Extension of distributions, scalings and renormalization of QFT on Riemannian manifolds.

Nguyen Viet Dang

Abstract. Let M be a smooth manifold and X a closed subset of M. In this paper, we introduce a natural condition of *moderate growth* along X for a distribution t in $\mathcal{D}'(M\setminus X)$ and prove that this condition is equivalent to the existence of an extension of t in $\mathcal{D}'(M)$ generalizing previous results of Meyer and Brunetti–Fredenhagen. When X is a closed submanifold of M, we show that our notion of moderate growth coincides with the weakly homogeneous distributions of Meyer defined in terms of scaling. Then using the whole analytical machinery developed, we give a simple existence proof of perturbative quantum field theories on Riemannian manifolds.

Mathematics Subject Classification (2010). Primary ; Secondary .

Keywords. Renormalization.

1. Introduction

Let us start with a simple example which is discussed in [29, Example 9 p. 140] and actually goes back to Hadamard. We denote by Θ the Heaviside function (the indicator function of $\mathbb{R}_{\geq 0}$), consider the function $x^{-1}\Theta(x)$ viewed as a distribution in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$. Obviously, the linear map

$$\varphi \longmapsto \int_0^\infty dx \frac{\varphi(x)}{x} \tag{1.1}$$

is ill-defined if $\varphi(0)\neq 0$ since the integral $\int_0^\infty \frac{dx}{x}$ diverges.

However, the integral $\int_0^\infty dx x^{-1}\varphi(x)$ **converges** if $\varphi(0)=0$ and an elementary estimate shows that $x^{-1}\Theta(x)$ defines a linear functional on the ideal of functions $x\mathcal{D}(\mathbb{R})$ vanishing at 0. A test function $\varphi\in\mathcal{D}(\mathbb{R})$ being given, note that the following expression

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 dx \frac{(\varphi(x)-\varphi(0))}{x} + \int_1^\infty dx\frac{\varphi(x)}{x} \tag{1.2}$$

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01).

converges.

We thus define a **renormalized** distribution:

$$x_+^{-1} = \lim_{\varepsilon\to 0}\int_\varepsilon^\infty dx x^{-1} + \log(\varepsilon)\delta \qquad (1.3)$$

where we subtracted the distribution $\log(\varepsilon)\delta$ supported at 0, which becomes singular when $\varepsilon \to 0$, called *local counterterm*. The renormalized distribution $x_+^{-1} \in \mathcal{D}'(\mathbb{R})$, called finite part of Hadamard, extends the linear functional $x^{-1}\Theta(x) \in (x\mathcal{D}(\mathbb{R}))'$. Our example shows the most elementary situation where we can extend a distribution by an *additive renormalization.*

In what follows, M will always denote a smooth, paracompact manifold. In our paper, motivated by the renormalization of quantum fields on Riemannian manifolds, we investigate the following problem which has simple formulation: we are given a manifold M and a closed subset $X \subset M$. We define a natural growth condition on $t \in \mathcal{D}'(M \setminus X)$ which measures the singular behaviour near X and we address the following problems:

1. can we find a distribution $\overline{t} \in \mathcal{D}'(M)$ s.t. the restriction of $\overline{t}$ on $M \setminus X$ coincides with t,
2. can we construct a linear extension operator $\mathcal{R}$, eventually give explicit formulas for $\mathcal{R}$,
3. can we classify the different extension operators.

In general, the extension problem has no positive answer for a generic distribution t in $\mathcal{D}'(M \setminus X)$ unless t has moderate growth when we approach the singular subset X.

Distributions having moderate growth along a closed subset $X \subset M$. If P is a differential operator with smooth coefficients on M, and $K \subset U$ a compact subset, we denote by $\|\varphi\|_P^K$ (resp $\|\varphi\|_P$) the seminorm $\sup_{x\in K}|P\varphi(x)|$ (resp $\sup_{x\in U}|P\varphi(x)|$). We also denote by d some arbitrary distance function induced by some choice of smooth metric on M. For every open set $V \subset M$, we denote by $\mathcal{T}_{M\setminus X}(V)$ the set of distributions in $\mathcal{D}'(V \setminus X)$ with moderate growth along X defined as follows:

Definition 1.1. A distribution $t \in \mathcal{D}'(V \setminus X)$ has moderate growth along X if for all open relatively compact $U \subset V$, there is a seminorm $\|.\|_P$ and a pair of constant $(C, s) \in \mathbb{R}^2_{\geq 0}$ such that

$$|t(\varphi)| \leq C(1 + d(\text{supp }\varphi, X)^{-s})\|\varphi\|_P. \qquad (1.4)$$

for all $\varphi \in \mathcal{D}(U \setminus X)$.

Remark: If t were in $\mathcal{D}'(M)$, we would have the same estimate without the divergent factor $(1 + d(\text{supp }\varphi, X)^{-s})$.

It should be emphasized that the property of having moderate growth is not local and that the space $\mathcal{T}_{M\setminus X}$ is intrinsically defined since all metrics on M are locally equivalent. The first part of our paper is devoted to give a detailed proof of the following:

Theorem 1.2. *Let M be a smooth manifold and X a closed subset of M. Then the three following claims are equivalent:*

1. *t has moderate growth along X,*
2. *$t \in \mathcal{D}'(M \setminus X)$ is extendible,*
3. *there is a family of functions $(\beta_\lambda)_{\lambda\in(0,1]} \subset C^\infty(M \setminus X)$, $\beta_\lambda = 0$ in a neighborhood of X, $\beta_\lambda \underset{\lambda\rightarrow 0}{\rightarrow} 1$ and a family of distributions $(c_\lambda)_{\lambda\in(0,1]}$* **supported on** *$X$ such that*

$$\lim_{\lambda\rightarrow 0} t\beta_\lambda - c_\lambda \tag{1.5}$$

exists and defines an extension of t in $\mathcal{D}'(M)$.

Our moderate growth condition is weaker than the hypothesis of [16, Lemma 3.3] and Theorem 1.2 can also be viewed as generalizations of [25, Theorem 2.1 p. 48] and [3, Theorem 5.2 p. 645] which only treat the extension problem in the case of a point. The third condition in the above Theorem is a generalization of Hadamard's definition of finite parts of distributions. This is beautifully explained in Yves Meyer's book [25] p.45 and also explains the appearance of local counterterms in the renormalization of Feynman amplitudes in QFT.

In the second part of the paper, we will study the easy case where X is a vector subspace of $M = \mathbb{R}^n$ and we compare the notion of moderate growth with conditions on distributions in terms of scalings, called *Steinman scaling degree* in the physics litterature, which is the relevant notion used to renormalize quantum fields on curved space times [3, 5.1 p. 644]. We prove in Theorem 3.1 that weakly homogeneous distributions in the sense of Meyer have moderate growth and are therefore extendible. In [9, Chapter 1], we proved that weakly homogeneous distributions along some vector subspace X are **invariant by diffeomorphisms preserving** X which implies that weakly homogeneous distributions along a submanifold $X \subset M$ can be intrinsically defined.

In the third part of our paper, we apply our extension techniques to establish in Theorem 4.2 that the product of distributions in $\mathcal{D}'(M)$ with functions which are tempered along X (see definition 4.1 for the algebra $\mathcal{M}(X,M)$ of tempered functions) is renormalizable which means that the space of extendible distributions or equivalently of distributions in $\mathcal{T}_{M\setminus X}$ is a left $\mathcal{M}(X,M)$-module (Theorem 4.4).

Finally we apply our analytic machinery to the study of perturbative QFT on Riemannian manifolds. In QFT, one is interested in making sense of correlation functions denoted by $\left\langle : \phi^{i_1} : (x_1) \cdots : \phi^{i_n} : (x_n) \right\rangle$ which are objects living on the configuration space M^n that can be expressed formally, using the Feynman rules, in terms of products of the form $\prod_{1\leq i<j\leq n} G(x_i,x_j)^{n_{ij}}$ where G is the Green function of $\Delta_g + m^2, m \geqslant 0$ where Δ_g is the Laplace Beltrami operator. A product $\prod_{1\leq i<j\leq n} G(x_i,x_j)^{n_{ij}}$ is called *Feynman amplitude* and is depicted pictorially by a graph with n labelled vertices $\{1,\dots,n\}$ where the vertices i and j are connected by n_{ij} lines. In the main Theorem

(Thm 5.5) of our paper, we prove that all Feynman amplitudes are renormalizable by a collection of extension maps $(\mathcal{R}_{M^n})_{n\in\mathbb{N}}$ where every map $\mathcal{R}_{M^n}$ extends Feynman amplitudes living on the configuration space M^n minus all *diagonals* to distributions on M^n and the maps $(\mathcal{R}_{M^n})_{n\in\mathbb{N}}$ satisfy some axioms given in definition 5.3 which are due to N. Nikolov [26]. This gives a different approach to Costello's existence Theorem [7] (see also [8]) for perturbative QFT on Riemannian manifolds.

Related works. To our knowledge, one of the first rigorous result on the renormalization of the ϕ^4 theory on curved Riemannian manifolds was given by Kopper–Müller [20] and is based on some perturbative implementation of the Wilson–Polchinsky equations to derive the renormalization group flow of the coupling constants. In his book [7], Costello gives a different approach to the first problem, first from any action functional of the form $S(\phi)=\int_M \phi\Delta_g\phi + I_{int}(\phi)$ where Δ_g is the Laplace–Beltrami operator and the interaction part I_{int} is at least cubic in ϕ, he defines a notion of effective field theory via the effective action:

$$\Gamma_\varepsilon(\chi)=\hbar\log\left(\int d\mu_{G_\varepsilon}(\phi)e^{\frac{iS(\phi+\chi)}{\hbar}}\right)$$

where $d\mu_{G_\varepsilon}$ is the Gaussian measure whose covariance is a regularized propagator G_ε, where $G_\varepsilon\to G$ when $\varepsilon\to 0$. He then proves that starting from any local action functional S, there is a local action functional S_ε^{CT} so that the limit

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(\chi)=\hbar\log\left(\int d\mu_{G_\varepsilon}(\phi)e^{\frac{i(S(\phi+\chi)+S_\varepsilon^{CT}(\phi+\chi))}{\hbar}}\right)$$

exists for every power of $\hbar$ [7, Theorems 9.3.1 and 10.1.1]. The important point being that S_ε^{CT} might contain infinitely many counterterms and that the limit theory can always be defined even for theories which are not renormalizable in the classical sense.

For quantum fields on curved Lorentzian spacetimes, a proof of renormalizability was first achieved by Brunetti–Fredenhagen [3], Hollands–Wald [13, 14] and relies on the Epstein–Glaser approach which is based on the idea that renormalization consists in an operation of extension of distributions which satisfies the physical constraint of causality. Recently this method was revisited in the very elegant work of Nikolov–Stora–Todorov which discusses Epstein–Glaser renormalization in the flat Minkowski space. Costello's approach is similar to the above methods because they both deal with Feynman amplitudes in position space and make sense of all quantum field theories, even nonrenormalizable in the classical sense.

Our goal in this paper is to give a simple existence proof of quantum field theories on arbitrary Riemannian manifolds following the Epstein–Glaser philosophy thus giving an alternative approach to the one by Costello. To reach our goal, we need to revisit some methods in analysis originally developped by H. Whitney [38] which were then improved by Malgrange and Lojasiewicz, to compare these techniques with the approach by scaling of Meyer [9, 25] and finally show their relevance in solving our renormalization problem.

In the mathematical litterature, the idea to consider extendible distributions really goes back to Lojasiewicz [21] and tempered functions already appear in the work of B. Malgrange [22, 23]. However, the first general definition of a tempered distribution on any open set U in some manifold M is due to M. Kashiwara, a distribution is tempered if it is extendible on $\overline{U}$ [16, Lemma 3.2 p. 332] (see also [5]) which implies by our Theorem 1.2 that these distributions are in $\mathcal{T}_{M\setminus\partial U}$ i.e. have moderate growth along ∂U. His work was then extended in [12, 18, 19]. Tempered functions and distributions were also recently studied in the context of real algebraic geometry [1, 5] with applications in representation theory. A different approach to the extension problem in terms of scaling was developped by Meyer in his book [25], his purpose was to study the singular behaviour at given points of irregular functions with applications in multifractal analysis [17].

Acknowledgements. I would like to thank Christian Brouder, Frédéric Hélein, Stefan De Bièvre, Laura Desideri, Camille Laurent Gengoux, Mathieu Stiénon for useful discussions and the Labex CEMPI for excellent working conditions.

2. The extension of distributions.

2.1. Proof of Theorem 1.2

Localization on open charts by a partition of unity. We shall reduce the proof of (1) $\Leftrightarrow$ (2) in Theorem 1.2 to the case where $M=\mathbb{R}^n$, X is a compact set contained in a larger compact K and $t\in\mathcal{D}'(\mathbb{R}^n\setminus X)$ vanishes outside K, this condition reads $t\in\mathcal{D}'_K(\mathbb{R}^n\setminus X)$. The first step is to localize the problem by a partition of unity. Choose a locally finite cover of M by relatively compact open charts $(U_i)_i$ and a subordinated partition of unity $(\varphi_i)_i$ s.t. $\sum\varphi_i=1$. Denote by t_i the restriction $t|_{U_i}$ and $K_i=\text{supp }\varphi_i\subset U_i$. For all $\varphi\in\mathcal{D}(U)$, $t\in\mathcal{D}'(U\setminus X)$ has moderate growth implies the same property for $t\varphi\in\mathcal{D}'(U\setminus X)$, therefore each $t\varphi_i|_{U_i\setminus X}$ is in $\mathcal{D}'_{K_i}(U_i\setminus(X\cap K_i))$, $t\varphi_i$ vanishes outside K_i and has moderate growth along X. Hence it suffices to extend $t\varphi_i|_{U_i\setminus X}$ in each U_i in such a way that the extension is supported by K_i. Call $\overline{t_i\varphi_i}$ such extension in $\mathcal{E}'(U_i)$ then the locally finite sum $\overline{t}=\sum_i\overline{t_i\varphi_i}\in\mathcal{D}'(M)$ is a well defined extension of t.

Working on $\mathbb{R}^n$. The second step is to use local charts to work on $\mathbb{R}^n$. On every open set (U_i), let $\psi_i:U_i\longmapsto V\subset\mathbb{R}^n$ denote the corresponding chart then the pushforward $\psi_{i*}(t\varphi_i)$ is in $\mathcal{D}'_{\psi_i(K_i)}(V\setminus\psi_i(X\cap K_i))$. Actually the compact set $\psi_i(X\cap K_i)$ is in the interior of V, since $(K_i\cap X)\subset\text{int}(U_i)$ and ψ_i is a diffeomorphism. Therefore the distribution $\psi_{i*}(t\varphi_i)$ is an element of $\mathcal{D}'_{K_i}(\mathbb{R}^n\setminus\psi_i(X\cap K_i))$ and we may reduce the proof of our theorem to the case where we have a distribution $t\in\mathcal{D}'_K(\mathbb{R}^n\setminus X)$ with moderate growth along X where $X\subset K$ are **compact subsets** of $\mathbb{R}^n$. In the sequel, we use the seminorms $\|\varphi\|_m=\sup_{x\in\mathbb{R}^n,|\alpha|\leq m}|\partial_x^\alpha\varphi(x)|$ and $\|\varphi\|_m^K=\sup_{x\in K,|\alpha|\leq m}|\partial_x^\alpha\varphi(x)|$ where K runs over the compact subsets of $\mathbb{R}^n$. Let $\mathcal{I}(X,\mathbb{R}^n)=\{\varphi\text{ s.t. supp }\varphi\cap X=$

$\emptyset\} \subset C^\infty(\mathbb{R}^n)$, since t vanishes outside some compact set K, the moderate growth condition now reads

$$\begin{aligned}\exists (C,s)\in\mathbb{R}^2_{\geq 0} \text{ and } \|.\|_m^K \text{ s.t. } \forall\varphi\in\mathcal{I}(X,\mathbb{R}^n),\\ |t(\varphi)|\leq C(1+d(\text{supp }\varphi,X)^{-s})\|\varphi\|_m^K.\end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $X\subset K$ be compact subsets of $\mathbb{R}^n$, then $t\in\mathcal{D}'_K(\mathbb{R}^n\setminus X)$ is extendible in $\mathcal{D}'_K(\mathbb{R}^n)$ if and only if t has moderate growth along X.*

Proof. We first prove a weaker equivalence: t is extendible iff the estimate (2.1) holds for some $m\in\mathbb{N}$ with $s=0$.

Assume the problem is solved and that we could find an extension $\bar{t}\in\mathcal{D}'_K(\mathbb{R}^n)$ of t. Observe that $\forall\varphi\in V, t(\varphi)=\bar{t}(\varphi)$ then by definition $\bar{t}$ is a linear continuous functional on $C^\infty(\mathbb{R}^n)$ equipped with the Fréchet topology, thus it induces a linear continuous map on the vector subspace $\mathcal{I}(X,\mathbb{R}^n)\subset C^\infty(\mathbb{R}^n)$:

$$\exists C\in\mathbb{R}_{\geq 0}, \|.\|_m^K \text{ s.t. } \forall\varphi\in\mathcal{I}(X,\mathbb{R}^n),\ \ |t(\varphi)|=|\bar{t}(\varphi)|\leq C\|\varphi\|_m^K.$$

Therefore, if t is extendible then estimate (2.1) is satisfied with $s=0$ and t has moderate growth along X.

Conversely, if $\exists C\in\mathbb{R}_{\geq 0}, \|.\|_m^K$ s.t. $\forall\varphi\in\mathcal{I}(X,\mathbb{R}^n), |t(\varphi)|\leq C\|\varphi\|_m^K$, then by the Hahn–Banach theorem [24, Thm 6.4 p. 46], we can extend t as a linear continuous mapping $\bar{t}$ on $C^\infty(\mathbb{R}^n)$ which satisfies the above estimate hence $\bar{t}\in\mathcal{D}'_K(\mathbb{R}^n)$. Therefore to prove that t has *moderate growth* implies that t is extendible in $\mathcal{D}'_K(\mathbb{R}^n)$, it suffices to show that

$$\begin{aligned}&\exists C\in\mathbb{R}_{\geq 0}, \|.\|_m^K \text{ s.t. } \forall\varphi\in\mathcal{I}(X,\mathbb{R}^n), |t(\varphi)|\leq C(1+d(\text{supp }\varphi,X)^{-s})\|\varphi\|_m^K\\ \Longrightarrow\ &\exists C'\in\mathbb{R}_{\geq 0}, \|.\|_{m'}^K \text{ s.t. } \forall\varphi\in\mathcal{I}(X,\mathbb{R}^n), |t(\varphi)|\leq C'\|\varphi\|_{m'}^K.\end{aligned}$$

Let us admit the following central technical Lemma whose proof will be given later:

Lemma 2.2. *For every integers $(d,m)\in\mathbb{N}^2$, let $\mathcal{I}^{m+d}(X,\mathbb{R}^n)$ denote the closed ideal of functions of regularity C^{m+d} which vanish at order $m+d$ on X. Then there is a function $\chi_\lambda\in C^\infty(\mathbb{R}^n)$ parametrized by $\lambda\in(0,1]$ s.t. $\chi_\lambda=1$ (resp $\chi_\lambda=0$) when $d(x,X)\leq\frac{\lambda}{8}$ (resp $d(x,X)\geq\lambda$) and the following estimate holds true:*

$$\exists\tilde{C},\forall\lambda\in(0,1],\forall\varphi\in\mathcal{I}^{m+d}(X,\mathbb{R}^n), \|\chi_\lambda\varphi\|_m^K\leq\tilde{C}\lambda^d\|\varphi\|_{m+d}^{K\cap\{d(x,X)\leq\lambda\}} \tag{2.2}$$

where the constant $\tilde{C}$ does not depend on φ,λ.

If $s=0$, then we know that there is an extension by Hahn Banach therefore we shall treat the case where $s>0$. Our idea is to absorb the divergence by a dyadic decomposition:

$$\begin{aligned}&\forall\varphi\in\mathcal{I}(X,\mathbb{R}^n),\exists N \text{ s.t. } \chi_{2^{-N}}\varphi=0\\ \Longrightarrow\ \ &t(\varphi)=t((1-\chi_{2^{-N}})\varphi)\\ \Longrightarrow\ \ &t(\varphi)=\sum_{j=0}^{N-1}t((\chi_{2^{-j}}-\chi_{2^{-j-1}})\varphi)+t((1-\chi_1)\varphi)\end{aligned}$$

We easily estimate $t((1-\chi_1)\varphi)$: $\forall\varphi\in C^\infty(\mathbb{R}^n), |t((1-\chi_1)\varphi)|\leq C\|\varphi\|_m^K$ for some constant C since the support of $1-\chi_1$ does not meet X. Choose $d\in\mathbb{N}^*$ such that $d-s>0$, then:

$$\begin{aligned}
|t(\chi_1\varphi)| &\leq \sum_{j=0}^{N-1}|t((\chi_{2^{-j}}-\chi_{2^{-j-1}})\varphi)| \\
&\leq C\sum_{j=1}^{N}(1+d(\text{supp }\varphi(\chi_{2^{-j}}-\chi_{2^{-j-1}}),X)^{-s})\|(\chi_{2^{-j}}-\chi_{2^{-j-1}})\varphi\|_m^K, \\
&\quad \text{by moderate growth} \\
&\leq C\sum_{j=1}^{N}(1+2^{s(j+4)})(2^{-jd}+2^{-(j+1)d})\tilde{C}\|\varphi\|_{m+d}^K, \text{ by Lemma 2.2} \\
&\leq C'\|\varphi\|_{m+d}^K
\end{aligned}$$

for $C'=\tilde{C}C(1+2^{-d})\underbrace{\sum_{j=1}^{\infty}2^{-jd}(1+2^{(j+4)s})}_{\text{convergent series since } d-s>0}<+\infty$ which is independent of N and φ. □

We now prove Lemma 2.2:

Proof. Choose $\phi\geq 0$ s.t. $\int\phi=1$, $\phi=0$ if $|x|\geq\frac{3}{8}$ then set $\phi_\lambda=\lambda^{-n}\phi(\lambda^{-1}.)$ and α_λ to be the characteristic function of the set $\{x \text{ s.t. } d(x,X)\leq\frac{\lambda}{2}\}$ then the convolution product $\phi_\lambda*\alpha_\lambda(x)=1$ if $d(x,X)\leq\frac{\lambda}{8}$ and equals 0 if $d(x,X)\geq\lambda$. Since by Leibniz rule one has

$$\partial^\alpha(\chi_\lambda\varphi)(x)=\sum_{|k|\leq|\alpha|}\begin{pmatrix}\alpha\\k\end{pmatrix}\partial^k\chi_\lambda\partial^{\alpha-k}\varphi(x),$$

it suffices to estimate each term $\partial^k\chi_\lambda\partial^{\alpha-k}\varphi(x)$ of the above sum.

For all multi-index k, there is some constant C_k such that $\forall x\in\mathbb{R}^n\setminus X, |\partial_x^k\chi_\lambda|\leq\frac{C_k}{\lambda^{|k|}}$ and $\text{supp }\partial_x^k\chi_\lambda\subset\{d(x,X)\leq\lambda\}$. Therefore for all $\varphi\in\mathcal{I}^{m+d}(X,\mathbb{R}^n)$, for all $x\in\text{supp }\partial_x^k\chi_\lambda\partial^{\alpha-k}\varphi$, for $y\in X$ such that $d(x,X)=|x-y|$, we find that $\partial^{\alpha-k}\varphi$ vanishes at y at order $|k|+d$ therefore:

$$\partial_x^{\alpha-k}\varphi(x)=\sum_{|\beta|=|k|+d}(x-y)^\beta R_\beta(x)$$

where the right hand side is just the integral remainder in Taylor's expansion of $\partial^{\alpha-k}\varphi$ around y. Hence:

$$|\partial^k\chi_\lambda\partial^{\alpha-k}\varphi(x)|\leq\frac{C_k}{\lambda^{|k|}}\sum_{|\beta|=|k|+d}|(x-y)^\beta R_\beta(x)|.$$

It is easy to see that R_β only depends on the Jets of φ of order $\leq m+d$. Hence

$$|\partial^k\chi_\lambda\partial^{\alpha-k}\varphi(x)| \leq C_k\lambda^d \sup_{x\in K, d(x,X)\leq\lambda} \sum_{|\beta|=|k|+d} |R_\beta(x)|$$

and the conclusion follows easily. □

Our partition of unity argument together with the result of Theorem 2.1 imply that (1) ⇔ (2) in Theorem 1.2.

2.2. Renormalizations and the Whitney extension Theorem

The goal of this subsection is to replace the use of Hahn Banach theorem by a more constructive argument. First, we discuss a particular case of extension where there is some canonical choice for $\bar{t}$.

Remark on the extension of positive measures with locally finite mass. The following proposition is inspired by some results of Skoda [35]. Let μ be a positive measure in $M\setminus X$, then we say that μ has locally finite mass if:

$$\forall K\subset M \text{ compact }, \exists C_K, \forall\varphi\in\mathcal{D}_K(M\setminus X), \varphi\geq 0, 0\leq\mu(\varphi)\leq C_K\|\varphi\|_0.$$

Proposition 2.3. *Let μ be a positive measure in $M\setminus X$. If μ has locally finite mass then μ has a canonical extension in the space of positive measures.*

Proof. By an obvious regularization argument, we can extend μ to the space $C^0_c(M\setminus X)$ of compactly supported functions of regularity C^0. Choose a family χ_λ as in the main technical Lemma 2.2 which satisfies $\chi_\lambda\geq 0, \chi_\lambda=1$ if $d(x,X)\leq\frac{\lambda}{8}$ and $\chi_\lambda=0$ when $d(x,X)\geq\lambda$. Then for all $\varphi\in C^0_c(M), \varphi\geq 0$, the sequence $\mu((1-\chi_{2^{-n}})\varphi)_n$ is increasing and bounded by $C_K\|\varphi\|_0$ where K is any compact set which contains the support of φ. Therefore for each $\varphi\geq 0$, $\lim_{n\rightarrow+\infty}\mu((1-\chi_{2^{-n}})\varphi)$ exists. It is easy to conclude using the fact that $C^0_c(M)$ is spanned by non negative functions. □

Constructive extension operator instead of Hahn Banach. Recall we denote by $\mathcal{I}(X,\mathbb{R}^n)$ the smooth functions vanishing in some neighborhood of X. In the proof of Theorem 2.1, we showed that if t were extendible equivalently if t satisfies the moderate growth condition then:

$$\exists(C,m), \forall\varphi\in\mathcal{I}(X,\mathbb{R}^n), |t(\varphi)|\leq C\|\varphi\|^K_m. \tag{2.3}$$

Therefore t defines a linear functional on $\mathcal{I}(X,\mathbb{R}^n)$ for the induced topology of $C^\infty(\mathbb{R}^n)$ and can be extended by Hahn Banach which is a non constructive argument and **does not imply** the existence of a linear extension operator $t\in\mathcal{D}'_K(\mathbb{R}^n\setminus X)\longmapsto\bar{t}\in\mathcal{D}'_K(\mathbb{R}^n)$.

Denote by $\mathcal{I}^m(X,\mathbb{R}^n)$ the space of C^m functions which vanish on X together with all their derivatives of order less than m, $\mathcal{I}^m(X,\mathbb{R}^n)$ is a closed ideal in $C^m(\mathbb{R}^n)$. To construct a linear extension operator, we have to prove first that t extends by continuity to some element t_m in the topological dual $\mathcal{I}^m(X,\mathbb{R}^n)'$ of $\mathcal{I}^m(X,\mathbb{R}^n)\subset C^m(\mathbb{R}^n)$.

Lemma 2.4. *A distribution t satisfies the estimate (2.3) if and only if t uniquely extends by continuity to an element t_m in $\mathcal{I}^m(X,\mathbb{R}^n)'$:*

$$\forall\varphi \in \mathcal{I}^m(X,\mathbb{R}^n), t_m(\varphi) \;=\; \lim_{\lambda\to 0}\lim_{\varepsilon\to 0} t((1-\chi_\lambda)\phi_\varepsilon * \varphi) \tag{2.4}$$

for the family of cut–off functions $(\chi_\lambda)_\lambda$ defined in Lemma 2.2 and a mollifier ϕ_ε.

Proof. It suffices to prove that the space of C^∞ functions whose support does not meet X is dense in $\mathcal{I}^m(X,\mathbb{R}^n)$ in the C^m topology. In fact, we prove more, let ϕ_ε be a smooth mollifier, then by a classical regularization argument, we have $\lim_{\varepsilon\to 0}(1-\chi_\lambda)\phi_\varepsilon * \varphi = (1-\chi_\lambda)\varphi$ in $C^m(\mathbb{R}^n)$ for all $\varphi\in C^m(\mathbb{R}^n)$ and $\lim_{\lambda\to 0}(1-\chi_\lambda)\varphi \to \varphi$ in $\mathcal{I}^m(X,\mathbb{R}^n)$. By the technical Lemma 2.2 (see [23] p. 11), we have

$$\forall\varphi \in \mathcal{I}^m(X,\mathbb{R}^n), \|\chi_\lambda\varphi\|_m^K \leq \tilde{C}\|\varphi\|_m^{K\cap\{d(x,X)\leq\lambda\}} \to 0$$

when $\lambda\to 0$ therefore $\varphi = \lim_{\lambda\to 0}(1-\chi_\lambda)\varphi$ in the C^m topology. Finally this proves $\mathcal{I}^m(X,\mathbb{R}^n)$ is the closure in $C^m(\mathbb{R}^n)$ of the space of C^∞ functions whose support does not meet X. □

Set $\beta_\lambda = 1-\chi_\lambda$, from the above Theorem we can make a notation abuse and say that $\lim_{\lambda\to 0} t\beta_\lambda \in \mathcal{I}^m(X,\mathbb{R}^n)'$ if t satisfies the estimate (2.3) (we just forget about the mollifier). The idea is to compose $\lim_{\lambda\to 0} t\beta_\lambda$ with a continuous projection $I_m : C^m(\mathbb{R}^n) \longmapsto \mathcal{I}^m(X,\mathbb{R}^n)$ so that $\lim_{\lambda\to 0} t\beta_\lambda\circ I_m$ defines an extension of t. Dually, every compactly supported distribution of order m induces by restriction a linear functional on $\mathcal{I}^m(X,\mathbb{R}^n)$, in other words we have a surjective linear map $p:\mathcal{E}'_m(\mathbb{R}^n)\mapsto\mathcal{I}^m(X,\mathbb{R}^n)'$. We want to construct a linear extension operator $\mathcal{R}$ from $\mathcal{I}^m(X,\mathbb{R}^n)'$ to $\mathcal{E}'_m(\mathbb{R}^n)$ such that $p\circ\mathcal{R}:\mathcal{I}^m(X,\mathbb{R}^n)'\mapsto\mathcal{I}^m(X,\mathbb{R}^n)'$ is the identity map. Then it is immediate to note that the transpose of $\mathcal{R}$ is the projection I_m. The following Proposition aims at classifying the extension operators $\mathcal{R}$. Denote by $\mathcal{E}^m(X)$ the space of differentiable functions of order m in the sense of Whitney [23, Definition 2.3 p. 3],[2, p. 146].

Proposition 2.5. *The three following sets are in bijection:*

- *the set of linear extension operators $\mathcal{R}$ from $\mathcal{I}^m(X,\mathbb{R}^n)'$ to $\mathcal{E}'_m(\mathbb{R}^n)$ such that $p\circ\mathcal{R}:\mathcal{I}^m(X,\mathbb{R}^n)'\mapsto\mathcal{I}^m(X,\mathbb{R}^n)'$ is the identity map,*
- *the set of closed subspaces B of $C^m(\mathbb{R}^n)$ such that $C^m(\mathbb{R}^n)=\mathcal{I}^m(X,\mathbb{R}^n)\oplus B$ which we call* renormalization scheme
- *the set of continuous linear splittings of the exact sequence*

$$0\longmapsto\mathcal{I}^m(X,\mathbb{R}^n)\longmapsto\mathcal{C}^m(\mathbb{R}^n)\overset{q}{\to}\mathcal{E}^m(X)\longmapsto 0. \tag{2.5}$$

Proof. The exactness of (2.5) and the existence of linear continuous splittings of (2.5) is a consequence of the Whitney extension theorem (see [23, p. 10], [2, Thm 2.3 p. 146]). Since (2.5) is a continuous exact sequence of Fréchet spaces, the dual sequence:

$$0\longmapsto\mathcal{E}'_{m,X}(\mathbb{R}^n)\longmapsto\mathcal{E}'_m(\mathbb{R}^n)\overset{p}{\to}\mathcal{I}^m(X,\mathbb{R}^n)'\longmapsto 0 \tag{2.6}$$

is exact [24, Prop 26.4 p. 308].

T is a linear splitting of (2.5)

- $\Leftrightarrow T \circ q$ is a continuous projector on the closed subspace $B = \text{ran}(T)$
- $\Leftrightarrow C^m(\mathbb{R}^n) = B \oplus \mathcal{I}^m(X, \mathbb{R}^n)$ where the projection $Id - T \circ q$ on $\mathcal{I}^m(X, \mathbb{R}^n)$ is denoted by I_m
- $\Leftrightarrow \mathcal{R} =^t I_m$ splits the dual exact sequence (2.6).

□

The above Proposition classifies the extension ambiguities in $\mathcal{E}'_m(\mathbb{R}^n)$ and the following summarizes all results of the above paragraph:

Proposition 2.6. *Let E be the vector space of all distributions $t \in \mathcal{E}'(\mathbb{R}^n \setminus X)$ which satisfies the estimate* 2.3*,*

$$F = \{\mathcal{P} \in Hom(E, \mathcal{E}'_m(\mathbb{R}^n)) \ s.t. \ \mathcal{P}(t)|_{\mathbb{R}^n \setminus X} = t\}$$

then F is in bijection with all three sets defined in Proposition 2.5*.*

The Whitney extension Theorem, formal neighborhoods and extendible distributions. Let us give several interpretations of the result of Proposition 2.5. First, the reader can think of the direct sum decomposition as a way to decompose a C^m function as a sum of a "Taylor remainder" which vanishes at order m on X and a "Taylor polynomial" in B. If X were a point, $\mathcal{E}^m(X)$ is isomorphic to the space $\mathbb{R}_m[X_1, ..., X_n]$ of polynomials of degree m in n variables, we can choose $B = \mathbb{R}_m[x_1, ..., x_n]$ and the decomposition $B + \mathcal{I}^m$ is given by Taylor's formula. For $\varphi \in C^m(\mathbb{R}^n)$, one can think of $q(\varphi) \in \mathcal{E}^m(X) \simeq C^m(\mathbb{R}^n)/\mathcal{I}^m(X, \mathbb{R}^n)$ as the restriction of φ to the *infinitesimal neighborhood of X of order m*. More generally, let $\mathcal{I}^\infty(X, \mathbb{R}^n)$ be the closed ideal of functions in $C^\infty(\mathbb{R}^n)$ which vanish at infinite order on X, this is a nuclear Fréchet space since it is a closed subspace of the nuclear Fréchet space $C^\infty(\mathbb{R}^n)$. We can think of the space $\mathcal{E}(X)$ of C^∞ functions in the sense of Whitney as some sort of ∞-jets in "the transverse directions" to X since by the Whitney extension theorem, we have a continuous exact sequence of nuclear Fréchet spaces:

$$0 \longmapsto \mathcal{I}^\infty(X, \mathbb{R}^n) \longmapsto C^\infty(\mathbb{R}^n) \longmapsto \mathcal{E}(X) \longmapsto 0 \quad (2.7)$$

which implies that $\mathcal{E}(X)$ is the quotient space $C^\infty(\mathbb{R}^n)/\mathcal{I}^\infty(X, \mathbb{R}^n)$. When X is a submanifold of $\mathbb{R}^n$, it is interesting to think of $\mathcal{E}(X)$ as smooth functions *restricted to the formal neighborhood of X*. And the *formal neighborhood of X* is then defined as the topological dual of $\mathcal{E}(X)$ which is nothing but the space of distributions $\mathcal{E}'_X(\mathbb{R}^n)$ with compact support contained in X and fits in the continuous dual exact sequence of DNF spaces [5, appendix A]:

$$0 \longmapsto \mathcal{E}'_X(\mathbb{R}^n) \longmapsto \mathcal{E}'(\mathbb{R}^n) \longmapsto \mathcal{E}'(X)/\mathcal{E}'_X(\mathbb{R}^n) \longmapsto 0 \quad (2.8)$$

where the quotient space $\mathcal{E}'(X)/\mathcal{E}'_X(\mathbb{R}^n)$ should be interpreted as the space of distributions in $\mathcal{D}'(\mathbb{R}^n \setminus X)$ which are **extendible** in $\mathcal{E}'(X)$ and the continuous map $\mathcal{E}'(\mathbb{R}^n) \longmapsto \mathcal{E}'(X)/\mathcal{E}'_X(\mathbb{R}^n)$ is in fact the transpose of the inclusion map $\mathbb{R}^n \setminus X \hookrightarrow \mathbb{R}^n$. Another nice consequence of the theory of nuclear Fréchet spaces is that the space of extendible distributions is a DNF space.

The renormalization group. We also define the renormalization group G as the collection of linear, continuous, bijective maps from $C^m(\mathbb{R}^n)$ to itself preserving $\mathcal{I}^m(X,\mathbb{R}^n)$. Note that $g \in G \implies g^{-1}$ is continuous by the open mapping theorem hence G is well defined as a group. Let $\mathcal{R}$ be a renormalization map corresponding to a projection I_m. For any element $g \in G$, we define the action of g on $\mathcal{R}$ as follows: $\forall t \in \mathcal{I}^m(X,\mathbb{R}^n)', g.\mathcal{R}t(\varphi) = \mathcal{R}t(g(\varphi)) = t(I_m \circ g(\varphi))$ where $\mathcal{R}t(g.) \in \mathcal{E}'(\mathbb{R}^n)$ is an extension of $t \in \mathcal{I}^m(X,\mathbb{R}^n)'$ since g preserves $\mathcal{I}^m(X,\mathbb{R}^n)$.

Renormalization as subtraction of counterterms. Assume we choose a renormalization scheme. We denote by $P_m = Id - I_m$ the projection from C^m to the closed subspace $B \subset C^m$ which plays the role of the *Taylor polynomials.* From the above theorem and recall $\beta_\lambda = 1 - \chi_\lambda$ where χ_λ is the function of Lemma 2.2

Proposition 2.7. *If t satisfies the estimate* 2.3 *then:*

$$\forall\varphi \in C^\infty(\mathbb{R}^n), \bar{t}(\varphi) = \lim_{\lambda\to 0} \underbrace{t(\beta_\lambda I_m\varphi)}_{\textit{finite part}} = \lim_{\lambda\to 0} t(\beta_\lambda\varphi) - \underbrace{t(\beta_\lambda P_m\varphi)}_{\textit{singular part}} \quad (2.9)$$

is a well defined extension of t.

We call such extension a **renormalization**. The divergences of $t(\beta_\lambda\varphi)$ come from the fact that $\varphi \notin \mathcal{I}^m(X,\mathbb{R}^n)$, however these divergences are local in the sense they can be subtracted by the counterterm $t(\beta_\lambda P_m\varphi)$ which becomes singular when $\lambda \to 0$ and only depends on **the restriction to X of the m-jets of** φ (since φ vanishes near X implies that $\varphi \in \mathcal{I}^m \implies P_m\varphi = 0$). By construction, the renormalization group G acts **on the space of all renormalizations** of t.

2.3. Going back to the manifold case

Difference between two extensions. Following the notations of 2.1, recall that $(U_i)_i$ was our locally finite open cover of M by relatively compact sets. On each open set U_i, we defined a chart $\psi_i : U_i \mapsto V \subset \mathbb{R}^n$ and we considered a partition of unity $(\varphi_i)_i$ subordinated to $(U_i)_i$. Let $t \in \mathcal{D}'(M \setminus X)$ be a distribution with moderate growth, then by Theorem 2.1 we may assume that:

$$\forall U_i, \exists m_i \in \mathbb{N}, \exists C_i > 0, \forall\varphi \in C^\infty(\mathbb{R}^n \setminus X \cap \text{supp } \varphi_i), |\psi_{i*}(t\varphi_i)(\varphi)| \leq C_i\|\varphi\|_{m_i}. \quad (2.10)$$

By Theorem 2.1, we may find an extension $\bar{t} = \sum_i \overline{t\varphi_i} \in \mathcal{D}'(M)$ in such a way that for every i, $\overline{t\varphi_i}|_{U_i}$ has order m_i. If we prescribe the **order of the extensions** on every U_i to be equal to $m_i \in \mathbb{N}$, then two extensions t_1, t_2 will differ on each U_i by a distribution $t_1 - t_2|_{U_i}$ of order m_i supported on $X \cap U_i$.

Renormalization in the manifold case. On each chart $\psi_i : U_i \mapsto V \subset \mathbb{R}^n$, we can extend $\psi_{i*}(t\varphi_i) \in \mathcal{D}'(V \setminus \psi_i(X \cap \text{ supp } \varphi_i))$ by renormalization. In other words, by Proposition 2.7, there is a family of functions $\beta_\lambda(i) \in C^\infty(\mathbb{R}^n), \beta_\lambda(i) \to 1$ and counterterms $c_\lambda(i) \in \mathcal{E}'_{\psi_i(X\cap \text{ supp } \varphi_i)}(\mathbb{R}^n)$ such that

$\lim_{\lambda\rightarrow 0}\psi_{i*}(t\varphi_i)\beta_\lambda(i)-c_\lambda(i)$ is an extension of $\psi_{i*}(t\varphi_i)$ in $\mathcal{E}'(\mathbb{R}^n)$. Then setting

$$\beta_\lambda=\sum_i \varphi_i\psi_i^*\beta_\lambda(i) \text{ and } c_\lambda=\sum_i\psi_i^*c_\lambda(i), \tag{2.11}$$

we find that:

$$t\beta_\lambda-c_\lambda=\sum_i t\varphi_i\psi_i^*\beta_\lambda(i)-\psi_i^*c_\lambda(i) \tag{2.12}$$

converges to some extension of t when $\lambda\rightarrow 0$. This proves (1) $\Leftrightarrow$ (3) in Theorem (1.2).

3. Moderate growth and scaling.

In this section, we compare two approaches that were developped to measure the singular behaviour of a distribution along a closed subset X: the moderate growth condition and the one used in [9, 25, 3] in terms of scaling. We show that both approach are equivalent when X is a submanifold of M.

3.1. Weakly homogeneous distributions have moderate growth

In this subsection, we work on $\mathbb{R}^n$ viewed as a product $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, n=n_1+n_2$ and we adopt the following splitting of variables $x\in\mathbb{R}^n=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Here we establish the relationship between our definition of moderate growth and the one used by Yves Meyer [25] and the author [9] in terms of scaling. First we scale in the transverse directions to a vector subspace $X=\mathbb{R}^{n_1}\times\{x_2=0\}$ of $\mathbb{R}^n$ with the maps $\Phi^\lambda:(x_1,x_2)\longmapsto(x_1,\lambda x_2)$. By definition, the scalings acts on $\mathcal{D}'(\mathbb{R}^n)$ by duality $\left(\Phi^{\lambda*}t\right)(\varphi)=\lambda^{-n_2}t(\Phi^{\lambda^{-1}*}\varphi)$. A distribution $t\in\mathcal{D}'(\mathbb{R}^n\setminus X)$ is said to be weakly homogeneous in $\mathcal{D}'(\mathbb{R}^n\setminus X)$ of degree s if the family of distributions $\lambda^{-s}\Phi^{\lambda*}t,\lambda\in(0,+\infty]$ is bounded in $\mathcal{D}'(\mathbb{R}^n\setminus X)$.

Theorem 3.1. *If t is weakly homogeneous of degree s in $\mathcal{D}'(\mathbb{R}^n\setminus X)$ then t has moderate growth along $X=\mathbb{R}^{n_1}\times\{x_2=0\}$. More precisely, for all compact subset $K\subset\mathbb{R}^n$ there is $(m,C)\in\mathbb{N}\times\mathbb{R}$ and a compact subset $B\subset\mathbb{R}^n$ containing K s.t.*

$$\forall\varphi\in\mathcal{D}_K(\mathbb{R}^n\setminus X), |t(\varphi)|\leq C(1+d(supp\ \varphi,X)^{s+n_2})\|\varphi\|_m^B. \tag{3.1}$$

It follows by Theorem 1.2 that such t has an extension in $\mathcal{D}'(\mathbb{R}^n)$. Note that when $s+n_2>0$, we are in a trivial situation of moderate growth since the r.h.s. does not diverge.

Proof. The proof relies on the existence of a continuous partition of unity,

$$\int_0^\infty\frac{d\lambda}{\lambda}\psi(\lambda^{-1}x_2)=\int_0^\infty\frac{d\lambda}{\lambda}\Phi^{\lambda^{-1}*}\psi=1$$

where $\psi(\lambda^{-1}x_2)$ is supported on the corona $\frac{\lambda}{2}\leq|x_2|\leq 2\lambda$. Indeed, let $\chi\in C^\infty(\mathbb{R}^{n_2})$ be a function s.t. $\chi=1$ (resp $\chi=0$) when $|x|\leq\frac{1}{2}$ (resp $|x|\geq 2$) then set $\psi=-x\frac{d\chi}{dx}$.

Fix a compact set $B=\{\sup_{i=1,2}|x_i|\leq L\}$, then for all test function $\varphi\in\mathcal{D}_B(\mathbb{R}^n\setminus X)$ we obviously have

$$\varphi=\int_\varepsilon^{2L}\frac{d\lambda}{\lambda}\left(\Phi^{\lambda^{-1}*}\psi\right)\varphi \text{ for } \varepsilon\leq\frac{d(\text{supp }\varphi,X)}{2},$$

since $\lambda\notin[\frac{d(\text{supp }\varphi,X)}{2},2L] \implies \text{supp }\left(\Phi^{\lambda^{-1}*}\psi\right)\cap\text{supp }(\varphi)=\emptyset$. Now it is obvious that

$$\begin{aligned} t(\varphi) &= \int_{\frac{d(\text{supp }\varphi,X)}{2}}^{2L}\frac{d\lambda}{\lambda}t\left(\left(\Phi^{\lambda^{-1}*}\psi\right)\varphi\right)\\ &= \int_{\frac{d(\text{supp }\varphi,X)}{2}}^{2L}\frac{d\lambda}{\lambda}\lambda^{s+n_2}\left(\lambda^{-s}\Phi^{\lambda*}t\right)\left(\psi\Phi^{\lambda*}\varphi\right)\\ \implies |t(\varphi)| &\leq ((2L)^{s+n_2}+\left(\frac{d(\text{supp }\varphi,X)}{2}\right)^{s+n_2})\sup_{\lambda\leq 2L}|\left(\lambda^{-s}\Phi^{\lambda*}t\right)\left(\psi\Phi^{\lambda*}\varphi\right)| \end{aligned}$$

A simple calculation proves that $\left(\psi\Phi^{\lambda*}\varphi\right)_{\lambda\leq 2L}\subset\mathcal{D}_{\tilde{K}}(\mathbb{R}^n\setminus X)$ for $\tilde{K}=\{(x_1,x_2)||x_1|\leq L,\frac{1}{2}\leq|x_2|\leq 2\}$, $\tilde{K}\cap X=\emptyset$ and that:

$$\forall m\in\mathbb{N},\exists C_m>0,\forall\lambda,\|\psi\Phi^{\lambda*}\varphi\|_m^{\tilde{K}}\leq C_m\|\varphi\|_m^B$$

therefore the family $\left(\psi\Phi^{\lambda*}\varphi\right)_\lambda$ is bounded in the Fréchet space $\mathcal{D}_{\tilde{K}}(\mathbb{R}^n\setminus X)$.

The family $\left(\lambda^{-s}\Phi^{\lambda*}t\right)$ is weakly bounded in $(\mathcal{D}_{\tilde{K}}(\mathbb{R}^n\setminus X))'$ thus strongly bounded by the uniform boundedness principle since $\mathcal{D}_{\tilde{K}}(\mathbb{R}^n\setminus X)$ is Fréchet ([31, Thm 2.5 p. 44]):

$$\exists C'>0,m\in\mathbb{N},\forall\lambda,\forall\varphi\in\mathcal{D}_{\tilde{K}}(\mathbb{R}^n\setminus X),|\left(\lambda^{-s}\Phi^{\lambda*}t\right)(\varphi)|\leq C'\|\varphi\|_m^{\tilde{K}}. \quad (3.2)$$

Therefore

$$\begin{aligned} \sup_{\lambda\leq 2L}|\left(\lambda^{-s}\Phi^{\lambda*}t\right)\left(\psi\Phi^{\lambda*}\varphi\right)| &\leq C'\|\psi\Phi^{\lambda*}\varphi\|_m^{\tilde{K}}\\ &\leq C'C_m\|\varphi\|_m^B\\ \implies |t(\varphi)| &\leq C(1+d(\text{supp }\varphi,X)^{s+n_2})\|\varphi\|_m^B \end{aligned}$$

for some $C>0$ independent of $\varphi\in\mathcal{D}_B(\mathbb{R}^n\setminus X)$. □

It was proved in [9] that the space of weakly homogeneous distributions of degree s along a closed embedded submanifold $X\subset M$ is invariant by the action of diffeomorphisms preserving X, therefore the above Theorem generalizes to the manifold case.

4. Renormalized products.

Let $X\subset\mathbb{R}^n$ be some closed subset. In this section, we first define the class $\mathcal{M}(X,\mathbb{R}^n)$ of tempered functions along X:

Definition 4.1. A function $f \in C^\infty(\mathbb{R}^n \setminus X)$ is tempered along X if for all compact $K \subset \mathbb{R}^n$,

$$\forall m \in \mathbb{N}, \exists (C_m, s) \in \mathbb{R}^2_{\geq 0}, \sup_{|\alpha|\leq m} |\partial^\alpha f(x)| \leq C(1 + d(x, X)^{-s}). \tag{4.1}$$

Tempered functions form an **algebra** by Leibniz rule. It is immediate that the definition 4.1 can be generalized to some closed subset X in a manifold M: we follow the notations of the partition of unity argument in 2.1, f is tempered along X i.e. $f \in \mathcal{M}(X, M)$ if in any local chart $\psi_i : U_i \subset M \mapsto V \subset \mathbb{R}^n$, $\psi_{i*}(\varphi_i f) \in \mathcal{M}(\psi_i(X), \mathbb{R}^n)$.

Then we establish a theorem about renormalized products:

Theorem 4.2. *Let M be a manifold and $X \subset M$ a closed subset. For all $f \in \mathcal{M}(X, M)$ and all $t \in \mathcal{D}'(M)$, there exists a distribution $\mathcal{R}(ft) \in \mathcal{D}'(M)$ which coincides with the regular product ft outside X.*

Thanks to the partition of unity argument of 2.1, we may reduce to the case where X is some closed subset of $M = \mathbb{R}^n$ hence $f \in \mathcal{M}(X, \mathbb{R}^n)$ and $t \in \mathcal{E}'(\mathbb{R}^n)$. By Theorem 2.1, distributions with moderate growth are extendible, therefore it suffices to prove that ft has moderate growth along X which is the content of the following proposition:

Proposition 4.3. *Let $t \in \mathcal{D}'_K(\mathbb{R}^n \setminus X)$ and $f \in C^\infty(\mathbb{R}^n \setminus X)$ such that (t, f) satisfy the estimates:*

$$\exists (C, s_1) \in \mathbb{R}^2_{\geq 0}, \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |t(\varphi)| \leq C(1 + d(supp\ \varphi, X)^{-s_1})\|\varphi\|^K_m \tag{4.2}$$

$$\exists (C_m, s_2) \in \mathbb{R}^2_{\geq 0}, \forall x \in K \setminus X, \sup_{|\alpha|\leq m} |\partial^\alpha f(x)| \leq C_m(1 + d(x, X)^{-s_2}). \tag{4.3}$$

Then ft satisfies the estimate:

$$\exists C', \forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |ft(\varphi)| \leq C'(1 + d(supp\ \varphi, X)^{-(s_1+s_2)})\|\varphi\|^K_m. \tag{4.4}$$

Proof. The claim follows from the estimate:

$$\begin{aligned}
\forall \varphi \in \mathcal{I}(X, \mathbb{R}^n), |ft(\varphi)| &\leq C(1 + d(\text{supp } \varphi, X)^{-s_1})\|f\varphi\|^K_m \\
&\leq CC_m 2^{mn}(1 + d(\text{supp } \varphi, X)^{-s_1})(1 + d(\text{supp } \varphi, X)^{-s_2})\|\varphi\|^K_m \\
&\leq \underbrace{4CC_m 2^{mn}}_{C'}(1 + d(\text{supp } \varphi, X)^{-(s_1+s_2)})\|\varphi\|^K_m.
\end{aligned}$$

□

Example. Our result shares some similarities with [25, Theorem 4.3 p. 85] where Meyer renormalizes the product of distributions $S_\gamma t$ at a point $x_0 \in \mathbb{R}^n$ where $S_\gamma(x) = fp|x - x_0|^\gamma$ (Hadamard's finite part), t is a distribution which is weakly homogeneous of degree s at x_0 and $s + \gamma \notin -\mathbb{N}$. He shows that the renormalized product $S_\gamma t$ is weakly homogeneous of degree $s + \gamma$ at x_0.

Let us recall that by Theorem 2.1, the space $\mathcal{T}_{\mathbb{R}^n \setminus X}(\mathbb{R}^n)$ of distributions with moderate growth along X corresponds to the quotient space $\mathcal{D}'(\mathbb{R}^n)/\mathcal{D}'_X(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n \setminus X$ extendible on $\mathbb{R}^n$. Therefore, we can reformulate Theorem 4.2 as follows:

Theorem 4.4. *$\mathcal{T}_{M\setminus X}(M)$ is a left $\mathcal{M}(X,M)$ module.*

This was also proved by Malgrange [22, Proposition 1 p. 4].

Let us consider a function $g\in C^\infty(\mathbb{R}^n)$, $X=\{g=0\}$ and $gC^\infty(\mathbb{R}^n)$ is a **closed ideal** of $C^\infty(\mathbb{R}^n)$, then a result of Malgrange [23, inequality (2.1) p. 88] yields that g satisfies the Lojasiewicz inequality:

$$\forall K \text{ compact },\exists (C,s)\in\mathbb{R}^2_{\geq 0},\forall x\in K, |g(x)|\geq Cd(x,X)^s. \qquad (4.5)$$

It follows by Leibniz rule that $f=g^{-1}$ must be tempered along X. We state and prove a specific case of "renormalized product" which is due to Malgrange [23, Thm 2.1 p. 100]:

Theorem 4.5. *Let M be a smooth paracompact manifold, let $f=g^{-1}$, $g\in C^\infty(M)$ such that the ideal $gC^\infty(M)$ is closed. Then*

$$\forall T\in\mathcal{D}'(M),\exists S\in\mathcal{D}'(M) \text{ s.t. } gS=T \qquad (4.6)$$

in particular, $S=fT$ outside X.

Beware that the renormalized product $S=fT$ is *not uniquely defined*, however it satisfies the equation $gS=T$ whereas without the closedness assumption on $gC^\infty(M)$, we would only have $gS=T$ modulo distributions supported by X.

Proof. By partition of unity, it suffices to prove that the linear map $m_g: t\in\mathcal{E}'(M)\longmapsto gt\in\mathcal{E}'(M)$ is onto if $gC^\infty(M)$ is closed in $C^\infty(M)$. We will establish that m_g has closed range and that $ran(m_g)$ is dense in $\mathcal{E}'(M)$.

$gC^\infty(M)$ is closed in $C^\infty(M)$ implies that the transposed map: $m_g^*: C^\infty(M)\longmapsto C^\infty(M)$ has closed range therefore m_g has closed range since $C^\infty(M)$ is Fréchet and $\mathcal{E}'(M)=C^\infty(M)'$ (see [24, Thm 26.3 p. 307]).

$gC^\infty(M)$ is closed in $C^\infty(M)$ hence it is Fréchet. By the open mapping Theorem [24, Thm 8.5 p. 60], $m_g: C^\infty(M)\mapsto gC^\infty(M)$ is a linear continuous, surjective map of Fréchet spaces hence m_g is **open**. In terms of estimates, this implies that for any continuous seminorm $\|.\|_m^K$ of $C^\infty(M)$, there is a continuous seminorm $\|.\|_{m'}^{K'}$ such that $\|\varphi\|_m^K\leq\|(g\varphi)\|_{m'}^{K'}$ (see [23, inequality (2.2) p. 88]), hence $g\varphi=0 \implies \varphi=0$. Then we conclude by the observation that $ran(m_g)^\perp=\{\varphi\in C^\infty(M) \text{ s.t. } \forall t\in\mathcal{E}'(M), gt(\varphi)=0\}=\{\varphi \text{ s.t. } g\varphi=0\}=\{0\}\implies ran(m_g)$ is everywhere dense in $\mathcal{E}'(\mathbb{R}^n)$. □

5. Renormalization of Feynman amplitudes in Euclidean quantum field theories.

5.1. Feynman amplitudes are extendible

We give the main application of our extension techniques. Our approach to renormalization follows the philosophy of Brunetti–Fredenhagen [3, 4, **?**], Nikolov–Stora–Todorov [26] which goes back to [10, 11], and is based on the concept of extension of distributions. However, we will use the beautiful formalism of *renormalization maps* of N. Nikolov [26, 27] which is closest in spirit to the present paper. In what follows, we will always assume that (M, g) is a smooth d-dimensional Riemannian manifold with Riemannian metric g. We denote by Δ_g the Laplace Beltrami operator corresponding to g, and we consider the *Green function* $G \in \mathcal{D}'(M \times M)$ of the operator $\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. G is the Schwartz kernel of the operator inverse of $\Delta_g + m^2$ ([34, Appendix 1]) which always exists when M is compact and $m^2 \notin \text{Spec}(\Delta_g)$. In the noncompact case, the general existence and uniqueness result for the Green function usually depends on the global properties of Δ_g and (M, g). If (M, g) has *bounded geometry* in the sense of [6, p. 33] and [30] (see also [34, Definition 1.1 Appendix 1],[33, Def 1.1 p. 3]), then one can find in [34, Appendix 1] conditions of spectral theoretic nature on Δ_g, m^2 that imply the existence of an operator inverse $\left(\Delta_g + m^2\right)^{-1} : L^p(M) \mapsto L^p(M), p \in (1, +\infty)$ whose Schwartz kernel is G.

However if G exists, then we recall a fundamental result about the asymptotics of G near the diagonal:

Lemma 5.1. *Let (M, g) be a smooth Riemannian manifold and Δ_g the corresponding Laplace operator. If $G \in \mathcal{D}'(M \times M)$ is the fundamental solution of $\Delta_g + m^2$, then G is tempered along $D_2 \subset M^2$.*

Proof. This follows from the estimate [37, (2.5) in Proposition 2.2] applied to the Green function G which is the Schwartz kernel of an elliptic pseudodifferential operator of degree -2 since G is a parametrix of the Laplace–Beltrami operator $\Delta_g + m^2$. □

Configuration spaces. For every finite subset $I \subset \mathbb{N}$ and open subset $U \subset M$, we define the configuration space $U^I = \text{Maps}\ (I \mapsto U) = \{(x_i)_{i\in I} \text{ s.t. } x_i \in U, \forall i \in I\}$ of $|I|$ particles in U labelled by the subset $I \subset \mathbb{N}$. In the sequel, we will distinguish two types of diagonals in U^I, the *big diagonal* $D_I = \{(x_i)_{i\in I} \text{ s.t. } \exists (i \neq j) \in I^2, x_i = x_j\}$ which represents configurations where at least two particles collide, and the *small diagonal* $d_I = \{(x_i)_{i\in I} \text{ s.t. } \forall (i, j) \in I^2, x_i = x_j\}$ where all particles in U^I collapse over the same element. The configuration space $M^{\{1,\dots,n\}}$ and the corresponding *big and small* diagonals $D_{\{1,\dots,n\}}, d_{\{1,\dots,n\}}$ will be denoted by M^n, D_n, d_n for simplicity. We also use the notation $d_{\{i,j\}}$ for the subset $\{x_i = x_j\}$ of the configuration space M^n.

Proposition 5.2. *Let (M, g) be a smooth Riemannian manifold, Δ_g the corresponding Laplace operator and G the Green function of $\Delta_g + m^2$. For any finite subset $I \subset \mathbb{N}$, we shall call* Feynman amplitude *all elements of the form $\prod_{(i<j)\in I^2} G^{n_{ij}}(x_i, x_j) \in C^\infty(M^I \setminus D_I), n_{ij} \in \mathbb{N}$. Then all* Feynman amplitudes *are extendible in $\mathcal{D}'(M^I)$.*

Proof. We assume w.l.o.g that $I = \{1, \dots, n\}$. For all $s \geq 0$, the inequality $d(x, d_{\{i,j\}})^{-s} \leq d(x, D_n)^{-s}$ follows from the inclusion $d_{\{i,j\}} \subset D_n$. The Green function $G(x_i, x_j)$ is tempered along $d_{\{i,j\}}$ and the above inequality imply that $G(x_i, x_j) \in \mathcal{M}(D_n, M^n)$. Since $\mathcal{M}(D_n, M^n)$ is an **algebra**, the element

$$\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i, x_j)$$

is also tempered along D_n and is therefore extendible on M^n by Theorem 4.2. □

5.2. Renormalization maps, locality and the factorization property

The vector subspace $\mathcal{O}(D_I,.)$ generated by Feynman amplitudes. In QFT, renormalization is not only extension of Feynman amplitudes in configuration space but our extension procedure should satisfy some consistency conditions in order to be compatible with the fundamental requirement of **locality**.

Recall that for any open subset $\Omega\subset M^I$, we denote by $\mathcal{M}(D_I,\Omega)$ the **algebra** of tempered functions along D_I. We introduce the vector space $\mathcal{O}(D_I,\Omega)\subset\mathcal{M}(D_I,\Omega)$ generated by the Feynman amplitudes

$$\mathcal{O}(D_I,\Omega)=\left\langle\left(\prod_{i<j\in I^2}G^{n_{ij}}(x_i,x_j)\right)_{n_{ij}}\right\rangle_{\mathbb{C}}. \tag{5.1}$$

Axioms for renormalization maps: factorization property as a consequence of locality. We define a collection of *renormalization maps* $(\mathcal{R}_{\Omega\subset M^I})_{\Omega,I}$ where I runs over the finite subsets of $\mathbb{N}$ and Ω runs over the open subsets of M^I which satisfy the following axioms which are simplified versions of those figuring in [27, 2.3 p. 12–14] [26, Section 5 p. 33–35]:

Definition 5.3.

1. For every $I\subset\mathbb{N},|I|<+\infty$, $\Omega\subset M^I$, $\mathcal{R}_{\Omega\subset M^I}$ is a **linear extension operator**:

$$\mathcal{R}_{\Omega\subset M^I}:\mathcal{O}(D_I,\Omega)\longmapsto\mathcal{D}'(\Omega). \tag{5.2}$$

2. For all inclusion of open subsets $\Omega_1\subset\Omega_2\subset M^I$, we require that:

$$\forall f\in\mathcal{O}(D_I,\Omega_2),\forall\varphi\in\mathcal{D}(\Omega_1)$$
$$\langle\mathcal{R}_{\Omega_2\subset M^I}(f),\varphi\rangle=\langle\mathcal{R}_{\Omega_1\subset M^I}(f),\varphi\rangle\,.$$

3. The renormalization maps satisfy the **factorization property**. If (U,V) are disjoint open subsets of M, and (I,J) are disjoint finite subsets of $\mathbb{N}$, $\forall(f,g)\in\mathcal{O}(D_I,U^I)\times\mathcal{O}(D_J,V^J)$:

$$\mathcal{R}_{(U^I\times V^J)\subset M^{I\cup J}}(f\boxtimes g)=\underbrace{\mathcal{R}_{U^I\subset M^I}(f)}_{\in\mathcal{D}'(U^I)}\boxtimes\underbrace{\mathcal{R}_{V^J\subset M^J}(g)}_{\in\mathcal{D}'(V^J)}\in\mathcal{D}'(U^I\times V^J)$$

The most important property is the factorization property (3) which is imposed in [26, equation (2.2) p. 5].

Remarks on the axioms of the Renormalization maps. To define $\mathcal{R}$ on M^I, it suffices to define $\mathcal{R}_{\Omega_i\subset M^I}$ for an open cover $(\Omega_i)_i$ of M^I (they do not necessarily coincide on the overlaps $\Omega_i\cap\Omega_j$) and glue the determinations by a partition of unity.

Uniqueness property of renormalization maps. The following Lemma is proved in [26, Lemmas 2.2, 2.3 p. 6] and tells us that if a collection of renormalization maps $(\mathcal{R}_{\Omega\subset M^I})_{\Omega,I}$ exists and satisfies the list of axioms of definition 5.3, then outside the small diagonal d_n, the restriction $\mathcal{R}_{M^n\setminus d_n\subset M^n}$ would be uniquely determined by the renormalizations $\mathcal{R}_{M^I}$ for all $|I|<n$ because of the factorization axiom.

Lemma 5.4. *Let $(\mathcal{R}_{\Omega\subset M^I})_{\Omega,I}$ be a collection of renormalization maps satisfying the axioms of definition* 5.3. *Then the renormalization map $\mathcal{R}_{M^n\setminus d_n\subset M^n}$ is uniquely determined by the renormalizations maps $\mathcal{R}_{M^I}$ for all $|I|<n$.*

Proof. See [26, p. 6-7] for the detailed proof. □

Beware that the above Lemma **does not imply the existence** of renormalization maps but only that they must satisfy certain consistency conditions if they exist.

5.3. The existence Theorem for renormalization maps

Now we give a short proof of the existence of renormalization maps on general Riemannian manifolds.

Theorem 5.5. *Let (M,g) be a smooth Riemannian manifold, Δ_g the corresponding Laplace operator, G the Green function of $\Delta_g+m^2, m\geqslant 0$ and for any configuration space M^I where I is a finite subset of $\mathbb{N}$, any open subset $\Omega\subset M^I$, recall $\mathcal{O}(D_I,\Omega)\subset\mathcal{M}(D_I,\Omega)$ is the vector space generated by the Feynman amplitudes of the form $\prod_{(i<j)\in I^2}G^{n_{ij}}(x_i,x_j), n_{ij}\in\mathbb{N}$.*

Then there exists a collection of renormalization maps $(\mathcal{R}_{\Omega\subset M^I})_{\Omega,I}$ where I runs over the finite subsets of $\mathbb{N}$ and Ω runs over the open subsets of M^I which satisfies the three axioms of definition 5.3.

Our proof relies on Lemmas (5.6) and (5.7) whose proof will be given later.

Proof. We proceed by induction on the number n of elements of the configuration space. For $n=2$, the renormalization map $\mathcal{R}_{M^2}:\mathcal{O}\left(D_2,M^2\right)\mapsto\mathcal{D}'(M^2)$ exists by Theorem 5.2.

Now assume that all renormalization maps $(\mathcal{R}_{\Omega\subset M^I})_{\Omega,I}$ for $|I|\leqslant n-1$ are constructed and satisfy the list of axioms of definition 5.3. The first step is to construct $\mathcal{R}_{M^n\setminus d_n}\left(\prod_{1\leq i<j\leq n}G^{n_{ij}}(x_i,x_j)\right)$ for generic Feynman amplitudes $\prod_{1\leq i<j\leq n}G^{n_{ij}}(x_i,x_j)\in\mathcal{O}(D_n,M^n)$. But by Lemma 5.6 below, $M^n\setminus d_n$ is covered by the open sets $C_I=M^n\setminus\left(\underset{i\in I,j\notin I}{\cup}d_{\{i,j\}}\right)$ where $I\subsetneq\{1,\dots,n\}$. Therefore it suffices to construct $\mathcal{R}_{C_I\subset M^n}$ for all $I\subsetneq\{1,\dots,n\}$ then glue them together with a partition of unity subordinated to the cover $(C_I)_I$. For every open subset $C_I\subset M^n\setminus d_n$, set $I^c=\{1,\dots,n\}\setminus I$, by the factorization property, the renormalization map $\mathcal{R}_{C_I}$ writes as a product:

$$\mathcal{R}_{C_I}\left(\prod_{1\leq i<j\leq n}G^{n_{ij}}(x_i,x_j)\right)=\underbrace{\underbrace{\mathcal{R}_{M^I}(G_I)}_{\in\mathcal{D}'(M^I)}\underbrace{\mathcal{R}_{M^{I^c}}(G_{I^c})}_{\in\mathcal{D}'(M^{I^c})}\prod_{(i,j)\in I\times I^c}G^{n_{ij}}(x_i,x_j)}_{\in\mathcal{M}(\partial C_I,M^n)}$$

$$G_I=\prod_{(i<j)\in I^2}G^{n_{ij}}(x_i,x_j),\qquad G_{I^c}=\prod_{(i<j)\in I^{c2}}G^{n_{ij}}(x_i,x_j)$$

Therefore the renormalization map $\mathcal{R}_{M^n\setminus d_n}$ is uniquely determined by the renormalization maps $\mathcal{R}_{M^I}$ for $|I|\leq n-1$ according to Lemma 5.4. Lemma 5.7 below yields a partition of unity $(\chi_I)_I$ of $M^n\setminus d_n$ subordinated to the open cover $(C_I)_I$ i.e. supp $\chi_I\subset C_I, \sum_I \chi_I=1$ such that each χ_I is tempered along d_n.

The product $\mathcal{R}_{M^I}(G_I)\mathcal{R}_{M^{I^c}}(G_{I^c})$ belongs to $\mathcal{D}'(M^n)$ and the product $\prod_{(i,j)\in I\times I^c} G^{n_{ij}}(x_i,x_j)$ is tempered along ∂C_I. It follows by corollary 4.2 that the distribution

$$\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)) = \underbrace{\prod_{(i,j)\in I\times I^c} G^{n_{ij}}(x_i,x_j)}_{\in\mathcal{M}(\partial C_I,M^n)}\underbrace{\mathcal{R}_{M^I}(G_I)\mathcal{R}_{M^{I^c}}(G_{I^c})}_{\in\mathcal{D}'(M^n)}\in\mathcal{D}'(C_I)$$

has an extension in $\mathcal{D}'(M^n)$ denoted by $\overline{\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j))}$. By construction, χ_I vanishes in some neighborhood of $\partial C_I\setminus d_n$ in $M^n\setminus d_n$ which implies that $\chi_I\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)) = \chi_I\overline{\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j))}$ in $\mathcal{D}'(M^n\setminus d_n)$. It follows that

$$\mathcal{R}_{M^n\setminus d_n}\left(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)\right) = \sum_I \chi_I\overline{\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j))}.$$

Again by Theorem 4.2, χ_I is tempered along d_n implies that the product $\chi_I\overline{\mathcal{R}_{C_I}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j))}$ is extendible in $\mathcal{D}'(M^n)$ and

$\mathcal{R}_{M^n\setminus d_n}\left(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)\right)$ is therefore extendible in $\mathcal{D}'(M^n)$. Then we define $\mathcal{R}_{M^n}(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j))$ to be any extension of $\mathcal{R}_{M^n\setminus d_n}\left(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)\right)$ in $\mathcal{D}'(M^n)$. □

An important remark is that the sequence of renormalization maps constructed in the above proof is not unique and has infinitely many degrees of freedom at each step of the induction since we can choose many possible extensions for the distribution $\mathcal{R}_{M^n\setminus d_n}\left(\prod_{1\leq i<j\leq n} G^{n_{ij}}(x_i,x_j)\right)$ and these are related to renormalization ambiguities which are often encountered in renormalization of QFT on curved space–times.

Covering lemma. The following simple Lemma is due to Popineau and Stora [26, Lemma 2.2 p. 6] [36, 28] and states that $M^n\setminus d_n$ can be partitioned as a union of open sets on which the renormalization map $\mathcal{R}_n$ can factorize.

Lemma 5.6. *Let M be a smooth manifold. For all subset $I\subsetneq\{1,\dots,n\}$, let $C_I=\{(x_1,\dots,x_n)$ s.t. $\forall i\in I, j\notin I, x_i\neq x_j\}\subset M^n$. Then*

$$\bigcup_I C_I = M^n\setminus d_n \tag{5.3}$$

where I runs over strict subsets of $\{1,\dots,n\}$.

Proof. The key observation is the following, if $(x_1,\dots,x_n)\notin d_n$, then at least two points (x_i,x_j) differ for $(i,j)\in\{1,\dots,n\}^2$ and it follows that $(x_1,\dots,x_n)\in C_I, I=\{j\in\{1,\dots,n\}:x_j=x_i\}$. □

Tempered partition of unity associated to the cover $(C_I)_I$.

Lemma 5.7. *Let M be a smooth manifold and let $(C_I)_I$ be the cover of $M^n\setminus d_n$ defined in Lemma* 5.6*, then there exists a partition of unity $(\chi_I)_I$ subordinated to $(C_I)_I$ such that every function χ_I is tempered along d_n.*

Proof. For every subset $I\subsetneq\{1,\dots,n\}$, let I^c denote its complement in $\{1,\dots,n\}$, then by definition of C_I the set $B_I=\cup_{(i,j)\in I\times I^c}d_{\{i,j\}}$ is the boundary of C_I in the configuration space M^n. For every I, [15, Corollary 1.4.11] yields the existence of a function $\psi_I\in C^\infty(M^n\setminus d_n)$ such that:

- $\psi_I=0$ in some neighborhood of $B_I\setminus d_n\subset M^n\setminus d_n$
- $\psi_I=1$ in some neighborhood of the closed set $((d_I\cup d_{I^c})\setminus d_n)\subset M^n\setminus d_n$
- ψ_I has moderate growth along d_n i.e. $\psi_I\in\mathcal{M}(d_n,M^n)$.

It follows that the family of functions $\left(\varphi_{1,I}=\frac{\psi_I^2}{\sum_{J\subsetneq\{1,\dots,n\}}\psi_J^2}\right)_I$ which is only defined on the open set $U=\cup_{J\subsetneq\{1,\dots,n\}}\{\psi_J>0\}$, forms a partition of unity subordinated to the cover $(C_I\cap U)_I$ where every $\varphi_{1,I}\in\mathcal{M}(d_n,U)$. By definition, U is a neighborhood of d_n and let $(\chi,1-\chi)$ be a partition of unity subordinated to the cover (U,U^c) of $M^n\setminus d_n$ and $(\varphi_{2,I})_I$ a partition of unity subordinated to the cover $(C_I\cap U^c)_I$ of U^c, we conclude that $(\chi_I=\chi\varphi_{1,I}+(1-\chi)\varphi_{2,I})_I$ is subordinated to $(C_I)_I$ and every χ_I belongs to $\mathcal{M}(d_n,M^n)$. □

References

[1] A. Aizenbud and D. Gourevitch, *Schwartz functions on Nash manifolds*, IMRN, **2008/5** (2008).

[2] E. Bierstone, *Differentiable functions*, Bulletin/Brazilian Mathematical Society **11** (1980), 139–189.

[3] R. Brunetti and K. Fredenhagen, *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds*, Commun. Math. Phys. **208** (2000), 623–61.

[4] R. Brunetti, K. Fredenhagen, and M. Köhler, *The Microlocal Spectrum Condition and Wick Polynomials of Free Fields on Curved Spacetimes*, Commun. Math. Phys. **180** (1996), 633–52.

[5] William Casselman, Henryk Hecht, and Dragan Milicić, *Bruhat filtrations and Whittaker vectors for real groups*, The Mathematical Legacy of Harish-Chandra: A Celebration of Representation Theory and Harmonic Analysis: an AMS Special Session Honoring the Memory of Harish-Chandra, January 9-10, 1998, Baltimore, Maryland 68 (2000), 151.

[6] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J.Diff.Geom. **17**, 15–53.

[7] K. Costello, *Renormalization and Effective Field Theory*, AMS, 2011.

[8] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory*, 2012.

[9] N. V. Dang, *Renormalization of quantum field theory on curved spacetimes, a causal approach*, Ph.D. thesis, Paris Diderot University, 2013, http://arxiv.org/abs/1312.5674.

[10] H. Epstein and V. Glaser, *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré **19** (1973), 211–95.

[11] H. Epstein, V. Glaser, and R. Stora, *General properties of the n-point functions in local quantum field theory*, Analyse structurale des amplitudes de collision (Amsterdam) (R. Balian and D. Iagolnitzer, eds.), Les Houches, North Holland, 1976, pp. 5–93.

[12] S. Guillermou and P. Schapira, *Construction of sheaves on the subanalytic site*, (2012), arXiv:1212.4326.

[13] S. Hollands and R. M. Wald. Local Wick polynomials and time ordered products of quantum fields in curved spacetime. *Commun. Math. Phys.*, 223:289–326, 2001.

[14] S. Hollands and R. M. Wald. Existence of local covariant time ordered products of quantum fields in curved spacetime. *Commun. Math. Phys.*, 231:309–45, 2002.

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vol.: 1.: Distribution Theory and Fourier Analysis.*

[16] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publications of the Research Institute for Mathematical Sciences **20** (1984), 319–365.

[17] S. Jaffard and Y. Meyer, *Wavelet methods for pointwise regularity and local oscillations of functions*, vol. 587, American Mathematical Soc., 1996.

[18] M. Kashiwara and P. Schapira, *Moderate and formal cohomology associated with constructible sheaves*, vol. 64, Mémoires Soc. Math. France, 1996.

[19] M. Kashiwara and P. Schapira, *Ind-sheaves*, vol. 271, Astérisque, Soc. Math. France, 2001.

[20] C. Kopper and V. F. Müller. Renormalization Proof for Massive ϕ^4_4 Theory on Riemannian Manifolds. *Commun. Math. Phys.*, 275:331-372, 2007.

[21] S. Lojasiewicz, *Sur le problème de division*, Studia Mathematica **18** (1959), 87–136.

[22] B. Malgrange, *Division des distributions. I: Distributions prolongeables*, Séminaire Schwartz **21** (1960), 1–5.

[23] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, 1966.

[24] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, 1997.

[25] Y. Meyer, *Wavelets, Vibrations and Scalings*, CRM Monograph Series, vol. 9, Amer. Math. Soc., Providence, 1998.

[26] Ivan Todorov, Nikolay M. Nikolov and Raymond Stora, *Renormalization of massless Feynman amplitudes in configuration space*, Reviews in Mathematical Physics, **26.04** (2014).

[27] Nikolay M. Nikolov, *Anomalies in quantum field theory and cohomologies of configuration spaces*, (2009), http://arxiv.org/abs/0903.0187.

[28] G. Popineau and R. Stora, 1982, A pedagogical remark on the main theorem of perturbative renormalization theory, (unpublished preprint).

[29] M. Reed and B. Simon, *Methods of modern mathematical physics. I: Functional analysis*, second ed., Academic Press, New York, 1980.

[30] J. Roe, *An index theorem on open manifolds I, II*, J. Diff. Geom. **27** (1988), 87–113, 115–136.

[31] W. Rudin, *Functional Analysis*, second ed., McGraw-Hill, Inc., 1991.

[32] N. Shimakura, *Partial Differential Operators of Elliptic Type*, Translations of Mathematical Monographs, vol. 99, AMS, 1992.

[33] M. Shubin, *Weak Bloch property and weight estimates for elliptic operators*, (1989-1990), Séminaire Équations aux dérivées partielles (dit "Goulaouic-Schwartz").

[34] M. Shubin, *Spectral theory of elliptic operators on noncompact manifolds*, Astérisque **207** (1992), 37–108.

[35] H. Skoda, *Prolongement des courants positifs fermés de masse finie*, Invent. Math. **66** (1982), 361–376.

[36] R. Stora, *Pedagogical experiments in renormalized perturbation theory, contribution to the conference* Theory of Renormalization and Regularization*, Hesselberg, Germany,*
http://wwwthep.physik.uni-mainz.de/ scheck/Hessbg02.html, 2002.

[37] M. E. Taylor, *Partial differential equations II*, second ed., Springer, 2011.

[38] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society **36.1** (1934), 63–89.

Nguyen Viet Dang
Laboratoire Paul Painlevé (U.M.R. CNRS 8524)
UFR de Mathématiques
Université de Lille 1
59 655 Villeneuve d'Ascq Cédex France.
e-mail: `dangnguyenviet20@gmail.com`