invariance equation by the action of diffeomorphisms: $\forall \Phi \in Diff(M), \mathcal{E}(u; g) = \mathcal{E}(u \circ \Phi; \Phi^* g)$ (see [37] p. 18-19 for the proof). Thus the Beltrami operator $\Box^g$ obeys the equation

$$\forall \Phi \in \text{Diff}(M), (\Box^g u) \circ \Phi = \Box^{\Phi^\star g} (u \circ \Phi) \tag{5.36}$$

The gradient operator ∇^g. We want to prove that gradient vector fields w.r.t. the metric g also behave in a natural way. Let $f \in C^\infty(M)$ then

$$\forall \Phi \in \text{Diff}(M), \forall f \in C^\infty(M), \nabla^{\Phi^\star g} (f \circ \Phi) = \Phi^\star (\nabla^g f) \tag{5.37}$$

$$\langle \nabla^g f, \nabla^g f \rangle_g = \left\langle \nabla^{\Phi^\star g} (f \circ \Phi), \nabla^{\Phi^\star g} (f \circ \Phi) \right\rangle_{\Phi^\star g} \tag{5.38}$$

The first equation is equivalent to the equation $\Phi_\star \left(\nabla^{\Phi^\star g} (f \circ \Phi) \right) = \nabla^g f$ ([47] p. 92–93). We use the coordinate convention:

$$\Phi : x^\alpha \in (M, \Phi^\star g) \mapsto \phi^\gamma(x) \in (M, g)$$

We start from the definition:

$$\nabla^{\Phi^\star g}(f\circ\Phi)=\left(g^{\gamma\delta}\frac{\partial x^\alpha}{\partial\phi^\gamma}\frac{\partial x^\beta}{\partial\phi^\delta}\right)\circ\Phi\frac{\partial(f\circ\Phi)}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}$$

$$=\left(g^{\gamma\delta}\frac{\partial x^\alpha}{\partial\phi^\gamma}\frac{\partial x^\beta}{\partial\phi^\delta}\frac{\partial f}{\partial\phi^\mu}\right)\circ\Phi\frac{\partial\phi^\mu}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}=\left(g^{\gamma\delta}\frac{\partial x^\beta}{\partial\phi^\delta}\frac{\partial f}{\partial\phi^\gamma}\right)\circ\Phi\frac{\partial}{\partial x^\beta}$$

then we push-forward this vector field

$$\Phi_\star\left(\nabla^{\Phi^\star g}(f\circ\Phi)\right)=\left(g^{\gamma\delta}\frac{\partial x^\beta}{\partial\phi^\delta}\frac{\partial f}{\partial\phi^\gamma}\circ\Phi\right)\circ\Phi^{-1}\frac{\partial\phi^\mu}{\partial x^\beta}\frac{\partial}{\partial\phi^\mu}$$

$$=g^{\gamma\delta}\frac{\partial f}{\partial\phi^\gamma}\frac{\partial}{\partial\phi^\delta}=\nabla^g f$$

The proof of the second identity can be simply deduced from the first one and one can also look at [37] p. 19 for a similar proof. In the sequel, we write ∇ instead of ∇^g where it will be obvious we take the gradient w.r.t. the intrinsic metric g which does not depend on the chart chosen. Recall that we denote by e_μ the orthonormal moving frame on M. We define two gradient operators ∇_1,∇_2 on M^2 as follows:

$$\forall f\in C^\infty(M^2),\nabla_1 f(p_1,p_2)=d_{p_1}f\left(e_\mu(p_1)\right)\eta^{\mu\nu}e_\nu(p_1) \tag{5.39}$$

$$\forall f\in C^\infty(M^2),\nabla_2 f(p_1,p_2)=d_{p_2}f\left(e_\mu(p_2)\right)\eta^{\mu\nu}e_\nu(p_2). \tag{5.40}$$

The exponential map and lifting on tangent spaces. Let us justify microlocally the philosophy of the Hadamard construction which consists in treating $Q^{-1}\circ F$ and $\log Q\circ F$ as distributions of p_2 where p_1 is viewed as a parameter: let $f\in\mathcal{D}'(\mathcal{V})$ be any distribution in $\mathcal{V}\subset M^2$. We fix $p_1\in M$, then we can make sense of the restriction of f, $f(p_1,.):=p_2\in M\mapsto f(p_1,p_2)$ as a distribution on $\{p_1\}\times M$ if

$$\text{Conormal }\left(\{p_1\}\times M\right)\bigcap WF(f)=\emptyset.$$

Let π_1 be the projection $\pi_1:=(p_1,p_2)\in M^2\mapsto p_1\in M$, if we have

$$\forall p_1\in M,\text{Conormal }\left(\{p_1\}\times M\right)\bigcap WF(f)=\emptyset,$$

then for any test density $\omega\in\mathcal{D}^{n+1}(M)$, the map $\pi_{1\star}(f\omega)$ defined by:

$$\left[\pi_{1\star}(\omega f)=p_1\mapsto\underbrace{\int_M\omega(p_2)f(p_1,p_2)}_{\text{partial integration}}\right]$$

is **smooth** since $WF(\pi_{1\star}f)=\emptyset$ by Proposition 1.3.4 in [17]. These conditions are satisfied in our case since the wave front set of $Q^{-1}\circ F$ and $\log Q\circ F$

are **transverse** to the conormal of $(\{p_1\} \times M)$ by Theorem 5.31. We pull back $f(p_1, .)$ on $\mathbb{R}^{n+1}$ by the map E_{p_1} defined as follows:

$$E_{p_1} : (h^\mu)_\mu \in \mathbb{R}^{n+1} \mapsto E_{p_1}(h) = \exp_{p_1}(h^\mu e_\mu(p_1))) \in M.$$

The orthonormal frame $(e_\mu(p_1))_\mu$ fixes the isomorphism beetween $T_{p_1}M$ and $\mathbb{R}^{n+1}$.

5.4.3 The function Γ and the vectors ρ_1, ρ_2.

In the Hadamard construction, everything is expanded in powers of the function Γ which is the "square of the pseudo Riemannian distance". Γ is a solution of the nonlinear equation (5.41). In the physics literature, the function Γ is called Synge world's function but the definition and the key equation (5.41) satisfied by Γ can already by found in Hadamard (see the equation (32) in [35] and the Lamé Beltrami differential parameters for Γ).

The function Γ.

We already defined the function $\Gamma(p_1, p_2) = \alpha^\mu_{p_1}(\exp^{-1}_{p_1}(p_2))\eta_{\mu\nu}\alpha^\nu_{p_1}(\exp^{-1}_{p_1}(p_2))$ in example (5.3.1). In the following proposition, we explain which differential equation this function satisfies.

Proposition 5.4.2 *Let us define the function*

$$\Gamma(p_1, p_2) = \left\langle \exp^{-1}_{p_1}(p_2), \exp^{-1}_{p_1}(p_2) \right\rangle_{g_{p_1}}$$

in $\mathcal{V} \subset M^2$. Then Γ satisfies the equation

$$\forall p_1, \langle \nabla_2 \Gamma, \nabla_2 \Gamma \rangle_{g(p_2)} (p_2) = 4\Gamma \tag{5.41}$$

Proof — Denote by $E^\star_{p_1} g$ the metric in the geodesic exponential chart centered at p_1. We give a purely pseudo Riemannian geometry proof of the claim. Since $\Gamma(p_1, p_2) = \left\langle \exp^{-1}_{p_1}(p_2), \exp^{-1}_{p_1}(p_2) \right\rangle_{g_{p_1}}$, we know that

$$\forall p_1 \in M, \forall h \in \mathbb{R}^{n+1}, E^\star_{p_1}\Gamma(p_1, \cdot)(h) = h^\mu \eta_{\mu\nu} h^\nu.$$

Then by equation (5.37), writing $\left(E^\star_{p_1} g\right)^{\mu\nu}(h) = \left(E^\star_{p_1} g\right)^{\mu\nu}$ for shortness:

$$\begin{aligned}
\forall p_1 \in M, \langle \nabla\Gamma(p_1, .), \nabla\Gamma(p_1, .)\rangle_g &= \left\langle \nabla_2^{E^\star_{p_1} g} \left(E^\star_{p_1}\Gamma\right), \nabla_2^{E^\star_{p_1} g} \left(E^\star_{p_1}\Gamma\right) \right\rangle_{E^\star_{p_1} g} \\
&= \left(E^\star_{p_1} g\right)^{\mu\nu} \partial_{h^\mu}(h^{\mu_1}\eta_{\mu_1\nu_1}h^{\nu_1})\partial_{h^\nu}(h^{\mu_2}\eta_{\mu_2\nu_2}h^{\nu_2}) \\
&= \left(E^\star_{p_1} g\right)^{\mu\nu} 2\delta^{\mu_1}_{\mu}\eta_{\mu_1\mu_2}h^{\mu_2} 2\delta^{\nu_1}_{\nu}\eta_{\nu_1\nu_2}h^{\nu_2} \\
&= 4(E^\star_{p_1} g)^{\mu\nu}(h)(E^\star_{p_1} g)_{\mu\mu_2}h^{\mu_2}(E^\star_{p_1} g)_{\nu\nu_2}h^{\nu_2} \\
&= 4(E^\star_{p_1} g)_{\mu_2\nu_2}h^{\mu_2}h^{\nu_2} = 4\eta_{\mu_2\nu_2}h^{\mu_2}h^{\nu_2},
\end{aligned}$$

by repeated application of the Gauss lemma: $(E^\star_{p_1} g)_{\mu\nu}h^\nu = \eta_{\mu\nu}h^\nu$. ■

The Euler fields defined by Hadamard. Once we have defined the geometric function Γ, we can define a pair of scaling vector fields:

Definition 5.4.1 *Let $(p_1, p_2) \in \mathcal{V} \subset M^2$, we define the pair of vector fields*

$$\rho_2 = \frac{1}{2}\nabla_1\Gamma = d_{p_2}\Gamma(e_\mu(p_2))\eta^{\mu\nu}e_\nu(p_2) \tag{5.42}$$

$$\rho_1 = \frac{1}{2}\nabla_2\Gamma = d_{p_1}\Gamma(e_\mu(p_1))\eta^{\mu\nu}e_\nu(p_1). \tag{5.43}$$

ρ_1, ρ_2 are Euler vector fields in the sense of Chapter 1 for the diagonal $d_2 \subset \mathcal{V}$. The situation is reminiscent of Morse theory. If we freeze the variable p_1, the vector field $\rho_1 = \frac{1}{2}\nabla_2\Gamma$ is the **gradient** (w.r.t. p_2 and metric g) of the **Morse function** $p_2 \mapsto \Gamma(p_1, p_2)$ which has a critical point at $p_1 = p_2$. The Hadamard equation (5.41) takes the simple form

$$\rho_2\Gamma(p_1, p_2) = \rho_1\Gamma(p_1, p_2) = 2\Gamma(p_1, p_2) \tag{5.44}$$

thus Γ is homogeneous of degree 2 with respect to the geometric scaling defined by these Euler vector fields.

Useful relations beetween Γ, ρ_2 and $Q^s \circ F$. Around p_1, the manifold M is locally parametrized by the map $E_{p_1} : h \in \mathbb{R}^{n+1} \mapsto \exp_{p_1}(h^\mu e_\mu(p_1))$. $\rho_2 = \nabla_2\Gamma$ is an Euler vector field in M and we want to study its pull-back $E_{p_1}^\star\rho_2$ by E_{p_1}.

Proposition 5.4.3 *We have the identity $\forall p_1 \in M, E_{p_1}^\star\rho_2 = 2h^j\partial_{h^j}$ and this identity is independent of the choice of orthonormal moving frame.*

Proof — Denote by $E_{p_1}^\star g$ the metric in the geodesic exponential chart centered at p_1. By naturality (5.37), we have setting $\left(E_{p_1}^\star g\right)^{\mu\nu}(h) = \left(E_{p_1}^\star g\right)^{\mu\nu}$

$$E_{p_1}^\star\rho_2 = E_{p_1}^\star\left(\nabla_2\Gamma\right) = \nabla\left(E_{p_1}^\star\Gamma\right)$$

$$= (E_{p_1}^\star g)^{\mu\nu}\partial_{h^\mu}\left(\eta_{kl}h^kh^l\right)\partial_{h^\nu} = (E_{p_1}^\star g)^{\mu\nu}\left(\eta_{kl}\delta_\mu^k h^l + \eta_{kl}h^k\delta_\mu^l\right)\partial_{h^\nu}$$

$$= 2(E_{p_1}^\star g)^{\mu\nu}\eta_{\mu l}h^l\partial_{h^\nu} = 2(E_{p_1}^\star g)^{\mu\nu}(E_{p_1}^\star g)_{\mu l}h^l\partial_{h^\nu} = 2h^\nu\partial_{h^\nu}$$

by application of the Gauss lemma. ■

This proposition allows us to interpret $\frac{1}{2}\nabla_2\Gamma$ as the vector $\dot\gamma(1)$ where $s \mapsto \gamma(s)$ is the unique geodesic with boundary condition $\gamma(0) = p_1, \gamma(1) = p_2$: in exponential chart, this geodesic is given by the simple equation $t \mapsto \gamma(t) = th^j$ and for all t the vector $\dot\gamma(t) = h^j\frac{\partial}{\partial h^j}$ is **parallel** along this geodesic. By symmetry of the whole construction, we can interchange the roles of p_1 and p_2 and we deduce that $\rho_1 \in T_{p_1}M, -\rho_2 \in T_{p_2}M$ are **parallel** vectors along γ (see the same remark in [82] p. 18). A similar proof can be found in [18] Lemma 8.4.

We denote by Γ^s the distribution $F^*\left((Q(. + i0\theta))^s\right)$ and observe that $\forall n \in \mathbb{N}$, $\Gamma^n = F^\star Q^n$.

Proposition 5.4.4 *The relation*

$$\forall s \in \mathbb{R}, \forall n \in \mathbb{N}, \Gamma^n\Gamma^s = \Gamma^{n+s} \tag{5.45}$$

holds in the distributional sense.

Proof — Recall $E_{p_1}^\star\Gamma^s(h) = (Q(h+i0\theta))^s$. For $\varepsilon > 0$,

$$Q^n(h)Q^s(h+i\varepsilon\theta)$$

$$= Q^{n+s}(h+i\varepsilon\theta) + ((Q^n(h) - (Q(h+i\varepsilon\theta))^n)\, Q^s(h+i\varepsilon\theta))$$

where $((Q^n(h) - Q^n(h+i\varepsilon\theta))\, Q^s(h+i\varepsilon\theta))$ is an error term which converges weakly to zero when $\varepsilon \to 0$. Thus we should have $Q^n(h)\,(Q(h+i0\theta))^s = (Q(h+i0\theta))^{s+n}$ in the distributional sense. ■

5.4.4 The main theorem.

We first prove a lemma which implies that $WF(\Delta_+)$ satisfies the soft landing condition.

Lemma 5.4.1 *Let Ξ be the wave front set of $F^*\left((Q(\cdot+i0\theta))^s\right)$ then Ξ satisfies the soft landing condition.*

Proof — First note that, by Theorem 5.3.1, $\Xi \cap T^\star_{d_2}M^2$ is contained in $(Td_2)^\perp$ and $T^\bullet M^2 \setminus d_2 \subset \Lambda$ hence it suffices to prove that the conormal Λ of the conoid $\{\Gamma = 0\}$ satisfies the soft landing condition. Let $p : x \in \Omega \mapsto p(x) \in M$ be a local parametrization of M, using the local diffeomorphism $(x,h) \in \Omega \times \mathbb{R}^{n+1} \mapsto \left(p(x), \exp_{p(x)}(h^\mu e_\mu(p(x)))\right) \in \mathcal{V}$ (recall that $(e_\mu)_\mu$ is the orthonormal moving frame), we can parametrize the neighborhood $\mathcal{V}$ of d_2 with some neighborhood of $\Omega \times \{0\}$ in $\Omega \times \mathbb{R}^{n+1}$. In coordinates (x,h), the conoid is parametrized by the simple equation $\eta_{\mu\nu}h^\mu h^\nu = 0$, thus it is immediate that its conormal $\{(x,h;0,\xi) | \eta_{\mu\nu}h^\mu h^\nu = 0, \xi_\mu = \lambda\eta_{\mu\nu}h^\nu, \lambda \in \mathbb{R}\}$ satisfies the soft landing condition. ■

From the previous Lemma, we deduce the main theorem of this chapter. The motivation for this theorem is that it proves that the two point function satisfies the hypothesis of Theorem 3.2.1 of Chapter 3 which allows us to initialize the inductive proof of Chapter 6 of renormalizability of all n-point functions. We denote by $\Gamma^{-1}, \log\Gamma$ the distributions $F^*Q^{-1}(\cdot+i0\theta)$, $F^*\log Q(\cdot+i0\theta)$. Recall for any open set U, $E_s^\mu(U)$ defined in 4.3.3 was the space of distributions microlocally weakly homogeneous of degree s.

Theorem 5.4.1 *For any pair U, V of smooth functions in $C^\infty(\mathcal{V})$, the distribution*

$$U\Gamma^{-1} + V\log\Gamma$$

is in $E^\mu_{-2}(\mathcal{V})$.

Proof — Let ρ be one of the Euler vector fields defined in (5.4.1). For any pair U, V of smooth functions in $C^\infty(\mathcal{V})$, by Theorem 4.3.2, it suffices to prove that the family

$$\lambda^2 e^{\log\lambda\rho*}\left(U\Gamma^{-1} + V\log\Gamma\right)_\lambda$$

is bounded in $\mathcal{D}'_\Xi$. First Γ^{-1} is homogeneous of degree -2 w.r.t. scaling: $\lambda^2 e^{\log\lambda\rho_2*}\Gamma^{-1} = \lambda^2\lambda^{-2}\Gamma^{-1} = \Gamma^{-1}$ and $\lambda e^{\log\lambda\rho_2*}\log\Gamma = \lambda\log\lambda^{-2}\Gamma = -2\lambda\log\lambda + \lambda\log\Gamma$. Then from these equations, we deduce that the families $\left(\lambda^2 e^{\log\lambda\rho_2*}\Gamma^{-1}\right)_{\lambda\in(0,1]}$ and $\left(\lambda^2 e^{\log\lambda\rho_2*}\log\Gamma\right)_{\lambda\in(0,1]}$ are bounded in $\mathcal{D}'_\Xi$. Finally, we use that U, V being smooth, the families $(U_\lambda)_\lambda, (V_\lambda)_\lambda$ are bounded in the C^∞ topology in the sense that on any compact set K, the sup norms of the derivatives of arbitrary orders of $(U_\lambda)_\lambda, (V_\lambda)_\lambda$ are bounded. We can conclude using the estimate 3.9 of Theorem 3.3.1 to deduce $(\lambda^2 U_\lambda\Gamma_\lambda^{-1})_\lambda = (U_\lambda\Gamma)$ and $(\lambda^2 V_\lambda\log\Gamma_\lambda)_\lambda = (\lambda^2 V_\lambda\log\Gamma + 2\lambda^2 V_\lambda\log\lambda)$ are bounded in $\mathcal{D}'_\Xi$. ■

Corollary 5.4.1 *Consequently, if* $\Delta_+ - \left(U\Gamma^{-1} + V\log\Gamma\right) \in C^\infty(\mathcal{V})$ *for some* U, V *in* $C^\infty(\mathcal{V})$ *then* $\Delta_+ \in E^\mu_{-2}(\mathcal{V})$.

Then we can construct the Hadamard Riesz coefficients from which we can deduce suitable U, V (see the above discussion on the Borel lemma), however this construction is really classical and one can look at [82] and [27] Chapter 5.2 for the construction of these coefficients.

Chapter 6

The recursive construction of the renormalization.

6.0.5 Introduction.

This chapter deals with the construction of a perturbative quantum field theory using the algebraic formalism developed in ([10],[9]) and proves their renormalisability using all the analytical tools developped in the previous chapters. In the first part, we describe the Hopf algebraic formalism for QFT relying heavily on a paper by Christian Brouder [10] and a paper by R. Borcherds [9]. The end goal of this first part is the construction of the operator product of quantum fields denoted by $\star$. Then in the second part, we introduce the important concept of causality which allows to axiomatically define the time ordered product denoted by T: T solves the causality equations and T satisfies the Wick expansion property which is a Hopf algebraic formulation of the Wick theorem. Once we have a T-product, we can define quantities such as $t_n = \langle 0|T\phi^{n_1}(x_1)\dots\phi^{n_k}(x_k)|0\rangle$ where t_n is a distribution defined on configuration space M^n. We prove that if T satisfies our predefined axioms, then the collection of distributions $(t_I)_I$ indexed by finite subsets I of $\mathbb{N}$ satisfies an equation which intuitively says that on the whole configuration space minus the thin diagonal $M^n \setminus d_n$, the distribution $t_n \in \mathcal{D}'(M^n \setminus d_n)$ can be expressed in terms of distributions $(t_I)_I$ for $I \subsetneq \{1,\dots,n\}$. However, this expression involves products of distributions, thus we prove a recursion theorem which states that these products of distributions are well defined and $t_n \in \mathcal{D}'(M^n \setminus d_n)$ can be extended in $\mathcal{D}'(M^n)$. This allows us to recursively construct all the distributions t_n for all configuration spaces $(M^n)_{n\in\mathbb{N}}$.

6.1 Hopf algebra, T product and $\star$ product.

In this part, we use the formalism of [10].

6.1.1 The polynomial algebra of fields.

The Hopf algebra bundle over M.

Let M be a smooth manifold which represents space time. We will denote by $H = \mathbb{R}[\phi]$ the **polynomial algebra** in the **indeterminate** ϕ and we use the notation $\underline{H}$ for the trivial bundle $\underline{H} = M \times \mathbb{R}[\phi]$. The space of sections

$$\Gamma\left(M, \underline{H}\right)$$

of this vector bundle will be denoted by the letter $\mathcal{H}$. ϕ is a formal **indeterminate** and we denote by $\underline{\phi}^n$ the section of $\underline{H}$ which is the **constant section equal** to ϕ^n. Any section of $\underline{H}$ (ie any element of $\mathcal{H}$) will be a finite combination $\sum_{n<+\infty} a_n\underline{\phi}^n$ where $a_n \in C^\infty(M)$. The space of section $\mathcal{H}$ is a Hopf **module** over the **algebra** $C^\infty(M)$. Actually, most of the theory of Hopf algebras is still valid on rings and does not require fields. In order to match with the physical convention, $\phi^n(x) := (x, \phi^n)$ denotes the section $\underline{\phi}^n = (x \mapsto \phi^n(x))$ evaluated at the point $x \in M$. $\underline{1}$ is the unit section of this module $\mathcal{H}$.

The module $\mathcal{H}$ has an algebra and coalgebra structure, the product and coproduct of $\mathcal{H}$ are induced from the product and coproduct of H, for instance the product $\underline{\phi}_1\underline{\phi}_2$ of two sections is just the product computed fiber by fiber in H, and the coproduct Δ in $\mathcal{H}$ is just the fiberwise coproduct.

The product. The rule for the product is simple

$$\underline{\phi}^k\underline{\phi}^l = \underline{\phi}^{k+l}$$

which means that the sections $\underline{\phi}^k$ and $\underline{\phi}^l$ multiply pointwise

$$\phi^k(x)\phi^l(x) = \phi^{k+l}(x)$$

The coproduct. The coproduct on the primitive element $\underline{\phi}$ is given by:

$$\Delta\underline{\phi} = \underline{1} \otimes_{C^\infty(M)} \underline{\phi} + \underline{\phi} \otimes_{C^\infty(M)} \underline{1}$$

and it can be extended to powers of the field $\underline{\phi}^n$ by the binomial formula:

$$\Delta\underline{\phi}^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \underline{\phi}^k \otimes_{C^\infty(M)} \underline{\phi}^{n-k}$$

Some comments and the Sweedler notation. A special case of coassociativity will be:

$$\sum a_{(11)} \otimes a_{(12)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(21)} \otimes a_{(22)} = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad (6.1)$$

in tensor notation this reads $(\Delta \otimes Id)\,\Delta = (Id \otimes \Delta)\,\Delta$ which justifies Sweedler's notation: $\Delta^{k-1}a = \sum a_{(1)} \otimes ... \otimes a_{(k)}$.

The counit The counit is the Hopf algebra analog of the vacuum expectation value in QFT:
$\varepsilon((x,\phi^n)) = \langle 0|\phi^n(x)|0\rangle = \delta_0^n$.

Definition 6.1.1 *The counit is a linear map $\varepsilon : H \mapsto C^\infty(M)$ which satisfies the following properties:*

- *ε is an algebra morphism: $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$*
- *$\varepsilon(\phi^n(x)) = \delta_{n0}\underline{1}$.*

$$\sum \varepsilon(a_1)a_2 = \sum a_1\varepsilon(a_2) = a. \tag{6.2}$$

We make the identification $\underline{\phi^0} = \underline{1}$.

Example 6.1.1 *We want to give an example of the defining equation*

$$\sum a_1\varepsilon(a_2) = a$$

for $a = \underline{\phi}^n$: $\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \underline{\phi}^{n-k} \underbrace{\varepsilon(\underline{\phi}^k)}_{=0 \text{ if } k\neq 0} = \underline{\phi}^n\varepsilon(\underline{1}) = \underline{\phi}^n$.

6.1.2 Comparison of our formalism and the classical formalism from physics textbooks.

In QFT textbooks, the fields ϕ are thought of as operator valued distributions. In our formalism, the field ϕ is merely an **indeterminate**. In QFT textbooks, the noncommutative operator product is defined first and the operator product of two fields $\phi(x)$ and $\phi(y)$ is written $\phi(x)\phi(y)$. Then using the representation of ϕ in terms of annihilation and creation operators, physicists define the normal ordered product denoted by $:\phi(x)\phi(y):$ which corresponds to the commutative product of the Hopf module $\mathcal{H}$. Whereas in our formalism, we start from the commutative product and then use a procedure called twisting to define the operator product $\star$.

	Standard QFT	Our approach	Borcherds
Commutative product	$:\phi(x)\phi(y):$	$\phi(x)\phi(y)$	$\phi(x)\phi(y)$
"Operator product"	$\phi(x)\phi(y)$	$\phi(x)\star\phi(y)$	
VEV	$\langle 0\| \ \|0\rangle$	ε	
Correlation functions	$\langle 0\|T\ldots\|0\rangle$	$t=\varepsilon\circ T$	Feynman measure ω
		Laplace coupling $(.\|.)$	Bicharacter Δ

6.1.3 Hopf algebra bundle over M^n.

A further step in the construction is to pass from the manifold M to the configuration space M^n of n points. In order to define products of quantum fields over n points, it is natural to construct an algebraic setting on configuration space M^n. We start again from the fiber $H = \mathbb{R}[\phi]$ and consider the n-fold tensor product $H^{\otimes n} = \mathbb{R}[\phi] \otimes \cdots \otimes \mathbb{R}[\phi]$. Then $H^{\otimes n}$ can be generated as a polynomial algebra by the n elements:

$$\phi \otimes 1 \otimes \cdots \otimes 1 = \phi_1$$

$$1 \otimes \phi \otimes 1 \otimes \cdots = \phi_2$$

$$\ldots$$

thus we deduce that $H^{\otimes n} \simeq \mathbb{R}[\phi_1, ..., \phi_n]$. Then we denote by $\underline{H^{\otimes n}}$ the bundle $M^n \times \mathbb{R}[\phi_1, ..., \phi_n]$ living over configuration space M^n. As we did in the previous part, we must consider a module over $C^\infty(M^n)$ which contains products of fields of the form

$$\underline{\phi}^{k_1} \otimes \cdots \otimes \underline{\phi}^{k_n},$$

hence we will consider the $C^\infty(M^n)$-module of sections $\Gamma\left(M^n, \underline{H^{\otimes n}}\right)$. This module over the ring $C^\infty(M^n)$ will be denoted $\mathcal{H}^n$. Similarly, for any finite subset I of the integers, let M^I be the set of maps from I to M, we define

$$\underline{H^{\otimes I}} = M^I \times \mathbb{R}[\phi_i]_{i \in I} = M^I \times \mathbb{R}[\phi_i]_{i \in I}.$$

Then $\mathcal{H}^I$ is defined as the $C^\infty(M^I)$-module of sections $\Gamma(M^I, \underline{H^{\otimes I}})$. To consider $\mathcal{H}^n$ over the ring $C^\infty(M^n)$ is not sufficient since in QFT textbooks, the operator product of fields denoted by $\star$ generates distributions as we can see in the following example:

Example 6.1.2 $\phi(x) \star \phi(y) = \Delta_+(x, y) + \phi(x)\phi(y)$.

We will have to extend the ring $C^\infty(M^n)$ of smooth functions living on configuration space M^n to a ring which contains **distributions**. In order to include sections of $\underline{H}$ with distributional coefficients, we use a tensor product technique. This idea already appeared in the previous work of Borcherds [8], in which he constructs a vertex algebra with value in some sort of ring with singular coefficients. If we have an algebra A of polynomials over a ring R and V a R-module, it is always possible to define the tensor product $A \otimes_R V$ over the ring R. Here we apply this construction: let V be a left $C^\infty(M^n)$-module of distributions, then the tensor product $\mathcal{H}^n \otimes_{C^\infty(M^n)} V$ makes sense. Warning: even if $\mathcal{H}^n$ is an algebra, it is no longer true that $\mathcal{H}^n \otimes_{C^\infty(M^n)} V$ is still an algebra since we cannot always multiply distributions.

The Rota Feynman convention. Following Rota and Feynman, we write $\underline{\phi}_1\underline{\phi}_2$ instead of $\phi(x)\otimes\phi(y)$. We drop the tensor product symbol $\otimes$, and the elements of $\mathcal{H}^n$ are $C^\infty(M^n)$-linear combinations of products of powers of fields $\underline{\phi}_1^{i_1}\dots\underline{\phi}_n^{i_n}$. Hence elements on the j-th factor of the tensor product is written $\underline{\phi}_j^{i_j}$. Sometimes, to make our proofs look even simpler, we write $a_1...a_n$ instead of $\underline{\phi}_1^{i_1}\dots\underline{\phi}_n^{i_n}$.

Extending the product and coproduct. To extend the product and coproduct to $\mathcal{H}^n$, we just compute products and coproducts "point by point".

Definition 6.1.2 *We give the formula of the product for the generators of* $\mathcal{H}^n$

$$\left(\underline{\phi}_1^{n_1}\dots\underline{\phi}_k^{n_k}\right)\left(\underline{\phi}_1^{l_1}\dots\underline{\phi}_k^{l_k}\right) = \left(\underline{\phi}_1^{n_1+l_1}\dots\underline{\phi}_k^{n_k+l_k}\right)$$

and the formula of the coproduct:

$$\Delta\left(\underline{\phi}_1^{n_1}\dots\underline{\phi}_k^{n_k}\right) = \Delta\underline{\phi}_1^{n_1}\dots\Delta\underline{\phi}_k^{n_k}$$

Although the definition is given in terms of sections $\underline{\phi}_i^{n_i}$, we will sometimes follow the physics folklore and write $\phi^{n_i}(x_i)$.

Fundamental example If we compute explicitly the coproduct for the generators, we obtain the formula:

$$\Delta\left(\underline{\phi}_1^{n_1}\dots\underline{\phi}_k^{n_k}\right) = \sum\begin{pmatrix} n_1 \\ i_1 \end{pmatrix}\dots\begin{pmatrix} n_k \\ i_k \end{pmatrix}\underline{\phi}_1^{n_1-i_1}\dots\underline{\phi}_k^{n_k-i_k}\otimes\underline{\phi}_1^{i_1}\dots\underline{\phi}_k^{i_k} \tag{6.3}$$

The counit and the vacuum expectation values. The counit is defined on $\mathcal{H}^n$ by extending the counit

$$\varepsilon:\mathcal{H}\to C^\infty(M^n)$$

to $\mathcal{H}^n$ by coalgebra morphism:

$$\varepsilon(uv)=\varepsilon(u)\varepsilon(v).$$

Example 6.1.3

$$\varepsilon(\underline{1})=1$$
$$\varepsilon(\underline{\phi}_1\underline{\phi}_2^2\underline{1}_3)=\varepsilon(\underline{\phi}_1)\varepsilon(\underline{\phi}_2^2)\varepsilon(\underline{1}_3)=0\times0\times1=0$$
$$\varepsilon(\underline{1}_1\underline{1}_2\underline{1}_3)=1\times1\times1=1$$

It is the Hopf algebraic version of the vacuum expectation value and is an essential ingredient if one wants to define "correlation fonctions" from product of fields.

6.1.4 Deformation of the polynomial algebra of fields.

The non commutative product of QFT.

Explanation on the notation of physicists. In this part, we will make the same notational abuse as physicists. Instead of writing products of section as $\underline{\phi}_1\underline{\phi}_2$ or the star product of sections as $\underline{\phi}_1 \star \underline{\phi}_2$, we prefer to adopt the conventional physicist notation $\phi(x_1)\phi(x_2)$ for the commutative product and $\phi(x_1) \star \phi(x_2)$ for the star product. The meaning of the formulas is changed, since in the physicist's notation, we multiply sections then evaluate them at points (x_1, x_2) of the configuration space M^2 whereas in the mathematical notation, we just multiply two sections $\underline{\phi}_1$ and $\underline{\phi}_2$.

Examples of Wick theorems coming from physics.

We give the general QFT formula for the star product in the notations of physicists

$$\phi_1^{n_1}(x_1) \star \cdots \star \phi_k^{n_k}(x_k)$$

$$= \sum \begin{pmatrix} n_1 \\ i_1 \end{pmatrix} \cdots \begin{pmatrix} n_k \\ i_k \end{pmatrix} \underbrace{\left\langle 0 \right| \left(\phi_1^{n_1-i_1}(x_1) \star \cdots \star \phi_k^{n_k-i_k}(x_k)\right) \left| 0 \right\rangle}_{\text{Distribution on } M^n} \phi_1^{i_1}(x_1) \dots \phi_k^{i_k}(x_k).$$

In Physics, the product of fields inside the $\langle 0| \dots |0\rangle$ is computed using Wick's theorem. Wick's theorem for time ordered product just means: $T(\phi_1 ... \phi_n) =$: all possible contractions : when we contract two fields, it just means we choose some pairs of fields in all possible ways and replace them by a propagator which is a distributional two point function Δ_+. We will represent a Wick contraction of two fields with the symbol $\overbrace{\phi(x_1)\phi(x_2)}$ and by definition $\overbrace{\phi(x_1)\phi(x_2)} = \Delta_+(x_1, x_2)$. We then give some simple examples of $\star$ products in order to illustrate the mechanism at work.

Example 6.1.4

$$\phi(x_1) \star \phi(x_2) = \phi(x_1)\phi(x_2) + \overbrace{\phi(x_1)\phi(x_2)}$$

$$= \phi(x_1)\phi(x_2) + \Delta_+(x_1, x_2)$$

$$\phi(x_1) \star \phi(x_2) \star \phi(x_3) = \phi(x_1)\phi(x_2)\phi(x_3) + \left(\overbrace{\phi(x_1)\phi(x_2)}\,\phi(x_3) + cyclic\right)$$

$$= \phi(x_1)\phi(x_2)\phi(x_3) + (\Delta_+(x_1, x_2)\phi(x_3) + cyclic)$$

$$\phi^2(x_1) \star \phi^2(x_2) = \phi^2(x_1)\phi^2(x_2) + 4\,\overbrace{\phi(x_1)\phi(x_2)}\,\phi(x_1)\phi(x_2) + 2\,\overbrace{\phi(x_1)\phi(x_2)}\,\overbrace{\phi(x_1)\phi(x_2)}$$

$$= \phi^2(x_1)\phi^2(x_2) + 4\Delta_+(x_1, x_2)\phi(x_1)\phi(x_2) + 2\Delta_+^2(x_1, x_2)$$

Functorial pull-back operation.

Let I be a finite subset of $\mathbb{N}$. Then we denote by M^I the configuration space of points labelled by I. In order to define the $\star$ product of fields, we need to define some operations which allows us to pull-back some products of fields living in configuration space $M^I, I \subset \{1, ..., n\}$, to the larger configuration space M^n.

Example 6.1.5 *Consider $\phi(x_1) \star \phi(x_2) \in \mathcal{H}^2$, we will illustrate the embedding of the element $\phi(x_1) \star \phi(x_2)$ in $\mathcal{H}^4$.*

$$p^*_{\{1234\}\mapsto\{12\}} \left(\phi(x_1) \star \phi(x_2)\right) = \left(\phi(x_1) \star \phi(x_2)\right) 1(x_3) 1(x_4)$$

If J is another finite subset of $\mathbb{N}$ such that $I \subset J$, then there is a canonical projection $p_{J\mapsto I} : M^J \mapsto M^I$ which induces by **pull-back** a morphism

$$\begin{array}{rccc} p^*_{J\mapsto I}: & C^\infty(M^I) & \mapsto & C^\infty(M^J) \\ & f(x_i)_{i\in I} & \mapsto & p^*_{J\mapsto I} f(x_j)_{j\in J} = 1(x_j)_{j\in J\setminus I} \otimes_{C^\infty(M^J)} f(x_i)_{i\in I}, \end{array}$$

p^* is an algebra morphism. To each configuration space M^I, we first define the bundle $\underline{H}^I = M^I \times \mathbb{R}[\phi_i]_{i\in I}$, and taking the sections of this bundle, we obtain the $C^\infty(M^I)$ module $\mathcal{H}^I = \Gamma\left(M^I, \underline{H}^I\right)$. If $I \subset J$, the idea is that the morphism $p^\star_{J\mapsto I}$ extends to Hopf modules by the pull-back operation $p^*_{J\mapsto I}$ lifts functorially to a map $\mathcal{H}^I \mapsto \mathcal{H}^J$ given by the formula:

$$\begin{array}{rccc} p^*_{J\mapsto I}: & \mathcal{H}^I & \mapsto & \mathcal{H}^J \\ & \bigotimes_{i\in I} \underline{a}_i & \mapsto & p^*_{J\mapsto I}\left(\bigotimes_{i\in I} \underline{a}_i\right) = \bigotimes_{j\in J\setminus I} \underline{1}_j \otimes_{C^\infty(M^J)} \bigotimes_{i\in I} \underline{a}_i \end{array}$$

where $\underline{1}_j$ is the unit section of the bundle $\underline{H}$ over the j-th factor manifold M.

This pull-back operation allows us to characterize collections $(T_I)_I$, where each T_I is a $C^\infty(M^I)$-module **morphism**: $T_I : \mathcal{H}^I \mapsto \mathcal{H}^I \otimes_{C^\infty(M^I)} V^I$, which satisfy some good compatibility relations with the collection of inclusions $p^*_{J\mapsto I} : \mathcal{H}^I \hookrightarrow \mathcal{H}^J$, which means that

$$\forall A_I \in \mathcal{H}^I, T_J\left(p^\star_{J\mapsto I} A_I\right) = p^\star_{J\mapsto I} T_I\left(A_I\right).$$

This can also be formulated as the commutativity of the diagram:

$$\begin{array}{rccc} p^*_{J\mapsto I}: & \mathcal{H}^I & \rightarrow & \mathcal{H}^J \\ & T_I \downarrow & & T_J \downarrow \\ & \mathcal{H}^I & \rightarrow & \mathcal{H}^J, \end{array}$$

for all $I \subset J$.

Example 6.1.6 $T_3(\phi^{i_1}(x_1) \otimes \phi^{i_2}(x_2) \otimes 1(x_3)) = T_2(\phi^{i_1}(x_1) \otimes \phi^{i_2}(x_2)) \otimes 1(x_3)$.

Domain of definition of the $\star$ product. For each $I \subset \mathbb{N}$, we need to twist $\mathcal{H}^I$ with a left $C^\infty(M^I)$ module $V^I \subset \mathcal{D}'(M^I)$ of distributional coefficients and we consider instead $\mathcal{H}^I \otimes_{C^\infty(M^I)} V^I$. For any finite subsets I, J of $\mathbb{N}$, such that $I \cap J = \emptyset$, our star product will be well defined as a bilinear map

$$\star : \mathcal{H}^I \times \mathcal{H}^J \mapsto \mathcal{H}^{I\cup J} \otimes_{C^\infty(M^{I\cup J})} V^{I\cup J}$$

where $V^I, V^J, V^{I\cup J}$ are respectively the left $C^\infty(M^I), C^\infty(M^J), C^\infty(M^{I\cup J})$-module which contains the distributional coefficients. The star product is supposed to satisfy the following rule

$$\forall (u,v) \in V^I \times V^J, \forall (P,Q) \in \mathcal{H}^I \times \mathcal{H}^J,$$

$$(uP) \star (vQ) = (p^*_{J\cup I\mapsto I} u)\,(p^*_{J\cup I\mapsto J} v)\,(P \star Q)$$

6.1.5 The construction of $\star$.

We will describe a general procedure called twisting, which allows to construct non commutative associative products from the usual commutative product of fields and an object called *Laplace coupling* $(.|.)$. The Laplace coupling is the Hopf algebraic machine which produces "the contractions of pairs of fields" that we need in order to reproduce the Wick theorem. In the sequel, we will use capital letters to denote strings of operators

Example 6.1.7 *$A = a_1 \dots a_n$ where $A \in \mathcal{H}^n$ and each $a_i \in \mathcal{H}^{\{i\}}$.*

And for $A = a_1 \dots a_n, B = b_1 \dots b_n$, the commutative product AB means the commutative product over each point $AB = (a_1 b_1) \dots (a_n b_n)$.

The Laplace coupling. For our Hopf algebras, the contraction operation of the Wick theorem in QFT is realised by the Laplace coupling:

Definition 6.1.3 *Let I, J be* ***finite disjoint*** *subsets of $\mathbb{N}$. The Laplace coupling is defined as a bilinear map $(.|.) : \mathcal{H}^I \otimes \mathcal{H}^J \mapsto V^{I\cup J}$ which satisfies the relations*

$$(\phi(x_1)|\phi(x_2)) = \Delta_+(x_1, x_2) \tag{6.4}$$
$$(AB|C) = \sum \left(A|C_{(1)}\right)\left(B|C_{(2)}\right) \tag{6.5}$$
$$(1|A) = (A|1) = \varepsilon(A) \tag{6.6}$$

more generally we have the coassociative version $\left(A^1 ... A^n|B\right) = \sum \Pi_{k=1}^n \left(A^k|B_{(k)}\right)$.

We notice that the Laplace coupling of two fields $\phi(x_1), \phi(x_2)$ is exactly the Wick contraction beetween these two fields: $(\phi(x_1)|\phi(x_2)) = \overbrace{\phi(x_1)\phi(x_2)} = \Delta_+(x_1, x_2)$.

Example 6.1.8

$$(\phi(x_1)|\phi(x_2)) = \Delta_+(x_1, x_2).$$
$$\big(\phi^2(x_1)|\phi^2(x_2)\big) = 2\Delta_+^2(x_1, x_2)$$
$$\big(\phi^2(x_1)|\phi(x_2)\phi(x_3)\big) = 2\Delta_+(x_1, x_2)\Delta_+(x_1, x_3).$$

Proposition 6.1.1 *Let* $(.|.)$ *be a Laplace coupling as in the definition (6.1.3). Then* $(.|.)$ *is entirely determined by the two point function* $(\phi(x_1)|\phi(x_2)) = \Delta_+(x_1, x_2)$*. Furthermore, we have the relation:* $\big(\phi^k(x_1)|\phi^l(x_2)\big) = \delta_{kl}k!\Delta_+^k(x_1, x_2)$*.*

Proof — See [10]. ■

The function $\Delta_+(x_1, x_2)$ appearing in the definition of the Laplace coupling should be a **propagator** for the Wave operator. In QFT, it is the *Wightman propagator* Δ_+ defined in chapter 5.

Definition 6.1.4 *The star product* $\star$ *is defined as follows. Let* I, J *be any finite disjoint subsets of* $\mathbb{N}$*, for all* $(A, B) \in \mathcal{H}^I \times \mathcal{H}^J$*:*

$$A \star B = \sum \big(A_{(1)}|B_{(1)}\big)\, A_{(2)}B_{(2)} \qquad (6.7)$$

where $(.|.)$ *denotes the Laplace coupling and* $A_{(2)}B_{(2)}$ *denotes the usual commutative product of fields.*

Example 6.1.9

$$\phi^3(x_1)\star\phi^3(x_2) = 6\Delta_+^3(x_1, x_2)+6\Delta_+^2(x_1, x_2)\phi(x_1)\phi(x_2)+3\Delta_+(x_1, x_2)\phi(x_1, x_2)+\phi^3(x_1)\phi^3(x_2).$$

$$\phi^2(x_1) \star (\phi(x_2)\phi(x_3))$$
$$= \phi^2(x_1)\phi(x_2)\phi(x_3)+2\phi(x_1)\phi(x_2)\Delta_+(x_1, x_3)+2\phi(x_1)\phi(x_3)\Delta_+(x_1, x_2)+2\Delta_+(x_1, x_2)\Delta_+(x_1, x_3)$$

From the last example, we notice the important fact that the star product $A \star B$ is not automatically well defined because the computation of the star product involves products of distributions and we have yet to prove that these products are well defined.

The counit ε.

As we already said, the counit plays the role of the vacuum expectation value in QFT. We first recall the most important result about the counit ε, it is the coassociativity equation:

$$A = \sum \varepsilon(A_{(1)})A_{(2)}$$

Example 6.1.10

$$\sum \varepsilon(\underline{\phi}^2_{(1)})\underline{\phi}^2_{(2)} = \varepsilon(\underline{\phi}^2)\underline{1} + 2\underline{\phi}\varepsilon(\underline{\phi}) + \underline{\phi}^2\varepsilon(\underline{1}) = 0 + 0 + \underline{\phi}^2 1 = \underline{\phi}^2$$

We give an example of the same quantity expressed in the language of Hopf algebras and the conventional QFT language so that the reader can compare:

Example 6.1.11

$$\varepsilon(\underline{\phi}_1^2 \star \underline{\phi}_2^2) = \varepsilon\left((\underline{1}|\underline{1})\,\underline{\phi}_1^2\underline{\phi}_2^2 + 4\left(\underline{\phi}_1|\underline{\phi}_2\right)\underline{\phi}_1\underline{\phi}_2 + \left(\underline{\phi}_1^2|\underline{\phi}_2^2\right)\right)$$

$$= \varepsilon(\underline{\phi}_1^2\underline{\phi}_2^2 + 4\Delta\underline{\phi}_1\underline{\phi}_2 + 2\Delta_+^2) = 0 + 0 + 2\Delta_+^2$$

$$\langle 0|\phi^2(x_1)\phi^2(x_2)|0\rangle = 2\Delta_+^2(x_1, x_2)$$

6.1.6 The associativity of $\star$.

For the moment, the $\star$ product we constructed is just bilinear. We have to prove it is associative. First, let us prove some lemmas.

Lemma 6.1.1 *The $\star$ product satisfies the identities:*

$$\Delta(a \star b) = \sum \left(a_{(1)} \star b_{(1)}\right) \otimes a_{(2)}b_{(2)} \tag{6.8}$$

$$(a \star b|c) = (a|b \star c) \tag{6.9}$$

$$\varepsilon\,(a \star b) = (a|b) \tag{6.10}$$

Note that Δ is the coproduct of the commutative product and not the coproduct of $\star$. *Proof* —

$$\Delta(a \star b) = \sum \left(a_{(1)}|b_{(1)}\right)\Delta(a_{(2)}b_{(2)}) = \sum \left(a_{(1)}|b_{(1)}\right) a_{(2)}b_{(2)} \otimes a_{(3)}b_{(3)}$$

$$= \sum \left(a_{(11)}|b_{(11)}\right) a_{(12)}b_{(12)} \otimes a_{(2)}b_{(2)} = \sum \left(a_{(1)} \star b_{(1)}\right) \otimes a_{(2)}b_{(2)}.$$

$$(a \star b|c) = \sum \left(a_{(1)}|b_{(1)}\right)\left(a_{(2)}b_{(2)}|c\right) = \sum \left(a_{(1)}|b_{(1)}\right)\left(a_{(2)}|c_{(1)}\right)\left(b_{(2)}|c_{(2)}\right)$$

$$= \sum \left(a_{(1)}|b_{(2)}\right)\left(a_{(2)}|c_{(2)}\right)\left(b_{(1)}|c_{(1)}\right)$$

because by cocommutativity of the field coproduct, we can permute $b_{(1)}, b_{(2)}$ and $c_{(1)}, c_{(2)}$.
$\varepsilon(a \star b) = \sum \varepsilon(\left(a_{(1)}|b_{(1)}\right) a_{(2)}b_{(2)}) = \sum \left(a_{(1)}\varepsilon(a_{(2)})|b_{(1)}\varepsilon(b_{(2)})\right) = \sum (a|b)$ ■

More generally, we have a distributed version of (6.8):

Proposition 6.1.2 *$\star$ satisfies the identity:*

$$\Delta(a_1 \star \cdots \star a_n) = \sum (a_{1(1)} \star \cdots \star a_{n(1)}) \otimes a_{1(2)}...a_{n(2)} \tag{6.11}$$

Theorem 6.1.1 *The product $\star$ is associative provided that the products of distributions coming from the Laplace couplings make sense.*

Proof —

$$(a\star b)\star c=\sum\left((a\star b)_{(1)}|c_{(1)}\right)(a\star b)_{(2)}c_{(2)}=\sum\left(a_{(1)}\star b_{(1)}|c_{(1)}\right)a_{(2)}b_{(2)}c_{(2)}$$

by (6.8)

$$=\sum\left(a_{(1)}|b_{(1)}\star c_1\right)a_{(2)}b_{(2)}c_{(2)}$$

by (6.9)

$$=\sum\left(a_{(1)}|(b\star c)_{(1)}\right)a_{(2)}(b\star c)_{(2)}$$

by (6.8)

$$=a\star(b\star c)$$

■

Corollary 6.1.1 *$a_1\star...\star a_n$ is well defined provided that the products of distributions coming from the Laplace couplings make sense.*

6.1.7 Wick's property.

We give a general QFT formula for the star product in the notations of physicists

$$\phi_1^{n_1}(x_1)\star\cdots\star\phi_k^{n_k}(x_k)$$

$$=\sum\begin{pmatrix}n_1\\ i_1\end{pmatrix}\cdots\begin{pmatrix}n_k\\ i_k\end{pmatrix}\underbrace{\langle 0|\phi^{n_1-i_1}(x_1)\star\cdots\star\phi^{n_k-i_k}(x_k)|0\rangle}_{\text{Distribution on }M^n}\phi^{i_1}(x_1)\dots\phi^{i_k}(x_k).$$

And we write the Hopf counterpart of this formula

$$\underline{a}_1\star\cdots\star\underline{a}_n=\sum\underbrace{\varepsilon(\underline{a}_{1(1)}\star\cdots\star\underline{a}_{n(1)})}_{\text{distributions}}\underline{a}_{1(2)}\cdots\underline{a}_{n(2)}.$$

We introduce a crucial definition which is the Hopf algebra counterpart of the Wick theorem of QFT. We call this property Wick's expansion. For any finite subsets I,J of $\mathbb{N}$, such that $I\cap J=\emptyset$, let $\star$ be any bilinear map

$$\star:\mathcal{H}^I\times\mathcal{H}^J\mapsto\mathcal{H}^{I\cup J}\otimes_{C^\infty(M^{I\cup J})}V^{I\cup J}.$$

Definition 6.1.5 *A bilinear map $\star$ as above satisfies the Wick expansion property if for $I\cap J=\emptyset$,*

$$\forall A=\left(\prod_{i\in I}a_i\right)\in\mathcal{H}^I\otimes_{C^\infty(M^I)}V^I,\forall B=\left(\prod_{j\in J}b_j\right)\in\mathcal{H}^J\otimes_{C^\infty(M^J)}V^J,$$

$$A\star B=\sum\varepsilon\left(A_{(1)}\star B_{(1)}\right)A_{(2)}B_{(2)}.\tag{6.12}$$

This property encodes in the Hopf algebraic language all the algebro combinatorial properties of the Wick theorem. We prove that our star product defined from the Laplace coupling does indeed satisfy the Wick property.

Theorem 6.1.2 *Let $\star$ be defined by*

$$A\star B=\sum\left(A_{(1)}|B_{(1)}\right)A_{(2)}B_{(2)} \tag{6.13}$$

where $(.|.)$ denotes the Laplace coupling, then $\star$ satisfies Wick's expansion:

$$\forall A=\prod_{i\in I}(a_i)\in\mathcal{H}^I\otimes_{C^\infty(M^I)}V^I,\forall B=\prod_{j\in J}(b_j)\in\mathcal{H}^J\otimes_{C^\infty(M^J)}V^J$$

$$A\star B=\sum\varepsilon(A_{(1)}\star B_{(1)})A_{(2)}B_{(2)}$$

Proof — By the identity (6.8), notice that $\varepsilon(A_{(1)}\star B_{(1)})=\left(A_{(1)}|B_{(1)}\right)$ which proves the claim. ■

The meaning of this theorem is that any associative product $\star$ constructed by the twisting procedure from the Laplace coupling $(.|.)$ should satisfy the Wick expansion property.

6.1.8 Recovering Feynman graphs.

Proposition 6.1.3 *For any $(p_1,...,p_n)$, $\varepsilon\left(\phi^{p_1}(x_1)\star...\star\phi^{p_n}(x_n)\right)=$*

$$p_1!...p_n!\sum_{\sum_{j=1}^n m_{ij}=p_i}\Pi_{1\leqslant i<j\leqslant n}\frac{\Delta_+^{m_{ij}}(x_i,x_j)}{m_{ij}!}, \tag{6.14}$$

where $(m_{ij})_{ij}$ runs over the set of all symmetric matrices with integer entries with vanishing diagonal and such that for all i, the sum of the coefficients on the i-th row is equal to p_i.

Note that $(m_{ij})_{ij}$ should be interpreted as the adjacency matrix of a Feynman graph. *Proof* — The sum is indexed by symmetric matrices with integer coefficicients vanishing diagonals. We will prove the theorem by recursion. We start by checking the formula at degree 2.

$$\varepsilon\left(\phi^{p_1}(x_1)\star\phi^{p_2}(x_2)\right)=\left(\phi^{p_1}(x_1)|\phi^{p_2}(x_2)\right)=p_1!\delta_{p_1p_2}\Delta_+^{p_1}(x_1,x_2)$$

$$=p_1!p_2!\sum_{p_{12}=p_1=p_2}\frac{\Delta_+^{p_{12}}(x_1,x_2)}{p_{12}!}.$$

Assume we know that $\frac{\varepsilon(\phi^{p_1}(x_1)\star...\star\phi^{p_k}(x_k))}{p_1!...p_k!}=\sum_{\sum_{j=1}^k m_{ij}=p_i}\Pi_{1\leqslant i<j\leqslant k}\frac{\Delta_+^{m_{ij}}(x_i,x_j)}{m_{ij}!}$ is true for any $k\leqslant n$. Set $A=\left(a^1\star...\star a^n\right)$ and $B=a^{n+1}$. We use the

identity $\varepsilon(A\star B)=(A|B)=\sum\varepsilon(A_{(1)})\left(A_{(2)}|B\right)$. We use the explicit formula for the coproduct of quantum fields

$$\Delta\phi^{p_j}(x_j)=\sum_{0\leqslant i_j\leqslant p_j}\begin{pmatrix} p_j \\ i_j\end{pmatrix}\phi^{i_j}(x_j)\otimes\phi^{p_j-i_j}(x_j)$$

and

$$\Delta^n\phi^{p_{n+1}}(x_{n+1})=\sum_{i_1+...+i_n=p_{n+1}}\begin{pmatrix} & p_{n+1} & \\ i_1 & ... & i_n\end{pmatrix}\phi^{i_1}(x_{n+1})\otimes\cdots\otimes\phi^{i_n}(x_{n+1})$$

to deduce

$$\varepsilon(A_{(1)})=\begin{pmatrix} p_1 \\ i_1\end{pmatrix}\dots\begin{pmatrix} p_n \\ i_n\end{pmatrix}\varepsilon\left(\phi^{p_1-i_1}(x_1)\star\cdots\star\phi^{p_n-i_n}(x_n)\right)$$

$$\left(A_{(2)}|B\right)=\left(A_{(2)}|\phi^{p_{n+1}}(x_{n+1})\right)$$

$$=\begin{pmatrix} & p_{n+1} & \\ i_1 & \dots & i_n\end{pmatrix}\frac{\Delta_+^{i_1}(x_1,x_{n+1})}{i_1!}\dots\frac{\Delta_+^{i_n}(x_n,x_{n+1})}{i_n!}$$

$$=p_{n+1}!\Delta_+^{i_1}(x_1,x_{n+1})\dots\Delta_+^{i_n}(x_n,x_{n+1}).$$

Each term $\varepsilon(A_{(1)})\left(A_{(2)}|B\right)$ has the form:

$$p_1!...p_{n+1}!\frac{\varepsilon\left(\phi^{p_1-i_1}(x_1)\star\cdots\star\phi^{p_n-i_n}(x_n)\right)}{(p_1-i_1)!\dots(p_n-i_n)!}\frac{\Delta_+^{i_1}(x_1,x_{n+1})}{i_1!}\dots\frac{\Delta_+^{i_n}(x_n,x_{n+1})}{i_n!},$$

which ends our proof because the product $(p_1-i_1)!\dots(p_n-i_n)!$ in the denominator kill the unwanted factors. The space of $n+1\times n+1$ symmetric matrices with fixed last row with coefficients $i_1,...,i_k$ and such that the sum of terms on the k-th line is equal to p_k is in bijection with the space of $n\times n$ symmetric matrices with sum of $k-th$ line equals p_k-i_k. ■

A word of caution and an introduction to the next section. From now on, the star product is **fixed** and is defined as above from the "twisting procedure" with the Laplace coupling defined by the Wightman propagator Δ_+. However, we *have not yet defined rigorously* the product $\star$ for elements

$$(A,B)\in\left(\mathcal{H}^I\otimes_{C^\infty(M^I)}V^I\right)\times\left(\mathcal{H}^J\otimes_{C^\infty(M^J)}V^J\right)$$

with **distributional coefficients**. We will construct a time ordered product T from $\star$ and we will prove that $T(AB)$ is well defined in the distributional sense. This is illustrated by one of our previous example:

Example 6.1.12

$$\phi^2(x_1)\star(\phi(x_2)\phi(x_3)) = \phi^2(x_1)\phi(x_2)\phi(x_3)+2\phi(x_1)\Delta(x_1,x_2)\phi(x_3)+2\phi(x_1)\Delta(x_1,x_3)\phi(x_2)$$
$$+2\underbrace{\Delta(x_1,x_2)\Delta(x_1,x_3)}_{\textit{product of distributions}}$$

In the next section, we are going to use $\star$ to define the time ordered product T which **only satisfies** the Wick expansion property $T(A)=\sum t(A_{(1)})A_{(2)}$ and the causality equation.

6.2 The causality equation.

The geometrical lemma The geometrical lemma (due to Popineau and Stora [57]) essentially states that we can partition the configuration space minus the thin diagonal $M^n\setminus d_n$, with open sets having nice properties from the point of view of causality.

Lemma 6.2.1 *Let $(M,\geqslant)$ be a causal Lorentzian manifold endowed with the canonical* poset *structure (i.e. a set equipped with a partial order) induced by the Lorentzian metric and the chronological causality on M: $x\leqslant y$ if y lies in the future cone of x. Define the relation $\nleqslant$ by: $x\nleqslant y$ if and only if $x\leqslant y$ does not hold Let $[n]=\{1,\dots,n\}$ and I a proper subset of $[n]$. If I^c is the complement of I in $[n]$ (i.e. $I\sqcup I^c=[n]$), we define the subset M_{I,I^c} of M^n by*

$$M_{I,I^c}=\{(x_1,\dots,x_n)\in M^n|\forall(i,j)\in I\times I^c, x_i\nleqslant x_j\}.$$

Then,

$$\bigcup_I M_{I,I^c} \quad = \quad M^n\backslash d_n, \tag{6.15}$$

where $d_n=\{x_1=\cdots=x_n\}$ is the thin diagonal of M^n and I runs over the proper subsets of $[n]$.

Proof — It is clear that, for all proper subsets I of $[n]$, we have $M_{I,I^c}\subset M^n\backslash d_n$, because if $(x_1,\dots,x_n)\in d_n$, then $x_i\geq x_j$ for all i and j in $[n]$. It remains to show that any $X=(x_1,\dots,x_n)\in M^n\backslash d_n$ belongs to some M_{I,I^c}. In fact we shall determine all the M_{I,I^c} to which a given X belongs. For all $X=(x_1,\cdots,x_n)\in M^n$, we define $\lambda(X)$ as the finite subset $\{a_1,\cdots,a_r\}\subset M$ s.t. $a\in\lambda(X)$ iff $\exists i\in[n], x_i=a$. To each $X\in M^n$, we associate a directed graph known as the *Hasse diagram* of X as follows. To each distinct $a\in\lambda(X)$, we associate a vertex and we draw a directed line from vertex a to vertex b if $a\leqslant b$, $a\neq b$ and no other $c\in\lambda(X)$, distinct from a and b, is such that $a\leqslant c\leqslant b$. All indices $i\in[n]$ such that $x_i=a$

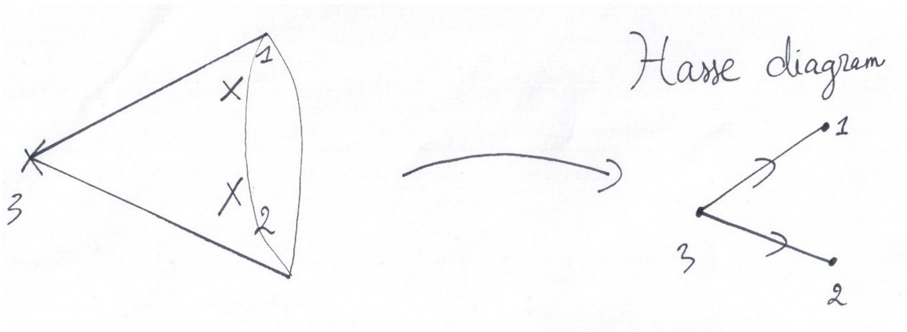

Figure 6.1: A configuration of three points in $C_{\{12\}} \subset M^3$ and the corresponding Hasse diagram.

decorate the same vertex. The Hasse diagram of X has a single vertex if and only if $X \in d_n$. Take $X \in M^n \backslash d_n$, its Hasse diagram has at least two vertices. If we pick up any vertex of the Hasse diagram, then any point x_j greater than a point x_i of this vertex is such that $x_j \geq x_i$. Thus, $j \in I$ if $i \in I$ and, to build a M_{I,I^c}, we can select a non-zero number of vertices of the diagram and add all the vertices that are greater than the selected ones. The points corresponding to all these vertices determine a subset I of $[n]$. If $I \neq [n]$, then $X \in M_{I,I^c}$ and it is always possible to find such a I by picking up a single maximal vertex in one connected component of the Hasse diagram. Conversely, any M_{I,I^c} is made of the points that are greater than their minima. To see this, consider a point $x_i \in M_{I,I^c}$ such that $i \in I$. Then, the set $S_i = \{x_j \in X | x_i \geq x_j\}$ is not empty because x_i belongs to it. Then, x_i is larger than a minimum of S_i, which is also a minimum of the Hasse diagram of X. ■

6.2.1 Definition of the time-ordering operator

In quantum field theory, the poset is the Lorentzian manifold M and the fields are, for example, $\phi^n(x)$. For any finite subset I of $\mathbb{N}$, we defined the configuration space M^I as the set of maps from I to M and we introduced some vector space of distributions V^I which contains the singularities of the Feynman amplitudes, then we introduced a module $\mathcal{H}^I \otimes_{C^\infty(M^I)} V^I$ associated to I. For all $A \in \mathcal{H}^I$, we will denote by $t_I(A)$ the element $\varepsilon(T_I(A))$ and $t : \mathcal{H}^I \mapsto V^I$.

Axioms for the time ordering operator. We are going to define the time-ordering operator as a collection $(T_I)_I$ of $C^\infty(M^I)$-module morphisms, for all finite subset I of $\mathbb{N}$, $T_I : \mathcal{H}^I \to \mathcal{H}^I \otimes_{C^\infty(M^I)} V^I$ with the following properties:

1. If $|I| \leqslant 1$, the restriction of T to $\mathcal{H}^I$ is the identity map,

2. T satisfies the Wick expansion property:

$$T(A) = \sum \varepsilon \circ T(A_{(1)})A_{(2)} \tag{6.16}$$

3. The causality equation. Let $A = a_1(x_1)\dots a_n(x_n) \in \mathcal{H}^n$. If there is a proper subset $I \subset \{1,\dots,n\}$ such that $x_i \not\leqslant x_j$ for $i \in I$ and $j \notin I$, denote $A_I = \prod_{i\in I} a_i(x_i)$ and $A_{I^c} = \prod_{j\in I^c} a_j(x_j)$ then

$$T(A) = T(A_I) \star T(A_{I^c}). \tag{6.17}$$

Remark: The equation $T(A) = \sum t(A_{(1)})A_{(2)}$ implies T is a comodule morphism, we denote by β the coaction defined as follows:

$$\forall (f,A) \in V^I \times \mathcal{H}^I, \beta(f\otimes A) = \sum (fA_1 \otimes A_2) = \sum (A_1 \otimes fA_2)\,.$$

$$\beta T(A) = \sum t(A_{(1)})A_{(21)} \otimes A_{(22)} = \sum t(A_{(1)})A_{(2)} \otimes A_{(3)}$$
$$= \sum t(A_{(11)})A_{(12)} \otimes A_{(2)} = \sum T(A_{(1)})A_{(2)} = (T\otimes Id)\,\beta A.$$

In fact, C Brouder communicated to us a proof of $T(A) = \sum t(A_{(1)})A_{(2)} \Leftrightarrow$ T is a comodule morphism.

What are we trying to construct ? We have a given star product which is the operator product of quantum fields. The idea is to construct all time ordered products satisfying the previous set of axioms, the most important being causality and the Wick expansion property. The T product is not unique, actually there are infinitely many T-products and there is an infinite dimensional group which acts freely and transitively on the space of all T-products (see equation (4.1) p. 17 in [10]). This group is the Bogoliubov renormalization group which was studied in Hopf algebraic terms by C. Brouder in [10] p. 17-20. in [9] The problem of construction of a QFT in our sense is reduced to the problem of constructing a T-product satisfying the axioms and to **make sense analytically** of this T-product. We will prove the existence of at least one T-product and we will show that it is analytically well defined. A crucial ingredient in the existence proof is to establish a recursion equation which expresses the T product $T_n \in Hom(\mathcal{H}^n,\mathcal{H}^n)$ in terms of the elements $T_I \in Hom(\mathcal{H}^I,\mathcal{H}^I)$ for $I \subsetneq \{1,\dots,n\}$. We will later see that the problem of defining the T-product reduces to a problem of making sense of **products of distributions** and a problem of **extension of distributions**. Our approach is related to the one of [9] but we use causality in a more explicit way following Epstein–Glaser. However, the strategy we will adopt make essential use of ideas of Raymond Stora which appeared in unpublished form ([57]).

6.2.2 The Causality theorem.

We give the main structure theorem for the amplitudes coming from perturbative QFT. This theorem relates T_n and all $(T_I)_I$ for $I \subsetneq \{1, \dots, n\}$ on the configuration space minus the thin diagonal $M^n \setminus d_n$.

Theorem 6.2.1 *Let T be a collection $(T_I)_I$ of $C^\infty(M^I)$-module morphisms $T_I : \mathcal{H}^I \to \mathcal{H}^I \otimes_{C^\infty(M^I)} V^I$ which satisfy the collection of axioms (6.2.1). Then for all $I \subsetneq \{1, ..., n\}$, $t = \varepsilon \circ T$ satisfies the equation:*

$$t(A) = \sum t(A_{I(1)}) t(A_{I^c(1)}) \left(A_{I(2)} | A_{I^c(2)}\right) \tag{6.18}$$

on *M_{I,I^c}. We call this equation the Hopf algebraic equation of causality.*

Proof — By definition $t = \varepsilon \circ T$,

$$t(A) = \varepsilon(T(A)) = \varepsilon(T(A_I(x_i)_{i \in I} A_{I^c}(x_i)_{i \in I^c}))$$

$$= \varepsilon(T(A_I) \star T(A_{I^c}))$$

because of the causality equation (6.17)

$$t(A) = (T(A_I) | T(A_{I^c})) = \sum \left(t(A_{I(1)}) A_{I(2)} | t(A_{I^c(1)}) A_{I^c(2)}\right)$$

because by Wick expansion property (6.16) $T(A_I) = \sum t(A_{I(1)}) A_{I(2)}$ and $T(A_{I^c}) = t(A_{I^c(1)}) A_{I^c(2)}$,

$$t(A) = \sum t(A_{I(1)}) t(A_{I^c(1)}) \left(A_{I(2)} | A_{I^c(2)}\right).$$

■

We notice some important facts: first, in Borcherds, the equation

$$t(A) = \sum t(A_{I(1)}) t(A_{I^c(1)}) \left(A_{I(2)} | A_{I^c(2)}\right) \tag{6.19}$$

is called the Gaussian condition for the Feynman measure t (Borcherds calls it ω), secondly beware that the above product is not a priori well defined since it is a product of distributions. Secondly, this theorem says that the T-product satisfying the axioms 6.2.1 **is not even well defined** on d_n. It is only well defined on each M_{I,I^c} thus on $M^n \setminus d_n$ because of Stora's geometrical Lemma (6.2.1). To explain the meaning of the causality equation, we shall quote Borcherds where we changed his notation to adapt to our case (and also inserted some comments): "We explain what is going on in this definition. We would like to define the value of the Feynman measure t to be a sum over Feynman diagrams, formed by joining up pairs of fields in all possible ways by lines, and then assigning a propagator to each line and taking the product of all propagators of a diagram. This does not work because of ultraviolet divergences: products of propagators need not be defined when points coincide. If these products were defined then they would satisfy the Gaussian condition 6.19, which then says roughly that if the set of vertices $\{1,\dots,n\}$ are divided into two disjoint subsets I and I^c, then a Feynman diagram can be divided into a subdiagram with vertices I, a subdiagram with vertices I^c, and some lines between I and I^c. The value $t(A_I A_{I^c})$ of the Feynman diagram would then be the product of its value $t_I(A_{I(1)})$ on I, the product $\big(A_{I(2)}|A_{I^c(2)}\big)$ of all the propagators of lines joining I and I^c, and its value $t_{I^c}(A_{I^c(1)})$ on I^c. The Gaussian condition 6.19 need not make sense if some point of I is equal to some point of I^c because if these points are joined by a line then the corresponding propagator may have a bad singularity [*however this never happens in the domain M_{I,I^c} defined in the geometrical lemma*], but does make sense whenever all points of I are not $\leqslant$ to all points of I^c [*this is exactly the definition of the domain M_{I,I^c}*]. The definition above says that a Feynman measure should at least satisfy the Gaussian condition in this case, when the product is well defined." The explanations of Borcherds show that the geometrical lemma gives a very convenient way of covering $M^n \setminus d_n$ by the sets M_{I,I^c}.

6.2.3 Consistency condition

The collection $(M_{I,I^c})_I$ forms an open cover of $M^n \setminus d_n$, thus there are open domains in which a given M_{I,I^c} will overlap with a given C_J and we must prove the causality equations give the same result on overlapping domains, which justify an eventual gluing by partitions of unity. We must check a sheaf consistency condition: if I_1 and I_2 are proper subsets of $\{1,\dots,n\}$ such that $C = C_{I_1} \cap C_{I_2}$ is not empty, then $T_{I_1}|_C = T_{I_2}|_C$. Let $u = v_1 w_1$ be the factorization of u corresponding to I_1 and $u = v_2 w_2$ the one corresponding to I_2. We define on C

$$
\begin{aligned}
a_{12} &= \prod_{k\in I_1\cap I_2} a^k(x_k),\\
a_{c2} &= \prod_{k\in I_1^c\cap I_2} a^k(x_k),\\
a_{1c} &= \prod_{k\in I_1\cap I_2^c} a^k(x_k),\\
a_{cc} &= \prod_{k\in I_1^c\cap I_2^c} a^k(x_k).
\end{aligned}
$$

Therefore, $v_1 = a_{12}a_{1c}$, $v_2 = a_{12}a_{c2}$, $w_1 = a_{c2}a_{cc}$ and $w_2 = a_{1c}a_{cc}$. We have

$$
T_{I_1}|_C(u) \;=\; T(v_1)\cdot T(w_1) = T(a_{12}a_{1c})\cdot T(a_{c2}a_{cc}).
$$

By definition of C_{I_2} we have $a_{1c} \not\geqslant a_{12}$ and $a_{cc} \not\geqslant a_{c2}$, so that

$$
T_{I_1}|_C(u) \;=\; T(v_1)\cdot T(w_1) = T(a_{12})\cdot T(a_{1c})\cdot T(a_{c2})\cdot T(a_{cc}).
$$

The indices k of a_{c2} are in I_1^c and those of a_{1c} are in I_1, thus $a_{c2} \not\geq a_{1c}$. On the other hand, the indices k of a_{c2} are in I_2 and those of a_{1c} are in I_2^c, thus $a_{1c} \not\geq a_{c2}$. In other words $a_{c2} \sim a_{1c}$ so that $T(a_{1c})$ and $T(a_{c2})$ commute. Therefore,

$$\begin{aligned} T_{I_1}|_C(u) &= T(a_{12}) \cdot T(a_{1c}) \cdot T(a_{c2}) \cdot T(a_{cc}) = T(a_{12}) \cdot T(a_{c2}) \cdot T(a_{1c}) \cdot T(a_{cc}) \\ &= T(a_{12}a_{c2}) \cdot T(a_{1c}a_{cc}) = T(v_2) \cdot T(w_2) = T_{I_2}|_C(u). \end{aligned}$$

So we have defined distributions $T_I(u)$ on each M_{I,I^c} in a consistent way. We must now show that these $T_I(u)$ extend to a distribution T on $M^n \backslash D_n$. If the test function f has its support in M_{I,I^c}, we can define $T(u(f)) = T_I(u(f))$. However, for a test function with a support not included in a single M_{I,I^c}, we need to patch different T_I. To do this we shall use a smooth partition of unity subordinate to M_{I,I^c}.

6.3 The geometrical lemma for curved space time.

In this part, we need to improve the geometrical lemma due to Stora. Why is the geometrical lemma not enough ? We first notice that the functions χ_I from the partition of unity $(\chi_I)_I$ subordinate to the open cover $(M_{I,I^c})_I$ of $M^n \setminus d_n$ given by the geometrical lemma (6.2.1) are **smooth** in $M^n \setminus d_n$ but **are not smooth** in M^n. However, we will see (see formulas 6.25,6.24) that we are supposed to multiply χ_I with some product of distributions $t_I t_{I^c} \prod \Delta_+^{m_{ij}}$ on $M^n \setminus d_n$, extend it on M^n and control the wave front set of the extension $\overline{\chi_I t_I t_{I^c} \prod \Delta_+^{m_{ij}}}$. Hence, in order to control the wave front set of the extension, we must show that χ_I is weakly microlocally bounded for some s. Otherwise if χ_I was badly behaving near d_n, we would not be able to control the wave front set of the extension $\overline{t_n}$! Actually, we explicitely prove that for each point $(x, \dots, x) \in d_n$, there is a neighborhood U^n of $(x, \dots, x)$ in M^n where we can construct $\chi_I \in C^\infty(U^n)$ homogeneous of degree 0 with respect to some specific Euler vector field ρ. χ_I is thus scale invariant which implies $\forall \lambda \in (0,1], \chi_{I,\lambda} = \chi_I$ which means that the family $(\chi_{I\lambda})_\lambda$ is bounded in $C^\infty(U^n \setminus d_n)$ hence in $D'_\emptyset(U^n \setminus d_n)$. We need these refined properties on $(\chi_I)_I$ since we will have to control the wave front set of products of distributions with these functions χ_I.

Lemma 6.3.1 *Let $(M_{I,I^c})_I$ be the open cover of $M^n \setminus d_n$ given by the geometrical lemma 6.2.1. Then there exists a refinement $(\tilde{M}_{I,I^c})_I$ of this cover and a subordinate partition of unity $(\chi_I)_I$ where for each $I, \chi_I \in C^\infty(M^n \setminus d_n) \bigcap L^1_{loc}(M^n)$ and for any Euler vector field ρ, $e^{\rho \log \lambda *}(\chi_I)_{\lambda \in (0,1]}$ is a bounded family in $\mathcal{D}'_\emptyset(M^n \setminus d_n)$.*

Note that for every I, χ_I is in $E_0(M^n)$.
Proof —

1. For $x_0 \in M$, we localize in a neighborhood of $(x_0, ..., x_0) \in d_n$. Using a local chart, we identify some neighborhood of x_0 with $U \subset \mathbb{R}^d$, on U the metric reads g. We pick coordinates $(x^\mu)_\mu$ on U in such a way that $g_{\mu\nu}(0)dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu$ (η is of signature $+,-,-,-$).

2. We are going to construct another poset structure on U^2. For every $x \in U$, we denote by $C_x = \{y \geqslant x\} \cap U$ the set of elements of U in the causal future of x. We consider the closed subset $\{x_i \leqslant x_j\} \cap U^2 \subset U^2$. This set fibers on U:

$$\{x_i \leqslant x_j\} \cap U^2 = \left(\bigcup_{x_i \in U} \{x_i\} \times C_{x_i} \right) \subset U \times U$$

Then in this local chart $U \subset \mathbb{R}^d$, set the quadratic form $Q = \eta_{\mu\nu}dx^\mu dx^\nu + c^2(dx^0)^2$ where the aperture of the future cone of Q depends on the parameter c. The metric g depends smoothly on x and thus satisfies the estimate $|g_{\mu\nu}(x) - \eta_{\mu\nu}| \leqslant C|x|$ on U. For any strictly positive $c > 0$, we have the following estimate at x_0:

$$\xi^0 > 0 \text{ and } g_{\mu\nu}(0)\xi^\mu\xi^\nu = \eta_{\mu\nu}\xi^\mu\xi^\nu \geqslant 0 \implies \eta_{\mu\nu}\xi^\mu\xi^\nu + c^2(\xi^0)^2 > 0$$

hence since $g_{\mu\nu}$ is continuous we can find U small enough and c large enough in such a way that

$$\xi^0 > 0, \sup_{x \in U} g_{\mu\nu}(x)\xi^\mu\xi^\nu \geqslant 0 \implies \eta_{\mu\nu}\xi^\mu\xi^\nu + c^2(\xi^0)^2 > 0. \quad (6.20)$$

Set $\tilde{C}$ the future solid cone defined by the constant metric $Q_c = \eta_{\mu\nu}dx^\mu dx^\nu + c(dx^0)^2$, $\tilde{C}$ is given by the equations:

$$x^0 \geqslant 0 \quad (6.21)$$
$$Q_c(x) \geqslant 0. \quad (6.22)$$

Intuitively, if $c \to \infty$, the future cone $\tilde{C}$ for the constant metric Q has solid angle which tends to 2π. Hence for c sufficiently large, equation (6.20) means that the future cone $\tilde{C}$ contains all future conoids C_x for all $x \in U$. Then:

$$\{x_i \leqslant x_j\} \subset \bigcup_{x_i \in U} \{x_i\} \times \tilde{C} \subset U \times U.$$

3. $\tilde{C}$ defines a new partial order relation $\tilde{\geqslant}$, hence a new poset structure on U defined as follows:

$$x_j \tilde{\geqslant} x_i \text{ if } x_j^0 - x_i^0 \geqslant 0 \text{ and } Q(x_j - x_i) \geqslant 0, \quad (6.23)$$

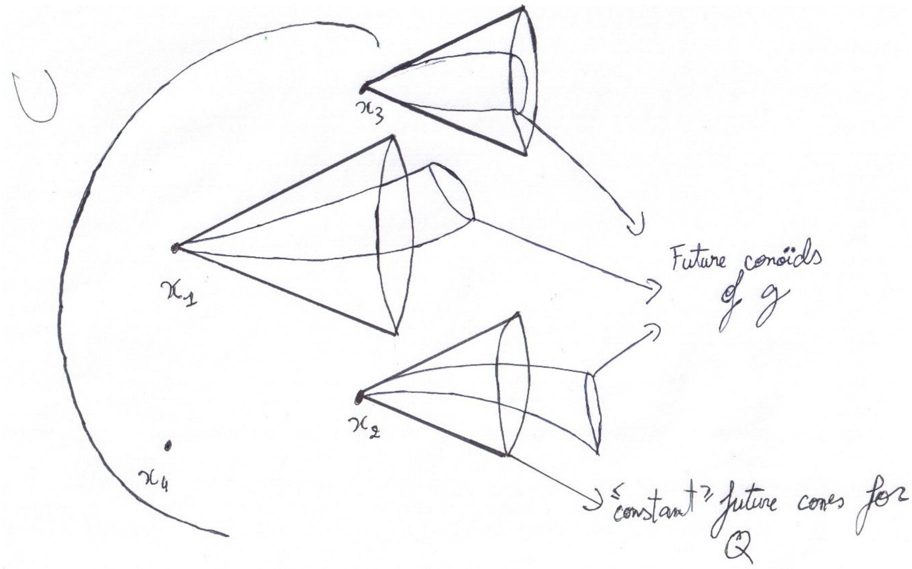

Figure 6.2: C_{123} for the partial order $\leqslant$ and for $\tilde{\leqslant}$

where both the cones $\tilde{C}$ and the corresponding partial order relation are invariant (in the configuration space $\mathbb{R}^{nd}$) under the action of the group $\mathbb{R}^* \ltimes \mathbb{R}^d$:

$$(\lambda, a) \in \mathbb{R}^* \ltimes \mathbb{R}^d : x \in \mathbb{R}^d \mapsto \lambda x + a \in \mathbb{R}^d.$$

Define for this new order relation new open sets $\tilde{M}_{I,I^c} = \{\forall (i,j) \in I \times I^c, x_i \tilde{\nleqslant} x_j\}$. Notice that if $x_i \leqslant x_j$ for the old order relation, then $x_i \tilde{\leqslant} x_j$ for the new order relation. Consequently, the sets M_{I,I^c} defined for the order relation $\leqslant$ are larger than the sets $\tilde{M}_{I,I^c}$ defined for $\tilde{\leqslant}$. Applying the geometrical lemma, we find:

$$U^n \setminus d_n \subset \bigcup_{I \subset \{1,\dots,n\}} \tilde{M}_{I,I^c}.$$

The group $\mathbb{R}^* \ltimes \mathbb{R}^d$ acts on the configuration space $\mathbb{R}^{dn}$, for $(\lambda, a) \in \mathbb{R}^* \ltimes \mathbb{R}^d$, we define the transformation:

$$(x_1, ..., x_n) \in \mathbb{R}^{dn} \mapsto (\lambda x_1 + a, ..., \lambda x_n + a) \in \mathbb{R}^{dn}.$$

4. We describe our construction in terms of **fibrations** of $\mathbb{R}^{nd} \setminus d_n$.

$$\begin{array}{ccccc} \mathbb{R}^{nd} \setminus d_n & \longrightarrow & \mathbb{R}^{d(n-1)} \setminus (0, \dots, 0) & \longrightarrow & \mathbb{S}^{(n-1)d-1} \\ (x_1, \dots, x_n) & \longmapsto & (h_2 = x_2 - x_1, \dots, h_n = x_n - x_1) & \longmapsto & (\frac{h_2}{\sum_{i=2}^n h_i^2}, \dots, \frac{h_n}{\sum_{i=2}^n h_i^2}) \end{array}$$

The first quotient is by the group of translation. The image of d_n by the first projection is the origin $(0,\dots,0)\in\mathbb{R}^{d(n-1)}$. The second quotient is by the group of dilations. We denote by π the projection $\pi:(x_1,\dots,x_n)\mathbb{R}^{nd}\setminus d_n\mapsto(\frac{h_2}{\sum_{i=2}^n h_i^2},\dots,\frac{h_n}{\sum_{i=2}^n h_i^2})\in\mathbb{S}^{(n-1)d-1}$. The open cover $(\tilde{M}_{I,I^c})_I$ are inverse images of some open cover $(\tilde{m}_{I,I^c})_I$ of the sphere $\mathbb{S}^{(n-1)d-1}$. Let $(\varphi_I)_I$ be a partition of unity subordinate to the open cover $(\tilde{m}_I)_I$ of $\mathbb{S}^{(n-1)d-1}$. Then we pull-back the functions $(\varphi_I)_I$ on $\mathbb{R}^{nd}\setminus d_n$ and set $\forall I,\chi_I=\pi^*\varphi_I$:

$$\chi_I(x_1,\cdots,x_n)=\varphi_I(\frac{x_2-x_1}{\sqrt{\sum_2^n(x_j-x_1)^2}},\cdots,\frac{x_n-x_1}{\sqrt{\sum_2^n(x_j-x_1)^2}}).$$

5. The collection of functions $(\chi_I)_I$ are both scale and translation invariant by the Euler vector field $\rho=\sum_{j=2}^n(x_j-x_1)\left(\partial_{x_j}-\partial_{x_1}\right)$. In the relative coordinate system $(x_1,h_{21}=x_2-x_1,...,h_{n1}=x_n-x_1)$, we notice that the collection $(\chi_I)_I$ only depends on the $(h_{i1})_{i\geqslant 2}$. χ_I is smooth in $\mathbb{R}^{nd}\setminus d_n$ hence $\chi_I\in\mathcal{D}'_\emptyset(U^n\setminus d_n)$. If we scale linearly, we notice $(\chi_I)_\lambda(h)=\chi_I(\lambda h)=\chi_I(h)$ thus the family $(\chi_I)_\lambda$ is bounded in $\mathcal{D}'_\emptyset(U^n\setminus d_n)$. However, we know that the boundedness of this family in $\mathcal{D}'_\emptyset(U^n\setminus d_n)$ and the degree of homogeneity does not depend on the choice of Euler vector field.

6. Let $(U_a)_{a\in A}$ be a locally finite cover of M then the collection of open sets $(U_a)^n_a$ forms an open cover of a neighborhood of d_n. Let φ_a be a partition of unity subordinate to the cover $(U_a)^n_a$. Then we can patch together the various functions $\chi_{I,a}$ constructed from the cover by the formula

$$\tilde{\chi}_I=\sum_a\frac{\chi_{I,a}\varphi_a^2}{\sum_J\sum_a\chi_{J,a}\varphi_a^2}$$

where the sum in the denominator is locally finite.

■

Remark. The fact that $\chi_I\in C^\infty(U^n\setminus d_n)$ does not immediately imply that the family $(\chi_I)_{\lambda,\lambda\in[0,1]}$ is **bounded** in $\mathcal{D}'_\emptyset(U^n\setminus d_n)$. For example, consider the function $\sin(\frac{1}{x})\in C^\infty(\mathbb{R}\setminus\{0\})$. For any interval $[a,b]\subset\mathbb{R}\setminus\{0\}$, we can construct a sequence λ_n which tends to 0 such that $\frac{d}{dx}\sin(\frac{1}{\lambda_n x})=\frac{1}{\lambda_n x^2}\cos(\frac{1}{\lambda_n x})\to\infty$ hence the family $\sin(\frac{1}{\lambda x})_\lambda$ is not bounded in $C^1[a,b]$ thus it is not bounded in $\mathcal{D}'_\emptyset(\mathbb{R}\setminus\{0\})$.

6.4 The recursion.

Notation, definitions. We denote by $x\simeq y$ if x and y in M are connected by a lightlike geodesic and $(x;\xi)\sim(y;\eta)$ if these two elements of the

cotangent are connected by a null bicharacteristic curve i.e. a Hamiltonian curve for the Hamiltonian $g_{\mu\nu}\xi^\mu\xi^\nu \in C^\infty(T^\star M)$.
We denote by $x > y$ if x is in the future cone of y and $x \neq y$.
Recall the configuration space M^I is the set of maps from I to M then the small diagonal d_I is just the subset of constant maps from I to M.
We denote by $M_{I,I^c}, I \sqcup I^c = [n]$ the set of all elements $(x_1, \dots, x_n) \in M^n$ s.t. $\forall (i,j) \in I \times I^c$ $x_i \not\leqslant x_j$. By the geometrical lemma the collection $(M_{I,I^c})_I$ forms an open cover of $M^n \setminus d_n$ and we denote by $(\chi_I)_I$ the subordinate partition of unity.
E_g^+ is the set of all elements in cotangent space having **positive energy**, the concept of positivity of energy being defined relative to the choice of Lorentzian metric g.

Definition 6.4.1 $E_g^+ = \{(x,\xi) | g_x(\xi,\xi) \geqslant 0, \xi_0 > 0\} \subset T^\bullet M$.

It is a **closed conic convex** set of $T^\bullet M$ and has the property that $E_g^+ \cap -E_g^+ = \emptyset$. We will denote by $E_{g,x}^+$ the component of E_g^+ living in the fiber $T_x^\bullet M$ over x.

Causality equation and wave front sets. The fact that for all n, $t_n \in Hom(H^n, \mathcal{D}'(M^n))$ satisfies the causality equation imposes some constraints on the wave front set of t_n. In M^n with coordinates $(x_i)_{i\in\{1,\dots,n\}}$, $(\chi_I)_I$ is the partition of unity subordinate to the cover $(M_{I,I^c})_I$ of $M^n \setminus d_n$ given by the improved geometrical lemma. For all n, $t_n(A) \in \mathcal{D}'(M^n \setminus d_n)$ satisfies the equation:

$$t_n(A) = \sum_{M_{I,I^c}} \sum \chi_I t_I(A_{I(1)}) t_{I^c}(A_{I^c(1)}) \left(A_{I(2)} | A_{I^c(2)}\right), \tag{6.24}$$

where $(\phi(x_i)|\phi(x_j)) = \Delta_+(x_i, x_j)$. For the sake of simplicity, each of the term $t_I(A_{I(1)}) t_{I^c}(A_{I^c(1)}) \left(A_{I(2)} | A_{I^c(2)}\right)$ in the above sum writes:

$$t_I \left(\prod_{ij \in I \times I^c} \Delta_+^{m_{ij}}(x_i, x_j) \right) t_{I^c}. \tag{6.25}$$

since each Laplace coupling $((A_I)_{(2)} | (A_{I^c})_{(2)}) = (\prod_{i\in I} \phi^{k_i}(x_i) | \prod_{j \in I^c} \phi^{k_j}(x_j))$ is a product of Wightman propagators: $\left(\prod_{ij\in I\times I^c} \Delta_+^{m_{ij}}(x_i,x_j)\right)$, $t_I = t_I(A_{I(1)})$ and $t_{I^c} = t_{I^c}(A_{I^c(1)})$. We now face the problem of defining t_n recursively by using the equation (6.25), the difficulty is to make sense of the r.h.s. of (6.25) on $M^n \setminus d_n$ which is a problem of multiplication of distributions and the second difficulty is to extend the distribution $t_n \in \mathcal{D}'(M^n \setminus d_n)$ (while retaining nice analytical properties) which is only defined on $M^n \setminus d_n$ to a distribution defined on M^n. We prove that renormalisability is local in M, for all $p \in M$ there exists an open neighborhood Ω of p on which all t_n are well defined as elements of $\mathcal{D}'(\Omega^n \setminus d_n)$ and can be extended as elements of $\mathcal{D}'(\Omega^n)$. In the sequel, using a local chart around p, we will identify Ω with an open set $U \subset \mathbb{R}^d$. In U, the metric reads g. The main theorem we prove is the following

Theorem 6.4.1 *The set of equations (6.24) can be solved recursively in n, where for each n, if all $t_I, I \subsetneq [n]$ are given then the product of distribution makes sense on $M^n \setminus d_n$ and defines a unique element $t_n \in \mathcal{D}'(M^n \setminus d_n)$ which has some extension in $\mathcal{D}'(M^n)$.*

We first treat the problem of multiplication of distributions outside d_n, to do this, we develop a machinery which allows us to describe wave front sets of Feynman amplitudes.

6.4.1 Polarized conic sets.

The idea of polarization is inspired by the exposition of Yves Meyer of Alberto Calderon's result on the product of Γ-holomorphic distributions ([52] p. 604 definition 1). In $\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$, the Γ-holomorphic distributions studied by Meyer are tempered distributions having their Fourier transform supported on a closed convex cone Γ in the Fourier domain which is contained in the upper half plane $\xi_n > 0$. The beautiful remark of Meyer is that Γ-holomorphic distributions can always be multiplied (the product extends to Γ-holomorphic distributions) and form an algebra for the extended product (because of the convexity of Γ the convolution product in the Fourier domain preserves is still supported on Γ)! For QFT, we are let to introduce the concept of **polarization** to describe subsets of the cotangent of configuration spaces $T^\bullet M^n$ for all n: this generalizes the concept of positivity of energy for the cotangent space of configuration space.

In order to generalize this condition to the wave front set of n-point functions, we define the right concept of positivity of energy which is adapted to conic sets in $T^\bullet M^n$:

Definition 6.4.2 *We define a* ***reduced polarized part*** *(resp* ***reduced strictly polarized part****) as a conical subset $\Xi \subset T^*M$ such that, if $\pi : T^*M \longrightarrow M$ is the natural projection, then $\pi(\Xi)$ is a finite subset $A = \{a_1, \cdots, a_r\} \subset M$ and, if $a \in A$ is maximal (in the sense there is no element $\tilde{a}$ in A s.t. $\tilde{a} > a$), then $\Xi \cap T_a^*M \subset (-E_g^+ \cup \{0\})$ (resp $\Xi \cap T_a^*M \subset (-E_g^+)$) where E_g^+ is the subset of elements of $T^\star M$ of positive energy.*

We define the trace operation as a map which associates to each element $p = (x_1, \dots, x_n; \xi_1, \dots, \xi_n) \in (T^*M)^k$ some finite part $Tr(p) \subset T^\star M$.

Definition 6.4.3 *For all elements $p = ((x_1, \xi_1), \cdots, (x_k, \xi_k)) \in T^*M^k$, we define the* ***trace*** *$Tr(p) \subset T^*M$ defined by the set of elements $(a, \eta) \in T^\star M$ such that $\exists i \in [1, k]$ with the property that $x_i = a$, $\xi_i \neq 0$ and $\eta = \sum_{i; x_i = a} \xi_i$.*

Then finally, we can define polarized subsets $\Gamma \subset T^*M^k$:

Definition 6.4.4 *A conical subset $\Gamma \subset T^*M^k$ is* ***polarized*** *(resp strictly polarized) if for all $p \in \Gamma$, its trace $Tr(p)$ is a reduced polarized part (resp reduced strictly polarized part) of T^*M.*

The union of two polarized (resp strictly polarized) subsets is polarized (resp strictly polarized) and if a conical subset is contained in a polarized subset it is also polarized.

The role of polarization is to control the wave front set of the distributions of the form $\langle 0|T\phi^{i_1}(x_1)\dots\phi^{i_n}(x_n)|0\rangle$.

The wave front set of Δ_+. In Theorem 5.3.1, we proved that for all $m \in \mathbb{N}$, $WF(\Delta_+^m)|_{U^2\setminus d_2} \subset \{\text{Conormal } \Gamma = 0\} \cap \left(-E_g^+ \times E_g^+\right)$ where E_g^+ is the set of elements of positive energy in $T^\bullet M$. Thus if $(x_1, x_2; \xi_1, \xi_2) \in WF(\Delta_+^m(x_1, x_2))|_{U^2\setminus d_2}$, two cases arise:

- if $x_1 \nleqslant x_2$ then we actually have $x_2 \leqslant x_1$ where $x_1 \in M$ is maximal in $\{x_1, x_2\}$ and $\xi_1 \in -E_{g,x_1}^+$ thus $WF(\Delta_+^m(x_1, x_2))|_{x_1 \nleqslant x_2}$ is strictly polarized,
- if $x_2 \nleqslant x_1$ then we actually have $x_1 \leqslant x_2$ where $x_2 \in M$ is maximal in $\{x_1, x_2\}$ and $\xi_2 \in E_{g,x_1}^+$ thus $WF(\Delta_+^m(x_1, x_2))|_{x_2 \nleqslant x_1}$ is **not polarized**.

Corollary 6.4.1 *For all $(i, j) \in I \times I^c, I \sqcup I^c = [n]$, $WF(\Delta_+^m(x_i, x_j))|_{M_{I,I^c}}$ is strictly polarized.*

We have to check that the conormals of the diagonals d_I are polarized since they are the wave front sets of counterterms from the extension procedure.

Proposition 6.4.1 *The conormal of the diagonal $d_I \subset M^I$ is polarized.*

Proof — Let $(x_i; \xi_i)_{i\in I}$ be in the conormal of d_I, let $a \in M$ s.t. $a = x_i, \forall i \in I$, and $\eta = \sum \xi_i = 0$ is in $-E_{g,a}^+ \cup \{0\}$. Thus the trace $Tr(x_i; \xi_i)_{i\in I} = (a; 0)$ of the element $(x_i; \xi_i)_{i\in I}$ in the conormal of d_I is a reduced polarized part of $T^\star M$. ■

Proposition 6.4.2 *For all $m \in \mathbb{N}$, if $t_2(\phi^m(x_1)\phi^m(x_2))$ satisfies the causality equation (6.24) on $U^2 \setminus d_2$ and $WF\left(t_2(\phi^m(x_1)\phi^m(x_2))\right)|_{d_2}$ is contained in the conormal of d_2, then the wave front set of $t_2(\phi^m(x_1)\phi^m(x_2))_{U^2}$ is polarized.*

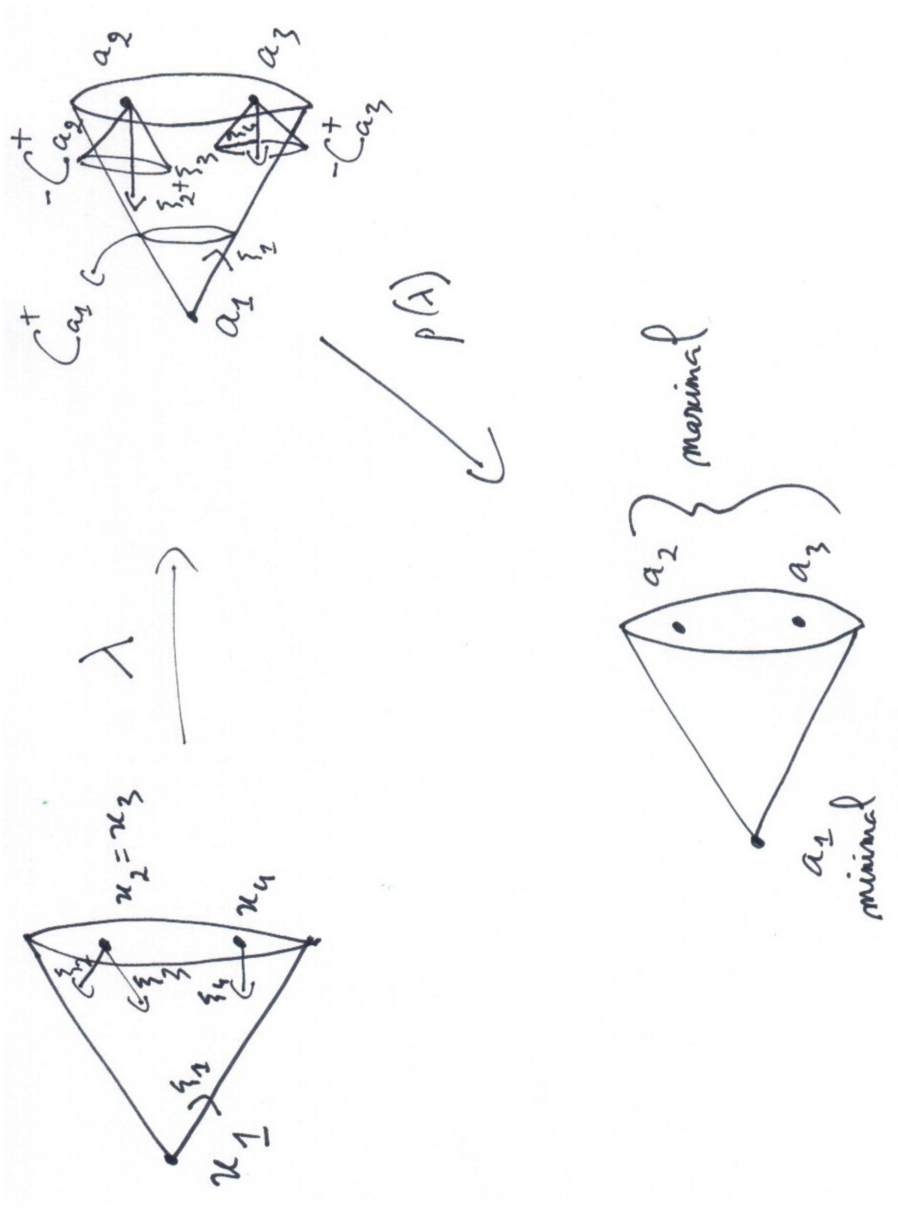

Figure 6.3: A polarized set, the trace Tr and the projection $\pi \circ Tr$.

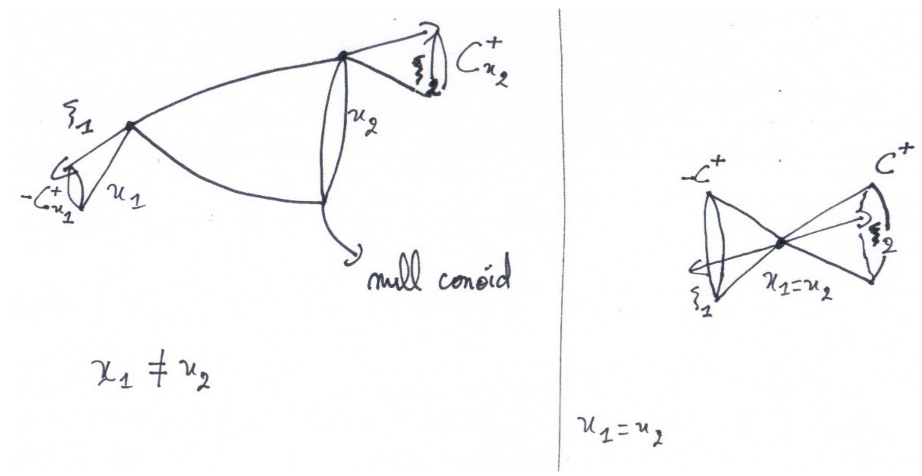

Figure 6.4: wave front set of Δ_+^m.

Notice that it is enough to prove the proposition for $t_2(\phi(x_1)\phi(x_2))$ since $t_2(\phi^m(x_1)\phi^m(x_2)) = m! t_2(\phi(x_1)\phi(x_2))^m$ on $U^2 \backslash d_2$ thus $WF\,(t_2(\phi^m(x_1)\phi^m(x_2))) \subset WF\,(t_2(\phi(x_1)\phi(x_2))) + WF\,(t_2(\phi(x_1)\phi(x_2)))$ on $U^2 \setminus d_2$.

Proof — Notice that if $x_1 \not\leq x_2$ then $T\phi(x_1)\phi(x_2) = \phi(x_1) \star \phi(x_2)$ by the definition of causality i.e. $T(AB) = TA \star TB$ if $A \not\leq B$. Thus the field $\phi(x_1)$ associated with the element x_1, where x_1 is not in the causal past of x_2, stands on the left of the product $\phi(x_1) \star \phi(x_2)$. Causality reads from right to left when we write products of fields i.e. $T(AB) = TA \star TB$ if $A \not\leq B$.

$$t_2(\phi(x_1)\phi(x_2)) = \varepsilon\,(T_2\phi(x_1)\phi(x_2)) = \varepsilon\,(\phi(x_1) \star \phi(x_2))$$

$$\begin{aligned} &= \Delta_+(x_1, x_2) \text{ if } x_1 \not\leq x_2 \\ &= \Delta_+(x_2, x_1) \text{ if } x_2 \not\leq x_1, \end{aligned}$$

which implies $WF\,(t_2)\,|_{U^2 \backslash d_2}$ is polarized. Using Proposition 6.4.1 and the fact that $WF(t_2)|_{d_2}$ is contained in the conormal of d_2, it is immediate to deduce $WF(t_2)$ is polarized. ■

Now we will prove the key theorem which allows to multiply two distributions under some conditions of polarization on their wave front sets and deduces specific properties of the wave front set of the product:

Theorem 6.4.2 *Let u, v be two distributions in $\mathcal{D}'(\Omega)$, for some subset $\Omega \subset M^n$, s.t. $WF(u) \cap T^\bullet\Omega$ is polarized and $WF(v) \cap T^\bullet\Omega$ is strictly polarized. Then the product uv makes sense in $\mathcal{D}'(\Omega)$ and $WF(uv) \cap T^\star\Omega$ is polarized. Moreover, if $WF(u)$ is also strictly polarized then $WF(uv)$ is strictly polarized.*

Proof — Step 1: we prove $WF(u)+WF(v)\cap T^\star\Omega$ does not meet the zero section. For any element $p=(x_1,\dots,x_n;\xi_1,\dots,\xi_n)\in T^\star M^n$ we denote by $-p$ the element $(x_1,\dots,x_n;-\xi_1,\dots,-\xi_n)\in T^\star M^n$. Let $p_1=(x_1,\dots,x_n;\xi_1,\dots,\xi_n)\in WF(u)$ and $p_2=(x_1,\dots,x_n;\eta_1,\dots,\eta_n)\in WF(v)$, necessarily we must have $(\xi_1,\dots,\xi_n)\neq 0,(\eta_1,\dots,\eta_n)\neq 0$. We will show by a contradiction argument that the sum $p_1+p_2=(x_1,\dots,x_n;\xi_1+\eta_1,\dots,\xi_n+\eta_n)$ does not meet the zero section. Assume that $\xi_1+\eta_1=0,\dots,\xi_n+\eta_n=0$ i.e. $p_1=-p_2$ then we would have $\xi_i=-\eta_i\neq 0$ for some $i\in\{1,\dots,n\}$ since $(\xi_1,\dots,\xi_n)\neq 0,(\eta_1,\dots,\eta_n)\neq 0$. We assume w.l.o.g. that $\eta_1\neq 0$, thus $Tr(p_2)$ is non empty ! Let $B=\pi(Tr(p_1)),C=\pi(Tr(p_2))$, we first notice $B=C$ since $p_2=-p_1 \implies Tr(p_1)=-Tr(p_2)\implies \pi\circ Tr(p_1)=\pi\circ Tr(p_2)$. Thus if a is maximal in B, a is also maximal in C and we have

$$0=\sum_{x_i=a}\xi_i+\eta_i=\sum_{x_i=a}\xi_i+\sum_{x_i=a}\eta_i\in\left(E^-_{g,a}\cup\{0\}+E^-_{g,a}\right)=E^-_{g,a},$$

where we denote $E^-_{g,a}=-E^+_{g,a}$ for notational clarity, (since p_1 is polarized and p_2 is strictly polarized) contradiction !

Step 2, we prove that the set

$$(WF(u)+WF(v))\cap T^\star\Omega$$

is strictly polarized. Recall $B=\pi\circ Tr(p_1)$, $C=\pi\circ Tr(p_2)$ and we denote by $A=\pi\circ Tr(p_1+p_2)$ hence in particular $A\subset B\cup C$. We denote by $\max A$ (resp $\max B,\max C$) the set of maximal elements in A (resp B,C). The key argument is to prove that $\max A=\max B\cap\max C$. Because if $\max A=\max B\cap\max C$ holds then for any $a\in\max A$, $\sum_{x_i=a}\xi_i+\eta_i=\sum_{x_i=a}\xi_i+\sum_{x_i=a}\eta_i\in -E^+_{g,a}$ since $a\in\max B\cap\max C$ and $Tr(p_1)$ is a reduced polarized part and $Tr(p_2)$ is reduced strictly polarized. Thus $\max A=\max B\cap\max C$ implies that p_1+p_2 is strictly polarized.
We first establish the inclusion $(\max B\cap\max C)\subset\max A$. Let $a\in\max B\cap\max C$, then $\sum_{x_i=a}\xi_i\in E^-_{g,a}\cup\{0\}$ and $\sum_{x_i=a}\eta_i\in E^-_{g,a}$. Thus $\sum_{x_i=a}\xi_i+\eta_i\in E^-_{g,a}\implies\sum_{x_i=a}\xi_i+\eta_i\neq 0$ so there must exist some i for which $x_i=a$ and $\xi_i+\eta_i\neq 0$. Hence $a\in A$. Since $A\subset B\cup C$, $a\in\max B\cap\max C$, we deduce that $a\in\max A$ (if there were $\tilde a$ in A greater than a then $\tilde a\in B$ or $\tilde a\in C$ and a would not be maximal in B and C).

We show the converse inclusion $\max A\subset(\max B\cap\max C)$ by contraposition. Assume $a\notin\max B$, then there exists $x_{j_1}\in\max B$ s.t. $x_{j_1}>a$ and $\xi_{j_1}\neq 0$. There are two cases

- either $x_{j_1}\in\max C$ as well, then $\sum_{x_{j_1}=x_i}\xi_i+\eta_i\in -E^+_{g,x_{j_1}}\implies\sum_{x_{j_1}=x_i}\xi_i+\eta_i\neq 0$ and there is some i for which $x_i=x_{j_1}$ and $\xi_i+\eta_i\neq 0$ thus $x_{j_1}\in A$ and $x_{j_1}>a$ hence $a\notin\max A$.
- or $x_{j_1}\notin\max C$ then there exists $x_{j_2}\in\max C$ s.t. $x_{j_2}>x_{j_1}$ and $\eta_{j_2}\neq 0$. Since $x_{j_1}\in\max B$, we must have $\xi_{j_2}=0$ so that $x_{j_2}\notin B$.

But we also have $\xi_{j_2} + \eta_{j_2} = \eta_{j_2} \neq 0$ so that $x_{j_2} \in A$. Thus $x_{j_2} \in A$ is greater than a hence $a \notin \max A$.

We thus proved

$$a \notin \max B \implies a \notin \max A$$

and by symmetry of the above arguments in B and C, we also have

$$a \notin \max C \implies a \notin \max A.$$

We established that $(\max B)^c \subset (\max A)^c$ and $(\max C)^c \subset (\max A)^c$, thus $(\max B)^c \cup (\max C)^c \subset (\max A)^c$ therefore $\max A \subset \max B \cap \max C$, from which we deduce the equality $\max A = \max B \cap \max C$ which implies that $WF(u) + WF(v)$ is strictly polarized and $WF(uv)$ is polarized. ∎

Lemma 6.4.1 *For all $I \sqcup I^c = [n]$, $(k_i)_{i\in I}, (k_j)_{j\in I^c}$ s.t. $\sum_{i\in I} k_i = \sum_{j\in I^c} k_j$ the Laplace coupling*

$$\left(\prod_{i\in I} \phi^{k_i}(x_i) | \prod_{j\in I^c} \phi^{k_j}(x_j)\right)$$

is well defined in the sense of distributions of $\mathcal{D}'(M_{I,I^c})$ and its wave front set is strictly polarized.

Proof — First the coupling $\left(\prod_{i\in I} \phi^{k_i}(x_i) | \prod_{j\in I^c} \phi^{k_j}(x_j)\right)$ is a finite sum of terms of the form $\prod_{(i,j)\in I\times I^c} \Delta_+^{m_{ij}}(x_i, x_j), m_{ij} \in \mathbb{N}$. However

$$WF(\prod_{(i,j)\in I\times I^c} \Delta_+^{m_{ij}}(x_i, x_j)|_{M_{I,I^c}})$$

is strictly polarized by application of lemma 6.4.2 since $WF(\Delta_+^{m_{ij}}|_{M_{I,I^c}})$ is strictly polarized. ∎

Lemma 6.4.2 *Let t_I, t_{I^c} be in $\mathcal{D}'(M^I), \mathcal{D}'(M^{I^c})$ respectively s.t. $WF(t_I)$ and $WF(t_{I^c})$ are polarized then $WF(t_I t_{I^c})|_{M_{I,I^c}}$ is polarized.*

Proof — For all $(x_i, x_j; \xi_i^I, \xi_j^{I^c})_{(i,j)\in I\times I^c} \in WF(t_I t_{I^c})|_{M_{I,I^c}}$, $Tr(x_i, x_j; \xi_i^I, \xi_j^{I^c}) = Tr(x_i; \xi_i^I) \cup Tr(x_j; \xi_j^{I^c})$ because for all $(x_1, \dots, x_n) \in M_{I,I^c}$ for all $(i,j) \in I \times I^c$, $x_i \neq x_j$. Then using the fact that $Tr(x_i; \xi_i^I)$ and $Tr(x_j; \xi_j^{I^c})$ are polarized, for all a maximal in $\pi \circ Tr(x_i, x_j; \xi_i^I, \xi_j^{I^c})$:

- either a is maximal in $Tr(x_i; \xi_i^I)$ in which case $\eta = \sum_{x_i=a} \xi_i^I \in -E_{g,a}^+ \cup \{0\}$ since $Tr(x_i; \xi_i^I)$ is polarized,

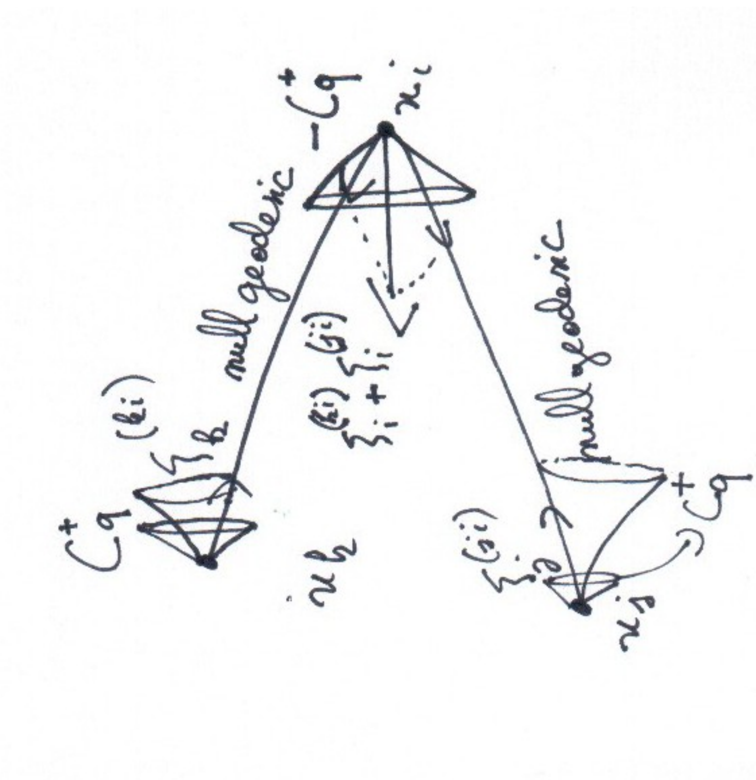

Figure 6.5: The Wavefront of Laplace couplings is strictly polarized.

- either a is maximal in $Tr(x_j;\xi_j^{I^c})$ and we deduce the same kind of result
$$\eta=\sum_{x_j=a}\xi_j^{I^c}\in -E_{g,a}^+\cup\{0\}$$
since $Tr(x_j;\xi_j^{I^c})$ is polarized.

■

Theorem 6.4.3 *Let t_I, t_{I^c} be distributions in $\mathcal{D}'_{\Gamma_I}, \mathcal{D}'_{\Gamma_{I^c}}$ where Γ_I, Γ_{I^c} are polarized in M^I and M^{I^c} and m_{ij} be a collection of integers. Then the product*
$$t_I t_{I^c}\prod_{(ij)\in I\times I^c}\Delta_+^{m_{ij}}(x_i,x_j)$$
is well defined as a distribution of $D'_{\Gamma_n}(M_{I,I^c})$ for
$$\Gamma_n=\sum_I\left(\Gamma_I^0+\Gamma_{I^c}^0+\sum_{ij}\Gamma_{ij}^0\right)\bigcap T^\bullet M_{I,I^c}$$
and Γ_n is polarized. Furthermore, t_n defined by the relation (6.24) is well defined in $\mathcal{D}'(U^n\setminus d_n)$ and its wave front set is polarized in $M^n\setminus d_n$.

Proof — $WF(t_I t_{I^c})$ is polarized in M_{I,I^c} by Lemma 6.4.2, each Laplace coupling is strictly polarized in M_{I,I^c} by Lemma 6.4.1 hence by Theorem 6.4.2 the product
$$t_I t_{I^c}\prod_{(ij)\in I\times I^c}\Delta_+^{m_{ij}}(x_i,x_j)$$
exists and its wave front set is polarized over M_{I,I^c}. We sum and multiply each term $\sum t_I(A_{I1})t_{I^c}(A_{I^c1})(A_{I2}|A_{I^c2})$ by the functions χ_I of the partition of unity from the geometrical lemma which does not affect the wave front set since they are smooth on $M^n\setminus d_n$, thus the wave front set of t_n defined by (6.24) is the finite union of polarized conical subsets thus polarized. ■

6.4.2 Localization and enlarging the polarization.

In the previous part, we were able to justify the products of distributions on $M^n\setminus d_n$ in equation 6.24 but have not yet extended the distribution t_n on M^n. The goal of this part is to prove that we can construct some polarized cone Γ_I, slightly larger than $WF(t_I)$, which is scale invariant for some family of linear Euler vector fields and satisfies the soft landing condition. The drawback of working with the cone $E_g^+\subset T^\bullet U$ is that the cones $E_{gx}^+\subset T_x^\bullet U$ depend on the point x. We will construct a larger closed convex conic E_q^+

for a constant metric q which contains E_g^+ and which has fibers E_{qx}^+ that do not depend on $x \in U$.

We identify an open set $\Omega \subset M$ with $U \subset \mathbb{R}^d$, in U the metric reads g. Then we soften the poset relation in a similar way to the step 2 and 3 in the proof of the improved geometrical lemma (6.3.1). We use a constant metric Q to define a new partial order denoted by $\tilde{\leqslant}$. Recall $E_g^+ \subset T^\star M$ is the subset of elements in cotangent space of positive energy. We prove a lemma which says we can localize in a domain $U \subset \mathbb{R}^d$ in which we can control the wave front set of the family $(\Delta_+)_\lambda$,

$$\forall \lambda \in (0,1], WF\left(\Delta_{+\lambda}\right) \subset \left(-E_q^+\right) \times \left(E_q^+\right)$$

by a scale and translation invariant set E_q^+ living in cotangent space $T^\bullet U$.

Lemma 6.4.3 *For any $x_0 \in U$, we can always make U smaller around x_0 so as to be able to construct a* ***closed conic convex*** *set $E_q^+ \subset T^\bullet U$ s.t. $E_g^+ \subset E_q^+$, E_q^+ does not depend on $x \in U$ and such that $E_q^+ \cap -E_q^+ = \emptyset$.*

Proof — We enlarge the cone of positive energy $E_g^+ \subset T^\bullet U$. Recall we defined E_g^+ as $E_g^+ = \{(x;\xi) | g_x(\xi,\xi) \geqslant 0, \xi_0 > 0\} \subset T^\bullet M$. But the drawback of this definition lies in the fact that the fibers $E_g^+{}_x$ of the set E_g^+ depend on the base point x since g is variable. We localize the construction in a sufficiently small open ball U in $\mathbb{R}^d$ and pick a constant metric q on this ball U in such a way that

$$\forall x \in U, g_x(\xi,\xi) \geqslant 0, \xi_0 > 0 \implies q(\xi,\xi) > 0. \qquad (6.26)$$

Such a metric is easy to construct, following the arguments of the proof of the improved geometrical lemma, we assume $g_{x_0}^{\mu\nu} = \eta^{\mu\nu}$ and by setting $q = \eta^{\mu\nu} + \lambda^2\delta^{00}$, we can always choose λ large enough so that the inequality (6.26) is satisfied for all $x \in U$.

Definition 6.4.5 *We set $E_q^+ = \{(x,\xi) | q(\xi,\xi) \geqslant 0, \xi_0 > 0, x \in U\}$.*

It is immediate by construction that our new **closed, conic, convex** set $E_q^+ \subset T^\bullet M$ contains the old set E_g^+. It is also obvious by construction that E_q^+ is both *scale and translation invariant* in U, since the metric q is constant in $\mathbb{R}^d$. ■

We have a new definition of polarization by applying Definition (6.4.2) for the new conic set E_q^+ and the partial order $\tilde{\leqslant}$ ($\tilde{\leqslant}$ affects the choices of maximal points). Hence the metric Q controls the order relation $\tilde{\leqslant}$ and exploits the finite propagation speed of light, whereas the metric q controls the cone of positive energy.

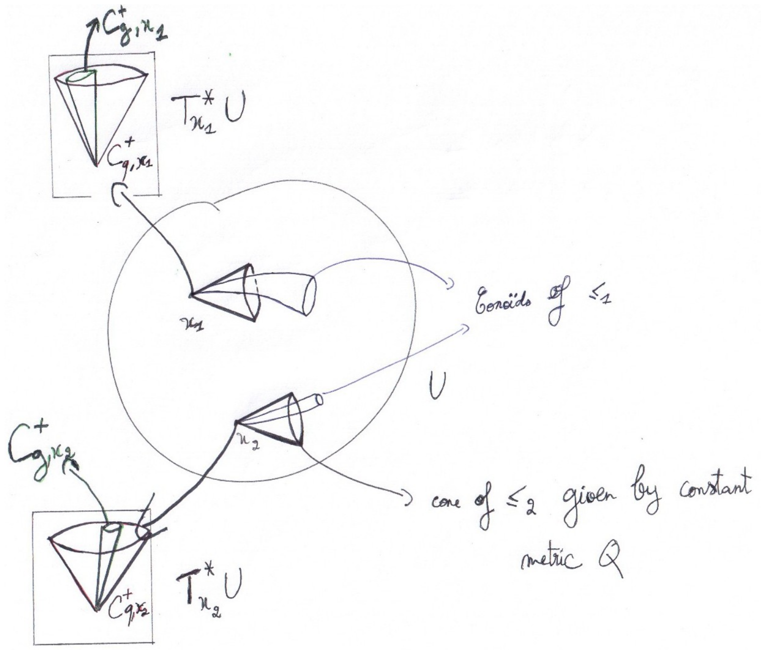

Figure 6.6: Picture of the new poset structure together with the new polarization.

Scaling in configuration spaces. On U^I, we denote the coordinates by $(x_i)_{i\in I}$, then we define the collection $\rho_{x_i}, i \in I$ of $|I|$ linear Euler fields $\rho_{x_i} = \sum_{j\neq i, j\in I}(x_j - x_i)\frac{\partial}{\partial x_j}$. ρ_{x_i} scales relative to the element x_i in configuration space U^I.

Example 6.4.1 *In U^n, the vector field $\sum_{j\neq 1}(x_j - x_1)\partial_{x_j}$ is Euler since $\left(\sum_{j\neq 1}(x_j - x_1)\partial_{x_j}(x_i - x_1)\right) - (x_i - x_1) = (x_i - x_1) - (x_i - x_1) = 0$ and this implies that $\sum_{j\neq 1}(x_j - x_1)\partial_{x_j} f - f \in \mathcal{I}^2$ for all $f \in \mathcal{I}$ the ideal of functions vanishing on d_n. If we scale by $f_\lambda(x_1, \dots, x_n) = f(x_1, \lambda(x_2 - x_1) + x_1, \dots, \lambda(x_n - x_1) + x_1)$ then this corresponds to the Euler vector field $\sum_{j=1}^{n-1} h_j \frac{\partial}{\partial h^j}$. The cotangent lift of this vector field equals*

$$\sum_{j\neq 1}(x_j - x_1)\partial_{x_j} - \xi_j(\partial_{\xi_j} - \partial_{\xi_1}).$$

The vector field $\sum_{j=2}^n \xi_j(\partial_{\xi_j} - \partial_{\xi_1})$ corresponds to the system of ODE's

$$\forall j \geqslant 2, \frac{d\xi^j}{dt} = \xi^j, \frac{d\xi^1}{dt} = \sum_{j=2}^n \xi^j,$$

thus integrating the vector field $\sum_{j\neq 1}(x_j - x_1)\partial_{x_j} - \xi_j(\partial_{\xi_j} - \partial_{\xi_1})$ in cotangent *space yields the flow :*

$$(x_1, \lambda(x_2 - x_1) + x_1, \dots; \xi_1 + (1 - \lambda^{-1}) \sum_{j=2}^{n} \xi_j, \lambda^{-1}\xi_2, \dots, \lambda^{-1}\xi_n).$$

Finally, we compute the coordinate transformation in cotangent space which passes from regular coordinates in cotangent space $T^\star U^n$ to the system of coordinates $(x, h; k, \xi)$ used in Chapters 1,2,3,4:

$$(x_1, \dots, x_n; \xi_1, \dots, \xi_n) \mapsto (x, h_1, \dots, h_{n-1}; k, \eta_1, \dots, \eta_{n-1}) \quad (6.27)$$

$$x_1 = x, h_j = x_{j+1} - x_1 \quad (6.28)$$

$$k = \sum_{i=1}^{n} \xi_i, \eta_j = \xi_{j+1}. \quad (6.29)$$

The soft landing condition on configuration space. We saw in Chapter 2 and 3 that the soft landing condition was an essential condition on the wave front set of a distribution which allows to control the wave front set of extensions of distributions. Before we state the soft landing condition in $T^\star U^n$, we first give the equation of the conormal of $d_n \subset U^n$ in coordinates $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$. The collection $dh_1 = dx_2 - dx_1, \dots, dh_{n-1} = dx_n - dx_1$ of 1-forms spans a basis of orthogonal forms to the tangent space of d_n, thus a 1-form $\xi_1 dx_1 + \dots + \xi_n dx_n$ belongs to the conormal if it writes $\sum_{i=2}^n a_i dh_i$ for some $(a_i)_i$ which implies $\xi_1 = -\sum_{i=2}^n \xi_i$, thus the equation of the conormal in U^n is $x_1 = x_2 = \dots = x_n, \xi_1 + \dots + \xi_n = 0$. If we write the equation of the soft landing condition in $T^\bullet U^n$ for the coordinates, we obtain

$$|\sum_{i=1}^{n} \xi_i| \leqslant \delta \left(\sum_{i=2}^{n} |x_1 - x_i|\right) \left(\sum_{i=2}^{n} |\xi_i|\right) \quad (6.30)$$

since $k = \sum_{i=1}^n \xi_i$ and $\forall i \geqslant 2, \eta_i = \xi_{i+1}$ by 6.27, the inequality 6.30 is clearly invariant by the flow $\lambda \mapsto (x_1, \lambda(x_2 - x_1) + x_1, \dots; \xi_1 + (1 - \lambda^{-1}) \sum_{j=2}^n \xi_j, \lambda^{-1}\xi_2, \dots, \lambda^{-1}\xi_n)$.

In configuration space $T^\star U^I$ with coordinates $(x_i; \xi_i)_{i\in I}$, the soft landing condition takes the following form: a conic set $\Gamma \subset T^\bullet U^I$ satisfies the **soft landing condition** w.r.t. to d_I if for all compact set $K \subset U^I$, there exists $\varepsilon > 0$ and $\delta > 0$, such that

$$\Gamma|_{K\cap\{\sum_{i\in I, i\neq j} |x_j - x_i| \leqslant \varepsilon\}} \subset \{|\sum_{i\in I} \xi_i| \leqslant \delta \left(\sum_{i\in I, i\neq j} |x_j - x_i|\right) \left(\sum_{i\in I; i\neq j} |\xi_i|\right)\}. \quad (6.31)$$

6.4.3 We have $\left(WF\left(e^{\log\lambda\rho_{x_i}*}\Delta_+\right) \bigcap T^\bullet U^2\right) \subset (-E_q^+) \times E_q^+$.

The next lemma aims to use our cone $E_q^+ \subset T^\bullet U$ to control the wave front set of the family $\left(e^{\log\lambda\rho_{x_i}\star}\Delta_+\right)_{\lambda\in(0,1]}, i = (1, 2)$.

Lemma 6.4.4 *We can choose q and U in such a way that*

$$\forall\lambda\in(0,1], \left(WF\left(e^{\log\lambda\rho_{x_i}*}\Delta_+\right)\bigcap T^\bullet U^2\right)\subset\left(-E_q^+\right)\times E_q^+.$$

Proof — By construction of E_q^+, $\left(WF\left(\Delta_+\right)\bigcap T^\bullet U^2\right)\subset\left(-E_q^+\right)\times E_q^+$. If $(x_1;\xi_1),(x_2;\xi_2)\in -E_q^+\times E_q^+$ then $\forall\lambda\in(0,1], (x_1;\xi_1+(1-\lambda)\xi_2),(\lambda^{-1}(x_2-x_1)+x_1;\lambda\xi_2)\in -E_q^+\times E_q^+$ by invariance and convexity of E_q^+ which immediately yields the result.

■

6.4.4 The scaling properties of translation invariant conic sets.

The next lemma we prove also has a geometric flavor.

Lemma 6.4.5 *Let $\Gamma_I\subset T^\bullet M^I$ be a* ***translation invariant*** *conic set. Then Γ_I is stable under $e^{\log\lambda\rho_i}$ for some $i\in I$ is equivalent to Γ_I is stable by $e^{\log\lambda\rho_i}$ for all $i\in I$.*

Proof — Following the approach of Chapter 1, we try to find a flow $\Phi(\lambda)$ relating the two linear scalings by ρ_{x_i} and ρ_{x_j}. This flow is given by the formula $\Phi(\lambda)=e^{-\log\lambda\rho_{x_i}}\circ e^{\log\lambda\rho_{x_j}}$ and the lifted flow $T^\star\Phi(\lambda)$ on cotangent space is given by the formula $T^\star\Phi(\lambda)=T^\star e^{-\log\lambda\rho_{x_i}}\circ T^\star e^{\log\lambda\rho_{x_j}}$. In our specific case, for each λ, $\Phi(\lambda)$ is a flow by linear translation. The map $\Phi(\lambda)$ results from the composition of two *scalings* relative to two elements (x_i,x_j) with ratio (λ,λ^{-1}) respectively. It can be computed explicitely

$$\Phi_\lambda : x\mapsto\lambda(x-x_i)+x_i$$

$$\mapsto\lambda^{-1}\left((\lambda(x-x_i)+x_i)-(\lambda(x_j-x_i)+x_i)\right)+(\lambda(x_j-x_i)+x_i)$$
$$=(x-x_j)+(\lambda(x_j-x_i)+x_i)=x+\underbrace{(\lambda-1)(x_j-x_i)}_{\text{translation vector}},$$

which proves $\Phi(\lambda)=e^{-\log\lambda\rho_{x_i}}\circ e^{\log\lambda\rho_{x_j}}$ is a translation of vector $(\lambda-1)(x_j-x_i)$. We also have $T^\star\Phi(\lambda):(x;\xi)\mapsto(x+(\lambda-1)(x_j-x_i);\xi)$. This computation proves the following fundamental fact: if a translation invariant set Γ_I is stable by the cotangent lift of scaling relative to one given $a\in\mathbb{R}^d$ then Γ_I is invariant by the cotangent lift of linear scalings relative to *any element* $a\in\mathbb{R}^d$ which implies the claimed result. ■

This lemma motivates the following definition: *a translation invariant conic set $\Gamma_I\subset T^\bullet M^I$ is said to be scale invariant if it is stable by scaling w.r.t. the vector field ρ_{x_i} for some $i\in I$.*

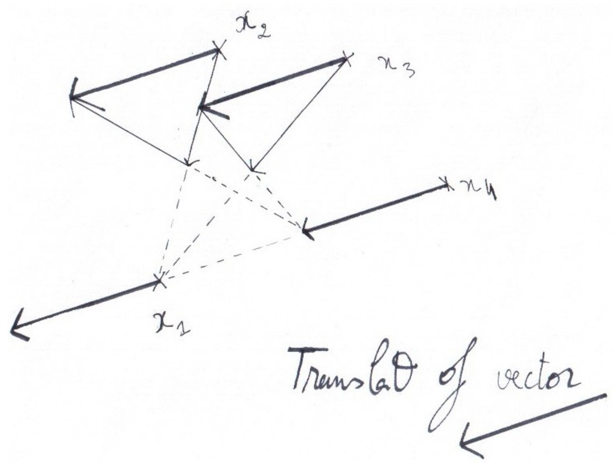

Figure 6.7: Action on configuration space $\left(\mathbb{R}^d\right)^4$ of the map $\Phi(\lambda) = e^{-\log\lambda\rho_{x_4}} \circ e^{\log\lambda\rho_{x_1}}$ for $\lambda = \frac{1}{2}$ as a translation.

6.4.5 Thickening sets.

Lemma 6.4.6 *If Γ_I satisfies the soft landing condition and is (strictly) polarized, then there exists a translation and scale invariant $\tilde{\Gamma}_I$ such that $\Gamma_I \subset \tilde{\Gamma}_I$, $\tilde{\Gamma}_I$ is still (strictly) polarized and satisfies the soft landing condition.*

We call *good*, any conic set that is translation invariant, scale invariant, polarized and satisfies the soft landing condition.
Proof — Notice that the formulation of the soft landing condition on configuration space by the equation

$$|\sum_{i\in I}\xi_i| \leqslant \delta\left(\sum_{i\in I, i\neq j}|x_j - x_i|\right)\left(\sum_{i\in I, i\neq j}|\xi_i|\right), \tag{6.32}$$

is clearly translation and scale invariant. But E_q^+ and $\tilde{\leqslant}$ are also translation and scale invariant thus the concept of polarization is translation and scale invariant. So if a set $\Gamma_I \subset T^\bullet U^I$ is polarized and satisfies the soft landing condition, then the union $\tilde{\Gamma}_I$ of all orbits of the group of translations and dilations which intersect Γ_I satisfies the same properties and contains Γ_I. ■

6.4.6 The μlocal properties of the two point function.

Let us consider the configuration space U^2 with coordinates (x_1, x_2). Let Ξ be the wave front set of Δ_+. In Chapter 5, we proved that

$$\Xi \subset \left(\Lambda \bigcup \{(x,x;-\eta,\eta) | g_x(\eta,\eta) \geqslant 0\}\right) \bigcap \{(x_1,x_2;\eta_1,\eta_2) | (\eta_2)_0 > 0\}$$

where Λ is the conormal bundle of the conoid $\Gamma = 0$ (Theorem 5.3.1) and we proved that Δ_+ is microlocally weakly homogeneous of degree -2, $\Delta_+ \in E^{\mu}_{-2}(U^2)$ (Theorem 5.4.1). Here, we initialize the recursion for $t_2(\phi^m(x_1)\phi^m(x_2)) = \varepsilon \circ T(\phi^m(x_1)\phi^m(x_2))$, and prove that $\lambda^{2m}t_{2,\lambda}$ is bounded in $\mathcal{D}'_{\Gamma_2}(U^2 \setminus d_2)$ where Γ_2 is a *good* cone (recall *good* means polarized, satisfies the soft landing condition, translation and scaling invariant). We denote by $(\chi_I)_I$ the partition of unity subordinate to the cover $\left(\tilde{M}_{I,I^c}\right)_I$ given by the improved geometrical lemma.

Theorem 6.4.4 *Let* $t_2(\phi^m(x_1)\phi^m(x_2)) = \chi_1\Delta_+^m(x_1,x_2) + \chi_2\Delta_+^m(x_2,x_1)$. *Then* $t_2 \in E^{\mu}_{-2m}(U^2 \setminus d_2)$ *and there exists a* good *cone* $\Gamma_2 \subset T^\bullet U^2$ *such that for each* $\rho_{x_i}, i = (1,2)$, *the family* $\left(\lambda^{-2m}e^{\log\lambda\rho_{x_i}*}t_2\right)_{\lambda\in(0,1]}$ *is bounded in* $\mathcal{D}'_{\Gamma_2}(U^2 \setminus d_2)$.

Proof — On the one hand $WF(\Delta_+^m)$ satisfies the soft landing condition by Lemma 5.4.1 which implies $WF(t_2)|_{U^2}$ also does. On the other hand, we already proved in proposition (6.4.2) that $WF(t_2)$ is polarized then applying Lemma 6.4.6, we find that the enveloppe Γ_2 of $WF(t_2)$ is a good cone. ■

6.4.7 Pull-back of *good* cones.

Since we always pull-back distributions living on configuration spaces U^I to higher configuration spaces U^n, we want the pull-back operation to preserve all the nice properties of the wave front set. Let $p_{[n]\mapsto I}$ be the canonical projection $p_{[n]\mapsto I} : U^n \mapsto U^I$.

Lemma 6.4.7 *If* $\Gamma_I \subset T^\bullet U^I$ *is a* good *cone then* $p^*_{[n]\mapsto I}\Gamma_I \subset T^\bullet U^n$ *is also a* good *cone.*

Proof — By definition $p^*_{[n]\mapsto I}\Gamma_I$ is polarized in $T^\bullet U^n$ since the trace $Tr(x_i;\xi_i)_{i\in I} \subset T^\bullet U$ of an element $(x_i;\xi_i)_{i\in I} \in \Gamma_I$ and of its pulled back element

$$((x_i;\xi_i),(x_j;0))_{i\in I, j\in I^c} \in p^*_{[n]\mapsto I}\Gamma_I$$

are the same. $p^*_{[n]\mapsto I}\Gamma_I$ is also translation, scale invariant by invariance of Γ_I and the projection $p_{[n]\mapsto I}$. The only subtle point is to prove that $p^*_{[n]\mapsto I}\Gamma_I$ still satisfies the soft landing condition. Start from the assumption that Γ_I satisfies the soft landing condition, then for all compact $K \subset U^I$, $\exists\varepsilon > 0, \exists\delta > 0$:

$$\Gamma|_{K\cap\{\sum_{i\in I, i\neq j}|x_j - x_i|\leqslant\varepsilon\}} \subset \{|\sum_{i\in I}\xi_i| \leqslant \delta\left(\sum_{i\in I, i\neq j}|x_j - x_i|\right)\left(\sum_{i\in I, j\neq i}|\xi_i|\right)\}$$

then notice

$$(x_i;\xi_i)_{i\in[n]} \in p^*_{[n]\mapsto I}\Gamma_I \implies (x_i;\xi_i)_{i\in I} \in \Gamma_I$$

$$\implies |\sum_{i=1}^{n} \xi_i| = |\sum_{i\in I} \xi_i| \leqslant \delta \left(\sum_{i\in I, i\neq j} |x_j - x_i|\right)\left(\sum_{i\in I, i\neq j} |\xi_i|\right)$$
$$\leqslant \delta \left(\sum_{i\in [n], i\neq j} |x_j - x_i|\right)\left(\sum_{i\in [n], i\neq j} |\xi_i|\right)$$

which implies $p^*_{[n]\mapsto I}\Gamma_I \subset \{|\sum_{i=1}^n \xi_i| \leqslant \delta \left(\sum_{i\in[n],i\neq j} |x_j - x_i|\right) \left(\sum_{i\in[n],i\neq j} |\xi_i|\right)\}$ which is exactly the soft landing condition. ■

In the sequel, we denote by Γ_I the set $p^*_{[n]\mapsto I}\Gamma_I$ making a slight notational abuse.

The soft landing condition is stable by summation: We proved in Proposition 4.2.1 that for Γ_1, Γ_2 two closed conic sets which both satisfy the **soft landing condition** and s.t. $\Gamma_1 \cap -\Gamma_2 = \emptyset$, the cone $\Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$ satisfies the **soft landing condition**.

For all subsets $I \subset \{1, \dots, n\}$, let $\Lambda_I \subset T^\bullet M^I$ be the set of all elements in $T^\bullet U^I$ **polarized** by E_q^+. Since the cone E_q^+, the partial order relation $\tilde{\leqslant}$ and the trace operation are translation and dilation invariant, by Definition 6.4.2, the subset Λ_I is also translation and dilation invariant. For any manifold M, for any closed cone $\Gamma \subset T^\bullet M$ in the cotangent cone $T^\bullet M$, we denote by $\Gamma^0 = \Gamma \cup \underline{0} \subset T^\star M$ where $\underline{0}$ is the zero section of $T^\star M$.

6.4.8 The wave front set of the product t_n is contained in a good cone Γ_n.

Theorem 6.4.5 *We assume the hypothesis of theorem (6.4.3) is valid and keep the same notations. If furthermore we assume all elements $\Gamma_I, I \subsetneq \{1, \dots, n\}$ are* good *conic sets then Γ_n is a* good *conic set.*

Proof — It is immediate since translation and scale invariance, the polarization property and the soft landing conditions are stable by sums. ■

6.4.9 We define the extension $\bar{t}_n$ and control $WF(\bar{t}_n)$.

We saw in Chapter 4 that the product of distributions satisfying the Hörmander condition was bounded: let Γ_1, Γ_2 be two cones, assume $\Gamma_1 \cap -\Gamma_2 = \emptyset$. Set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$, then the product

$$(t_1, t_2) \in D'_{\Gamma_1} \times D'_{\Gamma_2} \mapsto t_1 t_2 \in D'_\Gamma$$

is well defined and bounded (Theorem 4.2.1). We also concluded Chapter 4 with a general extension theorem (4.3.3): if $t \in E_s^\mu(U^n \setminus d_n)$ then an extension $\overline{t}$ exists in $E^{\mu}_{s'}(U^n)$ for all $s' < s$. Now we prove a theorem that gives conditions for which the extension $\overline{t}_n$ exists, has finite scaling degree and has *good* wave front set.

Theorem 6.4.6 *Assume that the assumptions of Theorems (6.4.3) and (6.4.5) are satisfied and that the family $\lambda^{-s_I} e^{\log\lambda\rho_{x_i}*} t_I$ is bounded in $\mathcal{D}'_{\Gamma_I}$ for some s_I where Γ_I is good. Then t_n has a well defined extension $\overline{t}_n$ in $\mathcal{D}'_{WF(t_n)\cup(Td_n)^\perp}(U^n)$ and there is a* good *conic set Γ_n such that for any $l \in \{1, \dots, n\}$, the family $\left(\lambda^{-s'} e^{\log\lambda\rho_{x_l}*}\overline{t_n}\right)_\lambda$, is bounded in $\mathcal{D}'_{\Gamma_n\cup(Td_n)^\perp}(U^n)$ for all $s' < s_I + s_{I^c} + \sum_{(i,j)\in I\times I^c} 2m_{ij}$.*

Proof — For any $l \in \{1, \dots, n\}$, the family

$$\lambda^{-s_I} e^{\log\lambda\rho_{x_l}*} t_I$$

is bounded in $\mathcal{D}'_{\Gamma_I}$ where Γ_I is a *good* cone. Let us set

$$\Gamma_n = \bigcup_I \left(\Gamma^0_I + \Gamma^0_{I^c} + \Gamma^0_{ij}\right)|_{\tilde{M}_{I,I^c}}. \tag{6.33}$$

Then the last step of the proof is a mere repetition of the proof of Theorems (6.4.3) and (6.4.5), but instead of considering a "static" product $t_I t_{I^c} \prod_{(i,j)\in I\times I^c} \Delta_+^{m_{ij}}(x_i, x_j)\chi_I$ on a given $\tilde{M}_{I,I^c}$, we will instead scale the whole product w.r.t. to some linear Euler vector field ρ_{x_l}:

$$\underbrace{\left(\lambda^{-s_I} e^{\log\lambda\rho_{x_l}*} t_I\right)}_{\text{bounded in } \mathcal{D}'_{\Gamma_I}(U^n\setminus d_n)} \underbrace{\left(\lambda^{-s_{I^c}} e^{\log\lambda\rho_{x_l}*} t_{I^c}\right)}_{\text{in } \mathcal{D}'_{\Gamma_{I^c}}(U^n\setminus d_n)}$$

$$\prod_{(i,j)\in I\times I^c} \underbrace{\left(\lambda^{-2m_{ij}} e^{\log\lambda\rho_{x_l}*} \Delta_+^{m_{ij}}(x_i, x_j)\right)}_{\text{in } \mathcal{D}'_{\Gamma_{ij}}(U^n\setminus d_n)} \underbrace{\chi_I}_{\text{in } \mathcal{D}'_{\emptyset}(U^n\setminus d_n)}.$$

Then we use the boundedness of the product (Theorem 4.2.1) to repeat the arguments of the proof of Theorem 6.4.3 for bounded families of distributions. Notice that it is very convenient for us that the functions χ_I constructed in the improved geometric lemma are smooth scale invariant functions since they are going to be bounded in $\mathcal{D}'_{\emptyset}(U^n \setminus d_n)$. The product

$$\lambda^{-s_I - s_{I^c} - 2\sum_{(ij)\in I\times I^c} m_{ij}} e^{\log\lambda\rho_{x_l}*} \left(t_I t_{I^c} \prod_{(i,j)\in I\times I^c} \Delta_+^{m_{ij}}(x_i, x_j)\right)_{\lambda\in(0,1]}$$

is well defined and bounded in $\mathcal{D}'_{\Gamma_n}(U^n \setminus d_n)$ (by Theorem 4.2.1) where

$$\Gamma_n = \bigcup_I \left(\Gamma^0_I + \Gamma^0_{I^c} + \Gamma^0_{ij}\right) \setminus \{\underline{0}\}|_{\tilde{M}_{I,I^c}}$$

is *good* by Theorem 6.4.5. Then the distribution

$$t_n = \left(t_I t_{I^c} \prod_{(i,j)\in I\times I^c} \Delta_+^{m_{ij}}(x_i, x_j) \right)$$

is in $E^\mu_{s_n}(U^n \setminus d_n)$ (since Γ_n satisfies the soft landing condition and the family of distributions $(\lambda^{-s_n} t_\lambda)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'_{\Gamma_n}$) for $s_n = s_I + s_{I^c} + 2\sum_{(ij)\in I\times I^c} m_{ij}$. We can conclude by the extension theorem (4.3.3), which provides an extension $\bar{t}_n$ in $E^\mu_{s'}(U^n)$ for all $s' < s_I + s_{I^c} + 2\sum_{(ij)\in I\times I^c} m_{ij}$ with the constraint $WF(\bar{t}_n) \subset WF(t_n) \bigcup (Td_n)^\perp$ on the wave front set of the extension. The wave front set $WF(t_n)$ is polarized and so is the conormal $(Td_n)^\perp$ hence the union $WF(t_n)\bigcup(Td_n)^\perp$ is also polarized. And the family $\left(\lambda^{-s'}\bar{t}_n\right)_{\lambda\in(0,1]}$ should be bounded in $\mathcal{D}'_{\Gamma_n\bigcup(Td_n)^\perp}(U^n)$ where $\Gamma_n\bigcup(Td_n)^\perp$ is a *good* conic set. ■

The last theorem allows to conclude the recursion since we were able to initialize the recursion at the step $n = 2$: $WF(t_2)$ is contained in a *good cone* Γ_2 and $\lambda^{2m} e^{\rho\log\lambda *} t_2(\phi^m\phi^m)$ is always bounded in $\mathcal{D}'_{\Gamma_2}(U^2 \setminus d_2)$, however beware that $t_2(\phi^m\phi^m)$ is in $E_{s'}(U^2)$ for all $s' < 2m$, hence repeated applications of theorem (6.4.6) allows to define all extensions $\bar{t}_n \in \mathcal{D}'(U^n)$ for all n.

Chapter 7

A conjecture by Bennequin.

7.1 Parametrizing the wave front set of the extended distributions.

In this short chapter, we solve a conjecture of Daniel Bennequin stating that the wave front set of the extensions $\overline{t_n}$ are singular Lagrangian manifolds.

Lagrangians often appears in quantum mechanics as the geometrical object living in cotangent space which represents the semiclassical limit of quantum states ([6] p. 16, 35, 60-63 and [84] p. 103). Our theorem might help us to give a similar geometric interpretation of the wave front set of n-point functions in quantum field theory: each element of the Lagrangian could represents the "trajectory of a process" in cotangent space. For instance:

1. an element of the wave front set of $t_2(\phi(x)\phi(y))$ represents a null geodesic lifted to the cotangent space,
2. an element of the wave front set of $t_3(\phi(x_1)\phi(x_2)\phi^3(y)\phi(x_3))$ represents the interaction of three null geodesics intersecting at one point.

The proof also clarifies the fact that the wave front set of these extensions can be parametrized by objects (generalizing the graph of a gradient) called Morse families which were introduced by Weinstein and Hörmander.

7.2 Morse families and Lagrangians.

Let us start by recalling some simple definitions. We introduce the concept (due to Weinstein see [6] Definition 4.17) of a Morse family (with some modifications of our own):

Definition 7.2.1 *A Morse family is a triple $\mathcal{S} = (\pi : B \mapsto M, S)$ satisfying the following conditions:*

- *a)* $(\pi : B \mapsto M)$ *is such that any connected component of* B *is of the form* $\left(\mathbb{R}^k \setminus \{0\}\right) \times \Omega$ *for some* k *and some set* $\Omega \subset M$*, this endows* B *with the structure of a smooth cone and the restriction of* π *to this connected component is the canonical projection,*
- *b)* $S \in C^\infty(B)$ *is homogeneous of degree* 1 *w.r.t. vertical scaling,*
- *c)* $dS \neq 0$*.*

Daniel Bennequin pointed out to us that this definition is actually very general since B is *not necessarily connected* thus we could have several connected components of B living over some given point in M, like branches of a cover. The second nice point of the definition of Alan Weinstein is that the map π is not necessarily surjective. Denote by x the coordinates in M and by $(x;\theta)$ the coordinates in B where θ is the vertical variable. Denote by $\Sigma_S = \{\frac{\partial S}{\partial \theta} = 0\} \subset B$ the critical set of S. The smooth projection π defines a set $\pi(\Sigma_S)$ which is the projection of the critical set.

Definition 7.2.2 *We denote by* $\mathcal{T}\pi(\Sigma_S)$ *the* ***tangent cone*** *of* $\pi(\Sigma_S)$ *which is defined as follows, for* $x \in \pi(\Sigma_S)$*,*

$$\mathcal{T}_x\pi(\Sigma_S) = \{d\pi|_{(x,\theta)}(X) | \exists \gamma \in C^1([0,1], \Sigma_S) \text{ s.t. } \gamma(0) = (x,\theta), \dot{\gamma}(0) = X\},$$

then $\mathcal{T}\pi(\Sigma_S) = \bigcup_{x\in\pi(\Sigma_S)} \mathcal{T}_x\pi(\Sigma_S)$*.*

Example 7.2.1 *For* $\mathcal{S} = \left(\mathbb{R}_{>0} \times (U^2 \setminus d_2) \mapsto (U^2 \setminus d_2), \theta\Gamma(x,y)\right)$*, the set* Σ_S *is equal to* $\left(\{\Gamma = 0\} \cap \left(U^2 \setminus d_2\right)\right) \times \mathbb{R}_{>0}$ *where* $\{\Gamma = 0\}$ *is the null conoid in* $U^2 \setminus d_2$ *i.e. the subset of pairs of points connected by a null geodesic. Thus* $\pi(\Sigma_S) = \{\Gamma = 0\}|_{U^2\setminus d_2}$ *is an open submanifold and* $\mathcal{T}\pi(\Sigma_S)$ *is just the tangent space to the submanifold* $\pi(\Sigma_S) = \{\Gamma = 0\} \cap U^2 \setminus d_2$*.*

It is possible to define a notion of tangent cone for very general sets but we will not need such theory here.

Definition 7.2.3 *We denote by* $N_{\pi(\Sigma_S)}$ *the* ***normal*** *to* $\pi(\Sigma_S)$ *which is defined as the subset* $\{(x,\xi) \in T^\star M | x \in \pi(\Sigma_S), \xi\left(\mathcal{T}_x\pi(\Sigma_S)\right) \geqslant 0\} \subset T^\star M$*.*

Throughout this section, for any cone C in a vector space E, we denote by C° the cone in dual space $E^\star$ defined as $\{\xi | \xi(C) \geqslant 0\}$ (it is sometimes called the *polar* of C). This definition can be extended to cones in tangent space and we denote by $\mathcal{T}\pi(\Sigma_S)^\circ$ the subset $\bigcup\limits_{x\in\pi(\Sigma_S)} (\mathcal{T}_x\pi(\Sigma_S))^\circ$ living in $T^\bullet M$. Geometrically, $N_{\pi(\Sigma_S)}$ is the **dual cone** $\mathcal{T}\pi(\Sigma_S)^\circ$ of the tangent cone $\mathcal{T}\pi(\Sigma_S)$. If π is a smooth embedding, N_π is just the conormal bundle of $\pi(\Sigma)$.

Definition 7.2.4 *We denote by* $\lambda_{\mathcal{S}}$ *the map* $\lambda_{\mathcal{S}} : (x;\theta) \in B \mapsto (x; d_xS)(x,\theta) \in T^\star M$*.*

In nice situations, $\lambda_S(\Sigma_S)$ is a smooth Lagrange immersion and coincides with $N_{\pi(\Sigma_S)}$. However in our general situation, we always have the following upperbound:

Proposition 7.2.1 $\lambda_S(\Sigma_S) \subset N_{\pi(\Sigma_S)}$.

Proof — Any vector field in $X \in Vect(B)$ decomposes uniquely as a sum $X = X_h + X_v = f^\mu \partial_{x^\mu} + f^i \partial_{\theta^i}$ where X_h is the horizontal part and X_v the vertical part since B is a trivial cone. Thus it suffices to prove that if $d(\partial_{\theta^i} S)(X)|_{\Sigma_S} = 0$ then $dS(X_h)|_{\Sigma_S} = 0$ because $dS(X_h)_{x,\theta} = d_x S(d\pi_{x,\theta}(X))$. The key observations are:

- a) $\frac{\partial S}{\partial \theta^i} = 0 \implies \theta^i \frac{\partial S}{\partial \theta^i} = S = 0$, since S is homogeneous of degree 1 in θ, thus $\Sigma_S \subset \{S = 0\}$ and $d(\partial_{\theta^i} S)(X)|_{\Sigma_S} = 0 \implies dS(X)|_{\Sigma_S} = 0$,
- b) for all vertical vector field X_v, $dS(X_v)|_{\Sigma_S} = 0$.

From these observations, we deduce that:

$$d\left(\frac{\partial S}{\partial \theta^i}\right)(X)|_{\Sigma_S} = 0 \implies dS(X)|_{\Sigma_S} = 0 \implies dS(X)|_{\Sigma_S}$$
$$= dS(X_h)|_{\Sigma_S} + \underbrace{dS(X_v)|_{\Sigma_S}}_{=0} = 0 \implies dS(X_h)|_{\Sigma_S} = 0.$$

■

We want to prove that $\lambda_S(\Sigma_S)$ is isotropic in the sense that the tangent cone of $\lambda_S(\Sigma_S)$ is symplectic orthogonal to itself. We denote by $\mathcal{T}_p(\lambda_S(\Sigma_S))$ the subset defined as

$$\{d\lambda_S|_{(x,\theta)}(X) | \exists \gamma \in C^1([0,1], \Sigma_S) \text{ s.t. } \gamma(0) = (x,\theta), \dot{\gamma}(0) = X\},$$

and $\mathcal{T}(\lambda_S(\Sigma_S)) = \bigcup_{p \in \lambda_S(\Sigma_S)} \mathcal{T}_p(\lambda_S(\Sigma_S))$. Let ω be the natural symplectic form in $T^\star M$:

Proposition 7.2.2 $\omega|_{\lambda_{\mathcal{S}}(\Sigma_{\mathcal{S}})} = 0$.

Proof — We actually prove that $\lambda_{\mathcal{S}}^\star \omega|_{\Sigma_{\mathcal{S}}} = 0$ which implies $\omega|_{\lambda_{\mathcal{S}}(\Sigma_{\mathcal{S}})} = 0$. Let us denote by $\alpha = \xi_i dx^i \in \Omega^1(T^\star M)$ the Liouville 1-form which is the primitive of ω i.e. $d\alpha = \omega$. We decompose uniquely the differential d acting on $\Omega^\bullet(B)$ as a sum $d = d_x + d_\theta$. The key observation is that $d_\theta S|_{\Sigma_{\mathcal{S}}=0}$.

$$\lambda_{\mathcal{S}}^\star \omega|_{\Sigma_{\mathcal{S}}} = \lambda_{\mathcal{S}}^\star d\alpha|_{\Sigma_{\mathcal{S}}} = d(\lambda_{\mathcal{S}}^\star \alpha)|_{\Sigma_{\mathcal{S}}} = d\left(\lambda_{\mathcal{S}}^\star \xi_i dx^i\right)|_{\Sigma_{\mathcal{S}}}$$
$$= d\left(\frac{\partial S}{\partial x^i} dx^i\right)|_{\Sigma_{\mathcal{S}}} = d(d_x S)|_{\Sigma_{\mathcal{S}}} = d(d_x S + d_\theta S)|_{\Sigma_{\mathcal{S}}}$$

since $d_\theta S|_{\Sigma_{\mathcal{S}}} = 0$

$$= d^2 S|_{\Sigma_{\mathcal{S}}} = 0.$$

■

This means that $\lambda_S(\Sigma_S)$ is **isotropic**. At each point $x \in \pi(\Sigma_S)$ where $\lambda_S(\Sigma_S)|_x = N_{\pi_S(\Sigma_S),x}$ we will say that $\lambda_S(\Sigma_S)$ is **Lagrangian** at x because it is **isotropic** of maximal dimension. If it is **Lagrangian** at every $x \in \pi(\Sigma_S)$ (or on an open dense subset of $\pi(\Sigma_S)$) then we call it **Lagrangian**, in nice situations this coincides with the usual notion of Lagrange immersion (see [40] vol 3 p. 291,292 and [6]). We will later consider Morse families $\mathcal{S}$ with the supplementary requirements that $\Sigma_S \subset B$ is a finite union of smooth submanifolds and $\lambda_S(\Sigma_S)$ is **Lagrangian**.

We work out a fundamental example of Morse family which generates the conormal bundle of a submanifold.

Example 7.2.2 *Let $I \subset M$ be a submanifold. We shall work in local chart where the manifold is given by a system of d equations $f_1 = \cdots = f_d = 0$. Then the Morse triple $((\mathbb{R}^d \setminus \{0\}) \times M \mapsto M, \sum_{i=1}^d \theta^i f_i)$ parametrizes the conormal bundle $(TI)^\perp$. Indeed, $\Sigma_S = \{f_i = 0\} \times (\mathbb{R}^d \setminus \{0\}) = I \times (\mathbb{R}^d \setminus \{0\})$ and $\lambda_S(\Sigma_S) = \{\theta_i df_i|_{\Sigma_S}, \theta \in \mathbb{R}^d \setminus \{0\}\}$. The key observation is that any element in the conormal of I should decompose in the basis of 1-forms $(df_i)_i$ thus $\lambda_S(\Sigma_S)$ parametrizes the conormal of I.*

An analytic interpretation of $\lambda_S(\Sigma_S)$. We interpret $\lambda_S(\Sigma_S)$ in terms of the wave front set of an oscillatory integral t. We can understand it as a parametrization of $WF(t)$ by the Morse family $\mathcal{S}$.

Proposition 7.2.3 *Let $\mathcal{S} = (\pi : M \times \mathbb{R}^k \mapsto M, S)$ be a Morse family over the manifold M and $(x;\theta)$ where $\theta \in \mathbb{R}^k$ a system of coordinates in $M \times \mathbb{R}^k$, for any asymptotic symbol a ([67] vol 2 p. 99):*

$$WF\left(\int_{\mathbb{R}^k} d\theta a(\cdot;\theta) e^{iS(\cdot,\theta)}\right) \subset \lambda_{\mathcal{S}} \Sigma_S.$$

Proof — In local coordinates (x, θ) for B, it is just a consequence of Theorem 9.47, p. 102 in [67]. ■

Functorial behaviour of Morse families. In microlocal geometry, we need the following fundamental operations on distributions

- the pull-back $t \mapsto f^\star t$ by a smooth map $f : M \to N$ which is not always well defined for distributions
- the exterior tensor product $(t_1, t_2) \mapsto t_1 \boxtimes t_2$ which is always well defined
- for our purpose, it will be important to add the product of distributions when it is well defined.

Assume that the wave front sets of given distributions t are parametrized by Morse families, we already know how the wave front sets transform under these functorial operations on distributions, the question is whether we can find a new Morse family to parametrize the wave front set of the distribution obtained by one of the previous operations. The functorial behaviour of Lagrangians under geometric transformations is already studied in [33] Chapter 4, however it is not described in terms of generating functions and our point of view is more explicit and more oriented towards applications.

Formal operations on Morse families.

First introduce operations on cones as follows. Let $B \mapsto M$ be a smooth cone, for any smooth map $f : N \mapsto M$, $f^\star B \mapsto f^\star M$ is a smooth cone (Appendix 2 of [33]) with fibers defined as follows $f^\star B|_x = B|_{f(x)}$. We also introduce a suitable generalization of the fiber product for cones, recall the fiber product of $\pi_1 : B_1 \mapsto M$ and $\pi_2 : B_2 \mapsto M$ denoted by $B_1 \times_M B_2$ is defined by $\{(p_1, p_2) \in B_1 \times B_2 | \pi_1(p_1) = \pi_2(p_2)\}$.

Definition 7.2.5 *Let B_1, B_2 be two smooth cones over a given base manifold M. Then we define the product $B_1 \overline{\times}_M B_2$ as the cone*

$$((B_1 \cup \underline{0}_1) \times_M (B_2 \cup \underline{0}_2)) \backslash (\underline{0}_1 \times_M \underline{0}_2) = (B_1 \times_M \underline{0}_2) \cup (\underline{0}_1 \times_M B_2) \cup (B_1 \times_M B_2)\,.$$

The key point of this product is that we add the zero section so that our trivial cones become trivial vector bundles we compute the fiber product and remove the zero section at the end.

The QFT case. In our recursion, we only need to pull-back by smooth projections. For instance, by the canonical projection maps $M^n \mapsto M^I$ for $I \subset [n]$. In this case, if we still denote by f the submersion $f : N \mapsto M$, the Morse family can be chosen extremely simple

Definition 7.2.6 *Let $\mathcal{S} = (\pi : B \mapsto M, S)$ be a Morse family over the manifold M, for any smooth **projection** $f : N \mapsto M$, we define the pulled back Morse family as the triple*

$$f^\star \mathcal{S} = (f^\star \pi : f^\star B \mapsto f^\star M, f^\star S). \tag{7.1}$$

It is obvious that $df^\star S \neq 0$ since $dS \neq 0$ and df is **surjective**. When f is a smooth map, we prove that the pull-back by f of $\lambda_{\mathcal{S}} \Sigma_{\mathcal{S}}$ is parametrized by the Morse family $f^\star \mathcal{S}$:

Proposition 7.2.4 *Let $f := N \mapsto M$ be a smooth projection and $\mathcal{S} = (\pi : B \mapsto M, S)$ a Morse family over the manifold M. Then:*

$$f^\star \lambda_{\mathcal{S}} \Sigma_{\mathcal{S}} = \lambda_{f^\star \mathcal{S}} \Sigma_{f^\star \mathcal{S}}. \tag{7.2}$$

Proof — We denote by $(y;\eta)$ the coordinates in $T^\star N$ and $(x;\xi)$ the coordinates in $T^\star M$. We have

$$f^*(\lambda_S\Sigma_S) = \{(y;\eta\circ df)|(f(y);\eta)\in\lambda_{\mathcal{S}}\Sigma_{\mathcal{S}}\}$$

by the definition of pull-back in [40] and [33]

$$\begin{aligned} &= \{(y; d_x S_{(f(y);\theta)}\circ df)|\, d_\theta S(f(y);\theta)=0\} \\ &= \{(y; d\,(S\circ f)_{(y;\theta)}\,|\, d_\theta\,(S\circ f)\,(y;\theta)=0\} \\ &= \lambda_{f^*\mathcal{S}}\Sigma_{f^*\mathcal{S}} \end{aligned}$$

by definition of $\lambda_{f^*\mathcal{S}}\Sigma_{f^*\mathcal{S}}$. ■

Proposition 7.2.5 *Under the assumptions of proposition (7.2.4), if $\lambda_S(\Sigma_S)$ is Lagrangian then $\lambda_{f^\star S}\Sigma_{f^\star S}$ is Lagrangian.*

Proof —

$$\begin{aligned} \lambda_{f^\star\mathcal{S}}\Sigma_{f^\star\mathcal{S}} &= f^\star\lambda_{\mathcal{S}}\Sigma_{\mathcal{S}} \text{ by the above proposition} \\ &= f^\star N_{\pi(\Sigma_S)} \text{ because } \lambda_{\mathcal{S}}\Sigma_{\mathcal{S}} \text{ Lagrangian} \\ &= N_{\pi(\Sigma_S)}\circ df \text{ by definition of the pull-back} \\ &= \mathcal{T}\pi(\Sigma_{f^\star S})^\circ\circ df \text{ by definition of } N_{\pi(\Sigma_S)} \\ &= \mathcal{T}\pi_{f^\star S}(\Sigma_{f^\star S})^\circ \text{ since } \mathcal{T}\pi_S(\Sigma_S) = Df\mathcal{T}\pi_{f^\star S}(\Sigma_{f^\star S}) \\ &= N_{\pi(\Sigma_{f^\star S})} \text{ by definition of } N_{\pi(\Sigma_{f^\star S})}. \end{aligned}$$

Finally, $\lambda_{f^\star\mathcal{S}}\Sigma_{f^\star\mathcal{S}} = N_{\pi(\Sigma_{f^\star S})}$ means, by definition, that $\lambda_{f^\star\mathcal{S}}\Sigma_{f^\star\mathcal{S}}$ is Lagrangian. ■

Proposition 7.2.6 *Under the assumptions of proposition (7.2.4), if Σ_S is a smooth submanifold (resp finite union of smooth submanifolds) in B then $\Sigma_{f^\star S}$ is also a smooth submanifold (resp finite union of smooth submanifolds) in $f^\star B$.*

Proof — This is immediate since $d_{y;\theta}(d_\theta(S\circ f))$ has the same rank as $d_{x,\theta}S$. ■

Let $\mathcal{S}_i = (\pi_i : B_i \mapsto M, S_i), i = (1,2)$ be a pair of Morse families over the manifold M, then we define the "sum of the Morse families" $\mathcal{S}_1+\mathcal{S}_2$ as the triple

$$\mathcal{S}_1+\mathcal{S}_2 = (\pi_1\overline{\times}_M\pi_2 : B_1\overline{\times}_M B_2 \mapsto M, S_1+S_2). \tag{7.3}$$

We put quotation marks "" to stress the fact that this operation still defines a triple (cone, base manifold, function) but this triple is not necessarily a Morse family since we do not know if $d(S_1+S_2)\neq 0$, we will see that a necessary and sufficient condition for $\mathcal{S}_1+\mathcal{S}_2$ to be a Morse family is that $\lambda_{S_1}\Sigma_{S_1}\cap -\lambda_{S_2}\Sigma_{S_2}=\emptyset$ which is the Hörmander condition.

Remark on sums of Morse families. Notice by definition that if the cone $B_i, i = (1,2)$ corresponding to the Morse family $\mathcal{S}_i$ has n_i connected components, then $B_1 \overline{\times}_M B_2$ has $(n_1+1)(n_2+1)-1$ connected components. An immediate recursion yields that the cone corresponding to the sum $\mathcal{S}_1 + \cdots + \mathcal{S}_k$ has $((n_1+1)\dots(n_k+1)) - 1$ connected components.

Transversality lemmas.

We recall the classical notion of transversality in differential geometry in our context (see [47] Definition 2.48 p. 80). Let $\Sigma_i, i = (1,2)$ be a pair of smooth manifolds and $\pi_i : \Sigma_i \mapsto M, i = (1,2)$ be a pair of smooth maps. In such case for every $x \in \pi_i(\Sigma_i)$, the tangent cones $\mathcal{T}_x\pi_i(\Sigma_i), i = (1,2)$ are **vector subspaces** of T_xM (a vector subspace has less structure than a cone).

Definition 7.2.7 *π_1 and π_2 are called* ***transverse*** *if for all $x \in \pi_1(\Sigma_1) \cap \pi_2(\Sigma_2)$, $\mathcal{T}_x\pi_1(\Sigma_1) + \mathcal{T}_x\pi_2(\Sigma_2) = T_xM$.*

Lemma 7.2.1 *Let $\Sigma_i, i = (1,2)$ be a pair of smooth submanifolds in B_i and $\pi_i : B_i \mapsto M, i = (1,2)$ be a pair of smooth maps. If π_1 and π_2 are* ***transverse*** *then $\Sigma_1 \times_M \Sigma_2$ is a smooth submanifold in $B_1 \times_M B_2$.*

Lemma 7.2.1 obviously generalizes to the case Σ_i is a finite union of submanifolds, in which case $\Sigma_1 \times_M \Sigma_2$ is a finite union of submanifolds.
Proof — Denote by Δ the diagonal in $M \times M$. Then $B_1 \times_M B_2$ can be identified with the inverse image $(\pi_1 \times \pi_2)^{-1}(\Delta) = B_1 \times_\Delta B_2 \subset B_1 \times B_2$ which is always a submanifold of $B_1 \times B_2$ and the fiber product $\Sigma_1 \times_M \Sigma_2$ is just the intersection $(\Sigma_1 \times \Sigma_2) \bigcap (B_1 \times_\Delta B_2)$ in $B_1 \times B_2$. So we view both $\Sigma_1 \times \Sigma_2$ and $B_1 \times_\Delta B_2$ as submanifolds sitting inside $B_1 \times B_2$, a sufficient condition for $(\Sigma_1 \times \Sigma_2) \bigcap (B_1 \times_\Delta B_2)$ to be a submanifold of $B_1 \times_\Delta B_2$ is that the intersection is transverse (it is a classical result of transversality theory that the transversal intersection of two submanifolds is a submanifold of the two initial submanifolds, it is a particular case of Theorem 2.47 in [47] for an embedding also see Theorem 3.3 p. 22 in [38]). It is immediate to check that at every point (p_1, p_2) of the intersection $(\Sigma_1 \times \Sigma_2) \bigcap (B_1 \times_\Delta B_2)$, $T_{p_1,p_2}(\Sigma_1 \times \Sigma_2) + T_{p_1,p_2}(B_1 \times_\Delta B_2) = T_{p_1,p_2}(B_1 \times B_2)$ since $D(\pi_1 \times \pi_2)(\Sigma_1 \times \Sigma_2) = T_x\Delta$ by transversality of $\pi_1(\Sigma_1), \pi_2(\Sigma_2)$ and $T_{p_1,p_2}(B_1 \times_\Delta B_2)$ spans the vertical tangent space of the bundle $B_1 \times B_2$. ■

For each smooth map $\pi : \Sigma_S \mapsto M$, we recall the definition of the normal to $\pi(\Sigma)$: $N_{\pi(\Sigma)} \subset TM$ as the subset $\bigcup_{x \in \pi(\Sigma)} \mathcal{T}_x\pi(\Sigma)^\circ$ in $T^\star M$ which is the dual cone in cotangent space of the tangent cone $\mathcal{T}\pi(\Sigma)$. We set $N^\bullet_{\pi(\Sigma)} = N_{\pi(\Sigma)} \cap T^\bullet M$.

Lemma 7.2.2 *Assume $\Sigma_i, i = (1,2)$ are smooth manifolds and $\pi_i : \Sigma_i \mapsto M$ are smooth maps, then π_1, π_2 are* ***transverse*** *if and only if $N^\bullet_{\pi_1} \cap -N^\bullet_{\pi_2} = \emptyset$.*

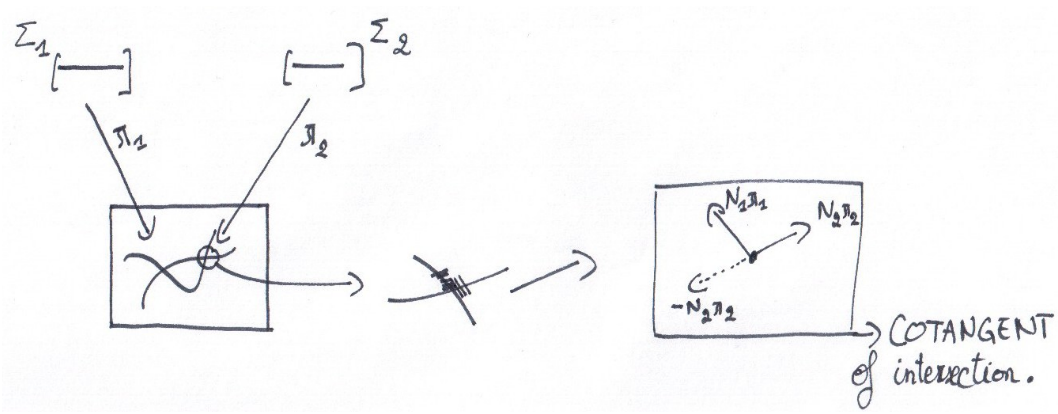

Figure 7.1: Transverse intersection of curves and their conormals.

Lemma 7.2.2 obviously generalizes to the case Σ_i is a finite union of submanifolds, in which case every submanifold in Σ_1 shall be transverse to any submanifold of Σ_2.
Proof — To prove the lemma, we just work infinitesimally. We fix a pair $(p_1, p_2) \in \Sigma_1 \times \Sigma_2$ such that $\pi_1(p_1) = \pi_2(p_2) = x$. π_1 and π_2 are transverse at $x \in M$ implies by definition that $\mathcal{T}_x\pi_1(\Sigma_1) + \mathcal{T}_x\pi_2(\Sigma_2) = T_xM$. Then by a classical result in the duality theory of cones,

$$\{0\} = \overline{T_xM}^\circ = \overline{\mathcal{T}_x\pi_1(\Sigma_1) + \mathcal{T}_x\pi_2(\Sigma_2)}^\circ$$

$$= \mathcal{T}_x\pi_1(\Sigma_1)^\circ \cap \mathcal{T}_x\pi_2(\Sigma_2)^\circ = N_{\pi_1} \cap -N_{\pi_2}.$$

∎

We illustrate the last lemma in the figure (7.1) for the case of two curves intersecting transversally in the plane and we represent the corresponding spaces N_{π_i}. The meaning of this lemma is that the condition $N^\bullet_{\pi_1} \cap -N^\bullet_{\pi_2} = \emptyset$ of Hörmander generalizes the classical differential geometric transversality when Σ_i are not necessarily smooth submanifolds in B_i.

Proposition 7.2.7 *Let $\mathcal{S}_i = (\pi_i : B_i \mapsto M, S_i), i = (1,2)$ be a pair of Morse families over the manifold M. If $\lambda_{\mathcal{S}_1}(\Sigma_{\mathcal{S}_1}) \bigcap (-\lambda_{\mathcal{S}_2}(\Sigma_{\mathcal{S}_2})) = \emptyset$, then $(\lambda_{\mathcal{S}_1}(\Sigma_{\mathcal{S}_1}) + \lambda_{\mathcal{S}_2}(\Sigma_{\mathcal{S}_2})) \cup \lambda_{\mathcal{S}_1}(\Sigma_{\mathcal{S}_1}) \cup \lambda_{\mathcal{S}_2}(\Sigma_{\mathcal{S}_2})$ is parametrized by the Morse family $\mathcal{S}_1 + \mathcal{S}_2 = (\pi_1 \overline{\times}_M \pi_2 : B_1 \overline{\times}_M B_2 \mapsto M, S_1 + S_2)$.*

Proof — It is sufficient to find the Morse family parametrizing $\lambda_{\mathcal{S}_1}(\Sigma_{\mathcal{S}_1}) + \lambda_{\mathcal{S}_2}(\Sigma_{\mathcal{S}_2})$. We will make some local computation in coordinates where we assume w.l.o.g. that B_i is equal to the cartesian product $M \times \Theta_i$ with coordinates (x, θ_i) where Θ_i is a vector space with the origin removed. Let us consider the Morse family $(\pi_1 \times_M \pi_2 : B_1 \times_M B_2 \mapsto M, S_1 + S_2)$, where we use the local coordinates $(x;\theta_1,\theta_2)$ for $B_1\times_M B_2$. Then the critical set of this Morse family is by definition $\{d_{\theta_1,\theta_2}(S_1+S_2)=0\}=\{d_{\theta_1}S_1=0\}\cap\{d_{\theta_2}S_2=0\}=\Sigma_{\mathcal{S}_1}\times_M\Sigma_{\mathcal{S}_2}\subset B_1\times_M B_2$, and the image of this subset by $\lambda_{S_1+S_2}$ is given by

$$\lambda_{S_1+S_2}\left(\Sigma_{\mathcal{S}_1}\times_M\Sigma_{\mathcal{S}_2}\right)=\left\{\left(x;d_x\left(S_1+S_2\right)\right)(x;\theta)|d_{\theta_1}S_1=0,d_{\theta_2}S_2=0\right\}$$

$$=\left\{\left(x;d_xS_1+d_xS_2\right)|(x;\theta_1,\theta_2)\in\Sigma_{\mathcal{S}_1}\times_M\Sigma_{\mathcal{S}_2}\right\}=\lambda_{\mathcal{S}_1}\Sigma_{\mathcal{S}_1}+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right),$$

which proves $(\pi_1\times_M\pi_2:B_1\times_M B_2\mapsto M,S_1+S_2)$ parametrizes $\lambda_{\mathcal{S}_1}\Sigma_{\mathcal{S}_1}+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)$, thus if we add all other components, $\lambda_{\mathcal{S}_1}\Sigma_{\mathcal{S}_1}+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)\cup\lambda_{\mathcal{S}_1}\Sigma_{\mathcal{S}_1}\cup\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)$ is parametrized by the family $\mathcal{S}_1+\mathcal{S}_2=(\pi_1\overline{\times}_M\pi_2:B_1\overline{\times}_M B_2\mapsto M,S_1+S_2)$.

It remains to prove that $d(S_1+S_2)\neq 0$ in $B_1\times_M B_2$. If both $d_{\theta_1}S_1(x;\theta_1)=0$ and $d_{\theta_2}S_2(x;\theta_2)=0$ then necessarily $d_x(S_1+S_2)(x;\theta_1,\theta_2)\neq 0$ since $\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)\bigcap-\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)=\emptyset$. ■

For the moment our results and statements are for general Morse families and we did not assume $\lambda_S\left(\Sigma_S\right)$ was Lagrangian (recall Lagrangian means $\lambda_S\left(\Sigma_S\right)=N_{\pi(\Sigma_S)}$ for us) nor that the critical set Σ_S was a finite union of submanifolds.

Proposition 7.2.8 *Under the assumptions of Proposition 7.2.7, if* $\left(\lambda_{S_i}\left(\Sigma_{S_i}\right)\right)_{i=(1,2)}$ *are* ***Lagrangians*** *then* $\lambda_{\mathcal{S}_1+\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_1+\mathcal{S}_2}\right)$ *is Lagrangian.*

Proof — One can check from the definitions that $\mathcal{T}((\pi_1\times_M\pi_2)(\Sigma_1\times_M\Sigma_2))=\mathcal{T}(\pi_1\Sigma_1)\cap\mathcal{T}(\pi_2\Sigma_2)$. Hence by linear algebra,

$$N_{(\pi_1\times_M\pi_2)(\Sigma_1\times_M\Sigma_2)}=\mathcal{T}((\pi_1\times_M\pi_2)(\Sigma_1\times_M\Sigma_2))^\circ=(\mathcal{T}(\pi_1\Sigma_1)\cap\mathcal{T}(\pi_2\Sigma_2))^\circ$$

$$=\overline{(\mathcal{T}(\pi_1\Sigma_1))^\circ+(\mathcal{T}(\pi_2\Sigma_2))^\circ}=\overline{N_{\pi_1(\Sigma_1)}+N_{\pi_2(\Sigma_2)}}=\overline{\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)},$$

finally $N_{(\pi_1\times_M\pi_2)(\Sigma_1\times_M\Sigma_2)}=\overline{\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)}$ means that

$$N_{(\pi_1\times_M\pi_2)(\Sigma_1\overline{\times_M}\Sigma_2)}$$

$$=N_{(\pi_1\times_M\pi_2)(\Sigma_1\times_M\Sigma_2)}\cup N_{(\pi_1\times_M\pi_2)(\Sigma_1\times_M\underline{0}_2)}\cup N_{(\pi_1\times_M\pi_2)(\underline{0}_1\times_M\Sigma_2)}$$

$$=\overline{\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)}\cup\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)\cup\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)$$

$$=\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)+\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)\cup\lambda_{\mathcal{S}_1}\left(\Sigma_{\mathcal{S}_1}\right)\cup\lambda_{\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_2}\right)$$

$$=\lambda_{\mathcal{S}_1+\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_1+\mathcal{S}_2}\right),$$

which by definition means $\lambda_{\mathcal{S}_1+\mathcal{S}_2}\left(\Sigma_{\mathcal{S}_1+\mathcal{S}_2}\right)$ is Lagrangian. ■

Proposition 7.2.9 *If under the assumptions of Proposition (7.2.8), each* Σ_{S_i} *is a finite union of smooth submanifolds in* B_i *then* $\Sigma_{\mathcal{S}_1} \times_M \Sigma_{\mathcal{S}_2}$ *is a finite union of smooth submanifolds of* $B_1 \times_M B_2$*.*

Proof — It suffices to recognize that the assumption $\lambda_{\mathcal{S}_1}(\Sigma_{\mathcal{S}_1}) \bigcap -\lambda_{\mathcal{S}_2}(\Sigma_{\mathcal{S}_2}) = \emptyset$ is equivalent to $N^\bullet_{\pi_1(\Sigma_{S_1})} \cap -N^\bullet_{\pi_2(\Sigma_{S_2})} = \emptyset$ (by our definition of being Lagrangian) which implies the transversality of the two maps $\pi_1 : \Sigma_{\mathcal{S}_1} \mapsto M$, $\pi_2 : \Sigma_{\mathcal{S}_2} \mapsto M$ by lemma (7.2.2), which means by application of lemma (7.2.1) that the fiber product $\Sigma_{\mathcal{S}_1} \times_M \Sigma_{\mathcal{S}_2}$ is a finite union of smooth submanifolds of $B_1 \times_M B_2$. ■

To summarize all the results we proved if t_1 and t_2 are distributions whith wave front set $WF(t_i)$ parametrized by the Morse family $\mathcal{S}_i$ and $(\lambda_{S_i}(\Sigma_{S_i}))_{i=(1,2)}$ satisfy the Hörmander condition $\lambda_{S_1}(\Sigma_{S_1}) \cap -\lambda_{S_2}(\Sigma_{S_2}) = \emptyset$ then the distributional product $t_1 t_2$ makes sense and has wave front set contained in the set $\lambda_{\mathcal{S}_1+\mathcal{S}_2}(\Sigma_{\mathcal{S}_1+\mathcal{S}_2})$ parametrized by the Morse family $\mathcal{S}_1 + \mathcal{S}_2$. Furthermore, we proved that if $(\lambda_{S_i}\Sigma_{S_i})_{i=(1,2)}$ are Lagrangians and $(\Sigma_{S_i})_{i=(1,2)}$ are finite union of smooth submanifolds then the same properties hold for the Morse family $\mathcal{S}_1 + \mathcal{S}_2$. If $f : N \mapsto M$ is a smooth submersion and $t \in \mathcal{D}'(M)$ whith wave front set $WF(t)$ parametrized by the Morse family $\mathcal{S}$ then the pull-back $f^\star t$ makes sense and has wave front set contained in the set $\lambda_{f^\star \mathcal{S}} \Sigma_{f^\star \mathcal{S}}$ parametrized by the Morse family $f^\star \mathcal{S}$. Furthermore, we proved that if $\lambda_S \Sigma_S$ is Lagrangian and Σ_S is a finite union of smooth submanifolds then the same properties hold for the Morse family $f^\star \mathcal{S}$.

Theorem 7.2.1 *Let* $\bar{t}_n$ *be the distributions defined by the recursion theorem. Then* $WF(\bar{t}_n)$ *is parametrized by a Morse family and is a union of smooth Lagrangian manifolds.*

Proof — We use the notation and formalism of the section 3 in Chapter 5. To inject this condition in our recursion theorem, it will be sufficient to check that $WF(\Delta_+)|_{C_i}, i \in \{1, 2\}$ or equivalently $WFt_2(\phi(x)\phi(y))|_{U^2 \setminus d_2}$ and all conormal bundles $(Td_I)^\perp$ are parametrized by Morse families. For $t_2(\phi(x)\phi(y))$, by Theorem 5.3.1 of Chapter 5 and causality,

$$WF(t_2(\phi(x)\phi(y))) = WF(\Delta_+(x, y))|_{x \geqslant y} \cup WF(\Delta_+(y, x))|_{y \geqslant x}$$

$$= \text{conormal } \{\Gamma = 0\} \cap \{(x, y; \xi, \eta) | (x^0 - y^0)\eta^0 > 0\}.$$

Thus we can write the Morse family in a local chart $U^2 \setminus d_2$:

$$\mathcal{S} = \left(\mathbb{R}_{>0} \times (U^2 \setminus d_2) \mapsto (U^2 \setminus d_2), \theta\Gamma(x, y)\right)$$

and the fact that it parametrizes $WF(t_2)$ results from the fact that:

$$\{(x, y; \theta d_x\Gamma, \theta d_y\Gamma | \Gamma(x, y) = 0, \theta > 0\} = \text{conormal } \{\Gamma = 0\} \cap \{(x, y; \xi, \eta) | (x^0 - y^0)\eta^0 > 0\}.$$

Furthermore the critical set

$$\Sigma_S = \{(x, y) \in U^2 \setminus d_2 | \Gamma(x, y) = 0\}$$

is a smooth submanifold and $\lambda_S (\Sigma_S) \subset T^\bullet(U^2 \setminus d_2)$ is **Lagrangian**. Also for the conormal of the diagonals, it was already treated in our examples, they can always be generated by Morse families. Then we inject these hypotheses in the recursion and we easily get the result. ■

Example 7.2.3 *In order to illustrate the mechanism at work, we choose to study the example of the wave front set of the product*

$$\delta_{x^1=0}\delta_{x^2=0}\delta_{x^3=0}(x^1, x^2, x^3)$$

of three delta functions $\delta_{x^i=0}, i = (1, 2, 3)$ in $\mathbb{R}^3$. Each $\delta_{x^i=0}$ is supported on the hyperplane $x^i = 0$. One should have in mind the boundary of a cube in a small neighborhood of one vertex ! Each $\delta_{x^i=0}$ has wave front set equal to the conormal bundle of the corresponding face $x^i = 0$ of a cube, which is parametrized by the Morse family

$$\mathcal{S}_i = \left((\theta_i; x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3 \mapsto x \in \mathbb{R}^3, S_i(x, \theta_i) = x^i \theta_i\right).$$

We represented in the figure some vectors $\nabla_x S_i$ standing for the momentum component of the conormal of the face $x^i = 0$. When two faces F_i, F_j are adjacent to an edge $F_i \cap F_j$, the convex sum of the wave front sets supported over the edge is the conormal of the edge (represented in the figure as a tube*) which is parametrized by the Morse family*

$$\Big((\theta_i, \theta_j; x) \in (\mathbb{R} \setminus \{0\})^2 \times \mathbb{R}^3 \mapsto x \in \mathbb{R}^3, (S_i + S_j)(x, \theta_i, \theta_j) = x^i \theta_i + x^j \theta_j\Big).$$

Finally the origin is a vertex adjacent to all faces and the wave front set over $(0, 0, 0)$ is parametrised by

$$\Big((\theta_1, \theta_2, \theta_3; x) \in (\mathbb{R} \setminus \{0\})^3 \times \mathbb{R}^3 \mapsto x \in \mathbb{R}^3, (S_1 + S_2 + S_3) = x^1\theta_1 + x^2\theta_2 + x^3\theta_3\Big),$$

and represents the conormal at the origin (represented in the figure as the sphere). In total, the Wavefronset has seven smooth components indexed by the strata of the cube boundary: (3 faces, 3 edges, 1 vertex). The reader can check that the wave front set of $\delta_{x_1=0}\delta_{x_2=0}\delta_{x_3=0}(x_1, x_2, x_3)$ is parametrized by the Morse family $\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$ (all seven cases are covered since by definition the sum of Morse families "contains zero sections") which is equal to

$$\{\pi : \left(\mathbb{R}^3 \setminus \{0, 0, 0\}\right) \times \mathbb{R}^3 \mapsto \mathbb{R}^3, S(x; \theta) = x^1\theta_1 + x^2\theta_2 + x^3\theta_3\}.$$

The morality of this example is that the conormal of a union of manifolds ***is not the union of the conormals!*** *One should take into account the informations contained in the "strata" and our formalism does it for the most elementary example.*

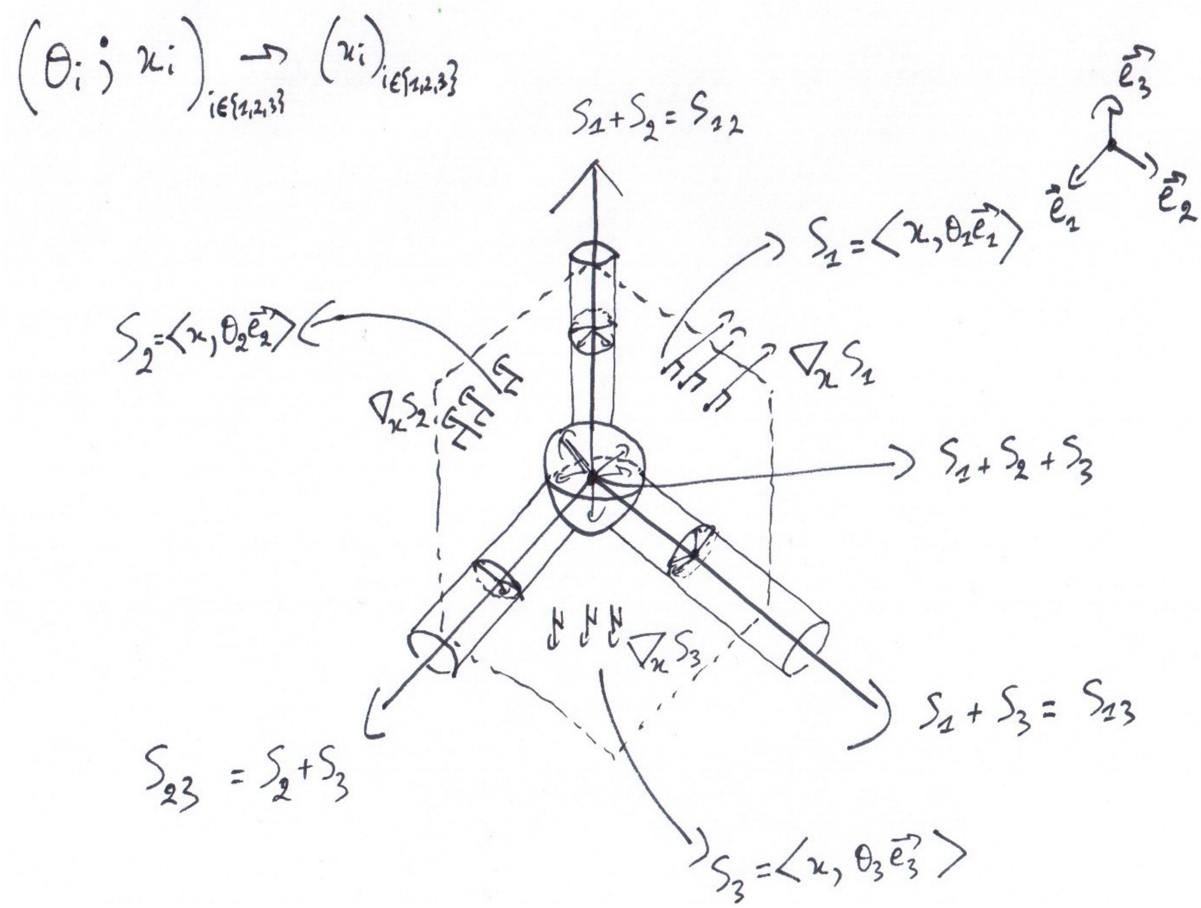

Figure 7.2: The wave front set of $\delta_{x_1=0}\delta_{x_2=0}\delta_{x_3=0}$ as a union of 7 Lagrange immersions.

7.3 A conjectural formula.

We conjecture a formula which should give an upper bound of the wave front set of any Feynman amplitude corresponding to a Feynman diagram Γ.

Let Γ be a graph with n vertices which are indexed by $[n]$. Let $E(\Gamma)$ denote the set of edges of Γ, to each element $e \in E(\Gamma)$ corresponds a unique injective map $e : \{1,2\} \mapsto [n]$ s.t. the edge e connects the vertices $e(1)$ and $e(2)$. To Γ, we associate the Morse family

$$\left(\pi : \left(\mathbb{R}_{\geqslant 0}^{E(\gamma)} \times \left(\mathbb{R}^d\right)^{\frac{n(n-1)}{2}} \times U^n\right) \setminus (\underline{0} \cup \{dS = 0\}) \mapsto U^n,\, S\right) \tag{7.4}$$

$$\pi : (\tau_e)_e, (\theta_{ij})_{ij}, (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n) \tag{7.5}$$

$$S = \sum_{e \in E(\Gamma)} \tau_e \Gamma(x_{e(1)}, x_{e(2)}) + \sum_{1 \leqslant i < j \leqslant n} \theta_{ij}.(x_i - x_j). \tag{7.6}$$

We conjecture that this Morse family parametrizes the wave front set of the Feynman amplitude corresponding to Γ.
We also conjecture that the wave front set of all n-point functions t_n are contained in the set parametrized by the Morse family:

$$\left(\pi : \left(\mathbb{R}_{\geqslant 0}^{\frac{n(n-1)}{2}} \times \left(\mathbb{R}^d\right)^{\frac{n(n-1)}{2}} \times U^n\right) \setminus (\underline{0} \cup \{dS = 0\}) \mapsto U^n,\, S\right) \tag{7.7}$$

$$\pi : (\tau_{ij}), (\theta_{ij})_{ij}, (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n) \tag{7.8}$$

$$S = \sum_{1 \leqslant i < j \leqslant n} \tau_{ij} \Gamma(x_i, x_j) + \theta_{ij}.(x_i - x_j). \tag{7.9}$$

Chapter 8

Anomalies and residues.

8.1 Introduction.

The plan of the chapter. First, we will generalize the notion of weak homogeneity of Yves Meyer [53] to the setting of currents, then show how the results of Chapter 1 naturally transfer to this new setting. However, we need to discuss the notion of Taylor expansion for test forms to give a suitable meaning to the notions of Taylor polynomial and Taylor remainder of a test form. We spend some time to discuss the notion of currents supported on a submanifold I and their *representation* in the current theoretic setting. Following physics terminology, we will call local counterterms the currents supported on I: actually in the causal approach to QFT, all ambiguities of the renormalization schemes can be described by **local counterterms**, more precisely the difference between two renormalizations is a current supported on I.

One natural example of ambiguity originates from the work of Yves Meyer [53]. We call R the composite operation of restriction of a distribution defined on M to $M \setminus I$ followed by any extension operation. We explain why this operation differs from the identity because of the non-uniqueness of the extension procedure. We describe explicitely the ambiguity of this operation R by giving an explicit formula for $T - RT$ and we show that this difference is a **local counterterm**. We give an interpretation of this ambiguity in terms of the notion of "generalized moment" for currents.

Then we will describe the dependance of the regularization operator R defined in Chapter 1, that might be called the *Hadamard regularization operator*, on the choice of bump function χ (which is equal to 1 in a neighborhood of I) and the choice of Euler vector field ρ. Without surprise, we will prove that a change in the function χ or the vector field ρ will result in a change of R by a **local counterterm**, these are explicit ambiguities. In QFT, a fundamental question is to ask if the symmetries or the exactness of currents can be preserved by the renormalization scheme. However since all continuous symmetries of QFT can be encoded by Lie algebras of vector fields it is natural to wonder if the Lie derivatives commute with the renormalization R. The symmetry is not always preserved and the quantity which measures this defect will be called **residue** of T. In the following, $\mathfrak{Res}$ is defined by generalizing Griffiths–Harris's definition ([36] p. 368) by the chain homotopy equation

$$dRT - RdT = \mathfrak{Res}[T] \tag{8.1}$$

and is a **local counterterm**. However $\mathfrak{Res}$ is a special type of counterterm since $\mathfrak{Res}$ is always closed in $\mathcal{D}'(M)$ and is exact when T is closed. We show that the regularization techniques of Meyer allows us to extend the notion of residues in the sense of Griffiths–Harris (see the section 3 in [36]) and our resulting definition has nothing to do with complex analysis. The residue in [36] is only well defined for *functions* $T \in L^q_{loc}(\mathbb{R}^n)$ ([36] p. 369) smooth outside a given singular set S, whereas our notion of residue works for distributions in E_s which are weakly homogeneous of degree s for arbitrary s. Somehow, our regularity hypothesis on the current T which guarantees the existence of residues is minimal because any current defined **globally** on M will live in some scale space E_s for some s. The residue theory provides a very flexible and general framework to study anomalies. We repeat the construction of geometric residues for infinite dimensional Lie algebras of symmetries, for X a vector field which commutes with ρ, we study the residue equation

$$L_X RT - RL_X T = \mathfrak{Res}_X[T]$$

and we interpret $\mathfrak{Res}_X[T]$ as an obstruction to the fact that quantization (in our sense quantization consists in an operation of extension of distributions) preserves classical symmetries. More precisely, if we assume that we have an infinite dimensional Lie algebra of vector fields $\mathfrak{g}$, and that $\forall X \in \mathfrak{g}, L_X T = 0$ ($\mathfrak{g}$ is the Lie algebra of classical symmetries) then $X \mapsto \mathfrak{Res}_X[T]$ is a coboundary for the infinite dimensional Lie algebra of vector fields. It can be thought in terms of a quantum version of the Noether theorem.

Physics terminology	Our interpretation
renormalization scheme	Extension operator $R : \mathcal{D}'(M \setminus I) \mapsto \mathcal{D}'(M)$
local counterterm	currents supported on I
ambiguity	$R_1 T - R_2 T$
Symmetry	Lie algebra of vector fields $\mathfrak{g}$
anomaly	residue $L_X R - RL_X$

Relationship to other work. During the preparation of our work appeared a very interesting preprint of Todorov, Nikolov and Stora [55] whose approach is close to the spirit of the present work. The difference is that the authors of [55] work on flat space time and deal with *associate homogeneous* *distributions* in the terminology of [43]. They found the same notion of residues as poles of the meromorphic regularization and as anomaly of the scaling equations. However their anomaly residue is not as general as ours since it only applies to associate homogeneous distributions whereas ours applies to **all weakly homogeneous** distributions and our formulation has a more homological flavour with the Schwartz, De Rham theory of currents. Our definition of anomaly is broader since it applies for all vector fields of symmetries and we make more explicit the connection with the concept of **periods**. This work complements nicely the work of Dorothea Bahns and Michal Wrochna [4] which gives very explicit anomaly formulas in Minkowski space-time. We also learned recently that the problem of extension of currents was also studied in Complex analytic geometry ([66, 16]).

8.2 Currents and renormalisation.

8.2.1 Notation and definitions.

Let us denote by $\mathcal{D}'_k(M)$ the topological dual of the space $\mathcal{D}^k(M)$ of compactly supported test forms of degree k. Elements of $\mathcal{D}'_k(M)$ are called currents. If $\alpha \in \Omega^{n-k}(M)$ is a smooth form of degree $n-k$, then **integration** on M gives a linear map $\omega \in \mathcal{D}^k(M) \mapsto \langle \alpha, \omega \rangle = \int_M \alpha \wedge \omega$ which allows to interpret α as an element of $\mathcal{D}'_k$. Thus we have the continuous injection $\Omega^{n-k}(M) \hookrightarrow \mathcal{D}'_k(M)$ and the symbol $\langle \alpha, \omega \rangle$ extends integration on M to arbitrary $\alpha \in \mathcal{D}'_k(M)$. Finally, an important structure theorem states that the topological dual space of the space of smooth compactly supported sections of a vector bundle E are just distributional sections of the dual bundle E', in our specific case $\mathcal{D}'_k(M) = \mathcal{D}'(M) \otimes_{C^\infty(M)} \Omega^{n-k}(M)$ (for more on distributional sections see [5, 32]). In the book of Laurent Schwartz [65], it is explained why currents can be treated as exterior forms, for instance the usual operations of contraction with a vector field (interior product), exterior differentiation, exterior product with a smooth form and Lie derivatives are well defined for currents. For $U \subset M$, we will denote by $H_k(\mathcal{D}'(U))$ the subspace of currents of $\mathcal{D}'_k(U)$ which are closed in U, and we denote by $B_k(\mathcal{D}'(U))$ the space of exact currents in U. We can define a differential d on the graded $C^\infty(U)$-module $H_\star(\mathcal{D}'(U))$ which extends the exterior derivative of smooth forms to currents, thus $(H_\star(\mathcal{D}'(U)), d)$ is a chain complex:

$$H_{\star+1}\left(\mathcal{D}'(U)\right) \overset{d}{\mapsto} H_\star\left(\mathcal{D}'(U)\right).$$

From t to vector valued currents. Let $\omega \in \mathcal{D}^k(M)$ be a test form, then the scaling of ω is defined by pull-back $\omega_\lambda = e^{\log \lambda \rho \star}\omega$. Therefore, we define scaling of currents by the following formula, for all current $T \in \mathcal{D}'_k(M)$ and test forms $\omega \in \mathcal{D}^k(M)$:

$$T_\lambda(\omega) = T(\omega_{\lambda^{-1}}).$$

Definition 8.2.1 *Let U be a ρ-convex subset of M. A current $T \in \mathcal{D}'_k(U)$ is in $E_s(\mathcal{D}'_k(U))$ iff for all test forms $\omega \in \mathcal{D}^k(U)$*

$$\sup_{\lambda \in (0,1]} |\lambda^{-s} T_\lambda(\omega)| < \infty.$$

fortunately, this definition coincides with the definition of [53] because in the work of Meyer: $\lambda^{-d}\int_{\mathbb{R}^d} T\varphi_{\lambda^{-1}} d^d x = \int_{\mathbb{R}^d} T\left(\varphi d^d x\right)_{\lambda^{-1}} = \int_{\mathbb{R}^d} T_\lambda \varphi d^d x$, Meyer views distributions as dual of test forms $\omega = \varphi d^d x$ and the theory of Chapter 1 applies verbatim to this case.

The Taylor formula for test forms. It is important to understand the formalism of Taylor expansion for currents because we need to subtract Taylor polynomials in order to define certain renormalized extensions of distributions. Let ω be a smooth test form in $\mathcal{D}^k(M)$, then for a given ρ using the normal form theorem of chapter 1, we find that there exists a local coordinate chart around each point of I in which $\rho = h^j \partial_{h^j}$ and $\omega = \sum_{|I|+|J|=k} \omega_{IJ}(x,h) dx^I \wedge dh^J$ where I, J are multi-indices. We immediately see that ω_{IJ} have various homogeneities w.r.t. ρ depending on the length $|J|$. Thus, it is wiser to view ω as a function of $(x,h;dx,dh)$ smooth in (x,h) and polynomial in the Grassmann variables (dx,dh) which are treated on an equal footing as the variables (x,h), a function ω is said to be homogeneous of degree n if $\omega(x,\lambda h, dx, \lambda dh) = \lambda^n \omega(x,h,dx,dh)$. Consider the decomposition:

$$\omega = \sum_{0 \leqslant n \leqslant m} \omega_n + I_m(\omega) = P_m(\omega) + I_m(\omega)$$

in the sense of the Taylor expansion of Chapter 1:

$$\omega_n = \frac{1}{n!}\left(\left(\frac{d}{dt}\right)^n e^{\log t\rho*}\omega\right)|_{t=0}$$

where ω_n is homogeneous of degree n. We also have the formula for the Taylor remainder:

$$I_m(\omega) = \frac{1}{m!}\int_0^1 dt(1-t)^m \left(\frac{d}{dt}\right)^{m+1}\left(e^{\log t\rho*}\omega\right)$$

Example 8.2.1 *In this formalism dh is homogeneous of degree 1, $\left(\left(\frac{d}{dt}\right)e^{\log t\rho*}dh\right)|_{t=0} = \frac{d}{dt}tdh|_{t=0} = dh$.*

Conceptual meaning of the Taylor expansion. We give an equivalent formula for ω_n due to F Hélein:

$$\omega_n = \lim_{t\to 0} \frac{1}{t^n n!} (\rho) ... (\rho - n + 1) e^{\log t\rho *}\omega = \lim_{t\to 0} t^{-n} \begin{pmatrix} n \\ \rho \end{pmatrix} e^{\log t\rho *}\omega \qquad (8.2)$$

which allows to give the following conceptual remark:

$$\lim_{t\to 0} \frac{1}{t^n n!} (\rho) ... (\rho - n + 1) e^{\log t\rho *}\omega(p) = \lim_{t\to 0} t^{-n} \begin{pmatrix} n \\ \rho \end{pmatrix} e^{\log t\rho *}\omega(p)$$

depends **linearly** on the n-jet of ω at the point $e^{\rho \log t}p$. But it also depends **polynomially** on the $(n-1)$-jet of the smooth Euler vector field ρ at the point $e^{\rho \log t}p$. Finally, ω_n depends **linearly** on the n-jet of ω, and depends polynomially on the $(n-1)$-jet of ρ at the point $\lim_{t\to 0} e^{\rho \log t}p \in I$. Since the n-jet of ω at the point $\lim_{t\to 0} e^{\rho \log t}p \in I$ is independent of ρ, we deduce that the Taylor polynomial $P_m(\omega) = \sum_{n\leqslant m} \omega_n$ depends linearly on the m-jet of ω along I, but it depends polynomially in the $(m-1)$-jets of ρ along I. As noticed by Hélein, in an arbitrary local chart, $P_m(\omega)$ is in general **not a polynomial** hence the term Taylor polynomial is somewhat abusive, however in the coordinates in which ρ takes the normal form $\rho = h^j \partial_{h^j}$, $P_m(\omega)$ is a genuine polynomial in the variables h^j, dh^j. Let us discuss the expression of the Taylor polynomial P in coordinates. Let ω be a test k-form which reads $\omega = \sum_{|I|+|J|=k} \omega_{IJ} dx^I \wedge dh^J$, then

$$P_m(\omega) = \sum_{|I|+|J|=k, |\gamma|+|J|\leqslant m} \frac{h^\gamma}{\gamma!} \partial_h^\gamma \omega_{IJ}(x,0) dx^I \wedge dh^J.$$

8.2.2 From Taylor polynomials to local counterterms via the notion of moments of a compactly supported distribution T.

The representation theorem. Before we discuss the results of Chapter 1 in the current theoretic setting, we would like to discuss the issue of **local counterterms**. But even before we discuss the problem of local counterterms, we must recall the representation theorem for currents supported on I (see [51]). For any distribution $t_{\alpha J} \in \mathcal{D}'(I)$, if we denote by $i : I \hookrightarrow M$ the canonical embedding of I in M then $i_\star t_{\alpha J}$ is the push-forward of $t_{\alpha J}$ in M:

$$\forall \varphi \in \mathcal{D}(M), \langle i_\star t_{\alpha J}, \varphi \rangle = \langle t_{\alpha J}, \varphi \circ i \rangle .$$

Let $I \subset M$ be a closed embedded submanifold of M.

Theorem 8.2.1 *Let us consider a current $t \in \mathcal{D}_*(M)$ supported on I. Then for any local system of coordinates $(h^j)_j$ transversal to I, t has a unique* *decomposition as locally finite linear combinations of transversal derivatives of push-forward to M of currents $t_{\alpha J}$ in $\mathcal{D}'_*(I)$:*

$$t = \sum_{\alpha,J} \partial_h^\alpha \left(i_\star t_{\alpha J}\right) \wedge dh^J. \tag{8.3}$$

Proof — We first use the decomposition of a current $t \in \mathcal{D}'_k(M)$ as a sum $t_{I,J} dx^I \wedge dh^J$ where $t_{I,J} \in \mathcal{D}'_0(M)$ are 0-currents (see [30] 2.3 p. 123 and [61] Chapter 3 p. 36). Then the 0-currents $t_{I,J}$ are in fact distributions supported on I, then we apply the structure theorem 37 p. 102 [65] which describes distributions supported on a submanifold, which gives the desired result (also see 2.3.1). ■

Let us explain the ideas of the concept of moments, first we fix a coordinate system which gives a basis dx^i, dh^j. Then we define the **moments** $c_{\alpha I} \in D'_*(I)$ of $T \in \mathcal{D}'_*(M)$ by the push-forward formula, if the projection $\pi : (x,h) \mapsto x$ is proper on supp T:

$$\forall \omega \in \mathcal{D}(I), \langle c_{\alpha I}(T), \omega \rangle = \int_I \int_h \left(T \wedge \frac{h^\alpha}{\alpha!} \left(\frac{\partial}{\partial h^I} \lrcorner dh^d \right) \wedge \omega(x) \right). \tag{8.4}$$

These moments are indexed by the multi-indices (α, I) and satisfy the identity

$$\langle T, P_m(\omega) \rangle = \sum_{|\alpha|+d-|I| \leqslant m} \left\langle c_{\alpha,I} \wedge dh^I \partial_h^\alpha \delta_I, \omega \right\rangle \tag{8.5}$$

In the case $n = 0$, and $I = \{0\}$ is the **origin** of $\mathbb{R}^d$ and $T(h)$ is an integrable function in $L^1(\mathbb{R}^d)$, this definition coincides with the moment of the function $T \in L^1(\mathbb{R}^d)$ (see [34] Proposition 6.3 p. 52). Now, we notice that when $t \in D'_*(M)$ is supported on I, the moments $c_{\alpha,J}(t)$ of t exactly coincide with the coefficients $t_{\alpha,J}$ in the representation (8.2.1). The concepts of moments are crucial when we wish to represent currents supported on I or residues.

8.2.3 The results of Chapter 1.

Now that we have the suitable language to describe local counterterms, we can recall the results of Chapter 1 in this new current theoretic setting:

Proposition 8.2.1 *Let $T \in E_s\left(\mathcal{D}'_k(M \setminus I)\right)$ and $p = \sup(0, k-n)$. If $s+p > 0$ then for all $\omega \in \mathcal{D}_k(M)$ and χ is some smooth function which is equal to 1 in a neighborhood of I:*

$$\lim_{\varepsilon \to 0} \left\langle T\left(\chi - e^{-\log \varepsilon \rho *}\chi\right), \omega \right\rangle \tag{8.6}$$

exists.
If $s + p \leqslant 0$ and let $m \in \mathbb{N}$ s.t. $-m-1 < s \leqslant -m$, then for all $\omega \in \mathcal{D}_k(M)$:

$$\lim_{\varepsilon \to 0} \left\langle T\left(\chi - e^{-\log \varepsilon \rho *}\chi\right), I_m(\omega) \right\rangle \tag{8.7}$$

exists where $I_m(\omega)$ *is the generalized Taylor remainder*

$$I_m(\omega) = \frac{1}{m!}\int_0^1 dt(1-t)^m \left(\frac{d}{dt}\right)^{m+1}\left(e^{\log t\rho*}\omega\right) \quad (8.8)$$

Proof — We decompose the test forms ω in local coordinates (x, h, dx, dh) then we reduce the proof exactly to the same proofs as in Chapter 1. There are differences involved because we are dealing with forms. In normal coordinates (x, h) for ρ If ω is a k form, in the decomposition $\omega = \sum_{|I|+|J|=k}\omega_{IJ}dx^I dh^J$ the length $|J|$ of the multi-index J is at least equal to $k-n$ because there are n coordinate functions $(x^i)_{i=1\cdots n}$. Thus ω is in fact weakly homogeneous of degree $k-n$ which explains the criteria $s+k-n>0$. Now the second case is simple since $I_m(\omega)$ is weakly homogeneous of degree $m+1$. ■

We would like to introduce a new notation for the operation of regularization, we call it R_ε, and we define it as follows:

Definition 8.2.2 *We define the continuous linear operator* R_ε *on* $E_s\left(\mathcal{D}'_k(M\setminus I)\right)$ *as follows. Let* $p = \sup(0, k-n)$.

- *If* $s+p>0$ *then for all* $\omega \in \mathcal{D}_k(M)$:

$$\langle R_\varepsilon T, \omega\rangle = \left\langle T\left(1 - e^{-\log\varepsilon\rho*}\chi\right), \omega\right\rangle, \quad (8.9)$$

  and $\lim_{\varepsilon\to 0} R_\varepsilon T = RT$ *exists in* $\mathcal{D}'_k(M)$ *and defines an extension* RT *of* T.

- *If* $s+p \leqslant 0$ *and let* $m\in\mathbb{N}$ *s.t.* $-m-1 < s \leqslant -m$, *then for all* $\omega\in\mathcal{D}_k(M)$:

$$\langle R_\varepsilon T, \omega\rangle = \left\langle T\left(\chi - e^{-\log\varepsilon\rho*}\chi\right), I_m(\omega)\right\rangle + \langle T(1-\chi), \omega\rangle, \quad (8.10)$$

  and $\lim_{\varepsilon\to 0} R_\varepsilon T = RT$ *exists in* $\mathcal{D}'_k(M)$ *and defines an extension* RT *of* T.

8.3 Renormalization, local counterterms and residues.

8.3.1 The ambiguities of the operator R_ε and the moments of a distribution T.

Actually, first notice that any current T in $\mathcal{D}'_*(M)$ is also an element of $\mathcal{D}'_*(M\setminus I)$ by the pull-back i^*T by the restriction map $i : M\setminus I \hookrightarrow M$. Thus we ask ourselves a very natural question, does the restriction followed by the extension operation allows to reconstruct the element T, in other words do we have $\lim_{\varepsilon\to 0} R_\varepsilon i^*T = T$? The answer is no ! A distribution supported on I is automatically killed by $R_\varepsilon, \forall\varepsilon > 0$ thus if T is supported on I $\lim_{\varepsilon\to 0} R_\varepsilon i^*T = 0$. This idea is strongly related to the discussion in [53] Chapter 1, let t be a tempered distribution, does the Littlewood–Paley series $\sum_{j=-N}^{\infty} \Delta_j(t)$ converges *weakly* to t when $N \to +\infty$? The answer is no! There is convergence modulo floating polynomials in Fourier space (see [53] Proposition 1.5 p. 15). The floating polynomials in Fourier space are in fact corrections that we have to subtract from the Littlewood–Paley series in order to make it convergent and these polynomials should be related to vanishing moments conditions (see Meyer chapter 2 p. 45). We introduce a linear operator A which describes the ambiguities of the restriction-extension operation on the distribution T.

Definition 8.3.1 *Let $T \in \mathcal{D}'_k(M)$, then we define the ambiguity as*

$$AT = \lim_{\varepsilon\to 0} (T - R_\varepsilon T).$$

The operator A depends on χ.

The ambiguity is a non trivial operator because of the example discussed previously. As usual, we motivate our theorem with the simplest fundamental example

Example 8.3.1 *$\delta \in \mathcal{D}'(\mathbb{R})$ is a well defined distribution. But $\forall\varepsilon > 0, R_\varepsilon\delta = 0$ because 0 never meets the support of the cut-off hence*

$$A\delta = \lim_{\varepsilon\to 0} (\delta - R_\varepsilon\delta) = \delta$$

We state a simple theorem which expresses the ambiguity A in terms of the moments of $T\chi$.

Theorem 8.3.1 *Let $T \in E_s\left(\mathcal{D}'_k(M)\right)$ where $-(m+1) < s \leqslant -m, m \in \mathbb{N}$, then the ambiguity AT is given by the following formula:*

$$\forall\omega \in \mathcal{D}^k(M), AT(\omega) = \langle T\chi, P_m(\omega)\rangle, \tag{8.11}$$

where $P_m(\omega) = \sum_{k\leqslant m} \omega_k$.

Proof — Yves Meyer defines the ambiguity by the Bernstein theorem. We will give a more direct in space proof which does not use the Fourier transform. The first idea is the concept of moments of a current $T\chi \in \mathcal{D}'_k(M)$. First write the duality coupling in simple form:

$$\langle T, \omega\rangle = \langle T(1-\chi), \omega\rangle + \langle T\chi, \omega\rangle = \langle T(1-\chi), \omega\rangle + \langle T\chi, P_m(\omega)\rangle + \langle T\chi, I_m(\omega)\rangle$$

where P is the Taylor polynomial $\sum_{k\leqslant m} \omega_k$. We remind the definition of $R_\varepsilon T$

$$\langle R_\varepsilon T, \omega\rangle = \langle T(1-\chi), \omega\rangle + \left\langle T\left(\chi - e^{-\log\varepsilon\rho*}\chi\right), I_m(\omega)\right\rangle.$$

Then we immediately find:

$$\langle T,\omega\rangle-\langle R_\varepsilon T,\omega\rangle=\langle T\chi,P_m(\omega)\rangle+\left\langle Te^{-\log\varepsilon\rho*}\chi,I_m(\omega)\right\rangle$$

now notice that

$$\left\langle Te^{-\log\varepsilon\rho*}\chi,I_m(\omega)\right\rangle=\left\langle\left(e^{\log\varepsilon\rho*}T\right)\chi,e^{\log\varepsilon\rho*}I_m(\omega)\right\rangle=\langle T_\varepsilon\chi,(I_m(\omega))_\varepsilon\rangle$$

where

$$\exists C>0,|\langle T_\varepsilon\chi,(I_m(\omega))_\varepsilon\rangle|\leqslant C\varepsilon^{s+m+1}\rightarrow 0$$

since $\chi(I_m(\omega))_\varepsilon$ is a bounded family of test forms, thus

$$AT(\omega)=\langle T\chi,P_m(\omega)\rangle$$

where $\omega=P_m(\omega)+I_m(\omega)$ and the final result follows from the definition of the notion of moment of the distribution $T\chi$. ■

The dependence of R on the choice of χ,ρ.

We would also like to describe the dependance of the operator R on the choice of χ and ρ. As usual, the result will be expressed in terms of **local counterterms**.

Changing χ. Let χ_1,χ_2 be two functions such that $\chi_i=1,i=1,2$ in a neighborhood of I and $\rho\chi_i$ is uniformly supported in an annulus domain of M. Let $R^i_\varepsilon,i=1,2$ be the corresponding regularization operators on $E_s\left(\mathcal{D}'_k(M\setminus I)\right)$ defined as follows: for $p=\sup(k-n,0)$, if $s+p\leqslant 0$ let $m\in\mathbb{N}$ s.t. $-m-1<s\leqslant m$, then the regularization operator R^i corresponding to each $\chi_i,i=(1,2)$ is given by the formula

$$\left\langle R^i_\varepsilon T,\omega\right\rangle=\left\langle T\left(\chi_i-e^{-\log\varepsilon\rho*}\chi_i\right),I_m\left(\omega\right)\right\rangle+\left\langle T\left(1-\chi_i\right),\omega\right\rangle,\qquad(8.12)$$

and $\lim_{\varepsilon\rightarrow 0}R^i_\varepsilon T=R^iT$ exists in $\mathcal{D}'_k(M)$ and defines an extension R^iT of T if otherwise $s+p\geqslant 0$ then $R^iT=\lim_{\varepsilon\rightarrow 0}T(1-\chi_{i\varepsilon^{-1}})$.

Theorem 8.3.2 *Let $T\in E_s\left(\mathcal{D}'_k(M\setminus I)\right)$. If $s+p>0$ then $R^1T=R^2T$ (i.e. R does not depend on the choice of χ). If $s+p\leqslant 0$ then*

$$\left\langle\left(R^1-R^2\right)T,\omega\right\rangle=\left\langle T\left(\chi_2-\chi_1\right),P_m(\omega)\right\rangle,\qquad(8.13)$$

where $m\in\mathbb{N}$ is s.t. $-m-1<s\leqslant -m$.

Proof — By definition, we have:

$$\left\langle R^i_\varepsilon T, \omega\right\rangle = \langle T(\chi_i - \chi_{i\varepsilon^{-1}}), I_m(\omega)\rangle + \langle T\left(1-\chi_i\right), \omega\rangle$$

The only thing we have to do is to compute the difference $\left(R^1_\varepsilon - R^2_\varepsilon\right) T$. First notice that

$$\langle T\left(1-\chi_1\right), \omega\rangle = \langle T\left(1-\chi_2\right), \omega\rangle + \langle T\left(\chi_2-\chi_1\right), \omega\rangle$$

$$= \langle T\left(1-\chi_2\right), \omega\rangle + \langle T\left(\chi_2-\chi_1\right), P_m(\omega)\rangle + \langle T\left(\chi_2-\chi_1\right), I_m(\omega)\rangle$$

thus

$$\left\langle R^1_\varepsilon T, \omega\right\rangle = \langle T\left(1-\chi_1\right), \omega\rangle + \langle T(\chi_1 - \chi_{1\varepsilon^{-1}}), I_m(\omega)\rangle$$

$$= \langle T\left(1-\chi_2\right), \omega\rangle + \langle T\left(\chi_2-\chi_1\right), P_m(\omega)\rangle + \langle T\left(\chi_2-\chi_1\right), I_m(\omega)\rangle + \langle T(\chi_1 - \chi_{1\varepsilon^{-1}}), I_m(\omega)\rangle$$

$$= \langle T\left(1-\chi_2\right), \omega\rangle + \langle T\left(\chi_2-\chi_1\right), P_m(\omega)\rangle + \langle T(\chi_2 - \chi_{1\varepsilon^{-1}}), I_m(\omega)\rangle$$

then computing the difference

$$\left\langle \left(R^1_\varepsilon - R^2_\varepsilon\right) T, \omega\right\rangle = \left\langle R^1_\varepsilon T, \omega\right\rangle - \left\langle R^2_\varepsilon T, \omega\right\rangle$$

$$= \langle T\left(1-\chi_2\right), \omega\rangle + \langle T\left(\chi_2-\chi_1\right), P_m(\omega)\rangle + \langle T(\chi_2 - \chi_{1\varepsilon^{-1}}), I_m(\omega)\rangle$$

$$- \langle T(\chi_2 - \chi_{2\varepsilon^{-1}}), I_m(\omega)\rangle - \langle T\left(1-\chi_2\right), \omega\rangle$$

$$= \langle T\left(\chi_2-\chi_1\right), P_m(\omega)\rangle + \langle T\left(\chi_2-\chi_1\right)_{\varepsilon^{-1}}, I_m(\omega)\rangle$$

As in the proof of theorem (8.3.1), we can take the limit $\varepsilon \to 0$ and we find that the term $\langle T\left(\chi_2-\chi_1\right)_{\varepsilon^{-1}}, I_m(\omega)\rangle$ will vanish when $\varepsilon \to 0$. ■

Changing ρ. We say that χ is compatible with ρ iff for each $p \in I$, there is a neighborhood V_p of p and a local chart $(x, h) : V_p \mapsto \mathbb{R}^{n+d}$ on this neighborhood on which $\rho = h^j \frac{\partial}{\partial h^j}$, $\chi = 0$ when $|h| \geqslant b$ and $\chi = 1$ when $|h| \leqslant a$ for some pair $0 < a < b$. Let ρ_1, ρ_2 be two Euler vector fields and χ which is *compatible* with ρ_1 and ρ_2. Let $R^i_\varepsilon, i = 1, 2$ be the corresponding regularization operators on $E_s\left(\mathcal{D}'_k(M \setminus I)\right)$ defined as follows: for $p = \sup(k-n, 0)$. If $s + p \leqslant 0$, let m s.t. $-m-1 < s \leqslant m$, the regularization operator R^i corresponding to each $\rho_i, i = (1, 2)$ is given by the formula

$$\left\langle R^i_\varepsilon T, \omega\right\rangle = \left\langle T\left(\chi - e^{-\log\varepsilon\rho_i *}\chi\right), I_{im}\right) + \langle T\left(1-\chi\right), \omega\rangle \tag{8.14}$$

where $\omega = P_{im}(\omega) + I_{im}(\omega), i = 1, 2$, $P_{im}(\omega)$ is the "Taylor polynomial of order m" of ω for the Euler vector field ρ_i and $\lim_{\varepsilon\to 0} R^i_\varepsilon T = R^i T$ exists in $\mathcal{D}'_k(M)$ and defines an extension $R^i T$ of T. Otherwise, if $s + p > 0$ then $R^i T = \lim_{\varepsilon\to 0} T(1 - e^{-\log\varepsilon\rho_i\star}\chi)$.

Theorem 8.3.3 *Let $T \in E_s\left(\mathcal{D}'_k(M \setminus I)\right)$ and $p = \sup(0, k-n)$. If $s+p > 0$ then $R^1T = R^2T$. If $s+p \leqslant 0$ let $m \in \mathbb{N}$ s.t. $-(m+1) < s \leqslant -m$, then for any Euler vector field ρ such that χ is* compatible *with ρ,*

$$\left\langle \left(R^1 - R^2\right)T, \omega\right\rangle = \lim_{\varepsilon\rightarrow 0}\left\langle T\left(\chi - e^{-\log\varepsilon\rho\star}\chi\right), P_{2m}(\omega) - P_{1m}(\omega)\right\rangle. \quad (8.15)$$

Notice that in the conclusion of this theorem the vector field ρ is chosen independently of ρ_1, ρ_2.
Proof — Before we prove our claim, we would like to give some important remarks.

- First, no matter what Euler vector field ρ_i we choose, the Taylor remainder $I_{im}(\omega)$ *always vanishes at order* m on the submanifold I. The key point is that if a smooth form ϖ vanishes at order m at I, then the limit $\lim_{\varepsilon\rightarrow 0}\left\langle T\left(\chi - e^{-\log\varepsilon\rho\star}\chi\right), \varpi\right\rangle$ does not depend on the choice of Euler vector field ρ provided χ is ρ admissible. Hence by choosing some Euler vector field ρ for which χ is ρ admissible, we still have

$$\forall i, \lim_{\varepsilon\rightarrow 0}\left\langle T(\chi - e^{-\log\varepsilon\rho_i}\chi), I_{im}(\omega)\right\rangle = \lim_{\varepsilon\rightarrow 0}\left\langle T(\chi - e^{-\log\varepsilon\rho}\chi), I_{im}(\omega)\right\rangle$$

$$= \lim_{\varepsilon\rightarrow 0}\left\langle T(\chi - \chi_{\varepsilon^{-1}}), I_{im}(\omega)\right\rangle \text{ where } \chi_{\varepsilon^{-1}} = e^{-\log\varepsilon\rho}\chi.$$

- Secondly, if we denote by $P_{im}(\omega), i = 1, 2$ (resp $I_{im}(\omega), i = 1, 2$) the "Taylor polynomials" (resp "Taylor remainders") associated with $\rho_i, i = 1, 2$, then from $\omega = P_{1m}(\omega) + I_{1m}(\omega) = P_{2m}(\omega) + I_{2m}(\omega)$ we deduce that $I_{1m}(\omega) - I_{2m}(\omega) = P_{2m}(\omega) - P_{1m}(\omega)$, hence $P_{2m}(\omega) - P_{1m}(\omega)$ depends only on some finite jet of ω, ρ_1, ρ_2 and vanishes at order m at I (it is in general **not a polynomial in arbitrary local charts**).

We can now compute $\left(R^1 - R^2\right)T$:

$$\left\langle \left(R^1 - R^2\right)T, \omega\right\rangle = \lim_{\varepsilon\rightarrow 0}\left\langle T(\chi - \chi_{\varepsilon^{-1}}), I_{1m}(\omega) - I_{2m}(\omega)\right\rangle.$$

Using $I_{1m}(\omega) - I_{2m}(\omega) = P_{2m}(\omega) - P_{1m}(\omega)$, we finally get:

$$\left\langle \left(R^1 - R^2\right)T, \omega\right\rangle = \lim_{\varepsilon\rightarrow 0}\left\langle T(\chi - \chi_{\varepsilon^{-1}}), P_{2m}(\omega) - P_{1m}(\omega)\right\rangle$$

where the above limit makes sense since $P_{2m}(\omega) - P_{1m}(\omega)$ vanishes at order m on the submanifold I. ■

8.3.2 The geometric residues.

The residues and the coboundary d of currents.

We want to describe the ambiguities of the restriction-extension operation on closed currents $T \in H_*\left(\mathcal{D}'_*(M \setminus I), d\right)$ defined on $M \setminus I$ and on exact currents $dT \in B_*\left(\mathcal{D}'_*(M \setminus I), d\right)$ defined on $M \setminus I$. In other words one could ask is how does our extension procedure behaves when applied to closed currents ? The notion of **residue** (following [36] and Eells–Allendoerfer [20]) that we define below answers this question, $\mathfrak{Res}[T]$ is defined as the solution of the chain homotopy equation:

$$\mathfrak{Res}[T] = dRT - RdT. \tag{8.16}$$

Recall $E_s\left(\mathcal{D}'_k(M \setminus I)\right)$ is the space of k-currents in $\mathcal{D}'_k(M \setminus I)$ which are weakly homogeneous of degree s and we work on $M \setminus I$ where $\dim M = n+d$ and $\dim I = n$.

Theorem 8.3.4 *Let $T \in E_s\left(\mathcal{D}'_k(M \setminus I)\right)$, and $p = \sup(0, k-n-1)$. If $s+p > 0$ then $\mathfrak{Res}[T] = 0$.*

Proof — The key remark is that $dT \in E_s\left(\mathcal{D}'_{k-1}(M \setminus I)\right)$ since d is scale invariant. The residue equals $d(RT) - R(dT)$ by definition. If $s+p > 0$ then by definition of R (8.2.2):

$$\langle d(RT) - R(dT), \omega\rangle = \lim_{\varepsilon \to 0} \langle d((1-\chi_{\varepsilon^{-1}})T) - (1-\chi_{\varepsilon^{-1}})(dT), \omega\rangle$$

since there are no counterterms to subtract

$$= -\lim_{\varepsilon \to 0} \langle d\chi_{\varepsilon^{-1}}, T \wedge \omega\rangle = 0.$$

Since $|\langle d\chi_{\varepsilon^{-1}}, T \wedge \omega\rangle| \leqslant C\varepsilon^{s+p}$ for some $C > 0$ by the hypothesis of homogeneity on T and the degree of T. ■

Let us give the fundamental example of residue from Griffiths–Harris see [36] p. 367 and Laurent Schwartz [65] p. 345-347.

Example 8.3.2 *Let H be the Heaviside function on $\mathbb{R}$. H is a smooth closed 0-form on $\mathbb{R} \setminus \{0\}$. The local integrability around 0 guarantees it extends in a unique way as a current denoted $RH \in \mathcal{D}'_1(\mathbb{R})$. By integration by parts and by the fact that $dH|_{\mathbb{R}\setminus\{0\}} = 0$ since H is **closed**, it is immediate that*

$$dRH - R\underbrace{dH}_{=0} = dRH = \delta_0(x)dx$$

So the current $\delta_0(x)dx \in \mathcal{D}'_0(\mathbb{R})$ is the residue of the Heaviside function H which is closed on $\mathbb{R} \setminus \{0\}$.

In the above example, the residue measures the jump at 0. However in the case of renormalization theory, our residues must generalize the "classical" notion of residue to take into account **more singular** distributions (see [36] p.369,371).

Theorem 8.3.5 *Let* $T \in E_s\left(\mathcal{D}'_k(M \setminus I)\right)$*, and* $p = \sup(0, k-n-1)$*. If* $s+p \leqslant 0$ *let for* $m \in \mathbb{N}$ *s.t.* $-m-1 < s \leqslant -m$*, then* $\mathfrak{Res}$ *is a* ***current supported on*** I *given by the formula*

$$\forall \omega \in \mathcal{D}^{k-1}(M), \mathfrak{Res}[T](\omega) = (-1)^{n-k-1}\left\langle T, d\chi \wedge P_m(\omega)\right\rangle . \tag{8.17}$$

Proof — Let T be a current in $\mathcal{D}'_k$ and $\omega \in \mathcal{D}^{k-1}(M)$ a $k-1$ test form. We want to compute the difference $\langle d\left(R_\varepsilon T\right), \omega\rangle - \langle (R_\varepsilon dT), \omega\rangle$. There are two cases for this theorem.

- Either both T and dT need a renormalization. We first treat this case. By definition of the coboundary d of a current ([65], [30]), we find that

  $$\langle d\left(R_\varepsilon T\right), \omega\rangle - \langle (R_\varepsilon dT), \omega\rangle = (-1)^{n-k-1}\left\langle R_\varepsilon T, d\omega\right\rangle - \left\langle R_\varepsilon dT, \omega\right\rangle .$$

  On the one hand, we have:

  $$\langle R_\varepsilon T, d\omega\rangle = \langle T, (1-\chi)d\omega\rangle + \left\langle T\left(\chi - e^{-\log\varepsilon\rho *}\chi\right), I_m(d\omega)\right\rangle$$

  $$= \langle T, (1-\chi)d\omega\rangle + \left\langle T\left(\chi - e^{-\log\varepsilon\rho *}\chi\right), dI_m(\omega)\right\rangle$$

  since $\frac{1}{m!}\int_\varepsilon^1 dt(1-t)^m\left(\frac{d}{dt}\right)^{m+1}\left(e^{\log t\rho *}d\omega\right) = d\frac{1}{m!}\int_\varepsilon^1 dt(1-t)^m\left(\frac{d}{dt}\right)^{m+1}\left(e^{\log t\rho *}\omega\right)$ because d commutes with the pull-back operator $e^{\log t\rho *}$. We hence notice the important fact that if we view I_m and P_m as projections in $Hom\left(\mathcal{D}^\star(M), \mathcal{D}^\star(M)\right)$, then they **commute** with d. On the other hand:

  $$\langle R_\varepsilon dT, \omega\rangle = \langle dT, (1-\chi)\,\omega\rangle + \left\langle dT, \left(\chi - e^{-\log\varepsilon\rho *}\chi\right) I_m(\omega)\right\rangle ,$$

  then following the definition of the coboundary d of a current, we differentiate the test form:

  $$\langle T, d\left((1-\chi)\,\omega\right)\rangle + \left\langle T, d\left(\left(\chi - e^{-\log\varepsilon\rho *}\chi\right) I_m(\omega)\right)\right\rangle = \langle T, (1-\chi)\, d\omega\rangle - \langle T, (d\chi)\wedge\omega\rangle$$

  $$+ \langle T, (d\chi)\wedge I_m(\omega)\rangle - \langle T, (d\chi)_{\varepsilon^{-1}}\wedge I_m(\omega)\rangle + \left\langle T, \left(\chi - e^{-\log\varepsilon\rho *}\chi\right) dI_m(\omega)\right\rangle .$$

  Thus

  $$(-1)^{n-k-1}\left\langle R_\varepsilon dT, \omega\right\rangle = \langle T, (1-\chi)\, d\omega\rangle - \langle T, (d\chi)\wedge P_m(\omega)\rangle$$

  $$- \langle T, (d\chi)_{\varepsilon^{-1}}\wedge I_m(\omega)\rangle + \left\langle T, \left(\chi - e^{-\log\varepsilon\rho *}\chi\right) dI_m(\omega)\right\rangle$$

where $\omega = P_m(\omega) + I_m(\omega)$ by the Taylor formula. Then we find:

$$\langle dR_\varepsilon T, \omega\rangle - \langle R_\varepsilon dT, \omega\rangle = (-1)^{n-k-1}\left(\langle T, (d\chi)\wedge P_m(\omega)\rangle + \langle T, (d\chi)_{\varepsilon^{-1}} \wedge I_m(\omega)\rangle\right).$$

Now notice that

$$\langle T, (d\chi)_{\varepsilon^{-1}} \wedge I_m(\omega)\rangle = \left\langle T, \left(e^{-\log\varepsilon\rho *} d\chi\right) \wedge I_m(\omega)\right\rangle$$

$$= \left\langle e^{\log\varepsilon\rho *} T, (d\chi) \wedge e^{\log\varepsilon\rho *} I_m(\omega)\right\rangle = \langle T_\varepsilon, (d\chi)\wedge I_m(\omega)_\varepsilon\rangle$$

and the above term satisfies the following estimate:

$$\exists C > 0, |\langle T_\varepsilon, d\chi \wedge I_m(\omega)_\varepsilon\rangle| \leqslant C\varepsilon^{s+m+1} \underset{\varepsilon\to 0}{\to} 0$$

since $-m-1 < s$, T is weakly homogeneous of degree s and the family of test forms $d\chi \wedge I_m(\omega)_\varepsilon, \varepsilon \in [0,1]$ is bounded. Thus

$$\lim_{\varepsilon\to 0} \langle (d\circ R_\varepsilon - R_\varepsilon \circ d)\, T, \omega\rangle = (-1)^{n-k-1} \langle T, (d\chi)\wedge P_m(\omega)\rangle .$$

Finally, we find

$$\mathfrak{Res}[T](\omega) = (-1)^{n-k-1} \langle T, d\chi \wedge P_m(\omega)\rangle .$$

- Either T is s.t. $s+\sup(0,k-n) > 0$ thus RT does not need a renormalization and $s + \sup(k-n-1, 0) \leqslant 0$ which implies that the definition of the extension RdT needs a renormalization and that $k-n-1 \geqslant 0$, thus $p = k-n-1$. Actually since $-p-1 < s \leqslant -p$, we must subtract a counterterm $P_p(\omega)$ to the $k-1$ form ω to define the extension: RdT. The key fact is to notice that $d\omega$ is polynomial in dh of degree at least $p+1$ thus $d\omega = I_p(d\omega) = d\omega - P_p(d\omega)$ and

$$\langle R_\varepsilon T, d\omega\rangle = \langle (1-\chi)T, d\omega\rangle + \langle (\chi - \chi_{\varepsilon^{-1}})T, I_p(\omega)\rangle$$

and we are reduced to the first case.

■

We give the most fundamental example illustrative of our approach

Example 8.3.3 *We set* $T = \frac{1}{|x|}$ *and we will show how to compute the residue for this simple example.* RT *is defined by the formula* $\langle RT, \varphi dx\rangle = \int_{-\infty}^{\infty} \frac{1}{|x|}\chi(x)(\varphi(x)-\varphi(0))dx + \int_{-\infty}^{\infty}\frac{1}{|x|}(1-\chi(x))\varphi(x)dx$. *The residue is given by the simple formula*

$$\mathfrak{Res}[\frac{1}{|x|}] = -\left(\int_{-\infty}^{\infty} \frac{1}{|x|}(\partial_x\chi)(x)dx\right)\delta_0.$$

We give a second example which illustrates the limit case where RT does not need a renormalization but RdT does.

Example 8.3.4 *Let us work in $\mathbb{R}^d$ and $n = 0$. Let T be a $d-1$ form in $\mathbb{R}^d \setminus \{0\}$ which is homogeneous of degree 0, i.e. $T \in E_0(\mathcal{D}'_1(\mathbb{R}^d \setminus \{0\}))$, then $p = \sup(0, 1-0) = 1$ and $s+1 = 0+1 > 0$ thus RT does not need a renormalization. dT is a d form which is still homogeneous of degree 0 but $dT \in E_0(\mathcal{D}'_0(\mathbb{R}^d \setminus \{0\}))$ thus $s+0 = 0+0 \leqslant 0$ and dT needs a renormalization with subtraction of the form ω_0.*

$$\langle RdT, \omega\rangle = \lim_{\varepsilon\rightarrow 0} \langle dT, (1-\chi_{\varepsilon^{-1}})\omega\rangle - \langle dT, (\chi - \chi_{\varepsilon^{-1}})\omega_0\rangle$$

but notice that $\langle dT, (\chi - \chi_{\varepsilon^{-1}})\omega_0\rangle = 0$ by scale invariance of ω_0 and dT thus in this example the ***counterterm vanishes****. Finally, the residue satisfies the simple equation:*

$$\lim_{\varepsilon\rightarrow 0} \langle d\left(T(1-\chi_{\varepsilon^{-1}})\right), \omega\rangle - \langle dT, (1-\chi_{\varepsilon^{-1}})\omega\rangle = \langle T, \omega_0 d\chi\rangle .$$

For T a closed current in $E_s(\mathcal{D}'_k(M \setminus I))$, we associated a current $\mathfrak{Res}[T] \in \mathcal{D}'_*(M)$ supported on I. If T is closed, what can be said about $\mathfrak{Res}[T]$?

Proposition 8.3.1 *Let V be some neighborhood of I, $\pi : V \setminus I \mapsto I$ a submersion and $T \in E_s(\mathcal{D}'_k(V \setminus I))$. If $T \in H_k\left(\mathcal{D}_*\left(V \setminus I\right), d\right)$ is a cycle in the complex of currents and π is proper on the support of T then $\mathfrak{Res}[T] \in B_*\left(\mathcal{D}'(M)\right)$.*

Proof — We first notice that if T is closed then

$$dRT - R\underbrace{dT}_{=0} = d\left(RT\right) = \mathfrak{Res}[T]$$

implies $\mathfrak{Res}[T] \in B_*\left(\mathcal{D}_{*k}(M), d\right)$ is an exact current. ■

Can we relate $\mathfrak{Res}[T] \in \mathcal{D}'_\star(M)$ with a current in $\mathcal{D}'_*(I)$ in the spirit of the representation theorem (8.2.1)? The answer is yes but the naive idea to "restrict" $\mathfrak{Res}[T]$ to the submanifold I does not make sense! We need another idea which is explained in the following example.

Example 8.3.5 *Let $\delta(h)d^dh$ be the current supported by the point 0. In this case, $I = \{0\} \subset \mathbb{R}^d$. Then the corresponding current of $\mathcal{D}'(I)$ is just the function 1, and it can be recovered by integrating over the "fiber" $\mathbb{R}^d$, $1 = \int_{\mathbb{R}^d} \delta(h)d^dh$.*

Let $N(I \subset M)$ be the normal bundle of I in M. We can identify the closed smooth forms in $H^\star(V, d)$, which are supported in some neighborhood V of I which is **homotopy retract** to I, with the closed smooth forms in $H^\star_v(N(I \subset M), d)$ which have compact vertical support (see [59] for more on these forms). The proof is a straightforward application of the tubular neighborhood theorem which gives a diffeomorphism between a neighborhood of the zero section of $N(I \subset M)$ and V and the fact that this diffeomorphism induces an isomorphism in cohomology $H^\star_v(N(I \subset M), d) \simeq H^\star(V, d)$. We denote by i the embedding $i : I \hookrightarrow M$. For any submersion $\pi : V \setminus I \mapsto I$ and any current $T \in \mathcal{D}'_\star(V)$ s.t. π is proper on its support, the push-forward $\pi_\star T$ is defined by the formula

$$\forall \omega \in \mathcal{D}(I), \langle \pi_\star T, \omega \rangle_I = \langle T, \pi^\star \omega \rangle_M .$$

Theorem 8.3.6 *Let V be some neighborhood of I, $\pi : V \setminus I \mapsto I$ a submersion and $T \in E_s(\mathcal{D}'_k(V \setminus I))$. If $T \in H_k\left(\mathcal{D}_*\left(V \setminus I\right), d\right)$ is a cycle in the complex of currents and π is proper on the support of T then the push-forward*

$$\pi_\star\left(\mathfrak{Res}[T]\right) \in B_\star(\mathcal{D}'(I), d).$$

In particular, the current $\mathfrak{Res}[T] \in \mathcal{D}'(M)$ is represented by the push-forward of $\pi_\star\left(\mathfrak{Res}[T]\right)$: $\mathfrak{Res}[T] = i_\star\left(\pi_\star\left(\mathfrak{Res}[T]\right)\right)$.

Remark that in this theorem, the map $T \mapsto \pi_\star\left(\mathfrak{Res}[T]\right)$ is the inverse of the Leray coboundary δ (see [56] p. 59–61).
Proof — Proposition 8.3.1 gave us the exactness of $\mathfrak{Res}[T]$. Thus by pull back on the normal bundle, $\mathfrak{Res}[T] \in B_\star(\mathcal{D}'_\star(N(I \subset M)))$ is exact and supported on the zero section of the normal bundle $N(I \subset M)$. Then we pushforward $\mathfrak{Res}[T]$ along the fibers of $\pi : N(I \subset M) \mapsto I$. Recall that pushforward $\pi_\star$ commutes with the coboundary operator d, hence $\pi_\star\left(\mathfrak{Res}[T]\right) = \pi_\star d\left(RT\right) = d\pi_\star\left(RT\right)$ by 8.3.1 which yields the result. ■

This means that the residue map induces a map on the level of cohomology.

The residues and symmetries.

The previous theorem gave us a formula which measured the defect of commutativity of the operator R with the coboundary operator d. Now we study the loss of commutativity of R with the operator of Lie derivation L_X for any vector field X such that $[X, \rho] = 0$ and X is tangent to I in the sense of Hörmander (Lemma 18.2.5 in [40] volume 3). We first notice that the vector space $\mathfrak{g}$ forms an **infinite dimensional Lie algebra**. However, despite the infinite dimensionality of this Lie algebra $\mathfrak{g}$, it has the following structure:

Proposition 8.3.2 *Let $\mathcal{A} \subset C^\infty(M)$ be the subalgebra of the algebra of smooth functions which are killed by ρ. Let us fix a local chart where $I = \{h = 0\} \subset \mathbb{R}^{n+d}$ in which the Euler vector field has the form $\rho = h^j \partial_{h^j}$. Then $\mathfrak{g}$ is a **finitely generated** left $\mathcal{A}$-**module** with generators $h^i \partial_{h^j}, \partial_{x^i}$.*

Any vector field X in $\mathfrak{g}$ is tangent to I thus it decomposes as $a_i^j h^i \partial_{h^j} + b^i \partial_{x^i}$ where a_i^j, b^i are smooth functions by Lemma 18.2.5 in [40]. Now if X commutes with ρ, an elementary computation forces the functions a_i^j, b^i to be ρ-invariant.

All our symmetries will be Lie subalgebras of $\mathfrak{g}$. As usual, we discuss here the most important example for QFT which comes from our understanding of an article of Hollands and Wald [39]. We study the neighborhood of the thin diagonal d_n of a configuration space M^n where (M, g) is a pseudoriemannian manifold of dimension $p+1$ and the signature of g is $(1, p)$. By the tubular neighborhood theorem, it is always possible to identify this neighborhood with a neighborhood of the zero section of the normal bundle $N(d_n \subset M^n)$. Another trick consists in using the exponential map (see Chapter 5 section 3) to identify the normal bundle with the metric vector bundle $\underbrace{TM \times_M \cdots \times_M M}_{(n-1) \text{ times}}$ of rank $(n-1)(p+1)$, the fiber of this bundle over x is $\underbrace{T_xM \times \cdots \times T_xM}_{n-1}$ which has a canonical metric γ_x of signature $n-1, (n-1)p$. Then the Lie algebra of infinitesimal gauge transformations of this vector bundle is the suitable Lie algebra of symmetries.

Example 8.3.6 *Let $\pi : (P, \gamma) \mapsto I$ be a metric vector bundle of rank d with metric γ on the fibers (in the Hollands Wald discussion P is the normal bundle $N(d_n \subset M^n)$ and $d = (n-1)(p+1)$). We construct a trivialisation of P by the moving frame technique. Let $U \subset I$ be an open set. Let $(e_0, ..., e_n)$ be* ***an orthonormal moving frame*** *$(\forall x \in U, \gamma_x(e_\mu, e_\nu) = \eta_{\mu\nu})$ and let*

$$(x, h) : \pi^{-1}(U) \to U \times \mathbb{R}^d$$

$$(p, v) \mapsto (x(p), h(p, v))$$

such that $v = \sum_0^d h^\mu(p, v) e_\mu(p)$, for $p \in U$ and $v \in \pi_p^{-1}(U)$. We use the coordinate system (x, h) on P. All orthonormal moving frames are related by gauge transformations which are maps in $C^\infty(I, O(\eta))$ where $O(\eta)$ is the orthogonal group of the quadratic form η. The gauge group $C^\infty(I, O(\eta))$ is a subgroup of the group of diffeomorphism of the total space M preserving the zero section $\underline{0}$ (the zero section $\underline{0}$ being isomorphic to I). The Euler vector field $\rho = h^j \frac{\partial}{\partial h^j}$ which scales linearly in the fibers w.r.t. the zero section $\underline{0}$ is canonically given and the gauge Lie algebra consists of vector fields of the form $\underbrace{a_{\mu\nu}}_{antisymmetric}(x) \quad (h^\mu \partial_h^\nu)$, where $\forall \nu, \partial_h^\nu = \gamma^{\mu\nu} \partial_{h^\mu}$, hence $(h^\mu \partial_h^\nu) - \left(h^\nu \partial_h^\mu\right)$ is an infinitesimal generator of the Lie algebra $o(\eta)$ which commutes with ρ and vanishes at $\underline{0}$.

Before we state and prove the residue theorem for vector fields with symmetries, let us pick again our simplest fundamental example (again due to Laurent Schwartz) to illustrate the anomaly phenomenon:

Example 8.3.7 *The Heaviside current $T = H(x)dx$ is smooth in $\mathbb{R} \setminus \{0\}$ and satisfies the symmetry equation $L_{\partial_x} T = 0$ on $\mathbb{R}\setminus\{0\}$, i.e. it is translation invariant outside the singularity. Again, let R be the extension operator, recall the extension RT is unique for this example and again by integration by parts, we obtain the residue equation:*

$$L_{\partial_x}(RT) - R(L_{\partial_x} T) = L_{\partial_x}(RT) = \delta_0 dx.$$

Recall $E_s(\mathcal{D}'_k(M \setminus I))$ is the space of k-currents in $\mathcal{D}'_k(M \setminus I)$ which are weakly homogeneous of degree s. For any vector field $X \in \mathfrak{g}$, we denote by L_X the operator of Lie derivation. We define the residue of T w.r.t. the vector field $X \in \mathfrak{g}$ as the current defined by the equation:

$$\mathfrak{Res}_X[T] = L_X(RT) - R(L_X T). \tag{8.18}$$

Theorem 8.3.7 *Let $T \in E_s(\mathcal{D}'_k(M \setminus I))$, $p = \sup(0, k-n)$ and $X \in \mathfrak{g}$. If $p + s \leqslant 0$, let $m \in \mathbb{N}$ s.t. $-m-1 < s \leqslant -m$, then we have the residue equation:*

$$\mathfrak{Res}_X[T](\omega) = (-1)^{n-k-1}\langle i_X(T \wedge P_m(\omega)), d\chi\rangle, \tag{8.19}$$

*where i_X denotes contraction of the current $(T \wedge P_m(\omega))$ with the vector field X. Note that $\mathfrak{Res}_X[T](\omega)$ is a local counterterm in the sense it is a **current supported on** I.*

The proof is exactly the same as in Theorem 8.3.5, just replace the boundary operator d by L_X and we obtain $\mathfrak{Res}_X[T] = (-1)\langle T(L_X\chi), P_m(\omega)\rangle$. Then we use exterior differential calculus to convert this expression

$$\langle T(L_X\chi), P_m(\omega)\rangle = \langle T i_X d\chi, P_m(\omega)\rangle$$

$$= \langle T \wedge P_m(\omega), i_X d\chi\rangle = (-1)^{n-k-1}\langle i_X(T \wedge P_m(\omega)), d\chi\rangle.$$

8.3.3 Stability of geometric residues.

Now the natural questions we should ask ourselves are: what are the conditions for which the residue vanishes ? Is the residue independent of χ ? In general, we would like to know what are the stability properties of residues. In the case of symmetries, what should replace the closed or exact currents in the De Rham complex of currents ?

There is a cohomological analogue of the De Rham complex in the case of symmetries generated by infinite dimensional Lie algebras of vector fields on M denoted by $\mathfrak{g}$. This is the theory of continuous cohomology of infinite dimensional Lie algebras developped by I M Gelfand and D Fuchs. Fortunately for us, we only need basic definitions of this theory following [24]. For any left $\mathfrak{g}$-module $\mathcal{M}$, we define the complex ([24] Chapter 1, "The standard chain complex of a Lie algebra", p. 137,138)

$$C^k(\mathfrak{g},\mathcal{M}) = Hom\left(\bigwedge^k \mathfrak{g},\mathcal{M}\right)$$

with the differential $\delta : C^k(\mathfrak{g},\mathcal{M}) \to C^{k+1}(\mathfrak{g},\mathcal{M})$ which for $k=0$ reads

$$\delta\Theta(X) = L_X\Theta,$$

$\Theta \in C^0(\mathcal{M}) \simeq \mathcal{M}$ and L_X denotes the left action of X on the module $\mathcal{M}$. $(C^\bullet(\mathfrak{g},\mathcal{M}),\delta)$ is called *the standard cochain complex of the Lie algebra* $\mathfrak{g}$ with coefficient in the module $\mathcal{M}$. Now, the choice of topological module $\mathcal{M}$ dictated by our problem is the space of currents $\mathcal{D}'_*(M)$ with the natural weak topology defined on it and the left action of $\mathfrak{g}$ on $\mathcal{D}'_*(M)$ is the action by **Lie derivatives**. Then without surprise, the formula for δ is the classical Cartan formula in differential geometry. The Lie algebra of smooth vector fields on M has a natural C^∞ topology, this topology induces on $\mathfrak{g}$ a C^∞ topology: the space of smooth vector fields is endowed with the topology of C^∞ convergence of the components and some finite number of derivatives over compact sets. Then we require our cochains $T \in C^\star(\mathfrak{g},\mathcal{M}) = Hom\left(\bigwedge^\star \mathfrak{g},\mathcal{M}\right)$ to be continuous for the C^∞ topology of $\mathfrak{g}$ and the weak topology of $\mathcal{M}$.

Theorem 8.3.8 *Let $T \in E_s\left(\mathcal{D}'_k(M\setminus I)\right)$ and $\omega \in \mathcal{D}^k(M)$. If $\exists X \in \mathfrak{g}$ such that $L_X\left(T\wedge P_m(\omega)\right) = 0$, then for all smooth closed forms $[C] \in H^1\left(\left(\Omega^*\left(M\setminus I\right),d\right)\right)$ such that $[C] = [-d\chi]$, we have the identity*

$$\mathfrak{Res}_X[T](\omega) = (-1)^{n-k}\left\langle i_X\left(T\wedge P_m(\omega)\right),[C]\right\rangle \tag{8.20}$$

*and $\mathfrak{Res}_X[T](\omega)$ is a **period**.*

Proof — If T is a current in $\mathcal{D}'_k(M\setminus I)$ and $\omega \in \mathcal{D}^k(M)$ is a test k-form, then the Taylor polynomial $P_m(\omega) \in \Omega^k(M)$ is also a smooth k-form but is no longer compactly supported. Thus the exterior product $T\wedge P_m(\omega)$ is well defined as a current in $\mathcal{D}'_0(M\setminus I)$ ([65] p. 341). Currents in $\mathcal{D}'_0(M\setminus I)$ are similar to **forms of maximal degree** and are always closed, thus $T\wedge P_m(\omega)$ is closed on supp $d\chi \subset (M\setminus I)$. But from the Lie Cartan formula for currents ([65]), $0 = L_X\left(T\wedge P_m(\omega)\right) = \left(i_Xd + di_X\right)\left(T\wedge P_m(\omega)\right) = di_X\left(T\wedge P_m(\omega)\right)$ because $T\wedge P_m(\omega)$ is closed. We find that $di_X\left(T\wedge P_m(\omega)\right) = 0$ which means $i_X\left(T\wedge P_m(\omega)\right)$ is a closed curent and $\mathfrak{Res}_X[T](\omega)$ is the **period** of the *closed form* $d\chi$ relative to *the cycle* $i_X\left(T\wedge P_m(\omega)\right)$ in the sense of Hodge and De Rham (see [61] p. 135 and [30] p. 585). ■

Corollary 8.3.1 *Under the assumptions of Theorem 8.3.8, $\mathfrak{Res}_X[T](\omega)$ **does not depend** on the choice of χ.*

Proof — $\mathfrak{Res}_X[T](\omega)$ does not depend on the choice of χ because if χ_1, χ_2 are two smooth functions such that $\chi_i = 1$ in a neighborhood of I, then $\chi_1 - \chi_2 = 0$ in a neighborhood of I, thus $[d\chi_1] - [d\chi_2] = [d(\chi_1 - \chi_2)] = 0$. ■

Theorem 8.3.9 *Let $T \in E_s\left(\mathcal{D}'_k(M \setminus I)\right)$ and $\omega \in \mathcal{D}^k(M)$. If $\exists X \in \mathfrak{g}$ such that $L_X\left(T \wedge P_m(\omega)\right) = 0$, then $\mathfrak{Res}_X[T]$ is local in the sense it is a **current supported on** I and it depends only on the restriction on I of **finite jets** of the vector field X.*

Proof — To prove the locality in the vector field X, the key point is to notice that $\forall \varepsilon > 0, [d\chi] = [d\chi_{\varepsilon^{-1}}]$ in $H^1(M \setminus I)$ since $d\chi - d\chi_{\varepsilon^{-1}} = d(\chi - \chi_{\varepsilon^{-1}})$ where $(\chi - \chi_{\varepsilon^{-1}}) \in C^\infty(M \setminus I)$ vanishes in a neighborhood of I thus

$$\forall \varepsilon > 0, \mathfrak{Res}_X[T](\omega) = (-1)^{n-k} \left\langle i_X\left(T \wedge P_m(\omega)\right), [-d\chi_{\varepsilon^{-1}}]\right\rangle.$$

Since $T \wedge P_m(\omega)$ is a distribution in $\mathcal{D}'_0(M \setminus I)$ we can assume it is a distribution of order m_i on each open ball U_i of a given cover $(U_i)_i$ of M. Let $(\varphi_i)_i$ be a partition of unity subordinated to the cover $(U_i)_i$. Then we decompose the duality coupling:

$$\langle T \wedge P_m(\omega), L_X\chi\rangle = \sum_i \langle T \wedge P_m(\omega), \varphi_i L_X \chi\rangle$$

On each ball U_i, the distribution $T \wedge P_m(\omega)$ can be represented as a continuous linear form ℓ_i acting on the m_i-jet of $\varphi_i L_X \chi$ (this is the structure theorem of Laurent Schwartz for distributions [65])

$$\langle T \wedge P_m(\omega), L_X\chi\rangle = \sum_i \ell_i\left(j^{m_i}(\varphi_i L_X \chi)\right)$$

Hence we deduce from this result that $\mathfrak{Res}_X[T]$ depends locally on finite jets of X. We can conclude by taking the limit

$$\langle T \wedge P_m(\omega), L_X\chi\rangle = \lim_{\varepsilon \to 0} \langle T \wedge P_m(\omega), L_X \chi_{\varepsilon^{-1}}\rangle$$

$$= \lim_{\varepsilon \to 0} \sum_i \ell_i\left(j^{m_i}(\varphi_i L_X \chi_{\varepsilon^{-1}})\right)$$

which localizes the dependence on the jets of X restricted on I. ■

We know that $\mathfrak{Res}_X[T]$ is a local coboundary supported on I, but we don't know if $\mathfrak{Res}_X[T]$ is the coboundary of a cochain supported on I. We prove a theorem which gives a cohomological formulation of the existence of a $\mathfrak{g}$-invariant extension of the current T in terms of the residue of the extension R.

Theorem 8.3.10 *Let $T \in E_s\left(\mathcal{D}'_0(M \setminus I)\right)$ and T is $\mathfrak{g}$ invariant i.e. $\forall X \in \mathfrak{g}, L_X T = 0$. Then there exists an extension $\overline{T}$ of T which is $\mathfrak{g}$-invariant* ***if and only if*** *$X \mapsto \mathfrak{Res}_X[T]$ is the 1-coboundary of a current supported on I.*

Proof — We just follow the definitions. We view the map $X \mapsto RT$ as an element in $C^0(\mathfrak{g}, \mathcal{M})$ because it does not depend on $\mathfrak{g}$. Then $\Theta = \delta RT$ is the coboundary of RT. Let $\overline{T}$ be a $\mathfrak{g}$ invariant extension of T. Then $c = \overline{T} - RT$ is a current supported by I.

$$\forall X \in \mathfrak{g}, L_X c = L_X \left(\overline{T} - RT\right) = -L_X RT$$

because $L_X\overline{T} = 0$. But this means that we were able to write Θ as minus the coboundary of the cochain c supported on I. Conversely, if Θ is the coboundary of a local cochain c supported on I, then setting $\overline{T} = RT - c$ gives a $\mathfrak{g}$-invariant extension of T. ■

Anomalies in QFT and relation with the work of Costello. The author wants to stress that the suitable language to speak about anomalies in QFT is to write them as cocycles for the Lie algebra $\mathfrak{g}$ of symmetries with value in a certain module $\mathcal{M}$ which depends on the formalism in which we work. Usually, the Lie algebra $\mathfrak{g}$ is infinite dimensional.

In recent works of Kevin Costello, anomalies appear under the form of a **character** χ and constitute a central extension of the Lie algebra $\mathfrak{g}$ of symmetries, this is the content of the "Noether theorem" for factorization algebras discovered by Costello Gwilliam. They also require that this cocycle be local ie the cocycle χ is bilinear in $\mathfrak{g}$ with value in the module $\mathcal{M}$ and is represented by integration against a Schwartz kernel.

$$\chi(X_1, X_2) = \int_{M^2} \langle \chi(x_1, x_2), X_1(x_1) \otimes X_2(x_2)\rangle$$

where $\chi(x_1, x_2)$ is **supported on the diagonal** $d_2 \subset M^2$. In our work, we exhibit a purely **analytic way** to produce such local cocycles as residues. The residue $\mathfrak{Res}_X[T]$ is **local** in the sense it is a **current supported on** I and it depends only on the restriction on the submanifold I of **finite jets** of the vector field X.

Chapter 9

The meromorphic regularization.

9.1 Introduction.

The plan of the chapter. In this part, we would like to revisit the theory of meromorphic regularization using the techniques of chapter 1. We will show the advantages of the continuous partition of unity over the dyadic methods because it allows us to define an extension of distributions, that we call Riesz extension, using meromorphic techniques as in the "dimensional regularization" used in physics textbooks. The first step is to define some suitable space of distributions on which we can apply the meromorphic regularization procedure. It was suggested to the author by L Boutet de Monvel that such spaces are the spaces of distributions having asymptotic expansions with moderate growth in the transversal directions to I.

Given the canonical Euler vector field ρ, we define a simple notion of constant coefficient Fuchsian differential equation and first order Fuchsian system P, the solutions t of the constant coefficient Fuchsian systems are vectors with distributional entries. For instance a Fuchsian operator P in the vector case is of the form $P=\rho-\Omega$ where Ω is a constant square matrix. These Fuchs operators are adaptation of the concept of Fuchsian systems appearing in complex analysis. We first motivate the reason why we have to introduce asymptotic expansions in the space of distributions and the relationship with Fuchsian systems.

QFT example of Δ_+ and motivations.

In curved space times, the Hadamard states $\Delta_+(x,y)$ viewed as a two point distribution in $\mathcal{D}'\left(M^2\right)$ is not an exact solution of any **constant coefficient** Fuchsian equation that would come to our mind. Actually, we would like to study Δ_+ and its powers Δ_+^k.

For the Euler vector field $\rho = \frac{1}{2}\nabla_x\Gamma$ we have the following asymptotic expansion of Δ_+:

$$\Delta_+ = \sum_{n=0}^{\infty} U_n\Gamma^{-1} + V_n \log\Gamma + W_n \tag{9.1}$$

where U_n, V_n, W_n are homogeneous of degree n wrt ρ.

Proposition 9.1.1 *Let Δ_+ be the Hadamard parametrix and $\rho = \frac{1}{2}\nabla_x\Gamma$, then Δ_+ satisfies the equation:*

$$(\rho+2)(\rho+1)\rho^2\Delta_+ \in E_0. \tag{9.2}$$

Proof — Notice that if U_n is homogeneous of degree n since Γ^{-1} is homogeneous of degree -2 then we must have $(\rho - n + 2)U_n\Gamma^{-1} = 0$ and also $\rho V_n \log\Gamma = nV_n\log\Gamma + 2V_n$ which implies $(\rho-n)^2 V_n\log\Gamma = 2(\rho-n)V_n = 0$. We deduce the system of equations:

$$(\rho+2)U_0\Gamma^{-1} = 0 \tag{9.3}$$
$$(\rho+1)U_1\Gamma^{-1} = 0 \tag{9.4}$$
$$\rho U_2\Gamma^{-1} = 0 \tag{9.5}$$
$$\rho^2 V_0\log\Gamma = 0. \tag{9.6}$$

Thus if we act on $\sum_{n=0}^{\infty} U_n\Gamma^{-1} + V_n\log\Gamma + W_n$ by the differential operator $(\rho+2)(\rho+1)\rho^2$, the above system of equations shows that we will kill all singular terms in the sum $\sum_{n=0}^{\infty} U_n\Gamma^{-1} + V_n\log\Gamma + W_n$. ■

From this typical quantum field theoretic example, we understand that it is not possible to find **constant coefficients** Fuchsian operators that kills exactly the Feynman amplitudes. However, we can kill them with constant coefficients Fuchsian operators modulo an *error term which lives in nicer space* and go on successively. We define the space F_Ω of **Fuchsian symbols** which consists of distributions t having asymptotic expansions of the form $t = \sum_0^\infty t_k$ i.e. $\exists s \in \mathbb{R}, \forall N, t - \sum_0^N t_k \in E_{s+N}$, where we used the property that the scale spaces E_s are filtered, $s' \geqslant s \implies E_{s'} \subset E_s$. Intuitively, we would say that these are spaces of distributions which are killed by constant coefficients Fuchsian operators modulo an error term which can be made "arbitrarily nice", the price to pay for a nice error term is that we must use constant coefficients Fuchsian operators of arbitrary order.

The meromorphic regularization and the Mellin transform. We modify the extension formula of Hörmander $\int_0^1 d\lambda\lambda^{-1}t\psi_{\lambda^{-1}} + (1-\chi)t$ and define a regularization of the extension $t^\mu = \int_0^1 d\lambda\lambda^{\mu-1}t\psi_{\lambda^{-1}}$ depending on a parameter μ. We relate the new regularization formula to the Mellin transform. The idea actually goes back to Gelfand who considered Mellin transform of functions averaged on hypersurfaces (see [43] (4.5) Chapter 3 p. 326 and [3] (7.2.1) p. 218). When $t \in E_s(U \setminus I)$, we prove that t^μ has an extension in $E_{s+\mu}$ and is holomorphic in μ for $Re(\mu)$ large enough, intuitively, when $Re(\mu)$ is large enough the integral $\int_0^1 d\lambda\lambda^{\mu-1}t\psi_{\lambda^{-1}}$ has better chances to converge. Moreover, we can already prove that if there is any meromorphic extension $\mu \mapsto t^\mu$, then **the tail of the Laurent series must be local counterterms**. Now if we know that $t \in F_\Omega$, which is a much stronger assumption than $t \in E_s$, we then establish a nice identity satisfied by the regularized extension

$$\forall N, \langle T^\mu, \varphi\rangle = \sum_{j\leqslant N} (\mu + j + \Omega)^{-1} \langle (T\varphi)_j, \psi\rangle + \langle (I_N(T\varphi))^\mu, \psi\rangle, \qquad (9.7)$$

where $I_N(T\varphi) = \frac{1}{N!}\int_0^1 ds(1-s)^N \left(\frac{\partial}{\partial s}\right)^{N+1} s^{-\Omega} (T\varphi)_s$ is the remainder of the expansion $(T\varphi)_s = \sum_{j\leqslant N} s^{j+\Omega}(T \wedge \omega)_j + I_N(T\varphi)_s$, and we prove that the regularization $\mu \mapsto t^\mu$ can be extended meromorphically in μ with poles located in $Spec\,(\Omega) + \mathbb{N}$. We write explicit formulas for the poles of t^μ.

The Riesz extension. To go back to the **interesting case**, we have to take the limit of t^μ when $\mu = 0$. However, if $\mu = 0$ is a pole of finite order of t^μ, then we must remove the tail of the Laurent series which are **local counterterms**, i.e. distributions supported on I. Then we will prove that the operation of meromorphic regularization then removing the poles at $\mu = 0$ and finally taking the limit $\mu \to 0$ defines an **extension operation** which is called the Riesz extension and is a specific case of all the extensions defined in Chapter 1. Then we will show that the Fuchsian symbols renormalized by the Riesz extension are still Fuchsian symbols. Finally, we will explain how to introduce a length scale ℓ in the Riesz extension and how the one parameter renormalization group emerges in this picture and involves only polynomials of $\log \ell$.

Relationship to other works. In this Chapter, we give general definitions of Fuchsian symbols which are adapted to QFT in curved space times as we illustrated in our example. To our knowledge, these definitions were first given by Kashiwara–Kawai [54]. They also appear in the work of Richard Melrose [58]. We undertake the task of meromorphic regularizing Fuchsian symbols which are asymptotic expansions of a more general nature than associate homogeneous distributions.

9.2 Fuchsian symbols.

In QFT, scalings of distributions is not necessarily homogeneous, there are log terms. Distributions encountered in QFT are not solutions of equations of the form $(\rho-d)t=0$ but they might be solutions of equations of the form $(\rho-d)^n t=0$. We work in flat space $\mathbb{R}^{n+d}$ with coordinates $(x,h)\in\mathbb{R}^n\times\mathbb{R}^d$ and where $I=\{h=0\}$. The scaling is defined by the Euler vector field $\rho=h^j\partial_{h^j}$.

9.2.1 Constant coefficients Fuchsian operators.

Given the canonical Euler vector field ρ, we give a simple definition of a **constant coefficient** Fuchsian differential operator of order n:

Definition 9.2.1 *A constant coefficient Fuchsian operator of degree n is an operator of the form $b(\rho)$ where $b\in\mathbb{C}[X]$ is a polynomial of degree n with* ***real roots****.*

In QFT, these roots will often be integers.

Example 9.2.1 *Consider the one variable case where $\rho=h\frac{d}{dh}$. The monomial h^d is solution of the equation $(\rho-d)h^d=0$, hence $b(X)=(X-d)$. On the other hand $\log h$ is solution of the equation $\rho^2\log h=0$ hence $b(X)=X^2$. Lastly, $h^d\log h$ is solution of the equation $(\rho-d)^2h^d\log h=0$.*

Next define first order **constant coefficient** Fuchsian operators of rank n:

Definition 9.2.2 *A Fuchsian system of rank n is a differential operator of the form $P=\rho-\Omega$ where $\Omega=(\omega_{ij})_{1\leqslant ij\leqslant n}\in M_n(\mathbb{C})$ is a* ***constant*** *$n\times n$ matrix with real eigenvalues.*

Example 9.2.2 *The column $\begin{pmatrix}\log h\\ 1\end{pmatrix}$ is solution of the system*

$$\rho\begin{pmatrix}\log h\\ 1\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\log h\\ 1\end{pmatrix}$$

Let U be an arbitrary open domain which is ρ-convex. For b a n-th order operator (resp $P=\rho-\Omega$ a system), we give a fairly general definition of some new subspaces $F_b(U)$ (resp $F_\Omega(U)$) which are associated to the differential operators b (resp P) and which are different from the space $E_s(U)$ defined by Yves Meyer. However their definition uses the spaces $E_s(U)$ defined by Meyer. We define the space $F_b(U)$ of **Fuchsian symbols** associated to a Fuchsian operator b:

Definition 9.2.3 *Let $b(\rho)$ be a constant coefficients Fuchsian differential operator of order n. Then the space $F_b(U)$ of Fuchsian symbols is defined as the space distributions t s.t. there exists some neighborhood V of $I\cap\overline{U}$ and a sequence $(t_k)_k$ of distributions such that*

$$\forall N, t=\sum_{k=0}^N t_k+R_N \tag{9.8}$$

$$\forall k, b(\rho-k)t_k|_V=0 \tag{9.9}$$

where $\forall N, R_N \in E_{s+N+1}(U)$*,* $s = \inf Spec(b)$*.*

Example 9.2.3 *Let us consider the series* $\sum_{k=0}^{\infty} a_k h^{d+k}$*, then each term* $a_k h^{d+k}$ *is killed by the operator* $(\rho - d - k)$*.*

Definition 9.2.4 *Let* $\Omega = (\omega_{ij})_{1\leqslant ij \leqslant n} \in M_n(\mathbb{C})$ *be a* $n \times n$ *matrix and* $P = \rho - \Omega$ *be a Fuchsian operator of first order and rank* n*. Then the space of Fuchsian symbols* $F_\Omega(U)$ *is the space of vector valued distributions* $t = (t_i)_{1\leqslant i\leqslant n}$ *such that there exists some neighborhood* V *of* $I \cap \overline{U}$ *and a sequence* $(t_k)_k$ *of distributions such that*

$$\forall N, t = \sum_{k=0}^{N} t_k + R_N \tag{9.10}$$

$$\forall k, (\rho - (\Omega + k))\, t_k|_V = 0 \tag{9.11}$$

where $\forall N, R_N \in E_{s+N+1}(U)$*,* $s = \inf Spec(\Omega)$*.*

Some remarks on scalings. Assume $t \in F_\Omega$. Notice that for all test functions φ, the function $\lambda \mapsto \lambda^{-\Omega} \langle t_\lambda, \varphi \rangle$ is smooth in $(0,1]$ since $\langle t_\lambda, \varphi\rangle = \lambda^{-d} \langle t, \varphi_{\lambda^{-1}}\rangle$ and has a **unique asymptotic expansion** at $\lambda = 0$,

$$\lambda^{-\Omega} \langle t_\lambda, \varphi \rangle \sim \sum_{k=0}^{\infty} \lambda^k \langle t_k, \varphi \rangle .$$

But this does not mean that $\lambda \mapsto \lambda^{-\Omega} \langle t_\lambda, \varphi\rangle$ is smooth at $\lambda = 0$ as the following counterexample illustrates:

Example 9.2.4 *The function* $f(\lambda) = e^{\frac{-1}{\lambda^2}} \sin(e^{\frac{1}{\lambda^2}})$ *has asymptotic expansion* $e^{\frac{-1}{\lambda^2}} \sin(e^{\frac{1}{\lambda^2}}) \sim 0$ *and is smooth in* $(0,1]$*, however it is not smooth in* $[0,1]$ *since the first derivative of this function does not converge to zero when* $\lambda \to 0$*.*

However, we have a condition which implies the smoothness on $[0,1]$:

Lemma 9.2.1 *Let* $\lambda \mapsto f(\lambda)$ *be a function which is smooth on* $(0,1]$ *and which has an asymptotic expansion at* $\lambda = 0$*. Then if* $\forall n$*,* $f^{(n)}$ *has asymptotic expansion at* 0 *which is obtained by formally differentiating* n *times the expansion of* f *then* f *extends smoothly at* $\lambda = 0$*.*

The proof can be found in [31] lemme 1 p. 120.

We want to remind the reader there is a standard way to go from Fuchsian differential operators of order n to 1st order Fuchsian systems of rank n, this is called the companion system (see [42] 19B p. 332, 19E p. 342 for this classical construction).

Asymptotic expansions. We explain the connection with asymptotic expansions of distributions.

Definition 9.2.5 *The distribution t admits an asymptotic expansion if $t \in E_s(U)$ and if there exists a strictly increasing sequence of real numbers $(s_i)_i$ such that $s \leq s_0$*

$$\exists (t_i)_i, t_i \in E_{s_i}(U) \tag{9.12}$$

$$\forall N, \left(t - \sum_{i=1}^{N} t_i\right) \in E_{s_{N+1}}(U) \tag{9.13}$$

In concrete applications, the sequence $(s_i)_i$ is equal to $A + \mathbb{N}$ where A is a finite set of real numbers. So we see that our space of Fuchsian symbols is just a subspace of the space of distributions having asymptotic expansions. However, these spaces are less general than the spaces E_s defined by Yves Meyer as we shall illustrate in the following example

Example 9.2.5 $\sin(\frac{1}{x})$ *is weakly homogeneous of degree* 0 *on* $\mathbb{R}$*, thus it lives in* $E_0(\mathbb{R})$*. However, it admits no asymptotic expansion !*

We want to insist on the fact that our spaces F_Ω are defined in the smooth category and does not require any analyticity hypothesis.

9.2.2 Fuchsian symbols currents.

For a given Fuchsian operator $P = \rho - \Omega$ of first order and rank n, $F_\Omega(U)$ is the space of vector valued currents T such that there exists a sequence $(T_k)_k$ of distributions such that in a certain neighborhood V of $I \cap U$

$$\forall N, T = \sum_{k=0}^{N} T_k + R_N \tag{9.14}$$

$$\forall k, (\rho - (\Omega + k))\, T_k = 0 \tag{9.15}$$

where $\forall N, R_N \in E_{s+N+1}(U)$, $s = \inf Spec(\Omega)$. Recall also that we are able to decompose test forms ω as a sum

$$\omega = \sum_{n=0}^{m} \omega_n + I_m(\omega)$$

where the ω_n are homogeneous of degree n.

Notice that for any compactly supported test form ω, the exterior product $T \wedge \omega$ is a Fuchsian symbol and $T_k \wedge \omega_n$ satisfies the following exact equation:

$$\rho\,(T_k\omega_n) = (n + k + \Omega)\, T_k\omega_n. \tag{9.16}$$

On the relationship with the standard notion of Fuchsian differential equations. The theory of Fuchsian differential equation has an old story which goes back to great names such as Poincaré, Riemann and Fuchs. More recently, there was a resurgence of activities around these equations in the context of PDE's with famous works in analysis by Malgrange, Kashiwara, Leray, Pham. Some very nice surveys and textbooks now exist on the subjects, and our work is particularly inspired by ([56, 83, 3, 42, 80]) which give very nice expositions of this topic. Distributions solution to Fuchsian differential operators have several names. They were called 'associate homogeneous distributions" by [43].

These distributions are also called "hyperfunctions of the Nilsson Class" by Pham [56], for instance a similar proof of Proposition (3.2) p. 18 in [55] can be found in [56] p. 153,154.

9.2.3 The solution of a variable coefficients Fuchsian equation is a Fuchsian symbol.

The idea is that we want to deal with perturbations of the Euler equation $(\rho - \Omega)t = 0$ where Ω is a constant matrix. Let $\mathcal{I} \subset C^\infty(M)$ denote the ideal of smooth functions vanishing on I. Let $\tilde{\Omega}$ be a perturbation of Ω: $\tilde{\Omega} - \Omega \in M_n(\mathcal{I})$, note that this implies $\tilde{\Omega}|_I$ is constant and equals Ω. We are then able to prove that solutions of the Fuchsian operator with variable coefficients $P = \rho - \tilde{\Omega}$ are Fuchsian symbols. The space of Fuchsian symbols is thus the natural space of solutions of perturbed Euler equation.

Let us work in a local chart in $\mathbb{R}^{n+d}$ with coordinates (x, h) where $I = \{h = 0\}$ and $\rho = h^j \frac{\partial}{\partial h^j}$. Let $P = \rho - \tilde{\Omega}$ where $\tilde{\Omega} \in \Omega + M_n(\mathcal{I})$ and $\rho - \Omega$ is a first order Fuchsian system of rank n with constant coefficients.

For any complex number λ and matrix Ω, we define λ^Ω by the equation

$$\lambda^\Omega = \exp(\log \lambda \Omega)$$

for the branch $0 \leqslant \arg \log < 2\pi$ of the logarithm.

Example 9.2.6 *Before we state and prove the theorem, let us give an example in the holomorphic case on $\mathbb{C}$. Assume $t(z)$ is holomorphic in $\mathbb{C}\setminus\{0\}$ and solves the equation $z\frac{d}{dz}t - (\Omega - zh(z))t = 0$ where h is holomorphic in a neighborhood of $\{0\}$. Then $f(z) = z^\Omega t(z)$ solves the equation $z\frac{d}{dz}f - zh(z)f = 0 \implies \frac{d}{dz}f - h(z)f = 0$. But this means that $f(z) = e^{\int_{z_0}^z h(t)dt} f(z_0)$ is* ***holomorphic*** *in a neighborhood of zero. Hence by the principle of analytic continuation, we can extend the function f holomorphically at 0 ! Finally, $t(z) = \sum_{k=0}^\infty \frac{1}{k!} f^{(k)}(0) z^{k+\Omega}$ has the asymptotic expansion of Fuchsian symbols.*

However, in contrast with the previous example our theorem **does not assume any hypothesis of analyticity** since our perturbed operator $\rho - \Omega$ is an operator with smooth coefficients.

Theorem 9.2.1 *Let $\tilde{\Omega} \in M_n(C^\infty(M))$ s.t. there exists $\Omega \in M_n(\mathbb{C})$ with real roots satisfying $\tilde{\Omega} - \Omega|_I = 0$. If $t \in \mathcal{D}'(U \setminus I)$ is a solution of the equation $(\rho - \tilde{\Omega})t = 0$ then t is a Fuchsian symbol in the space $F_\Omega(U \setminus I)$ and $t = \sum_0^\infty t_k$ where $(t_k)_\lambda = \lambda^{\Omega+k} t_k$.*

Proof — The idea consists in proving that $\lambda^{-\Omega} t_\lambda$ is smooth in λ, then the Taylor expansion about $\lambda = 0$ of $\lambda^{-\Omega} t_\lambda$ will give us the expansion as Fuchsian symbol. We restrict to a set $K' = \{(x,h) || h| \leqslant R\}$ which is stable by scaling. We can pick a function χ which vanishes outside a compact neighborhood K of K', $\chi|_{K'} = 1$, then the distribution $t\chi$ equals t on K' and is an element of the dual space $(C^m(K))'$ of the *Banach space* $C^m(K)$ where m is the order of the distribution t (see Eskin theorem 6.4 page 22). The *topological dual* $(C^m(K))'$ of the Banach space $C^m(K)$ is also a *Banach space* for the operator norm. We want to prove that $\|\lambda^{-\Omega} t_\lambda \chi\|_{(C^m(K))'}$ is bounded for the Banach space norm $\|.\|_{(C^m(K))'}$ of $(C^m(K))'$ and we also want to prove that the map $\lambda \mapsto \lambda^{-\Omega} t_\lambda \chi$ is a smooth map for $\lambda \in [0,1]$ with value in the *Banach space* $(C^m(K))'$. We must precise the regularity of $\lambda^{-\Omega} t_\lambda \chi$ in $\lambda \in (0,1]$. From the identity $\langle t_\lambda \chi, \varphi \rangle = \langle t, \chi_{\lambda^{-1}} \varphi_{\lambda^{-1}} \rangle$ we can easily prove the C^0 regularity on $\lambda \in (0,1]$ with value distribution of order m. Then the derivative in λ is given by the formula $\partial_\lambda \left(\lambda^{-\Omega} t_\lambda \chi\right) = \lambda^{-1-\Omega}\left((\rho - \Omega) t_\lambda\right) \chi$ where $(\rho t_\lambda)\chi$ is of order $m+1$. This implies $\lambda \in (0,1] \mapsto \lambda^{-\Omega} t_\lambda \chi \in C^1\left((0,1], (C^{m+1}(K))'\right)$ then by recursion $\lambda \in (0,1] \mapsto \lambda^{-\Omega} t_\lambda \chi \in C^k\left((0,1], (C^{m+k}(K))'\right)$ where t is a distribution of order m. We see that at each time we increase the order of regularity in λ of one unit, we lose regularity of $\lambda^{-\Omega} t_\lambda \chi$ as a compactly supported distribution. For the moment, we know $\lambda^{-\Omega} t_\lambda$ is smooth in $\lambda \in (0,1]$ with value distribution but the difficulty is to prove that there is no blow up at $\lambda = 0$ and that it has a C^∞ extension for $\lambda \in [0,1]$. The idea is to exploit the fact it satisfies a differential equation and use a version of the Gronwall lemma for Banach space valued ODE. $f_\lambda = \lambda^{-\Omega} t_\lambda \chi$ is a solution of the linear ODE

$$\frac{d}{d\lambda} f_\lambda = \frac{\left(\tilde{\Omega} - \Omega\right)_\lambda}{\lambda} f_\lambda, f_1 = t\chi \tag{9.17}$$

where $\frac{\left(\tilde{\Omega}-\Omega\right)_\lambda}{\lambda} = \frac{e^{\log \lambda \rho \star}\left(\tilde{\Omega}-\Omega\right)}{\lambda}$ is smooth in $(\lambda, x, h) \in [0,1] \times \mathbb{R}^{n+d}$ since $\tilde{\Omega} - \Omega \in M_n(\mathcal{I})$. We want to prove that there is no blow up at $\lambda = 0$ which would give a unique extension of $\lambda^{-\Omega} t_\lambda \chi$ to $\lambda \in [0,1]$ by ODE uniqueness. We notice that there exists a constant C such that

$$\forall \lambda \in [0,1], \| \frac{\left(\tilde{\Omega} - \Omega\right)_\lambda}{\lambda} \lambda^{-\Omega} t_\lambda \chi \|_{(C^m(K))'} \leqslant C \|\lambda^{-\Omega} t_\lambda \chi\|_{(C^m(K))'}$$

since $\tilde{\Omega} - \Omega \in M_n(\mathcal{I})$ which means $\left(\tilde{\Omega} - \Omega\right)_\lambda = O(\lambda)$ and $\frac{\left(\tilde{\Omega}-\Omega\right)_\lambda}{\lambda}$ is bounded in λ in the space of smooth functions for usual C^∞ topology. Actually, we only need the simple estimate $\forall\lambda\in[0,1], \sup_{\lambda\in[0,1]}\|\frac{(\tilde{\Omega}-\Omega)_\lambda}{\lambda}\|_{C^m(K)}<\infty$, thus

$$f_\tau = f_1 + \int_1^\tau d\lambda \frac{\left(\tilde{\Omega}-\Omega\right)_\lambda}{\lambda} f_\lambda$$

and

$$\|f_\tau\|_{(C^m(K))'} \leqslant \|f_1\|_{(C^m(K))'} + \|\int_1^\tau d\lambda \frac{\left(\tilde{\Omega}-\Omega\right)_\lambda}{\lambda} f_\lambda\|_{(C^m(K))'}$$

by the triangle inequality

$$\|f_\tau\|_{(C^m(K))'} \leqslant \|f_1\|_{(C^m(K))'} + \int_\tau^1 d\lambda \|\frac{\left(\tilde{\Omega}-\Omega\right)_\lambda}{\lambda} f_\lambda\|_{(C^m(K))'}$$

by Minkowski inequality

$$\|f_\tau\|_{(C^m(K))'} \leqslant \|f_1\|_{(C^m(K))'} + C\int_\tau^1 d\lambda \|f_\lambda\|_{(C^m(K))'}$$

and we can conclude by an application of the Gronwall lemma. We deduce that $\forall\lambda\in[0,1], \|f_\lambda\|_{(C^m(K))'}\leqslant e^{C(1-\lambda)}\|f_1\|_{(C^m(K))'}$. Hence f_λ exists on $[0,1]$ (for more on Gronwall see [73] Theorem 1.17 p. 14) otherwise there would be blow up at $\lambda=0$ but the Gronwall lemma prevents f_λ from blowing up at $\lambda=0$. Since the ODE (9.17) has smooth coefficients the value of its solution is smooth in λ. To conclude, we Taylor expand $\lambda^{-\Omega}t_\lambda\chi$ in λ

$$\lambda^{-\Omega}t_\lambda\chi = \sum_{k=0}^\infty \frac{\lambda^k}{k!}u_k$$

hence using $\chi|_K = 1$:

$$t_\lambda|_K = \sum_{k=0}^\infty \frac{\lambda^{k+\Omega}}{k!}u_k|_K.$$

Hence we deduce the conclusion with $t_k|_K = \frac{u_k}{k!}$. ■

9.2.4 Stability of the concept of approximate Fuchsians.

First, the space F_Ω is stable by left product with elements in $C^\infty(M)$, the proof is simple by Taylor expanding the smooth function. Let G be the space of diffeomorphisms of M **fixing** I. Before we end this section, let us prove a theorem which shows that the space $F_\Omega(U)$ of Fuchsian symbols is stable by action of G. This result will imply that $F_\Omega(U)$ does not depend on the choice of Euler ρ. Before proving the theorem we give some useful lemmas:

Lemma 9.2.2 *Let* $\Phi(\lambda) = S(\lambda)^{-1} \circ \Phi \circ S(\lambda)$ *where* $S(\lambda) = e^{\log \lambda \rho}$ *and* $\Phi = e^X$ *for some vector field* X *which vanishes on* I*. Then* $\Phi(\lambda)$ *is smooth in* $\lambda \in [0,1]$ *and* $\Phi(0)$ *is a diffeomorphism fixing* I *which commutes with* ρ *and* $\Phi, \Phi(0)$ *have the same 1-jet on* I*.*

Proof — Let $\Phi(\lambda) = S(\lambda)^{-1} \circ \Phi \circ S(\lambda)$. We assume $\Phi = e^X \in G$ where $X \in \mathfrak{g}$ is a vector field vanishing on I thus $\Phi(\lambda) = S(\lambda)^{-1} \circ \Phi \circ S(\lambda) = S(\lambda)^{-1} \circ e^X \circ S(\lambda) = e^{S(\lambda)^{-1} \circ X \circ S(\lambda)} = e^{X(\lambda)}$ where $X(\lambda) = S(\lambda)^{-1} \circ X \circ S(\lambda)$. $\lim_{\lambda \to 0} X(\lambda) = X(0)$ exists since $X = h^i a_i^j(x,h)\partial_{h^j} + h^i b_i^j(x,h)\partial_{x^j}$ hence $X(\lambda) = h^i a_i^j(x,\lambda h)\partial_{h^j} + \lambda h^i b_i^j(x,\lambda h)\partial_{x^j}$ and $X(0) = h^i a_i^j(x,0)\partial_{h^j}$. We recall the following important fact, $X(0)$ is in fact scale invariant i.e. it commutes with ρ. thus $\Phi(0) = e^{X(0)}$ commutes with ρ. Moreover an easy computation:

$$\begin{aligned}(X - X(0))\, h^i H_i(x,h) \\ = \left(h^i (a_i^j(x,h) - a_i^j(x,0))\partial_{h^j} + h^i b_i^j(x,h)\partial_{x^j}\right) h^i H_i(x,h)\end{aligned}$$

and the fact that $a_i^j(x,h) - a_i^j(x,0) \in \mathcal{I}$ prove that $(X - X(0))h^i H_i(x,h) = O(|h|^2)$. Thus $(X - X(0))\mathcal{I} \subset \mathcal{I}^2$ which implies $\left(e^X - e^{X(0)}\right)^\star \mathcal{I} = (\Phi - \Phi(0))^\star \mathcal{I} \subset \mathcal{I}^2$. This is enough to prove that Φ and $\Phi(0)$ have same 1-jet along I. ■

Lemma 9.2.3 *Under the hypothesis of the above lemma, the pull-back operator* $\Phi(\lambda)^\star$ *admits a Taylor expansion of the following form:*

$$\Phi(\lambda)^\star = \sum_{k=0}^{N} \frac{\lambda^k}{k!} \mathbb{D}_k \Phi_0^* + I_N(\Phi, \lambda)^\star$$

where $\mathbb{D}_k$ *is a differential operator which depends polynomially on finite jets of* X *and* ρ *at* I*.*

Proof — We start from the identity $\lambda \frac{d}{d\lambda} X(\lambda) = \lambda \frac{d}{d\lambda} Ad_{S(\lambda)} X = -[\rho, X(\lambda)]$. This implies

$$\begin{aligned}\partial_\lambda^i X(\lambda) &= \frac{1}{\lambda^i} \lambda^i \partial_\lambda^i X(\lambda) = \frac{1}{\lambda^i i!} \lambda \frac{d}{d\lambda} \dots \left(\lambda \frac{d}{d\lambda} - i + 1\right) X(\lambda) \\ &= \frac{1}{\lambda^i i!} (-ad_\rho) \dots (-ad_\rho - i + 1) X(\lambda) \\ \implies \partial_\lambda^i X(0) &= \lim_{\lambda \to 0} \frac{1}{\lambda^i i!} (-ad_\rho) \dots (-ad_\rho - i + 1) X(\lambda).\end{aligned}$$

Hence the derivatives $\partial_\lambda^i X(0)$ only depend polynomially on finite jets of X and ρ at $(x,0)$. Then we Taylor expand the map $\Phi(\lambda)$ at $\lambda = 0$:

$$\Phi(\lambda) = \sum_{k \leqslant N} \frac{\lambda^k}{k!} \left(\partial_\lambda^k e^{X(\lambda)\star}\right)_{\lambda=0} + I_N(\Phi, \lambda)^\star$$

by definition of the exponential map and successive differentiation, the terms $\left(\partial_\lambda^k e^{X(\lambda)\star}\right)_{\lambda=0}$ are all of the form $\mathbb{D}_k\Phi_0^*$ where each $\mathbb{D}_k$ is a differential operator in $\mathbb{C}\left\langle\partial_\lambda^i X(0)\right\rangle_i$, for instance:

$$\mathbb{D}_1 = \partial_\lambda X(0), \mathbb{D}_2 = \partial_\lambda^2 X(0) + (\partial_\lambda X)^2(0).$$

■

A consequence of the above lemma is that for all distribution t, for all N, λ, the pull-back $I_N(\Phi,\lambda)^\star t$ exists and we can bound its wave front set:

$$WF(I_N(\Phi,\lambda)^\star t) \subset \Phi(0)^\star WF(t) \cup \Phi(\lambda)^\star WF(t).$$

Theorem 9.2.2 *Let $t \in F_\Omega^\rho$ for a choice of ρ, t has the asymptotic expansion $t = \sum_l t_l$, and $\Phi = e^X \in G$ for X vanishing on I. Then we have $\Phi^\star t \in F_\Omega^\rho$ and $\Phi^\star t = \sum_{n=0}^\infty \tilde{t}_n$ where $\tilde{t}_n$ depends only on $t_l, l \leqslant n$ and polynomially on finite jets of ρ, X at I.*

Proof — Since $\Phi(\lambda)$ depends smoothly in λ and $\lambda^{-\Omega}t_\lambda$ admits an asymptotic expansion at $\lambda = 0$, the pulled back family $\Phi(\lambda)^*(\lambda^{-\Omega}t_\lambda) = \lambda^{-\Omega}(\Phi^* t)_\lambda$ admits an asymptotic expansion at $\lambda = 0$. In order to conclude, we expand $\lambda^{-\Omega}t_\lambda = \sum_{l=0}^\infty \lambda^{-\Omega+l}t_l$ and $\Phi(\lambda) = \sum_{k=0}^\infty \frac{\lambda^k}{k!}\mathbb{D}_k\Phi_0^*$ and we obtain the general expansion

$$\Phi(\lambda)^*\left(\lambda^{-\Omega}t_\lambda\right) = \sum_{n=0}^\infty \lambda^{-\Omega+n}\sum_{k+l=n}\frac{1}{k!}\mathbb{D}_k\Phi_0^* t_l.$$

■

We keep the notation and hypothesis of the above theorem

Corollary 9.2.1 *Let Γ be a cone in $T^\bullet(M \setminus I)$. If $\forall k$, $WF(t_k) \subset \Gamma$ then $\forall n, WF(\tilde{t}_n) \subset \Phi_0^\star\Gamma$.*

We deduce from the previous theorem an important corollary which is that the class of Fuchsian symbols F_Ω is **independent** of the choice of Euler vector field.

Corollary 9.2.2 *Let $t \in F_\Omega^\rho$ for a choice of ρ, then for any other generalized Euler $\tilde{\rho}$, we have $t \in F_\Omega^{\tilde{\rho}}$.*

Proof — By the result of chapter 1, for any other vector $\tilde{\rho}$, we have $\Phi^{-1*}\tilde{\rho} = \rho$ for a diffeomorphism Φ fixing I.

$$0 = \rho t - \Omega t = \Phi^{-1*}\tilde{\rho}\Phi^* t - \Phi^{-1*}\Omega\Phi^* t \implies \tilde{\rho}\Phi^* t - \Omega\Phi^* t = 0$$

this means $\Phi^* t$ is killed by the Fuchsian operator $\tilde{\rho} - \Omega$ thus $\Phi^* t \in F_\Omega^{\tilde{\rho}}$. ■

9.3 Meromorphic regularization as a Mellin transform.

In this section, for pedagogical reasons, we work in local charts in order to make as explicit as possible the relationship with the Mellin transform. More precisely, we work in a given fixed compact subset $K = K_1 \times K_2 \subset \mathbb{R}^{n+d}$, the compact set is geodesically convex for $\rho = h^j\partial_{h^j}$. All test functions are supported in K. Let $\chi \in C_0^\infty(\mathbb{R}^{n+d})$, $\chi \geqslant 0$ and $\chi|_{K\cap\{|h|\leqslant a\}} = 1, \chi|_{K\cap\{|h|\geqslant b\}} = 0$ where $b > a > 0$.

$$\langle T, \omega\rangle = \int_0^1 \frac{d\lambda}{\lambda} \langle T\psi_{\lambda^{-1}}, \omega\rangle + \langle T(1-\chi), \omega\rangle \tag{9.18}$$

The meromorphic regularization formula. We modify the extension formula of Hörmander by introducing a weight λ^μ in the integral over the scale λ:

$$\langle T^\mu, \omega\rangle = \int_0^1 \frac{d\lambda}{\lambda}\lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega\rangle, \tag{9.19}$$

this defines a regularization of the extension depending on a parameter μ. We would like to call the attention of the reader on the fact that if the test form ω was not supported on I, we would have a well defined extension at the limit $\mu \to 0$.

The philosophy of meromorphic regularization. The goal is to prove that T^μ can be extended to a family of current in $\mathcal{D}'_k(U)$ depending holomorphically in μ for $Re(\mu)$ large enough. Then under the hypothesis that T is a Fuchsian symbol, T^μ should extend **meromorphically** in μ with poles at $\mu = 0$ which are currents supported on I (ie local counterterms). Then the meromorphic regularization will be given by the formula

$$\lim_{\mu\to 0} \left(T^\mu + T(1-\chi) - \text{poles at } \mu = 0 \text{ with value current supported on } I\right) \tag{9.20}$$

Definition 9.3.1 *A family $(T^\mu)_\mu$ of currents in $\mathcal{D}'_k(U)$ is said to be holomorphic (resp meromorphic) in μ iff for all test forms $\omega \in \mathcal{D}^k(U)$, $\mu \mapsto \langle T^\mu, \omega\rangle \in \mathbb{C}$ is holomorphic (resp meromorphic).*

If $\mu \mapsto T^\mu$ is holomorphic in a domain $B_r(\mu_0) \setminus \{\mu_0\}$, for all test functions φ, the map $\mu \mapsto \langle T^\mu, \varphi\rangle$ has an expansion in Laurent series in μ around μ_0, $\langle T^\mu, \varphi\rangle = \sum_{k=-\infty}^{k=+\infty}(\mu-\mu_0)^k \left\langle T^{\mu_0(k)}, \varphi\right\rangle$ where each coefficient of the Laurent series is a distribution tested against φ (there is a similar discussion in [43] Chapter 1 appendix 2).

Proof — By the Cauchy formula and by the holomorphicity of $\langle T^\mu, \varphi\rangle$, for all test function φ, we must have

$$\forall k \in \mathbb{Z}, \left\langle T^{\mu_0(k)}, \varphi \right\rangle = \frac{1}{2i\pi} \int_{\partial B_r(\mu_0)} \frac{d\mu}{(\mu-\mu_0)^{k+1}} \langle T^\mu, \varphi \rangle .$$

Thus we define $T^{\mu_0(k)} = \frac{1}{2i\pi}\int_{\partial B_r(\mu_0)} \frac{d\mu}{(\mu-\mu_0)^{k+1}} T^\mu$ which is a linear map on $\mathcal{D}(U)$. To prove the continuity, we just use the Banach Steinhaus theorem, for all compact $K \subset U$, there exists $C > 0$ and a seminorm π_m s.t. for all $\varphi \in \mathcal{D}_K(U)$

$$\forall \mu \in \partial B_r(\mu_0), |\langle T^\mu, \varphi\rangle| \leqslant C\pi_m(\varphi),$$

thus

$$\forall \varphi \in \mathcal{D}_K(U), |\left\langle T^{\mu_0(k)}, \varphi \right\rangle| \leqslant C r^{-k} \pi_m(\varphi),$$

which proves the continuity of $T^{\mu_0(k)}$ for all k. ■

Thus we can write the Laurent series expansion of $\mu \mapsto T^\mu$ around μ_0 as a series in powers of $(\mu - \mu_0)$ with distributional coefficients:

$$T^\mu = \sum_{k=-\infty}^{k=+\infty} (\mu-\mu_0)^k T^{\mu_0(k)}.$$

Definition 9.3.2 *We say that $\mu \mapsto T^\mu$ is meromorphic with poles of order N at μ_0 when $\mu \mapsto T^\mu$ is holomorphic in a domain $B_r(\mu_0) \setminus \{\mu_0\}$ and $T^\mu = \sum_{k=-N}^{k=+\infty} (\mu-\mu_0)^k T^{\mu_0(k)}$.*

Using this definition, it makes sense to speak about the support of the poles, it just means the support of the distributions $T^{\mu_0(k)}$ for $k < 0$.

The holomorphicity theorem.

Recall that T^μ is defined by the formula $\langle T^\mu, \omega\rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega\rangle$.

Lemma 9.3.1 *If $T \in E_s\left(\mathcal{D}'_k\left(U \setminus I\right)\right)$, then T^μ has a well defined extension in $\mathcal{D}'_k(U)$ for $Re(\mu) + s + k - n > 0$ and $T^\mu \in E_{s+Re(\mu)}(\mathcal{D}'_k(U))$.*

Proof — We keep the notation of the proof of theorem (1.2) and we recall the main facts. In the proof of theorem (1.2), we proved that if $(c_\lambda)_\lambda$ is a bounded family of distributions supported on a fixed annulus $a \leqslant |h| \leqslant b$, then $\lambda^{-d} c_\lambda(., \lambda.)$ is a bounded family of distributions. Hence from the boundedness of the family $(c_\lambda = \lambda^{-s} t_\lambda \psi)_\lambda$, we deduced the boundedness of the family $(\lambda^{-d} c_\lambda(., \lambda.) = \lambda^{-s-d} t\psi_{\lambda^{-1}})_\lambda$. By reasoning as in the proof of theorem (1.2) in Chapter 1, the function $\lambda \mapsto f(\lambda) = \lambda^{-s-(k-n)} \langle T\psi_{\lambda^{-1}}, \omega\rangle$ is a bounded function supported on the interval $[0, 1]$. Thus we find

$$\langle T^\mu, \omega\rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega\rangle$$

$$= \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\mu+s+k-n} \lambda^{-s-(k-n)} \langle T\psi_{\lambda^{-1}}, \omega \rangle = \int_0^{+\infty} \frac{d\lambda}{\lambda} \lambda^{\mu+s+k-n} f(\lambda)$$

The last integral **converges** when $Re(\mu) + s + k - n > 0$ because f is bounded on $[0, 1]$. This already tells us that the family of currents $(T^\mu)_\mu$ is **well defined** in $\mathcal{D}'_k(U)$ when $Re(\mu) + s + k - n > 0$. To prove that $T^\mu \in E_{s+Re(\mu)}$, we use the theorem (2.1) proved in Chapter 1 for the bounded family of currents $(c_\lambda = \lambda^{-s} T_\lambda \psi)_\lambda$ supported on a fixed annulus. ■

We establish a neat result namely that the function $\lambda \mapsto \langle T\psi_{\lambda^{-1}}, \omega \rangle$ is in fact always smooth in $\lambda \in (0, 1]$. But of course that does not mean it should be L^1_{loc} at $\lambda = 0$.

Lemma 9.3.2 $\lambda \mapsto \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega \rangle$ *is smooth in* $0 < \lambda \leqslant 1$.

Proof — There is a compact set $K = \text{supp } \omega$ such that if $x \notin K$, $\psi_{\lambda^{-1}}\omega(x) = 0, \forall \lambda \in (0, 1]$. Also $\lambda \mapsto \psi_{\lambda^{-1}}\omega$ is smooth in λ. Then the result follows from application of Theorem 2.1.3 in [40]. ■

Theorem 9.3.1 *We keep the notation and hypothesis of lemma (9.3.1), then* $\forall \omega \in \mathcal{D}^k(U)$ *(resp* $\omega \in \mathcal{D}^k(U \setminus I)$*), the map* $\mu \mapsto \langle T^\mu, \omega \rangle$ *is holomorphic in the* ***half-plane*** $Re(\mu) + s + k - n > 0$ *(resp holomorphic in* $\mathbb{C}$*).*

Proof — We relate the regularization formulas to the Mellin transform. By definition, the Mellin transform of a distribution $f \in \mathcal{D}'(\mathbb{R}^+)$ is given by the formula (see “The Mellin Transformation and Other Useful Analytic Techniques” by Don Zagier in [81] p. 305 and [44])

$$\tilde{f}(\mu) = \int_0^\infty \frac{d\lambda}{\lambda} \lambda^\mu f(\lambda). \tag{9.21}$$

Actually, in the notation of Zagier, we study the half-Mellin transform:

$$\tilde{f}_{\leqslant 1}(\mu) = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu f(\lambda) \tag{9.22}$$

The regularization formula (9.19) is the **Mellin transform** of the function $\lambda \mapsto \langle T\psi_{\lambda^{-1}}, \omega \rangle \chi_{[0,1]}$, where χ is the characteristic function of the interval $[0, 1]$. The function $\lambda \mapsto f(\lambda) = \lambda^{-s-(k-n)} \langle T\psi_{\lambda^{-1}}, \omega \rangle \chi_{[0,1]}$ is a function in $C^\infty(0, 1] \cap L^\infty[0, 1]$ (however, it is not smooth at 0), $\langle T^\mu, \omega \rangle$ is thus reinterpreted as the Mellin transform $\Gamma_f(\mu + s + k - n)$ of $f \in C^\infty(0, 1] \cap L^\infty[0, 1] \implies f \in L^1[0, 1]$. Then we use the classical holomorphic properties of the Mellin transform as explained in [74] appendix A p. 308,309. To understand the holomorphicity properties of the Mellin transform, we relate the Mellin transform with the Fourier Laplace transform in the complex plane by the variable change $e^t = \lambda$ (see [74] appendix A formula A.18 p. 308)

$$\int_0^1 \frac{d\lambda}{\lambda}\lambda^s f(\lambda) = \int_{-\infty}^0 dte^{ts} f(e^t) = \int_{-\infty}^{\infty} dte^{-ts} f(e^{-t})H(t)$$

where H is the Heaviside function and where $t \mapsto f(e^{-t})H(t)$ is bounded. For any $\varepsilon > 0$, $t \mapsto e^{-t\varepsilon}H(t)f(e^{-t})$ is in $L^p(\mathbb{R}), \forall p \in [1,\infty]$, especially in $L^2(\mathbb{R})$ hence

$$s \mapsto \int_{-\infty}^{\infty} dte^{-t(s+\varepsilon)} f(e^{-t})H(t)$$

is holomorphic in s for $Re(s) \geqslant 0$ by the properties of the holomorphic Fourier transform. As this is true for any $\varepsilon > 0$, the Mellin transform is holomorphic on $Re(s) > 0$. ■

Let us keep the notations of the previous theorem and consider the family $\mu \mapsto T^\mu$ holomorphic for $Re(\mu) + s + k - n > 0$. We prove a lemma which states that if there is a meromorphic extension of the holomorphic family $\mu \mapsto T^\mu$, then this meromorphic extension must have poles supported on I (ie locality of counterterms).

Lemma 9.3.3 *If $\mu \mapsto T^\mu$ is a meromorphic extension of the holomorphic family $\mu \mapsto T^\mu$, then the poles of T^μ are distributions in $\mathcal{D}'(U)$* ***supported*** *on $U \cap I$ i.e. they are local counterterms.*

Proof — $\forall \omega \in \mathcal{D}^k(U), \mu \mapsto \langle T^\mu, \omega\rangle$ is holomorphic in the **half-plane** $Re(\mu) + s + k - n > 0$. Let us notice that if $\omega \in \mathcal{D}^k(U \setminus I)$, the function $\lambda \mapsto \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega\rangle$ is smooth in λ and vanishes in a neighborhood of $\lambda = 0$, hence the formula (9.19) makes sense for all $\mu \in \mathbb{C}$ and is holomorphic in μ. If T^μ had a meromorphic expansion, then we write the Laurent series expansion of $\mu \mapsto T^\mu$ around some value $\mu_0 \in \mathbb{C}$:

$$T^\mu = \sum_{k=-N}^{k=+\infty} (\mu - \mu_0)^k T^{\mu_0(k)}$$

but for all ω supported on $U \setminus I$, $\langle T^\mu, \omega\rangle$ is holomorphic at μ_0 thus all the poles $(\langle T^{\mu_0(k)}, \omega\rangle)_{k<0}$ must vanish ! $\forall \omega \in \mathcal{D}^k(U \setminus I), \forall k < 0, \langle T^{\mu_0(k)}, \omega\rangle = 0$ which means $\forall k < 0, \text{supp } T^{\mu_0(k)}$ does not meet $U \setminus I$ which yields the conclusion. ■

9.3.1 The meromorphic extension.

We set the stage for our next theorem which states that if T is a Fuchsian symbol, then the holomorphic regularization formula of Hörmander $\mu \mapsto T^\mu$

has a **meromorphic extension** in the complex parameter μ. Let $T \in \mathcal{D}'_k(U \setminus I)$ and if $T \in F_\Omega(U \setminus I)$ then we have by definition $T = \sum_0^N T_k + R_N$ where the error term $R_N \in E_{s+N+1}$ where $s = \inf Spec(\Omega)$. Notice that for any compactly supported test form ω, the current $T \wedge \omega$ is also a Fuchsian symbol, and we have the expansion $\forall N, (T \wedge \omega) = \sum_{j \leqslant N} (T \wedge \omega)_j + I_N(T \wedge \omega)$ where $(T \wedge \omega)_{js} = s^{j+\Omega}(T \wedge \omega)_j$ and the remainder $I_N(T \wedge \omega) \in E_{s+N+1}$. Following the notations of Chapter 1, we denote by ψ the function $(-\rho\chi)$.

Theorem 9.3.2 *If $T \in F_\Omega(U \setminus I)$ then $\mu \mapsto T^\mu$ has an extension as a distribution in $\mathcal{D}'(U)$ and depends meromorphically in μ with poles in $-Spec(\Omega) - \mathbb{N}$.*

$$\forall p, \exists N, \langle T^\mu, \omega \rangle = \sum_{j \leqslant N} (\mu + j + \Omega)^{-1} \langle (T \wedge \omega)_j, \psi \rangle + \left\langle I_N^\mu(T \wedge \omega), \psi \right\rangle \quad (9.23)$$

where the identity is meromorphic in the domain $\{Re(\mu) + p > 0\}$.

Proof — Before we start proving anything, let us make a small comment on the principle used here. The key idea is **analytic continuation**, when two holomorphic functions f_1, f_2 defined on respective domains U_1, U_2 coincide on an open set, then there is a **unique** function f (unique in the sense that any analytic continuation of $f_i, i = 1, 2$ must coincide with f on their common domain of definition) defined on $U_1 \bigcup U_2$ which extends f_1, f_2. Recall that the exterior product $(T \wedge \omega)$ is a Fuchsian symbol since $T \in F_\Omega$ is Fuchsian and ω is a smooth test form. Thus $\lambda^{-\Omega}(T \wedge \omega)_\lambda$ has an asymptotic expansion in λ. We expand $(T \wedge \omega)$ in order to extract the relevant first terms and the remainder of the asymptotic expansion.

$$T \wedge \omega = \sum_{k=0}^N \underbrace{(T \wedge \omega)_k}_{\text{killed by } \rho - k - \Omega} + \underbrace{I_N(T \wedge \omega)}_{\in E_{N+\Omega+1}}$$

we replace this decomposition in the integral formula $\int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle T \psi_{\lambda^{-1}}, \omega \rangle$. The computation gives:

$$\forall N, \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle T \psi_{\lambda^{-1}}, \omega \rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle (T \wedge \omega), \psi_{\lambda^{-1}} \rangle$$

$$= \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle (T \wedge \omega)_\lambda, \psi \rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left(\sum_{j \leqslant N} \langle (T \wedge \omega)_{j\lambda}, \psi \rangle + \langle (I_N(T \wedge \omega))_\lambda, \psi \rangle \right)$$

$$= \sum_{j \leqslant N} \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\mu + \Omega + j} \langle (T \wedge \omega)_j, \psi \rangle + \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle (I_N(T \wedge \omega))_\lambda, \psi \rangle .$$

Then for $Re(\mu)$ large enough, the first $N+1$ integrals converge and can be computed

$$= \sum_{j\leqslant N} \underbrace{(\mu+\Omega+j)^{-1}}_{\text{poles when }\det(\mu+\Omega+j)=0} \langle (T\wedge\omega)_j, \psi\rangle + \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \underbrace{\langle (I_N(T\wedge\omega))_\lambda, \psi\rangle}_{O(\lambda^{N+1+\Omega+Re(\mu)})}$$

where the remainder is integrable and holomorphic in μ in the half plane $Re(\mu)+N+1+\Omega>0$ by theorem (9.3.1). Finally for all N, $\langle T^\mu, \omega\rangle$ has meromorphic continuation on $Re(\mu)+N+1+\Omega>0$ hence it has meromorphic continuation everywhere on $\mathbb{C}$. ■

By a matrix conjuguation, we can always reduce Ω to its Jordan normal form $\Omega = G^{-1}(D+N)G$ where D is diagonal and N is a nilpotent matrix which commutes with D. We set $(-d_i, n_i)_{i\in I}$ the eigenvalues of Ω with their respective multiplicities, hence D is a diagonal matrix with eigenvalues $(-d_i)_i$. Note that if $0\in -Spec(\Omega)-\mathbb{N}$, then $\mu=0$ is a pole of the meromorphic extension: $0=d_i-j$ where $j\in\mathbb{N}$ and d_i is an eigenvalue of Ω with multiplicity n_i.

Proposition 9.3.1 *Let $\Omega\in M_n(\mathbb{C})$ and $T\in F_\Omega(U\setminus I)$. If $Spec\,(\Omega)\cap -\mathbb{N}=\emptyset$ then T^μ is holomorphic at $\mu=0$. If $Spec\,(\Omega)\cap -\mathbb{N}\neq\emptyset$ then T^μ has a pole at $\mu=0$ of order at most n.*

Proof — We assume that $d_i-j=0$ for some eigenvalue $d_i\in Spec(\Omega)$ and some integer j. Up to conjuguation and projection, the proof reduces to an elementary computation in a generalized eigenspace E_i of dimension n_i associated to the eigenvalue $-d_i$ s.t. $d_i-j=0$. Indeed, $\Omega|_{E_i}=-d_i+N_i$ where N_i is a *nilpotent* matrix of fixed order n_i. $(\mu+\Omega+j)^{-1}|_{E_i}=(\mu+N_i)^{-1}=\mu^{-1}\left(\sum_{k=0}^{n_i-1}(-1)^k\mu^{-k}N_i^k\right)=\mu^{-1}+\cdots+\mu^{-n_i}(-1)^{n_i-1}N_i^{n_i-1}$, so the worst singularity is a pole of order at most n_i in μ. ■

Proposition 9.3.2 *The extension T^μ defined in the previous theorem satisfies the property $T^\mu\in F_{\Omega+\mu}$.*

Proof — To prove that $T^\mu\in F_{\Omega+\mu}$, it is enough to prove that if T is a solution of $(\rho-\Omega)T=0$, then the meromorphic extension T^μ is solution of the equation $(\rho-\Omega-\mu)T^\mu=0$ **on the domain** $\chi=1$. We try to scale T^μ and we compute $\tau^{-\Omega-\mu}T^\mu(.,\tau.)$ where $T\in\mathcal{D}'_k(U\setminus I)$ is exact Fuchsian $T_\lambda=\lambda^\Omega T$. First, it is not true that T^μ will scale exactly like $T^\mu_\tau=\tau^{\Omega+\mu}T^\mu$ everywhere in $U\setminus I$. However, in any ρ-stable domain U for $\rho=h^j\partial_{h^j}$ in which $\chi|_U=1$, we will be able to find that $\forall\tau\in(0,1], T^\mu_\tau|_U=\tau^{\Omega+\mu}T^\mu|_U$. This can be understood in terms of section $T^\mu|_U$ of the **sheaf of currents** over the open set U. A typical example of such nice domains would be

$K \times \{|h| \leqslant a\} \subset \mathbb{R}^n \times \mathbb{R}^d$ in the local chart $\mathbb{R}^{n+d}$ where the plateau function χ satisfies the support condition:

$$\chi_{K \times \{|h| \leqslant a\}} = 1, \chi_{K \times \{|h| \geqslant b\}} = 0 \tag{9.24}$$

for $0 < a < b$. We pick a test form $\omega \in \mathcal{D}'(U)$.

$$\forall 0 < \tau \leqslant 1, \tau^{-\Omega-\mu} \langle T^\mu_\tau, \omega \rangle = \tau^{-\Omega-\mu} \langle T^\mu, \omega_{\tau^{-1}} \rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \tau^{-\Omega-\mu} \langle T\psi_{\lambda^{-1}}, \omega_{\tau^{-1}} \rangle$$

$$= \int_0^1 \frac{d\lambda}{\lambda} \left(\frac{\lambda}{\tau}\right)^\mu \tau^{-\Omega} \langle T_\lambda \psi, \omega_{\lambda\tau^{-1}} \rangle = \int_0^1 \frac{d\lambda}{\lambda} \left(\frac{\lambda}{\tau}\right)^\mu \langle T_{\lambda\tau^{-1}} \psi, \omega_{\lambda\tau^{-1}} \rangle$$

because T is exact Fuchsian. Then by a change of variable, we obtain

$$\tau^{-\Omega-\mu} \langle T^\mu_\tau, \omega \rangle = \int_0^{\frac{1}{\tau}} \frac{d\lambda}{\lambda} \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega \rangle$$

We notice that the condition on the support of χ implies $\psi = -\rho\chi$ is supported in $\{a \leqslant |h| \leqslant b\} \cap U$. Since ψ is supported in $\{a \leqslant |h| \leqslant b\} \cap U$, $\psi_{\lambda^{-1}}$ is supported in $\{\lambda a \leqslant |h| \leqslant \lambda b\} \cap U$. However, we also recall that ω is supported inside the domain $\{|h| \leqslant a\}$. $T\psi_{\lambda^{-1}}$ is supported in $\{\lambda a \leqslant |h| \leqslant \lambda b\}$ hence $\langle T\psi_{\lambda^{-1}}, \omega \rangle$ vanishes when $\lambda \geqslant 1$. Finally:

$$\tau^{-\Omega-\mu} \langle T^\mu_\tau, \omega \rangle = \int_0^{\frac{1}{\tau}} \frac{d\lambda}{\lambda} \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega \rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega \rangle = \langle T^\mu, \omega \rangle$$

Notice that for $Re(\mu)$ large enough, all our integrals make sense when $\tau > 0$ because the integrand viewed as a function of λ is in $L^1([0,1])$. Then by the principle of analytic continuation

$$\rho T^\mu - (\Omega + \mu) T^\mu = 0 \text{ on } U$$

for $Re(\mu)$ large enough thus the same equation is satisfied by any meromorphic continuation of T^μ and the r.h.s. of the equation 9.23 satifies the Fuchs equation $\rho T^\mu - (\Omega + \mu) T^\mu = 0$. ■

9.4 The Riesz regularization.

Preliminary discussion.

Up to now, the meromorphic regularization operation seems not very interesting since it does not define an extension of the original current $T \in F_\Omega(U \setminus I)$ from which we started. In order to recover a genuine extension, we must somehow make μ tend to 0 in the meromorphic regularization of Hörmander. In order to do this, we will have to subtract poles but fortunately these poles are local counterterms hence the subtraction operation does not affect the extension outside the submanifold I. The procedure we are going to describe will be called Riesz regularization. Let us consider a given $T \in F_\Omega(U \setminus I)$. If $-m-1 < s \leqslant -m$, the extension procedure defined in Chapter 1 which could be called the Hadamard finite part procedure is given by

$$\left\langle \overline{T}_{\text{Hadamard}}, \omega \right\rangle = \lim_{\varepsilon \to 0} \langle T(\chi - \chi_{\varepsilon^{-1}}), I_m(\omega) \rangle + \langle T(1-\chi), \omega \rangle \tag{9.25}$$

whereas in the Riesz regularization, we first extend meromorphically in μ, then we subtract the poles at $\mu = 0$, and finally take the limit $\mu \to 0$.

Fundamental example.

Example 9.4.1 *To illustrate this section, we give our favorite example: we are going to Riesz regularize the function $\frac{1}{h^n}$ following the classical approach of [43]. First, we regularize by the formula*

$$\int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left\langle \frac{1}{h^n} \psi_{\lambda^{-1}}, \varphi \right\rangle + \left\langle \frac{1}{h^n}(1-\chi), \varphi \right\rangle$$

where $\mu \in \mathbb{C}$. We shall concentrate only on the term $\int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left\langle \frac{1}{h^n} \psi_{\lambda^{-1}}, \varphi \right\rangle$:

$$\int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left\langle \frac{1}{h^n} \psi_{\lambda^{-1}}, \varphi \right\rangle = \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\mu-n+1} \left\langle \frac{1}{h^n} \psi, \varphi_\lambda \right\rangle$$

$$= \int_0^1 \frac{d\lambda}{\lambda} \sum_{k=0}^N \frac{\lambda^{\mu-n+1+k}}{k!} \left\langle \frac{1}{h^n} \psi, h^k \partial_h^k \varphi(0) \right\rangle + \int_0^1 \frac{d\lambda}{\lambda} \lambda^{\mu-n+1} \left\langle \frac{1}{h^n} \psi, I_{N,\lambda} \right\rangle$$

Then for $Re(\mu)$ small enough, we can integrate the first N terms:

$$\int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left\langle \frac{1}{h^n} \psi_{\lambda^{-1}}, \varphi \right\rangle + \left\langle \frac{1}{h^n}(1-\chi), \varphi \right\rangle$$

$$= \sum_{k=0}^N \frac{1}{(\mu - n + 1 + k)k!} \left\langle \frac{1}{h^n} \psi, h^k \partial_h^k \varphi(0) \right\rangle + \text{ nice terms .}$$

At $\mu = 0$, when $k = n-1$, we have a pole $\frac{1}{\mu(n-1)!} \left\langle \frac{1}{h} \psi, \partial_h^{n-1} \varphi(0) \right\rangle$ of the Laurent series, and subtracting it allows us to define the regularization:

$$\lim_{\mu \to 0} \int_0^1 \frac{d\lambda}{\lambda} \lambda^\mu \left\langle \frac{1}{h^n} \psi_{\lambda^{-1}}, \varphi \right\rangle - \frac{1}{\mu(n-1)!} \left\langle \frac{1}{h} \psi, \partial_h^{n-1} \varphi(0) \right\rangle + \left\langle \frac{1}{h^n}(1-\chi), \varphi \right\rangle .$$

We recall that if T^μ is meromorphic at $\mu = 0$ then the pole has order at most n and T^μ is holomorphic in $B_r(0) \setminus \{0\}$ for r small enough (since the poles of T^μ are located in $-Spec(\Omega) - \mathbb{N}$), then $T^\mu = \sum_{k=-n}^{+\infty} \mu^k T^k$ where $\forall k \in \mathbb{Z}, T^k = \frac{1}{2i\pi} \int_{\partial B_r(0)} \frac{d\mu}{\mu^{k+1}} T^\mu$.

Definition 9.4.1 *Let $T \in \mathcal{D}'_k(U \setminus I)$ and $T \in F_\Omega(U \setminus I)$. Then T^μ is* ***meromorphic*** *in μ by Theorem 9.3.2 and the Riesz regularization is defined as*

$$\langle R_{Riesz}T, \omega\rangle = \lim_{\mu\to 0}\left(\langle T^\mu, \omega\rangle - \sum_{k=-n}^{-1}\mu^k\left\langle T^k, \omega\right\rangle\right) + \langle T(1-\chi), \omega\rangle\,. \quad (9.26)$$

It is not completely obvious from its definition that R_{Riesz} defines an extension operator.

Proposition 9.4.1 *For all $T \in \mathcal{D}'_k(U \setminus I) \cap F_\Omega(U \setminus I)$, $R_{Riesz}T$ is an extension of T.*

Proof — Let ω be a test form supported in $U \setminus I$. Then by lemma 9.3.3, all poles of $\langle T^\mu, \omega\rangle$ vanish hence $\langle T^\mu, \omega\rangle$ is holomorphic in μ and

$$\langle R_{Riesz}T, \omega\rangle = \lim_{\mu\to 0}\left(\langle T^\mu, \omega\rangle - \sum_{k=-n}^{-1}\mu^k\left\langle T^k, \omega\right\rangle\right) + \langle T(1-\chi), \omega\rangle$$

$$= \lim_{\mu\to 0}\left(\langle T^\mu, \omega\rangle\right) + \langle T(1-\chi), \omega\rangle = \langle T\chi, \omega\rangle + \langle T(1-\chi), \omega\rangle = \langle T, \omega\rangle\,,$$

since $\lim_{\mu\to 0}\int_0^1 \frac{d\lambda}{\lambda}\lambda^\mu \langle T\psi_{\lambda^{-1}}, \omega\rangle = \int_0^1 \frac{d\lambda}{\lambda}\langle T\psi_{\lambda^{-1}}, \omega\rangle = \langle T\chi, \omega\rangle$. ■

The anomalous scaling. Our next theorem is fundamental for quantum field theory since it implies that if T is a Fuchsian symbol then its extension $R_{Riesz}T$ is also a Fuchsian symbol.

Theorem 9.4.1 *Let $\Omega \in M_n(\mathbb{C})$ where $Spec(\Omega) \in \mathbb{R}$. For all $T \in \mathcal{D}'_k(U \setminus I) \cap F_\Omega(U \setminus I)$, if $(\rho - \Omega)T = 0$ then $R_{Riesz}T$ satisfies the equation $(\rho - \Omega)R_{Riesz}T = 0$ when $Spec(\Omega) \cap -\mathbb{N} = \emptyset$ and $(\rho - \Omega)^{n+1}R_{Riesz}T = 0$ when $Spec(\Omega) \cap -\mathbb{N} \neq \emptyset$.*

Proof — By the proof of 9.3.2, we know that $(\rho - \Omega)T = 0$ implies

$$(\rho - \mu - \Omega)T^\mu = 0 \quad (9.27)$$

on some neighborhood V of I provided V is stable by scaling and $\chi|_U = 1$. Then the trick consists in replacing T^μ by its Laurent series expansion in equation 9.27.

$$(\rho - \Omega - \mu)T^\mu = (\rho - \Omega - \mu)\left(\sum_{k=-n}^{+\infty}\mu^k T^k\right)$$

$$= (\rho - \Omega - \mu)\left(\sum_{k=-n}^{-1}\mu^k T^k + T^0 + O(\mu)\right) = 0. \quad (9.28)$$

Notice that the constant term in the Laurent series expansion $T^0 = R_{Riesz}T - T(1-\chi)$ therefore on V, we have $T^0 = R_{Riesz}T$ since $1-\chi|_V = 0$. By **uniqueness of the Laurent series expansion**, we expand the equation (9.28) in powers of μ:

$$(\rho-\Omega)T^{-n}\mu^{-n} + \sum_{k=-n+1}^{0} \mu^k \left((\rho-\Omega)T^k - T^{k-1}\right) + O(\mu) = 0$$

and we require that all coefficients of the Laurent series expansion should vanish. Hence we find a system of equations:

$$(\rho-\Omega)T^{-n} = 0 \tag{9.29}$$

$$\forall k, -n+1 \leqslant k \leqslant 0, \left((\rho-\Omega)T^k - T^{k-1}\right) = 0. \tag{9.30}$$

Then for $T^0 = R_{Riesz}T$ on V, we have $(\rho-\Omega)T^0 = (\rho-\Omega)R_{Riesz}T = T^{-1}$. Also note that on the complement of V, $(\rho-\Omega)R_{Riesz}T = 0$ since $R_{Riesz}T = T$ because $R_{Riesz}T$ is an extension of T. Thus we have globally $(\rho-\Omega)R_{Riesz}T = T^{-1}$. Now the key fact is that if $Spec(\Omega)\cap -\mathbb{N} = \emptyset$ then $T^{-1} = 0$ since T^μ has no poles at $\mu = 0$. Finally, if $Spec(\Omega)\cap -\mathbb{N} \neq \emptyset$ then by an easy recursion:

$$(\rho-\Omega)^{n+1}R_{Riesz}T = (\rho-\Omega)^n T^{-1} = (\rho-\Omega)^{n-1}T^{-2} = \cdots = (\rho-\Omega)T^{-n} = 0,$$

which is the final equation we wanted to find. ■

Example 9.4.2 *We pick again our example of $T = \frac{1}{h^n}$, its Riesz extension satisfies the differential equations*

$$(\rho+n)R_{Riesz}T = \left\langle \frac{1}{h}, \psi \right\rangle \frac{1}{(n-1)!}\partial_h^{n-1}\delta_0$$

and

$$(\rho+n)^2 R_{Riesz}T = 0.$$

The residue equation. A small comment before we state anything. The role of the poles seems to disappear since we subtract them in order to define the Riesz regularization, however they come back with a revenge when we compute the residue or anomaly of the Riesz regularization. Following the philosophy of Chapter 8, we **define** the residues of R_{Riesz} for the vector field ρ by the simple equation: $\mathfrak{Res}_\rho[T] = \rho(R_{Riesz}T) - R_{Riesz}(\rho T)$.

Theorem 9.4.2 *Let $T \in F_\Omega(U\setminus I)$ and T^{-1} is the coefficient of μ^{-1} in the Laurent series expansion of the meromorphic function T^μ around $\mu = 0$. Then R_{Riesz} satisfies the residue equation*

$$\mathfrak{Res}_\rho[T] = T^{-1}. \tag{9.31}$$

In particular the residue vanishes when $Spec(\Omega)\cap -\mathbb{N} = \emptyset$.

Comment: the residue $\mathfrak{Res}[T]$ is the *holomorphic residue* of T^μ at $\mu = 0$.
Proof — By Proposition (9.3.1), the residue vanishes if $-Spec(\Omega) \cap \mathbb{N} = \emptyset$ because T_k^μ admits no pole at $\mu = 0$ thus $R_{Riesz}T_k$ satisfies the same equation as T_k, thus $(\rho - \Omega - k)R_{Riesz}T_k = 0 = \rho R_{Riesz}T_k - R_{Riesz}\rho T_k$. If $k \in -Spec(\Omega) \cap \mathbb{N}$, then by equation 9.29, $\rho R_{Riesz}T_k - R_{Riesz}\rho T_k = (\rho - \Omega - k)R_{Riesz}T_k = T_k^{-1}$ which yields the result. ■

9.5 The $\log$ and the 1-parameter RG.

Let us fix ρ and a current $T \in \mathcal{D}'_k(U \setminus I) \cap F_\Omega(U \setminus I)$. Once we fix the function χ and the Euler vector field ρ, we can renormalize following the Riesz extension since $T \in F_\Omega(U \setminus I)$, this is called choosing a *renormalization scheme*. But in contrary to the flat case, if we change the Euler field ρ and the function χ, we change the renormalization scheme, and the extensions will differ by a **local counterterm** which is a distribution supported on I. We thus have some infinite dimensional space of choices. But if χ, ρ and the extension R_{Riesz} is choosed, then we still have a one dimensional degree of freedom left when we **scale the cut-off function** χ by the flow $\chi \mapsto e^{\rho \log \ell *}\chi, \ell \in \mathbb{R}^{+\star}$ which changes the *length scale* of our renormalization. The idea of scaling the function χ by the one parameter group $e^{\log \ell \rho}$ was inspired by the reading of unpublished lecture notes of John Cardy [12] and [13] Chapter 5 section (5.2). The mechanism we are going to explain allows to relate the Bogoliubov, Epstein-Glaser technique with the 1-parameter renormalization group of Bogoliubov Shirkov.

Example 9.5.1 *Let us give some important comment on the physical meaning of the variable ℓ in the case where the manifold is a configuration space M^2 and $I = d_2$ is the diagonal of M^2. When $\ell \to \infty$, the function χ_ℓ will have a* ***support shrinking*** *to the diagonal d_2. This means that we must think of ℓ^{-1} in terms of characteristic length beetween pair of points $(x, y) \in M^2$ (think of them in terms of particles in the hard ball model, see [13] p. 88). Then according to this interpretation $\ell \to \infty$ should be called UV flow whereas $\ell \to 0$ is the IR flow. We describe the simple example of the amplitude $\langle \phi^2(x)\phi^2(y) \rangle$ in the flat Euclidean case:*

Cardy poor man's renorm	*Our approach*	*Costello Heat kernel*
$\int_{M^2 \setminus \{\lvert x-y \rvert \geqslant \ell\}} \Delta^2(x,y)g(x)g(y)d^4xd^4y$	$\langle R^\ell \Delta^2, g \otimes g \rangle$	$\frac{1}{2}\int_\ell^\infty \frac{dt}{t}t^2 \langle K_t, g \otimes g \rangle$

In Costello's approach ([14] (4.2) p. 43), K_t is the Heat kernel and the UV regularized two point function in the massless case is given by the formula $\int_\ell^\infty dtK_t$.

Let T be a given current $T \in \mathcal{D}'_k(U \setminus I)$. For each function χ such that $\chi = 1$ in a neighborhood of I and vanishes outside a tubular neighborhood of I, we denote by R^{ℓ}_{Riesz} the corresponding Riesz regularization operator constructed with χ_ℓ:

$$\left\langle R^{\ell}_{Riesz}T,\omega\right\rangle=\lim_{\mu\to 0}\left(1-\sum_{k=-n}^{-1}\int_{\partial B(0,r)}\frac{d\mu}{2i\pi\mu^{k+1}}\right)\int_0^1\frac{d\lambda}{\lambda}\lambda^{\mu}T\psi_{\ell\lambda^{-1}}+T(1-\chi_\ell).$$

We shall state the renormalization group flow theorem for the Riesz regularization. The residue $\mathfrak{Res}$ appears when we scale the bump function χ.

Theorem 9.5.1 *Let $T\in F_\Omega(U\setminus I)$ and $\forall\ell\in\mathbb{R}_{>0}$, the residue $\mathfrak{Res}_\rho[T](\ell)=\rho R^{\ell}_{Riesz}T-R^{\ell}_{Riesz}\rho T$. Then both $R^{\ell}_{Riesz},\mathfrak{Res}_\rho[T](\ell)$ satisfy the differential equations*

$$\ell\frac{d}{d\ell}R^{\ell}_{Riesz}T=\mathfrak{Res}_\rho[T](\ell) \tag{9.32}$$

$$\left(\ell\frac{d}{d\ell}\right)^n\mathfrak{Res}_\rho[T](\ell)=0. \tag{9.33}$$

Thus $R^{\ell}_{Riesz}T$ scales like a polynomial of $\log\ell$ of degree n:

$$R^{\ell}_{Riesz}T=R^{1}_{Riesz}T+\sum_{k=1}^{n}\frac{(\log\ell)^k}{k!}\left(\ell\frac{d}{d\ell}\right)^k\mathfrak{Res}_\rho[T](1) \tag{9.34}$$

where the divergent part is a polynomial of degree n in $\log\ell$ with coefficients local counterterms.

Proof — From the decomposition $T=\sum_0^\infty T_j$ where $\forall j,(\rho-\Omega-j)T_j=0$, by linearity of the Riesz extension and by the fact that $Res_\rho[T_j]$ vanishes for j large enough, we can reduce the proof to an element $T\in F_\Omega(U\setminus I)$ killed by $\rho-\Omega$.

$$\ell\frac{d}{d\ell}\left(T^{\mu,\ell}+T(1-\chi_\ell)\right)=\ell\frac{d}{d\ell}T^{\mu,\ell}$$
$$=\ell\frac{d}{d\ell}\int_0^1\frac{d\lambda}{\lambda}\lambda^{\mu}T\psi_{\ell\lambda^{-1}}=\int_0^1\frac{d\lambda}{\lambda}\lambda^{\mu}T(\rho\psi)_{\lambda^{-1}\ell}$$
$$=\int_0^1\frac{d\lambda}{\lambda}\lambda^{\mu}\rho(T\psi)_{\lambda^{-1}\ell}-\int_0^1\frac{d\lambda}{\lambda}\lambda^{\mu}(\rho T)\psi_{\lambda^{-1}\ell}$$
$$=\rho T^{\mu\ell}-\Omega T^{\mu,\ell}=(\Omega+\mu)T^{\mu,\ell}-\Omega T^{\mu\ell}=\mu T^{\mu\ell}.$$

We obtain the simple equation $\ell\frac{d}{d\ell}T^{\mu,\ell}=\mu T^{\mu,\ell}$. Expanding the l.h.s and the r.h.s. of this equation in Laurent series and identifying the different terms in the Laurent series expansion,

$$\sum_{k=-n}^{+\infty}\ell\frac{d}{d\ell}T^{k,\ell}\mu^k=\sum_{k=-n}^{+\infty}T^{k,\ell}\mu^{k+1}$$

we deduce a system of linear equations:

$$\forall k \geqslant -n+1, \ell\frac{d}{d\ell}T^{k,\ell} = T^{k-1,\ell} \text{ and } \ell\frac{d}{d\ell}T^{-n,\ell} = 0. \quad (9.35)$$

But since $\ell\frac{d}{d\ell}T^{0,\ell} = \ell\frac{d}{d\ell}R^{\ell}_{Riesz}T$ and from the fact that $\left(\ell\frac{d}{d\ell}\right)^{n+1}T^{0,\ell} = \left(\ell\frac{d}{d\ell}\right)^{n}T^{-1,\ell} = \left(\ell\frac{d}{d\ell}\right)^{n}\mathfrak{Res}_\rho[T](\ell) = \cdots = \ell\frac{d}{d\ell}T^{-n,\ell} = 0$, we must have $\left(\ell\frac{d}{d\ell}\right)^{n+1}R^{\ell}_{Riesz}T = 0$ which implies $R^{\ell}_{Riesz}T$ scales like a *polynomial* of $\log\ell$ of degree n. ■

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