The extension of distributions on manifolds, a microlocal approach.

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Abstract

Let M be a smooth manifold, $I \subset M$ a closed embedded submanifold of M and U an open subset of M. In this paper, we find conditions using a geometric notion of scaling for $t \in \mathcal{D}'(U \setminus I)$ to admit an extension in $\mathcal{D}'(U)$. We give microlocal conditions on t which allow to control the wave front set of the extension. Furthermore, we show that there is a subspace of extendible distributions for which the wave front of the extension is minimal which has applications for the renormalization of quantum field theory on curved spacetimes.

Contents

Introduction.

From the early days of quantum field theory, it has been known [16, 15, 3] that QFT calculations are plagued with infinities arising from the integration of divergent Feynman amplitudes in momentum space. The recipe devised to subtract these divergences is called the renormalization algorithm [9]. When one generalizes QFT to curved Lorentzian spacetimes [21, 49, 2], a simple observation is that both the conventional axiomatic approach to quantum field theory following Wightman's axioms [48] or the usual textbook approach based on the representation of Feynman amplitudes in momentum space, completely break down for the obvious reasons that there is no Fourier transform on curved spacetime and the spacetime is no longer Lorentz invariant.
This motivates to look at the renormalization problem of Feynman amplitudes from the point of view of the position space and this problem was solved in the seminal work of Brunetti and Fredenhagen [5]. The starting point of [5] was to follow one of the very first approach to QFT due to Stueckelberg and his collaborators (D. Rivier, T.A. Green, A. Petermann), which is based on the concept of causality.

The ideas of Stueckelberg were first understood and developed by Bogoliubov and his school ([3]) and then by Epstein-Glaser ([17], [18]) (on flat spacetime). In these approaches, one works directly in position space and the renormalization is formulated as a problem of extension of distributions. Somehow, this point of view based on the S-matrix formulation of QFT was almost completely forgotten by people working on QFT at the exception of some works [36, 37, 38, 43, 47]. However, in 1996, a student of Wightman, M. Radzikowsky revived the subject. In his thesis [34, 35, 44], he used microlocal analysis for the first time in this context and introduced the concept of *microlocal spectrum condition*, a condition on the wave front set of the distributional two-point function which represents the quantum states of positive energy (named Hadamard states) on curved spacetimes [22, 23, 24, 50]. In 2000, in a breakthrough paper, Brunetti and Fredenhagen were able to generalize the Epstein-Glaser theory on curved spacetimes by relying on the fundamental contribution of Radzikowski. These results were further extended in some exciting recent works [14, 26, 28, 29, 30, 31, 42] where the formalism of algebraic QFT now includes the treatment of gauge theories like Yang-Mills fields [27, 51], and also incorporates the Batalin Vilkovisky formalism [19, 20] in order to perturbatively quantize gravitation [6, 7, 45].

All the above works rely on a formalism for renormalization theory which consists in a recursive procedure of extension of products of distributions representing Feynman amplitudes on configuration space. More precisely, if we denote by Δ_F the Feynman propagator which is a fundamental solution of the Klein Gordon operator $\Box+m^2$ with a specific wave front set, then a Feynman amplitude will be a product of the form $\prod_{1<i\leqslant j<n}\Delta_F^{n_{ij}}(x_i,x_j)$, this product of distribution is well defined on the configuration space M^n minus all diagonals $x_i=x_j$ since all the wave front sets are transverse and renormalization consists in extending the above product on the whole configuration space M^n in a way which is compatible with the physical axiom of causality. The central technical ingredient of the recursive procedure is to control the wave front set and the *microlocal scaling degree* of the renormalized products in such a way that we can construct all Feynman amplitudes on all configuration spaces M^n by induction on $n\in\mathbb{N}$. In the present paper, which is an outgrowth of [12], our goal is to build some scale spaces of distributions on manifolds, study their intrinsic property then discuss the operations of extension and renormalization of products relying on recent works on the functional analytic properties of the space $\mathcal{D}'_\Gamma$ of distributions with given wave front set [4, 10, 11]. An interesting perspective for future investigations is to study how our renormalization preserves or breaks symmetries of distributions in the spirit of [1].

The following section is a detailed overview of our results and can be read independently from the rest.

Main results of our paper.

In our paper, we investigate the following problem which has simple formulation: we are given a manifold M and a closed submanifold $I\subset M$. We have a distribution t defined on $M\setminus I$ and we would like to find under what reasonable conditions on t,

1. we can construct an extension $\overline{t}$ of t defined on the whole manifold M,

2. we can control the wave front set of the extension.

The first problem has been addressed in greater generality in [13] where we found necessary and sufficient conditions for a distribution $t \in \mathcal{D}'(M \setminus I)$ where I is a **closed subset** of M, to be extendible. However, our method which uses distance functions, is only adapted to Euclidean QFT. Actually, for QFT on curved Lorentzian spacetimes, it is crucial to find estimates on the wave front set of the extension. This is what we do in the present paper which is focused entirely on the microlocal approach.

In general, the extension problem has no positive answer for a generic distribution t in $\mathcal{D}'(M \setminus I)$ unless t has moderate growth when we approach the singular submanifold I. In the work of Yves Meyer [41] where the manifold M is flat space $\mathbb{R}^n$ and $I = \{0\}$, the distributions having this property are called *weakly homogeneous distributions.* A first difficulty is to extend the definition of Meyer to the case of manifolds. In order to generalize the definition of scaling to measure the growth of distributions, we introduce a class of vector fields called Euler vector fields associated to the submanifold I:

Definition 0.1. *Let M be a smooth manifold, I a submanifold of M and U some open subset of M. Set $\mathcal{I}(U)$ to be the ideal of functions vanishing on I and $\mathcal{I}^k(U)$ its k-th power. A vector field ρ locally defined on U is called Euler if*

$$\forall f \in \mathcal{I}(U), \rho f - f \in \mathcal{I}^2(U). \quad (1)$$

The above definition is obviously intrinsic. In particular, when $M = \mathbb{R}^d, I = \{0\}$ then $\rho = h^j \frac{d}{dh^j}$ is Euler.

In section 1, the main properties of Euler vector fields are studied. They satisfy a property of diffeomorphism invariance

Theorem 0.1. *Let M, M' be two smooth manifolds, $I \subset M, I' \subset M'$ smooth embedded submanifolds and $\Phi := U \longmapsto U'$ a local diffeomorphism such that $\Phi(I \cap U) = I' \cap U'$. Then for any Euler vector field ρ defined on U, the pushforward $\Phi_*\rho$ is Euler.*

And that the flow generated by Euler vector fields are always locally conjugate.

Theorem 0.2. *Let ρ_1, ρ_2 be two Euler vector fields defined in some neighborhood of $p \in I$. Then there is some germ of diffeomorphism Φ at p such that $\rho_1 = \Phi_*\rho_2$.*

In the sequel, once we are given an Euler vector field ρ, let $(e^{t\rho})_t$ be the one parameter group of diffeomorphisms generated by ρ then we will be interested by the one parameter group of scaling flows $(e^{\log\lambda\rho})_{\lambda\in(0,1]}$ and the open subsets U which are stable by the flow $(e^{\log\lambda\rho})_{\lambda\in(0,1]}$ are called ρ *convex.*

For every manifold M and $I \subset M$ a closed embedded submanifold, we construct in section 2 a collection of spaces $(E_{s,I}(U))_U$, indexed by open subsets of M, of *weakly homogeneous distributions of degree* s where $E_{s,I}(U) \subset \mathcal{D}'(U)$, with the following properties:

1. $E_{s,I}$ satisfies a restriction property, if $V \subset U$ then the restriction of $E_{s,I}(U)$ on V is $E_{s,I}(V)$ and satisfies the following gluing property, if $\cup_i V_i$ is an open cover of U s.t. $\cup_i int\left(\overline{V_i}\right)$ is a neighborhood of U, then for $t \in \mathcal{D}'(\cup_i V_i)$, $t \in E_{s,I}(V_i), \forall i \implies t \in E_{s,I}(U)$.

2. $E_{s,I}$ has the important property of diffeomorphism invariance:

   Theorem 0.3. *Let M, M' be two smooth manifolds, $I \subset M, I' \subset M'$ smooth embedded submanifolds and $\Phi := U \longmapsto U'$ a local diffeomorphism such that $\Phi(I\cap U) = I'\cap U'$. Then $\Phi^* E_{s,I}(U') = E_{s,I}(U)$.*

3. The following proposition gives a concrete characterization of elements in $E_{s,I}(U)$ for arbitrary open sets U which could be used as a definition of $E_{s,I}(U)$:

   Proposition 0.1. *t belongs to the local space $E_{s,I}(U)$ if and only if for all $p \in int\left(\overline{U}\right) \cap I$, there is some open chart $\psi : V_p \subset int\left(\overline{U}\right) \mapsto \mathbb{R}^{n+d}$, $\psi(I) \subset \mathbb{R}^n \times \{0\}$ where $\lambda^{-s}(\psi_*t)(x, \lambda h)$ is bounded in $\mathcal{D}'(\psi(V_p \cap U))$.*

However, the property of diffeomorphism invariance imply that $E_{s,I}$ **does not depend on the choice of Euler vector fields.** In particular in the flat case where $M = \mathbb{R}^{n+d}$ with coordinates $(x,h) = (x^i, h^j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d}$ and $I = \{h = 0\}$, $t \in E_{s,I}(\mathbb{R}^{n+d})$ if $(\lambda^{-s}t(x,\lambda h))_{\lambda\in(0,1]}$ is a bounded family of distributions in $\mathcal{D}'(V)$ where $\overline{V}$ is some neighborhood of I.

4. The collection $(E_{s,I})_{s\in\mathbb{R}}$ is filtered, $s' \geqslant s \implies E_{s,I} \subset E_{s',I}$ and $E_{s,I}$ satisfies an extension property (section 4):

   Theorem 0.4. *Let $U \subset M$ be some open set. If $t \in E_{s,I}(U \setminus I)$ then t is extendible. Conversely, if $t \in \mathcal{D}'(M)$ then for any bounded open set $U \subset M$, $t \in E_{s,I}(U)$ for some $s \in \mathbb{R}$.*

   Moreover,

   Theorem 0.5. *For all $s \in \mathbb{R}$, there is a linear map*

$$t \in E_{s,I}(U \setminus I) \longmapsto \overline{t} \in E_{s',I}(U)$$

   where $s' = s$ if $s + d \notin -\mathbb{N}$ and $s' < s$ otherwise.

Using diffeomorphism invariance and locality of $E_{s,I}$, the proof of the above property is a consequence of the microlocal extension Theorem 0.6 proved in the flat case. The space $E_{s,I}$ only takes into account the growth of distributions along the submanifold I which is not enough for many applications, in particular in quantum field theory where we need to know the wave front set of the extension $\overline{t}$ since we must *multiply distributions to define Feynman amplitudes.* Therefore, we need to refine the definition of weakly homogeneous distributions, let us introduce the necessary definitions to state our theorem. We work in $\mathbb{R}^{n+d}$ with coordinates $(x,h) = (x^i, h^j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant d}$, $I = \mathbb{R}^n \times \{0\}$ is the linear subspace $\{h = 0\}$. We assume U to be of the form $U_1 \times U_2$ where U_1 (resp U_2) is an open subset of $\mathbb{R}^n$ (resp $\mathbb{R}^d$) s.t. $\lambda U_2 \subset U_2, \forall\lambda \in (0,1]$.

We denote by $(x,h;\xi,\eta)$ the coordinates in cotangent space T^*U, where ξ (resp η) is dual to x (resp h). $T^\bullet U$ denotes the cotangent T^*U minus the zero section $\underline{0}$. If U is *convex*, then a set $\Gamma \subset T^\bullet U$ is *stable by scaling* if

$$\forall\lambda \in (0,1], \left(\{(x,\lambda^{-1}h;\xi,\lambda\eta); (x,h;\xi,\eta) \in \Gamma\} \cap T^\bullet U\right) \subset \Gamma. \tag{2}$$

For Γ a closed conic set in $T^\bullet U$, $\mathcal{D}'_\Gamma(U)$ is the space of distributions in $\mathcal{D}'(U)$ with wave front set in Γ. For U a convex set and $\Gamma \subset T^\bullet U$ a closed conic set stable by scaling, we denote by $E_s(\mathcal{D}'_\Gamma(U))$ the locally convex space of *weakly homogeneous distributions of degree s in $\mathcal{D}'_\Gamma(U)$* defined as follows: $t \in E_s(\mathcal{D}'_\Gamma(U))$ if $(\lambda^{-s}t(x,\lambda h))_{\lambda\in(0,1]}$ is a bounded family of distributions in $\mathcal{D}'_\Gamma(U)$.

We denote by $N^*(I)$ the conormal bundle of I. The central result of our paper is a general extension theorem (subsection 3.3) for distributions in flat space with control of the wave front set:

Theorem 0.6. *Let $U \subset \mathbb{R}^{n+d}$ be of the form $U_1 \times U_2$ where U_1 (resp U_2) is an open subset of $\mathbb{R}^n$ (resp $\mathbb{R}^d$) s.t. $\lambda U_2 \subset U_2, \forall\lambda \in (0,1]$ and Γ some closed conic set in $T^\bullet U$. Set $\Xi = \{(x,0;\xi,\eta) | (x,h;\xi,0) \in \Gamma\} \subset T^*_I U$. For all $s \in \mathbb{R}$ there exists a linear, bounded map $t \in E_s(\mathcal{D}'_\Gamma(U \setminus I)) \longmapsto \overline{t} \in E_{s'}(\mathcal{D}'_{\Gamma\cup\Xi\cup N^*(I)}(U))$, where $s' = s$ if $s + d \notin -\mathbb{N}$ and $s' < s$ otherwise.*

An immediate corollary of the above theorem is the bound $WF(\overline{t}) \subset (WF(t) \cup \Xi \cup N^*(I))$ on the wave front of the extension. The central ingredients of the proof are: a partition of unity formula which is a continuous analog of the Littlewood–Paley decomposition used by Meyer [41], to consider $(\lambda, x, h) \mapsto \lambda^{-s}t(x,\lambda h)$ as a distribution on the extended space $\mathbb{R} \times \mathbb{R}^{n+d}$ and a new integral formula for the extension which reduces the bounds on the wave front set as applications of the Theorems in [4].

A particular case of the above theorem was proved by Brunetti and Fredenhagen [5] when the closure $\overline{\Gamma}$ of Γ over I is contained in $N^*(I)$. In that case, one can choose an extension $\overline{t}$ such that $WF(\overline{t}) \subset WF(t) \cup N^*(I)$ and $\overline{t} \in E_{s'}(\mathcal{D}'_{\Gamma\cup N^*(I)}(U))$. The important condition $\left(\overline{\Gamma} \cap T^\bullet_I M\right) \subset N^*(I)$ called *conormal landing condition* is *intrinsic* and generalizes in a straightforward way to manifolds. It is a kind of microlocal regularity condition and ensures that the wave front set of the extension is minimal.

Motivated by this intrinsic geometric condition and the result of Theorem 0.6, we construct in section 5 a subspace $E_{s,N^*(I)} \subset E_{s,I}$ which satisfies the following properties:

1. $E_{s,N^*(I)}$ satisfies the same restriction and gluing properties as $E_{s,I}$

2. $E_{s,N^*(I)}$ has the important property of diffeomorphism invariance:

   Theorem 0.7. *Let M, M' be two smooth manifolds, $I \subset M, I' \subset M'$ smooth embedded submanifolds and $\Phi := U \longmapsto U'$ a local diffeomorphism such that $\Phi(I \cap U) = I' \cap U'$. Then $\Phi^* E_{s,N^*(I)}(U') = E_{s,N^*(I)}(U)$.*

   A consequence of the above diffeomorphism invariance is that the definition of $E_{s,N^*(I)}$ **does not depend on the choice of Euler vector fields.**

3. The collection of spaces $(E_{s,N^*(I)})_{s\in\mathbb{R}}$ is filtered, $s' \geqslant s \implies E_{s,N^*(I)} \subset E_{s',N^*(I)}$

The subspace $E_{s,N^*(I)}$ satisfies an extension theorem (section 6)

Theorem 0.8. *Let $U \subset M$ be some open neighborhood of I. If $t \in E_{s,N^*(I)}(U \setminus I)$ then there exists an extension $\bar{t}$ with $WF(\bar{t}) \subset WF(t) \cup N^*(I)$ and $\bar{t} \in E_{s',N^*(I)}(U)$, where $s' = s$ if $s + d \notin -\mathbb{N}$ and $s' < s$ otherwise.*

The main interest of this subspace is that the wave front set $WF(\bar{t})$ of the extension $\bar{t}$ is *minimal* in the sense we only add the conormal $N^*(I)$ to $WF(t)$. Then in section 7, we present an application of the above theorem to **renormalize products of distributions**, we denote by $E_s^\rho(\mathcal{D}'_\Gamma(U))$ the space of distributions t s.t. the family $\left(\lambda^{-s} e^{\log\lambda\rho *} t\right)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'_\Gamma(U)$ for some ρ-convex set U and some cone Γ stable by scaling:

Theorem 0.9. *Let ρ be some Euler vector field, U some neighborhood of I, (Γ_1, Γ_2) two cones in $T^\bullet(U \setminus I)$ which satisfy the conormal landing condition and $\Gamma_1 \cap -\Gamma_2 = \emptyset$. Set $\Gamma = (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$. If $\Gamma_1 + \Gamma_2$ satisfies the conormal landing condition then there exists a bilinear map $\mathcal{R}$ satisfying the following properties:*

- $\mathcal{R} : (u_1, u_2) \in E^\rho_{s_1}\left(\mathcal{D}'_{\Gamma_1}(U \setminus I)\right) \times E^\rho_{s_2}\left(\mathcal{D}'_{\Gamma_2}(U \setminus I)\right) \mapsto \mathcal{R}(u_1 u_2) \in E_{s,N^*(I)}(U), \forall s < s_1 + s_2$
- $\mathcal{R}(u_1u_2) = u_1u_2$ *on* $U \setminus I$
- $\mathcal{R}(u_1u_2) \in \mathcal{D}'_{\Gamma\cup N^*(I)}(U)$.

The above actually means that $\mathcal{R}(u_1u_2) \in \mathcal{D}'_{\Gamma\cup N^*(I)}(U)$ is a **distributional extension** of the Hörmander product $u_1u_2 \in \mathcal{D}'_\Gamma(U \setminus I)$.

In the last section 8 of our paper, we study the renormalization ambiguities which aim to classify the various extensions we constructed.

1 Scaling on manifolds.

Introduction. To solve the extension problem for distributions on manifolds, we define in 0.1 a class of Euler vector fields which scale transversally to a given fixed submanifold $I \subset M$. In this section, we discuss the most important properties of this class of vector fields and their flows.

Example 1.1. *If $M = \mathbb{R}^{n+d}$ and I is the vector subspace which is the zero locus $\{h^j = 0\}$ of the collection of coordinate functions $(h^j)_j$, then $h^j\partial_{h^j}$ is Euler. Indeed, by application of Hadamard's lemma, if $f \in \mathcal{I}$ then $f = h^j H_j$ where the H_j are smooth functions, which implies $\rho f = f + h^i h^j \partial_{h^j} H_i \implies \rho f - f = h^i h^j \partial_{h^j} H_i \in \mathcal{I}^2$.*

Euler vector fields satisfy the following nice properties:

- Given I, the set of *global* Euler vector fields defined on some open neighborhood of I is **nonempty**.
- For any local Euler vector field $\rho|_U$, for any open set $V \subset U$ there is an Euler vector field ρ' defined on a **global neighborhood** of I such that $\rho'|_V = \rho|_V$.

Proof. These two properties result from the fact that one can glue together Euler vector fields by a partition of unity subordinated to some cover of some neighborhood N of I. By paracompactness of M, we can pick an arbitrary locally finite open cover $\cup_{a\in A}V_a$ of M by open sets V_a, define the subset $J\subset A$ such that for each $a\in J$, $V_a\cap I\neq\emptyset$, there is a local chart $(x,h): V_a\mapsto\mathbb{R}^{n+d}$ where the image of I by the local chart is the subspace $\{h^j=0\}$. For such charts which have non empty overlaps with I, we can define an Euler vector field $\rho|_{V_a}$, it suffices to consider the vector field $\rho=h^j\partial_{h^j}$ in each local chart $V_a, a\in J$ and by the example 1.1 this is Euler. The vector fields $\rho_a=\rho|_{V_a}$ do not necessarily coincide on the overlaps $V_a\cap V_b$. However, for any partition of unity $(\alpha_a)_a$ subordinated to $(V_a)_a$, the vector field ρ defined by the formula

$$\rho=\sum_{a\in J}\alpha_a\rho_a \tag{3}$$

is Euler since $\forall f\in\mathcal{I}(U), \rho f-f=\sum_{a\in J}\alpha_a\rho_a f-\sum_{a\in A}\alpha_a f=\sum_{a\in J}\alpha_a\left(\rho_a f-f\right)-\sum_{a\in A\setminus J}\alpha_a f\in\mathcal{I}^2(U)$ since every α_a for $a\in A\setminus J$ vanishes on some neighborhood of I. □

We can find the general form for all possible Euler vector fields ρ in arbitrary coordinate system (x,h) where $I=\{h=0\}$.

Lemma 1.1. *$\rho|_U$ is **Euler** if and only if for all $p\in I\cap U$, in **any arbitrary** local chart (x,h) centered at p where $I=\{h=0\}$, ρ has the standard form*

$$\rho=h^j\frac{\partial}{\partial h^j}+h^iA_i^j(x,h)\frac{\partial}{\partial x^j}+h^ih^jB_{ij}^k(x,h)\frac{\partial}{\partial h^k} \tag{4}$$

where A,B are smooth functions of (x,h).

Proof. The proof is straightforward by noticing that

$$\forall j, \rho h^j-h^j=o(\|h\|^2) \tag{5}$$
$$\forall(i,j),(\rho x^ih^j)-x^ih^j=o(\|h\|^2), \tag{6}$$

from the definition of ρ being an Euler vector field. □

1.1 The diffeomorphism invariance of Euler vector fields.

The class of Euler vector fields enjoys many interesting properties, the first being diffeomorphism invariance. From the introduction, let us recall the statement of Theorem 0.1:

Theorem 1.1. *Let M,M' be two smooth manifolds, $I\subset M, I'\subset M'$ smooth embedded submanifolds and $\Phi:=U\longmapsto U'$ a local diffeomorphism such that $\Phi(I\cap U)=I'\cap U'$. Then for any Euler vector field ρ defined on U the pushforward $\Phi_*\rho$ is Euler.*

Proof. Let G be the pseudogroup of local diffeomorphisms of M (i.e. an element Φ in G is defined over an open set $U\subset M$ and maps it diffeomorphically to an open set $\Phi(U)\subset M$) such that $\forall p\in I\cap U,\forall\Phi\in G,\Phi(p)\in I$. Then it suffices to establish that for all Euler vector field ρ, for all $\Phi\in G$, $\Phi_*\rho$ is **Euler**. In the sequel, we shall identify vector fields X with the associated Lie derivative L_X acting on functions, then the identity $\forall f\in C^\infty(U),(\Phi_*\rho)f=\Phi^{-1*}(\rho(\Phi^*f))$ holds true, it follows from the well–known functorial identity $\Phi_*(\rho f)=(\Phi_*\rho)(\Phi_*f)$ [39, Proposition 2.80 p. 93]. Now if we choose f to be an arbitrary function in $\mathcal{I}$ then we get

$$\forall\Phi\in G,\forall f\in\mathcal{I},(\Phi_*\rho)f-f=\Phi^{-1*}\left(\rho\left(\Phi^*f\right)-\left(\Phi^*f\right)\right). \tag{7}$$

Since $\Phi(I)\subset I$, we have actually $\Phi^*f\in\mathcal{I}$ hence $\left(\rho\left(\Phi^*f\right)-\left(\Phi^*f\right)\right)\in\mathcal{I}^2$ and we deduce that

$$\Phi^{-1*}\left(\rho\left(\Phi^*f\right)-\left(\Phi^*f\right)\right)\in\Phi^{-1*}\mathcal{I}^2=\mathcal{I}^2.$$

□

1.2 Local conjugations of scalings.

We work at the level of germs, a germ of Euler vector field at p is some Euler vector field defined on some neighborhood of p. A germ of diffeomorphism (resp smooth family of germs) at p fixing p is some smooth map $\Phi \in C^\infty(U, M)$ (resp $\Phi \in C^\infty([0,1] \times U, M)$) where U is some neighborhood of p, assume there is a coordinate chart $(x^i, h^j)_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant d} : U \mapsto \mathbb{R}^{n+d}$ such that $I \cap U = \{h^j = 0, 1 \leqslant j \leqslant d\}$ and $|\det d_{x,h}\Phi| > 0$ (resp $\inf_{\lambda \in [0,1]} |\det d_{x,h}\Phi(., \lambda)| > 0$) on U. On the one hand, we saw that the class of Euler vector fields is invariant by the action of G, on the other hand we will prove that for any two germs of Euler vector fields ρ_1, ρ_2 at p, there is a germ of diffeomorphism Ψ at p such that $\Psi_*\rho_1 = \rho_2$.

Denote by $S(\lambda) = e^{\log \lambda \rho}$ the scaling operator defined by the Euler ρ, $S(\lambda)$ satisfies the identity $S(\lambda_1) \circ S(\lambda_2) = S(\lambda_1 \lambda_2)$.

Proposition 1.1. *Let p in I, ρ_1, ρ_2 be two germs of Euler vector fields at p and $S_a(\lambda) = e^{\log \lambda \rho_a}, a = 1, 2$ the corresponding scalings. Then there is a smooth family $(\Phi(\lambda))_{\lambda \in [0,1]}$ of germs of diffeomorphisms at p such that:*

$$S_2(\lambda) = S_1(\lambda) \circ \Phi(\lambda).$$

Proof. We use a local chart $(x, h) : U \mapsto \mathbb{R}^{n+d}$ centered at p, where $I = \{h = 0\}$. We set $\rho = h^j \partial_{h^j}$ which generates the flow $S(\lambda) = e^{\log \lambda \rho}$ and we construct two germs of diffeomorphisms $\Phi_a(\lambda), a = 1, 2$ at p such that

$$\forall \lambda \in (0, 1], \Phi_a(\lambda) = S_a^{-1}(\lambda) \circ S(\lambda), a = 1, 2. \tag{8}$$

Then the germ of diffeomorphism $\Phi(\lambda) = \Phi_1(\lambda) \circ \Phi_2^{-1}(\lambda)$ is a solution of our problem.

Let us construct $\Phi_a(\lambda)$ as a solution of the differential equation obtained by differentiating 8:

$$\lambda \frac{\partial}{\partial \lambda} \Phi_a(\lambda) = \left(\rho - S^{-1}(\lambda)_* \rho_a\right)(\Phi_a(\lambda)) \text{ with } \Phi_a(1) = Id \tag{9}$$

Let f be a smooth function and X a vector field, then the pushforward of fX by a diffeomorphism Φ is:

$$\Phi_*(fX) = (\Phi_* f)(\Phi_* X). \tag{10}$$

We use the general form (4) for Euler vector fields:

$$\rho_a = h^j \frac{\partial}{\partial h^j} + h^i A_i^j(x, h) \frac{\partial}{\partial x^j} + h^i h^j B_{ij}^k(x, h) \frac{\partial}{\partial h^k}$$

hence we apply formula (10):

$$\begin{aligned} S^{-1}(\lambda)_* \rho_a &= \lambda h^j \lambda^{-1} \partial_{h^j} + \lambda h^i A_i^j(x, \lambda h) \frac{\partial}{\partial x^j} + \lambda^2 h^i h^j B_{ij}^k(x, \lambda h) \lambda^{-1} \frac{\partial}{\partial h^k} \\ &= h^j \partial_{h^j} + \lambda h^i A_i^j(x, \lambda h) \frac{\partial}{\partial x^j} + \lambda h^i h^j B_{ij}^k(x, \lambda h) \frac{\partial}{\partial h^k} \\ \implies \rho - S_*^{-1}(\lambda) \rho_a &= -\lambda \left(h^i A_i^j(x, \lambda h) \frac{\partial}{\partial x^j} + h^i h^j B_{ij}^k(x, \lambda h) \frac{\partial}{\partial h^k} \right). \end{aligned}$$

If we define the vector field $X(\lambda) = -\left(h^i A_i^j(x, \lambda h) \frac{\partial}{\partial x^j} + h^i h^j B_{ij}^k(x, \lambda h) \frac{\partial}{\partial h^k} \right)$ then

$$\frac{\partial \Phi_a}{\partial \lambda}(\lambda) = X(\lambda, \Phi_a(\lambda)) \text{ with } \Phi_a(1) = Id. \tag{11}$$

$\Phi_a(\lambda)$ satisfies a non autonomous ODE where the vector field $X(\lambda, .)$ depends smoothly on (λ, x, h). Note that for all $\lambda \in [0, 1]$, the vector field $X(\lambda)$ vanishes at p, therefore by choosing some sufficiently small open neighborhood U of p, there is a smooth map $\Phi(\lambda, p)$ which integrates the differential equation (11) on the interval $[0, 1]$. ∎

We keep the notations and assumptions of the above proposition, we give a simple proof of Theorem 0.2 which states that Euler vector fields are always locally conjugate:

Theorem 1.2. *Let ρ_1, ρ_2 be two germs of Euler vector fields at $p \in I$. Then there is a germ of diffeomorphism Ψ at p such that $\rho_1 = \Psi_* \rho_2$.*

Proof. To prove the above claim, it suffices to construct Ψ in such a way that $S_1(\lambda) = \Psi \circ S_2(\lambda) \circ \Psi^{-1}$ where $S_a(\lambda) = e^{\log \lambda \rho_a}, a = (1, 2)$. In local coordinates $(x^i, h^j)_{ij}$ around p where $I = \{h = 0\}$, let $\rho = h^j \partial_{h^j}$ be some Euler vector field (canonically associated to the choice of coordinates), $S(\lambda) = e^{\log \lambda \rho}$ the corresponding scaling and $\Phi_a(\lambda)$ the family of diffeomorphisms $\Phi_a(\lambda) = S^{-1}(\lambda) \circ S_a(\lambda)$ which has a **smooth limit** $\Psi_a = \Phi_a(0)$ when $\lambda \to 0$ by Proposition 1.1. Start from the identity:

$$\Phi_a(\lambda) \circ S(\mu) = \left(S_a^{-1}(\lambda) \circ S(\lambda)\right) \circ S(\mu) = S_a^{-1}(\lambda) \circ S(\lambda\mu)$$

$$= S_a(\mu) \circ S_a^{-1}(\lambda\mu) \circ S(\lambda\mu) = S_a(\mu) \circ \Phi_a(\lambda\mu),$$

Hence $\forall (\lambda, \mu), \Phi_a(\lambda) \circ S(\mu) = S_a(\mu) \circ \Phi_a(\lambda\mu) \implies \Phi_a(0) \circ S(\mu) = S_a(\mu) \circ \Phi_a(0)$ at the limit when $\lambda \to 0$ where the limit makes sense because Φ_a is smooth in λ at 0. Hence we find that $S_1(\lambda) = \Psi_1 \circ \Psi_2^{-1} \circ S_2(\lambda) \circ \Psi_2 \circ \Psi_1^{-1}$ and the germ of diffeomorphism $\Psi = \Psi_1 \circ \Psi_2^{-1}$ solves our problem. □

2 The space $E_{s,I}(U)$.

In this section, I is a closely embedded submanifold of M and we use Euler vector fields to scale distributions along I and to define scale spaces of distributions. First a set U is called ρ-convex if U is stable by the flow $(e^{-t\rho})_{t>0}$. We give a definition of weakly homogeneous distributions on manifolds but this definition is ρ dependent:

Definition 2.1. *Let U be a ρ-convex open set. The set $E_s^\rho(U)$ is defined as the set of distributions $t \in \mathcal{D}'(U)$ such that*

$$\forall \varphi \in \mathcal{D}(U), \exists C, \sup_{\lambda \in (0,1]} |\left\langle \lambda^{-s} t_\lambda, \varphi \right\rangle| \leqslant C.$$

We next define the space $E_{s,p}^\rho$ of distributions which are locally weakly homogeneous of degree s at $p \in I$.

Definition 2.2. *A distribution t belongs to $E_{s,p}^\rho$ if there exists an open ρ-convex set $U \subset M$ such that $\overline{U}$ is a neighborhood of p and such that $t \in E_s^\rho(U)$.*

A key locality theorem. The next Theorem proves a crucial result that if t is locally $E_{s,p}^\rho$ for some Euler vector field ρ then it is locally $E_{s,p}^\rho$ for **all Euler vector fields** ρ.

Theorem 2.1. *Let $p \in I$, if t belongs to $E_{s,p}^\rho$ for some Euler vector field ρ, then it is so for any other Euler vector field.*

Proof. It suffices to prove the equality $E_{s,p}^{\rho_1} = E_{s,p}^{\rho_2}$ for any pair ρ_1, ρ_2 of Euler vector fields at p. Recall there is a smooth family of germs $\Phi(\lambda)_\lambda$ satisfying $\Phi(\lambda) = S_1^{-1}(\lambda) \circ S_2(\lambda)$ where $(S_a(\lambda) = e^{\log \lambda \rho_a})_{a \in \{1,2\}}$, by Proposition 1.1. Then $\lambda^{-s} S_2(\lambda)^* t = \lambda^{-s} \Phi(\lambda)^* \left(S_1(\lambda)^* t\right)$ is a bounded family of distribution in $\mathcal{D}'(V)$ for some neighborhood V of p implies that $\lambda^{-s}\left(S_1(\lambda)^* t\right)$ is also a bounded family of distribution in $\mathcal{D}'(V')$ for some smaller neighborhood V' of p. □

A comment on the statement of the theorem, first the definition of ρ-convexity is $\forall p \in U, \forall \lambda \in (0, 1], S(\lambda, p) \in U$, the fact that we let λ to be positive allows U to have *empty intersection* with I. The previous theorem allows to give a definition of the spaces of distributions $E_{s,p}$ and $E_{s,I}(U)$ which makes *no mention of the choice of Euler vector field*:

Definition 2.3. *A distribution t belongs to $E_{s,p}$ if t belongs to $E_{s,p}^\rho$ for some ρ. We define $E_{s,I}(U)$ as the space of all distributions $t \in \mathcal{D}'(U)$ such that $t \in E_{s,p}^\rho$, $\forall p \in I \cap int(\overline{U})$.*

An equivalent definition of the space $E_{s,I}(U)$ is the following: *t belongs to the local space $E_{s,I}(U)$ if and only if for all $p \in I \cap int(\overline{U})$, there is some open chart $\psi : V_p \mapsto \mathbb{R}^{n+d}$, $\psi(I) \subset \mathbb{R}^n \times \{0\}$ where $\lambda^{-s}(\psi_* t)(x, \lambda h)$ is bounded in $\mathcal{D}'(\psi(V_p \cap U))$.*

It is immediate that $E_{s,I}$ satisfies the restriction property: if $V \subset U$ then $p \in int(\overline{V}) \cap I \implies p \in int(\overline{U}) \cap I$ and therefore the restriction of $E_{s,I}(U)$ on V is $E_{s,I}(V)$.

A consequence of Theorem 2.1 is the following properties of $E_{s,I}$ under gluings:

Theorem 2.2. *$E_{s,I}$ satisfies the following gluing property: if $\cup_i V_i$ is an open cover of U s.t. $\cup_i int\left(\overline{V_i}\right)$ is a neighborhood of U then for $t \in \mathcal{D}'(\cup V_i)$, $t \in E_{s,I}(V_i), \forall i \implies t \in E_{s,I}(U)$*

Proof. It suffices to prove that $t \in E_{s,p}(U)$ for all $p \in int(\overline{U}) \cap I$. Let $p \in int\left(\overline{U}\right) \cap I$, then obviously $p \in \cup_i int\left(\overline{V_i}\right)$ since $\cup_i int\left(\overline{V_i}\right)$ is a neighborhood of U. Then by definition of $t \in E_{s,p}(V_i)$, there is some neighborhood V_p of p s.t. $V_p \subset int\left(\overline{V_i}\right)$ for some i and $\lambda^{-s} e^{\log \lambda \rho *} t$ is bounded in $\mathcal{D}'(V_p \cap V_i)$ which implies in particular that $\lambda^{-s} e^{\log \lambda \rho *} t$ is bounded in $\mathcal{D}'(\tilde{V_p} \cap U)$ where $\tilde{V_p} = V_p \cap int\left(\overline{U}\right)$ is a neighborhood of p in $int\left(\overline{U}\right)$, therefore $t \in E_{s,p}(U)$. $\square$

We prove Theorem 0.3 which claims that $E_{s,I}(U)$ satisfies a property of diffeomorphism invariance:

Theorem 2.3. *Let I (resp I') be a closed embedded submanifold of M (resp M'), $U \subset M$ (resp $U' \subset M'$) open and $\Phi : U' \mapsto U$ a diffeomorphism s.t. $\Phi(U' \cap I') = I \cap U$. Then $\Phi^* E_{s,I}(U) = E_{s,I'}(U')$.*

Proof. By Theorem 2.1, we can localize the proof at all points $p \in int\left(\overline{U}\right) \cap I$. Let $p \in int\left(\overline{U}\right) \cap I$, then $t \in E_{s,I}(U)$ implies by definition that $t \in E^{\rho}_{s,p}$ for some ρ which means that:

$$\lambda^{-s} e^{\log \lambda \rho *} t \text{ bounded in } \mathcal{D}'(V), int(\overline{V}) \text{ neighborhood of } p$$

$$\Leftrightarrow \lambda^{-s} \Phi^* e^{\log \lambda \rho *} \Phi^{-1*}(\Phi^* t) \text{ bounded in } \mathcal{D}'(\Phi^{-1}(V))$$

because the pull–back by a diffeomorphism is bounded from $\mathcal{D}'(V)$ to $\mathcal{D}'(\Phi^{-1}(V))$ [4, Prop 6.1],

$$\Leftrightarrow \lambda^{-s} e^{\log \lambda (\Phi_*^{-1} \rho) *}(\Phi^* t) \text{ bounded in } \mathcal{D}'(\Phi^{-1}(V))$$

where the vector field $\Phi_*^{-1}\rho$ is a germ of Euler field near p by 0.1. Therefore $\Phi^* t$ is in $E^{\Phi_*^{-1}\rho}_{s,p'}$ where $p' = \Phi^{-1}(p)$ and repeating the proof for all $p \in int\left(\overline{U}\right) \cap I$ yields the claim. $\square$

3 The extension problem on flat space.

Formulation of the problem. We work in $\mathbb{R}^{n+d}$ with coordinates (x, h), $I = \mathbb{R}^n \times \{0\}$ is the linear subspace $\{h = 0\}$. In the sequel, unless it is specified otherwise, we will always assume that we work with open sets U of the form $U_1 \times U_2$ where U_1 (resp U_2) is an open subset of $\mathbb{R}^n$ (resp $\mathbb{R}^d$) s.t. $\lambda U_2 \subset U_2, \forall \lambda \in (0, 1]$ in particular such U is *convex* meaning that:

$$(x, h) \in U \implies \forall \lambda \in (0, 1], (x, \lambda h) \in U. \tag{12}$$

We reformulate the extension problem on flat space:

Definition 3.1. *We are given a convex open set $U \subset \mathbb{R}^{n+d}$ and $I = \mathbb{R}^n \times \{0\}$. We have a distribution $t \in \mathcal{D}'(U \setminus I)$ and we would like to find under what reasonable conditions on t one can construct an extension $\overline{t} \in \mathcal{D}'(U)$.*

3.1 Construction of a formal extension.

In this subsection, we construct a candidate for the formal extension.

Defining a smooth partition of unity. A partition of unity will provide us with some family of smooth functions supported in $\mathbb{R}^{n+d} \setminus I$ approximating the constant function $1 \in C^\infty(\mathbb{R}^{n+d} \setminus I)$.

Definition 3.2. *A smooth partition of unity is a function* $\Psi \in C^\infty\left((0,\infty), \mathbb{R}^{n+d} \setminus I\right)$ *such that* $\forall \Lambda \in (0,\infty), \Psi_\Lambda = 0$ *in some neighborhood of* I *and* $\Psi_\Lambda \underset{\Lambda\to\infty}{\to} 1$ *for the Fréchet topology of* $C^\infty(\mathbb{R}^{n+d} \setminus I)$.

Motivated by the above definition, we choose a function χ such that $\chi = 1$ in a neighborhood of I and the projection $\pi : \mathbb{R}^n \times \mathbb{R}^d \longmapsto \mathbb{R}^n \times \{0\}$ is proper on the support of χ. This implies χ satisfies the following constraint: for all compact set $K \subset \mathbb{R}^n, \exists (a,b) \in \mathbb{R}^2$ such that $b > a > 0$ and $\chi|_{(K\times\mathbb{R}^d)\cap\{|h|\leqslant a\}} = 1, \chi|_{(K\times\mathbb{R}^d)\cap\{|h|\geqslant b\}} = 0$. We set $\Psi(\Lambda, x, h) = 1 - \chi(x, \Lambda h)$ and it is a simple exercice to verify that this defines a partition of unity of $\mathbb{R}^{n+d} \setminus I$.

A candidate for the extension.

Proposition 3.1. *Let* U *be an open set of* $\mathbb{R}^{n+d}$, *if* $t \in \mathcal{D}'(U \setminus I)$ *then for any smooth partition of unity* Ψ_Λ

$$t = \lim_{\Lambda\to+\infty} t\Psi_\Lambda \tag{13}$$

as distribution on $U \setminus I$.

From the above proposition, we deduce that if $\lim_{\Lambda\to+\infty} t\Psi_\Lambda$ **converges in** $\mathcal{D}'(U)$ the limit defines an extension of t. So this raises the question, for all test function $\varphi \in \mathcal{D}(U)$, does the limit $\lim_{\Lambda\to+\infty} \langle t, \Psi_\Lambda \varphi \rangle$ exist ? To study this question, we introduce a continuous decomposition of our partition of unity formula

$$\begin{aligned} 1 - \chi(x, \Lambda h) &= 1 - \chi(x,h) + \chi(x,h) - \chi(x,\Lambda h) \\ &= \int_{\Lambda^{-1}}^1 \frac{d\lambda}{\lambda} \lambda \frac{d}{d\lambda} \left[\chi(x, \lambda^{-1} h)\right] + 1 - \chi(x,h) \\ &= \int_{\Lambda^{-1}}^1 \frac{d\lambda}{\lambda} \left(-\rho\chi\right)(x, \lambda^{-1}h) + 1 - \chi(x,h), \end{aligned}$$

where ρ is the Euler vector field $\sum_{j=1}^d h^j \frac{\partial}{\partial h^j}$ which scales tranversally to $I = \{h^j = 0, 1 \leqslant j \leqslant d\}$. In the sequel, we will write $\rho = h\frac{\partial}{\partial h}$ for brevity. Set $\psi = -\rho\chi$ and define the scaling by a factor $\lambda \in (0,1]$:

$$\psi_\lambda(x,h) = \psi(x, \lambda h).$$

In these notations, the partition of unity formula simply writes:

$$\chi - \chi_\Lambda = \int_{\Lambda^{-1}}^1 \frac{d\lambda}{\lambda} \psi_{\lambda^{-1}}. \tag{14}$$

In the sequel, instead of studying the limit $\Lambda \to \infty$, we will set $\varepsilon^{-1} = \Lambda$ and study instead the limit $\varepsilon \to 0$. We will also denote by π the projection $(x,h) \in \mathbb{R}^n \times \mathbb{R}^d \longmapsto x \in \mathbb{R}^n$ and χ will always designate a smooth function such that $\chi = 1$ in some neighborhood of I and π is proper on supp χ. In what follows, we will study the behaviour of

$$t(1 - \chi_{\varepsilon^{-1}}) = \int_\varepsilon^1 \frac{d\lambda}{\lambda} t\psi_{\lambda^{-1}} + t(1-\chi). \tag{15}$$

when $\varepsilon \to 0$.

3.2 The extension theorems.

3.2.1 Some definitions and notations.

Let us introduce the terminology needed to state our theorems. Let $U \subset \mathbb{R}^{n+d}$ be an open set, we denote by $(x, h; \xi, \eta)$ the coordinates in cotangent space T^*U, where ξ (resp η) is dual to x (resp h). $T^\bullet U$ denotes the cotangent TU minus the zero section $\underline{0}$.

A set $\Gamma \subset T^\bullet U$ is *stable by scaling* if

$$\forall \lambda \in (0,1], \left(\{(x, \lambda^{-1}h; \xi, \lambda\eta); (x,h;\xi,\eta) \in \Gamma\} \cap T^\bullet U\right) \subset \Gamma. \tag{16}$$

Concisely, if we denote by $\Phi_\lambda^*\Gamma$ the pull–back of Γ by Φ_λ [4], we require that $\forall \lambda \in (0,1], (\Phi_\lambda^*\Gamma \cap T^\bullet U) \subset \Gamma$. We also denote by $T_I^*\mathbb{R}^{n+d}$ the restriction of $T^*\mathbb{R}^{n+d}$ on I and $N^*(I)$ the conormal bundle of I. As we explained in the introduction, the extension theorem has no positive solution for arbitrary distributions in $U \setminus I$. However, if we impose that the distribution has "moderate growth" in terms of scaling then we will be able to solve it. The scaling of distribution is defined by duality

$$\forall \varphi \in \mathcal{D}(U), \langle t_\lambda, \varphi \rangle = \lambda^{-d} \langle t, \varphi_{\lambda^{-1}} \rangle$$
$$\text{where } \varphi_{\lambda^{-1}} = \varphi(x, \lambda^{-1}h).$$

In the sequel, for a given open set U, a compact set $K \subset U$, we will denote by $(\pi_{m,K})_{m\in\mathbb{N}}$ the collection of continuous seminorms on the Fréchet space $\mathcal{D}_K(U)$ of test functions supported on K defined as:

$$\forall \varphi \in \mathcal{D}_K(U), \pi_{m,K}(\varphi) = \sup_{|\alpha| \leqslant m, x \in K} |\partial^\alpha \varphi(x)|.$$

Weakly homogeneous distributions in $\mathcal{D}'_\Gamma$. Let us formalize this notion of distribution having nice behaviour under scaling by defining the main space of distributions for which the extension problem has a positive answer.

Using the recent work [10, 6.3], we can characterize bounded sets in $\mathcal{D}'_\Gamma$ by duality pairing. A set $B \subset \mathcal{D}'_\Gamma(U)$ is bounded if for every $v \in \mathcal{E}'_\Lambda(U)$ where Λ is an open cone s.t. $\Lambda \cap -\Gamma = \emptyset$, we have

$$\sup_{t\in B} | \langle t, v \rangle | < +\infty.$$

Definition 3.3. *Let $\Gamma \subset T^\bullet U$ be a closed conic set stable by scaling. A distribution t is weakly homogeneous of degree s in $\mathcal{D}'_\Gamma(U)$, if for all distribution $v \in \mathcal{E}'_\Lambda(U)$ where $\Lambda = -\Gamma^c$,*

$$\sup_{\lambda\in[0,1]} | \left\langle \lambda^{-s} t_\lambda, v \right\rangle | < +\infty.$$

We denote this space by $E_s(\mathcal{D}'_\Gamma(U))$ and we endow it with the locally convex topology generated by the seminorms

$$P_B(t) = \sup_{\lambda\in[0,1], v\in B} | \left\langle \lambda^{-s} t_\lambda, v \right\rangle |$$

for B equicontinuous [4, lemma 6.3] in $\mathcal{E}'_\Lambda$.

We recover the definition of Yves Meyer in the particular case where $\Gamma = T^\bullet\mathbb{R}^{n+d}$ in which case $\mathcal{D}'_\Gamma = \mathcal{D}'$.

A key conceptual step in our approach is to think of $\lambda^{-s}t(x, \lambda h)$ as a distribution of the three variables (λ, x, h). Let us define the map

$$\Phi : (\lambda, x, h) \in \mathbb{R} \times \mathbb{R}^{n+d} \longmapsto (x, \lambda h) \in \mathbb{R}^{n+d}. \tag{17}$$

Theorem 3.1. *Let $s \in \mathbb{R}$ s.t. $s + d > 0$, $\Gamma \subset T^\bullet U$ a closed conic set stable by scaling. If $t \in \mathcal{D}'(U \setminus I)$ is weakly homogeneous of degree s in $\mathcal{D}'_\Gamma(U \setminus I)$, then $\bar{t} = \lim_{\varepsilon\to 0} t(1 - \chi_{\varepsilon^{-1}})$ is a well defined extension of t and $WF(\bar{t}) \subset WF(t) \cup N^*(I) \cup \Xi$ where*

$$\Xi = \{(x, 0; \xi, \eta) | \exists (x, h; \xi, 0) \in \Gamma \cap T^*_{supp\ \psi} U\}.$$

Before we prove the theorem, let us show why the set $WF(t) \cup N^*(I) \cup \Xi$ is a closed conic set. Recall that $U = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^d$ where $\lambda U_2 \subset U_2, \forall \lambda \in (0, 1]$. We may assume w.l.o.g that U_2 contains a set of the form $\{0 < |h| \leqslant \varepsilon\}$. There is nothing to prove over $U \setminus I$ since $WF(t)$ is closed in $T^\bullet(U \setminus I)$, therefore we study the closure of $WF(t) \cup N^*(I) \cup \Xi$ in T_I^*U. Let $(x, 0; \xi, \eta)$ be in its closure $\overline{WF(t)\cup N^*(I)\cup\Xi}$. If $\xi=0$ then $(x,0;0,\eta)\in N^*(I)$. Otherwise $\xi\neq 0$, there is a sequence $(x_n,h_n;\xi_n,\eta_n)\rightarrow(x,0;\xi,\eta)$ where $(x_n,h_n;\xi_n,\eta_n)\in WF(t)$ and $h_n\rightarrow 0$. But since $WF(t)\subset\Gamma$ and since Γ is scale invariant then $(x_n,\varepsilon\frac{h_n}{|h_n|};\xi_n,\varepsilon^{-1}|h_n|\eta_n)\in\Gamma$. By compactness of the unit sphere, we can extract a convergent subsequence for $\varepsilon\frac{h_n}{|h_n|}$ and the limit $(x,h;\xi,0)$ is in Γ. Therefore by definition of Ξ, we will have $(x,0;\xi,\eta)\in\Xi$ and this implies that $WF(t)\cup N^*(I)\cup\Xi$ is closed.

Proof. We have to establish the convergence of $t(1-\chi_{\varepsilon^{-1}})$ in $\mathcal{D}'_\Lambda(U)$ when $\varepsilon\rightarrow 0$ for $\Lambda=WF(t)\cup N^*(I)\cup\Xi$. Our proof is divided in three parts, in the first, we prove that the limit exists in $\mathcal{D}'(U)$ with arguments similar to [41] but in our setting of continuous partition of unity. Then in the second part, we derive a new integral formula for $t(1-\chi_{\varepsilon^{-1}})$, and we shall use the integral formula to show that the family $(t(1-\chi_{\varepsilon^{-1}}))_\varepsilon$ is bounded in $\mathcal{D}'_\Lambda(U)$ using the behaviour of the WF under the fundamental operations on distributions [4]. Then $\lim_{\varepsilon\rightarrow 0}t(\chi-\chi_\varepsilon)$ converges in $\mathcal{D}'(U)$ and is bounded in $\mathcal{D}'_\Lambda(U)$ implies that $\lim_{\varepsilon\rightarrow 0}t(\chi-\chi_\varepsilon)$ converges in $\mathcal{D}'_\Lambda(U)$.

Step 1. We prove that $\lim_{\varepsilon\rightarrow 0}t(1-\chi_{\varepsilon^{-1}})$ exists in $\mathcal{D}'(U)$ when $\varepsilon\rightarrow 0$. Let us give a different analytical expression using the partition of unity formula,

$$t(1-\chi_{\varepsilon^{-1}})=\int_\varepsilon^1\frac{d\lambda}{\lambda}t\psi_{\lambda^{-1}}+t(1-\chi).$$

Therefore:

$$\langle t(\chi-\chi_{\varepsilon^{-1}}),\varphi\rangle \quad = \quad \int_\varepsilon^1 d\lambda\lambda^{s+d-1}\left\langle(\lambda^{-s}t_\lambda)\psi,\varphi_\lambda\right\rangle. \tag{18}$$

It follows that the r.h.s of 18 has a limit when $\varepsilon\rightarrow 0$ since λ^{s+d-1} is integrable on $[0,1]$. It remains to prove that the limit is a distribution. $(\lambda^{-s}t_\lambda)_{\lambda\in(0,1]}$ is bounded in $\mathcal{D}'(U\setminus I)$ therefore for all compact subset $K\subset U\setminus I$:

$$\exists C_K,\forall\varphi\in\mathcal{D}_K(U),\ \sup_{\lambda\in(0,1]}|\left\langle\lambda^{-s}t_\lambda,\varphi\right\rangle|\leqslant C_K\pi_{m,K}(\varphi).$$

For all compact subset $K'\subset\mathbb{R}^{n+d}$ and for all $\varphi\in\mathcal{D}_{K'}(U)$, the family $(\psi\varphi_\lambda)_\lambda$ has fixed compact support which does not meet I and is bounded in $\mathcal{D}_K(U\setminus I)$ for some compact set K:

$$\forall\lambda\in(0,1],\pi_{m,K}(\psi\varphi_\lambda)\leqslant C_2\pi_{m,K'}(\varphi).$$

The two above bounds easily imply that:

$$\begin{aligned}
\forall\varphi\in\mathcal{D}_{K'}(U),\ \sup_{\lambda\in(0,1]}|\left\langle\lambda^{-s}t_\lambda,\psi\varphi_\lambda\right\rangle| &\leqslant C_KC_2\pi_{m,K'}(\varphi)\\
\implies |\langle t(\chi-\chi_{\varepsilon^{-1}}),\varphi\rangle| &\leqslant |\int_\varepsilon^1 d\lambda\lambda^{s+d-1}\left\langle(\lambda^{-s}t_\lambda)\psi,\varphi_\lambda\right\rangle|\\
&\leqslant \frac{1-\varepsilon^{s+d}}{s+d}C_KC_2\pi_{m,K'}(\varphi)\\
\implies \lim_{\varepsilon\rightarrow 0}|\langle t(\chi-\chi_{\varepsilon^{-1}}),\varphi\rangle| &\leqslant \frac{C_KC_2\pi_{m,K'}(\varphi)}{s+d}
\end{aligned}$$

The above bound means that $\lim_{\varepsilon\rightarrow 0}t(\chi-\chi_{\varepsilon^{-1}})$ is well defined in $\mathcal{D}'(U)$. But the difficult point is to control the wave front set of the limit over the subspace $I=\{h=0\}$.

Step 2 We just proved that $\lim_{\varepsilon\rightarrow 0}t(\chi-\chi_\varepsilon)$ converges in $\mathcal{D}'(U)$. In order to control the WF of the limit, it suffices to prove that the family $t(\chi-\chi_\varepsilon)_\varepsilon$ is bounded in $\mathcal{D}'_\Lambda(U),\Lambda=WF(t)\cup N^*(I)\cup\Xi$. We propose a simple method which consists in giving a new integral formula for the identity 18. We double the space $\mathbb{R}^{n+d}$ and transform the formula $\int_\varepsilon^1\frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle(\lambda^{-s}t_\lambda)\psi,\varphi_\lambda\right\rangle$ into an integral formula on $\mathbb{R}\times\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$. We work in $\mathbb{R}\times\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$ with coordinates (λ,x,h,x',h'). We denote by $\delta\in\mathcal{D}'(\mathbb{R}^{n+d})$ the delta distribution supported at $(0,0) \in \mathbb{R}^{n+d}$ and $\delta_\Delta(.,.)$ the delta distribution supported by the diagonal $\Delta \subset \mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$ where we have the relation $\delta_\Delta((x,h),(x',h')) = \delta(x-x',h-h')$. Thus $\langle t(\chi-\chi_{\varepsilon^{-1}}),\varphi\rangle$

$$\begin{aligned} &= \int_\varepsilon^1 \frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle (\lambda^{-s}t_\lambda)\psi,\varphi_\lambda\right\rangle \\ &= \int_{\mathbb{R}^{n+d}} dx'dh' \int_{\mathbb{R}\times\mathbb{R}^{n+d}} \frac{d\lambda}{\lambda} dxdh 1_{[\varepsilon,1]}(\lambda)\lambda^{s+d}\lambda^{-s}t(x,\lambda h)\psi(x,h)\delta(x-x',\lambda h-h')\varphi(x',h'). \end{aligned}$$

Finally, we end up with the integral formula:

$$t(\chi-\chi_{\varepsilon^{-1}})(x',h') = \int_{\mathbb{R}\times\mathbb{R}^{n+d}} d\lambda dxdh 1_{[\varepsilon,1]}(\lambda)\lambda^{s+d-1}\lambda^{-s}t(x,\lambda h)\psi(x,h)\delta(x-x',\lambda h-h'). \quad (19)$$

It suffices to estimate Λ over I since we already know that the family $t(\chi-\chi_{\varepsilon^{-1}})_\varepsilon$ is bounded in $\mathcal{D}'_{WF(t)}(U\setminus I)$ i.e. $\Lambda\cap T^*(U\setminus I) = WF(t)$. We want to calculate the WF of the r.h.s of (19) in $T^*_I U$.

1. decompose the r.h.s of (19) in two blocks

$$\underbrace{1_{[\varepsilon,1]}(\lambda)\lambda^{s+d-1}\lambda^{-s}t(x,\lambda h)\psi(x,h)}_{B_{1,\varepsilon}}\underbrace{\delta(x-x',\lambda h-h')}$$

2. $1_{[\varepsilon,1]}(\lambda)\lambda^{s+d-1} \in L^1(\mathbb{R})$ and $t\in E_s(\mathcal{D}'_\Gamma(U))$ hence by Lemma 9.3 proved in appendix, the block $\left(B_{1,\varepsilon} = 1_{[\varepsilon,1]}(\lambda)\lambda^{s+d-1}\lambda^{-s}t(x,\lambda h)\psi(x,h)\right)_\varepsilon$ is a bounded family in $\mathcal{D}'_V(\mathbb{R}\times U)$ when $\varepsilon\in(0,1]$ and where

$$V = \left\{\begin{pmatrix} \lambda & ; & \widehat{\lambda} \\ x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta}\end{pmatrix} | \begin{pmatrix} x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta}\end{pmatrix} \in \Gamma\cup\underline{0}, (x,h)\in \text{supp }\psi\right\}. \quad (20)$$

We evaluate the wave front set of the family of products of distributions $\left(B_{1,\varepsilon}(\lambda,x,h)\delta(x-x',\lambda h-h')\right)_\varepsilon$ in $T^*(\mathbb{R}\times U\times\mathbb{R}^{n+d})$ using the functional properties of the Hörmander product [4, Theorem 7.1]. We start with the wave front set of the various distributions involved in formula (19), the family $B_{1,\varepsilon}(\lambda,x,h)\otimes 1(x',h')$ is bounded in $\mathcal{D}'_{\Lambda_1}(\mathbb{R}\times\mathbb{R}^{n+d}\times\mathbb{R}^{n+d})$ where:

$$\begin{aligned} \Lambda_1 &= \left\{\begin{pmatrix} \lambda & ; & \widehat{\lambda} \\ x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \\ x' & ; & 0 \\ h' & ; & 0\end{pmatrix} | \begin{pmatrix} x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta}\end{pmatrix} \in \Gamma\cup\underline{0}, (x,h)\in\text{supp }\psi\right\} \\ WF\left(\delta_\Delta(\Phi,.)\right) \subset \Lambda_2 &= \left\{\begin{pmatrix} \lambda & ; & -\langle h,\eta\rangle \\ x & ; & -\xi \\ h & ; & -\lambda\eta \\ x' & ; & \xi \\ h' & ; & \eta\end{pmatrix} |(x,\lambda h) = (x',h') \text{ and } (\xi,\eta)\neq(0,0)\right\}. \end{aligned}$$

Note that $\Lambda_1\cap -\Lambda_2 = \emptyset$ which implies by hypocontinuity of the Hörmander product [4, Theorem 7.1] that the products $\left(B_{1,\varepsilon}(\lambda,x,h)\delta(x-x',\lambda h-h')\right)_\varepsilon$ are bounded in $\mathcal{D}'_{\Lambda_1+\Lambda_2\cup\Lambda_1\cup\Lambda_2}$.

The projection

$$\pi_3 := (\lambda,x,h,x',h') \longmapsto (x',h')$$

is proper on the support of u therefore the pushforward of $B_{1,\varepsilon}(\lambda,x,h)\delta(x-x',\lambda h-h')$ by π_3, which equals the integral $\int_{\mathbb{R}\times U} d\lambda dxdh B_{1,\varepsilon}(\lambda,x,h)\delta(x-x',\lambda h-h')$, exists in the distributional sense. By continuity hence boudedness of the pushforward [4, Theorem 7.3], we find that the family $\left(t(\chi-\chi_{\varepsilon^{-1}})\right)_\varepsilon$ is bounded in $\mathcal{D}'_\Lambda$ where

$$(\Lambda\cap T^*_I U) \subset \pi_{3*}\left((\Lambda_1+\Lambda_2)\cup\Lambda_1\cup\Lambda_2\right)\cap T^*_I U. \quad (21)$$

We study the closed conic set $\pi_{3*}(\Lambda_1+\Lambda_2)\cap T^*_I U$:

$$\begin{array}{c} (x',0;\xi,\eta)\in\pi_{3*}(\Lambda_1+\Lambda_2) \\ \Leftrightarrow \left\{\begin{array}{c}\widehat{\lambda}-\langle h,\eta\rangle=0\\ \widehat{\xi}-\xi=0\\ \widehat{\eta}-\lambda\eta=0\end{array}\right. \text{ s.t. } (x,\lambda h)=(x',0),(x,h)\in\text{supp }\psi, \left(\begin{array}{ccc} x & ; & \widehat{\xi}\\ h & ; & \widehat{\eta}\end{array}\right)\in\Gamma\cup\underline{0}\Bigg\} \end{array}$$

has a solution. Note that $\left\{\begin{array}{c}(x,h)\in\text{supp }\psi\\ (x,\lambda h)=(x',0)\\ \widehat{\eta}-\lambda\eta=0\end{array}\right\} \Longrightarrow |h|\neq 0,\lambda=0,\widehat{\eta}=0$. Therefore

$$\begin{array}{cc} & (x',0;\xi,\eta)\in\pi_{3*}(\Lambda_1+\Lambda_2)\\ \Leftrightarrow & \left\{\left(\begin{array}{ccc} x & ; & \xi\\ h & ; & 0\end{array}\right)\in\Gamma\cup\underline{0},(x,h)\in\text{supp }\psi\right\}\\ \Leftrightarrow & \pi_{3*}(\Lambda_1+\Lambda_2)\cap T^*_I U\subset\Xi. \end{array}$$

It is immediate that $\pi_{3*}\Lambda_1=\emptyset$, finally

$$\begin{array}{ccc} \left(\begin{array}{ccc} x' & ; & \xi\\ 0 & ; & \eta\end{array}\right)\in\pi_{3*}\Lambda_2\cap T^*_I U & \Leftrightarrow & \left\{\begin{array}{c}\langle h,\eta\rangle=0\\ \xi=0\\ \lambda\eta=0\end{array}\right\} \text{ for } (x,\lambda h)=(x',0),(x,h)\in\text{supp }\psi\\ & \Longrightarrow & \xi=0\Longrightarrow \pi_{3*}\Lambda_2\cap T^*_I U\subset N^*(I). \end{array}$$

Finally, we can summarize the bounds that we obtained:

$$\Lambda\cap T^*_I\mathbb{R}^{n+d} \quad\subset\quad \Xi\cup N^*(I) \tag{22}$$

which establishes the claim of our theorem. □

Now we prove that under the assumptions of Theorem 3.1, the extension $\bar{t}$ constructed is weakly homogeneous of degree s in $\mathcal{D}'_{\Gamma\cup N^*(I)\cup\Xi}(U)$.

Theorem 3.2. *Let $s\in\mathbb{R}$ s.t. $s+d>0$, $\Gamma\subset T^\bullet U$ a closed conic set stable by scaling. Then the extension $\bar{t}=\lim_{\varepsilon\rightarrow 0}t(1-\chi_{\varepsilon^{-1}})$ is in $E_s\left(\mathcal{D}'_{\Xi\cup\Gamma\cup N^*(I)}(U)\right)$ for $\Xi=\{(x,0;\xi,\eta)|(x,h)\in \text{supp }\psi,\ \ (x,h;\xi,0)\in\Gamma\}$.*

Proof. For all test function φ, we study the family $\left(\langle\mu^{-s}\bar{t}_\mu,\varphi\rangle\right)_{\mu\in(0,1]}$. But since $\bar{t}=\lim_{\varepsilon\rightarrow 0}t(1-\chi_{\varepsilon^{-1}})$, it suffices to study the family $\mu^{-s}\left(t(1-\chi_{\varepsilon^{-1}})\right)_{\varepsilon,\mu}$ for $\varepsilon\leqslant\mu$.

A simple calculation using variable changes gives:

$$\begin{aligned} \forall 0<\varepsilon\leqslant\mu\leqslant 1,\mu^{-s}\left\langle\left(t(1-\chi_{\varepsilon^{-1}})\right)_\mu,\varphi\right\rangle &= \int_\varepsilon^1\frac{d\lambda}{\lambda}\mu^{-s-d}\left\langle t\psi_{\lambda^{-1}},\varphi_{\mu^{-1}}\right\rangle+\left\langle\mu^{-s}t_\mu(1-\chi_\mu),\varphi\right\rangle\\ &= \int_\varepsilon^1\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle\lambda^{-s}t_\lambda\psi,\varphi_{\frac{\lambda}{\mu}}\right\rangle+\left\langle\mu^{-s}t_\mu(1-\chi_\mu),\varphi\right\rangle\\ &= \int_{\frac{\varepsilon}{\mu}}^{\frac{1}{\mu}}\frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle(\lambda\mu)^{-s}t_{\lambda\mu}\psi,\varphi_\lambda\right\rangle+\left\langle\mu^{-s}t_\mu(1-\chi_\mu),\varphi\right\rangle. \end{aligned}$$

First, note that the family $(\mu^{-s}t_\mu)_{\mu\in(0,1]}$ is bounded in $\mathcal{D}'_\Gamma(U\setminus I)$ and $(1-\chi_\mu)\rightarrow 0$ when $\mu\rightarrow 0$ therefore the family $\left(\mu^{-s}t_\mu(1-\chi_\mu)\right)_{\mu\in(0,1]}$ is bounded.

The next thing we show is that the integral $\int_{\frac{\varepsilon}{\mu}}^{\frac{1}{\mu}}\frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle(\lambda\mu)^{-s}t_{\lambda\mu}\psi,\varphi_\lambda\right\rangle$ does not blow up because its integrand vanishes when λ is large enough. Let K be a compact subset of $\mathbb{R}^{n+d}$.

$$\begin{aligned} \varphi\in\mathcal{D}_K(U) &\Longrightarrow \exists R>0 \text{ s.t. supp }\varphi\subset\{|h|\leqslant R\}\\ &\Longrightarrow \text{supp }\varphi_\lambda\subset\{|h|\leqslant\lambda^{-1}R\}. \end{aligned}$$

Recall that π was the projection $\pi := (x,h) \in \mathbb{R}^{n+d} \mapsto x \in \mathbb{R}^n$.

$$\begin{aligned} & \pi \text{ is proper on supp } \psi \text{ and } \pi(\text{supp } \varphi) \subset \mathbb{R}^n \text{ compact} \\ \implies & \text{supp } t_{\lambda\mu}\psi|_{(K\times\mathbb{R}^d)\cap U} \subset \{a \leqslant |h| \leqslant b\} \text{ for } 0 < a < b \\ \implies & \left\{\lambda \geqslant \frac{R}{a} \implies \langle t_{\lambda\mu}\psi, \varphi_\lambda\rangle = 0\right\} \\ \implies & \forall \mu \in (0,1], \varepsilon \leqslant \mu, \int_{\frac{\varepsilon}{\mu}}^{\frac{1}{\mu}} \frac{d\lambda}{\lambda}\lambda^{s+d}\left\langle(\lambda\mu)^{-s}t_{\lambda\mu}\psi, \varphi_\lambda\right\rangle = \int_0^{+\infty} \frac{d\lambda}{\lambda} 1_{\{\frac{\varepsilon}{\mu}\leqslant\frac{R}{a}\}}(\lambda)\lambda^{s+d}\left\langle(\lambda\mu)^{-s}t_{\lambda\mu}\psi, \varphi_\lambda\right\rangle . \end{aligned}$$

For all t, we define $t^\mu(x,h) = \mu^{-s}t(x,\mu h)$ and we consider the family of distributions $B = (t^\mu)_{\mu\in(0,1]}$ which is bounded in $E_s(\mathcal{D}'_\Gamma(U))$. Therefore the result of Lemma (9.3) implies that the family

$$\left(1_{\{\frac{\varepsilon}{\mu}\leqslant\frac{R}{a}\}}(\lambda)\lambda^{s+d-1}\lambda^{-s}t^\mu(x,\lambda h)\psi(x,h)\right)_{0<\varepsilon\leqslant\mu\leqslant1}$$

is **bounded** in $\mathcal{D}'_\Lambda([0,\frac{R}{a}]\times(U\backslash I))$, for $\Lambda = \{(\lambda,x,h;\tau,\xi,\eta) \in \dot{T}^*([0,\frac{R}{a}]\times(U\backslash I)) | (x,h) \in \text{supp } \psi, (x,h;\xi,\eta) \in \Gamma\cup\underline{0}\}$. Therefore, we can repeat the proof of Theorem 3.1 for the family

$$\int_{\mathbb{R}\times\mathbb{R}^{n+d}} d\lambda dx dh 1_{\{\frac{\varepsilon}{\mu}\leqslant\frac{R}{a}\}}(\lambda)\lambda^{s+d-1}\lambda^{-s}t^\mu(x,\lambda h)\psi(x,h)\delta_\Delta(\Phi(\lambda,x,h),.) \tag{23}$$

Using the fact that

1. the Hörmander product of $1_{\{\frac{\varepsilon}{\mu}\leqslant\frac{R}{a}\}}(\lambda)\lambda^{s+d-1}\lambda^{-s}t^\mu(x,\lambda h)\psi(x,h)$ with $\delta_\Delta(\Phi(\lambda,x,h),.)$ is hypocontinuous [4, Thm 7.1]
2. the push–forward of $1_{\{\frac{\varepsilon}{\mu}\leqslant\frac{R}{a}\}}(\lambda)\lambda^{s+d-1}\lambda^{-s}t^\mu(x,\lambda h)\psi(x,h)\delta_\Delta(\Phi(.),.)$ by the projection π_3 is continuous in the normal topology hence bounded [4, Thm 7.3],

we obtain the desired result. □

3.2.2 Optimality of the wave front set of the extension.

We show with an example how our technique gives an **optimal bound** for the wave front set of the extension of distributions in a situation where the assumptions of the results of Brunetti–Fredenhagen [5, Lemma 6.1] are not satisfied.

The wave front set of an example of extension not handled by Brunetti–Fredenhagen's method. We work in $T^*\mathbb{R}^3$ with variables $(x_1,x_2,h;\xi_1,\xi_2,\eta)$ and I is the plane $\left(\mathbb{R}^2\times\{0\}\right) = \{h=0\}$. Let $f \in C^\infty(\mathbb{R}\setminus\{0\})\cap L^\infty(\mathbb{R}), f > 0$ which is nonsmooth at the origin and let us consider the function $f(x_1)$ as a distribution in the vector space $\mathbb{R}^3\setminus I$. Then we prove the following claim:

Proposition 3.2. *Let $\chi \in C^\infty(\mathbb{R})$ be a smooth function s.t. $\chi(h) = 1$ when $h \le 1$ and $\chi(h) = 0$ when $h \geqslant 2$. Then the family of distributions $f(x_1)(\chi(\varepsilon^{-1}h)-\chi(h))_\varepsilon$ converges to $f(x_1)$ when $\varepsilon \to 0$ in $\mathcal{D}'_V$ where $V = N^*(\{x_1=0\})\cup N^*I\cup(N^*(\{x_1=0\})+N^*I)$.*

In fact, for all $\varepsilon > 0$, the wave front set of the distribution $f(x_1)(\chi(\varepsilon^{-1}h)-\chi(h))$ is in $N^*(\{x_1=0\})$ therefore it does not satisfy the assumption that the closure of $WF(f(x_1)(\chi(\varepsilon^{-1}h)-\chi(h)))$ should be contained in the conormal $N^*(I)$ which is an important assumption of Theorem 6.9 in the paper [5] of Brunetti Fredenhagen.

Proof. Let V be the smallest closed conic set such that the family $f(x_1)(\chi(\varepsilon^{-1}h)-\chi(h))_{\varepsilon\in(0,1]}$ is bounded in $\mathcal{D}'_V$. It is obvious that outside $\{h=0\}$ the cone V equals $N^*(\{x_1=0\})$. It suffices to calculate V over $\{h=0\}$.

To estimate V over $\{h=0\}$, there are two cases to study: $x_1 = 0$ and $x_1 \neq 0$ (x_2 is arbitrary). We start with the case $x_1 \neq 0$. Let φ be a test function:

$$\begin{aligned} & \mathcal{F}\left(f(x_1)\varphi(x_1,x_2)(\chi(\varepsilon^{-1}h)-\chi(h))\right) \\ = & \widehat{f\varphi}(\xi_1,\xi_2)\left(\widehat{\chi(\varepsilon^{-1}.)}(\eta)-\widehat{\chi}(\eta)\right) \\ = & \widehat{f\varphi}(\xi_1,\xi_2)\left(\varepsilon\widehat{\chi}(\varepsilon\eta)-\widehat{\chi}(\eta)\right) \end{aligned}$$

Since $\widehat{\chi} \neq 0$ and is even analytic, we have $\forall R > 0, \sup_{|\eta|\geqslant R} |\widehat{\chi}(\eta)| = C(R) > 0$. This gives us the estimate

$$\begin{aligned} \forall R > 0, \sup_{\eta}(1+|\eta|)^N\varepsilon|\widehat{\chi}(\varepsilon\eta)| & \geqslant \left(1+\frac{R}{\varepsilon}\right)^N \varepsilon C(R) \\ & \geqslant \varepsilon^{-N+1}R^N C(R) \rightarrow_{\varepsilon\rightarrow 0} \infty \end{aligned}$$

This implies that $(\chi(\varepsilon^{-1}.)-\chi)$ is **bounded in** $\mathcal{D}'_{N^*(I)}$, therefore using the fact that $f > 0$, we find that V corresponds with the conormal $N^*(I)$ of I as long as f is smooth hence outside $x_1 = 0$.

We conclude by studying the case where $x_1 = 0$. Since f is singular at $x_1 = 0$ and that $ss(f) = \{0\} = \pi_{T^*\mathbb{R}\mapsto\mathbb{R}}(WF(f))$, the wave front set of f in the fiber $T^*_0\mathbb{R}$ over $x_1 = 0$ is non empty and we deduce there is a function φ of the two variables (x_1, x_2) such that $\widehat{f\varphi}(\xi_1,\xi_2)$ has slow decrease in the direction $(\xi_1, 0)$. The Fourier transform of $f\varphi(\chi(\varepsilon^{-1}.)-\chi)$ w.r.t (x_1,x_2,h) equals $\widehat{f\varphi}(\xi_1,\xi_2)\left(\widehat{\chi(\varepsilon^{-1}.)}(\eta)-\widehat{\chi}(\eta)\right)$ from which one easily concludes that $\{(0,x_2,0;\xi_1,0,\eta)\} = N^*(I)+N^*(\{x_1=0\}) \subset V$. □

The singular case. In the next part, we will deal with the singular case where $-m-1 < s+d \leqslant -m, m \in \mathbb{N}$. Instead of calculating the pairing $\langle t(\chi-\chi_{\varepsilon^{-1}}),\varphi\rangle$, we will subtract from φ its Taylor polynomial in the h variable to a sufficient order, therefore we will pair $t(\chi-\chi_{\varepsilon^{-1}})$ with the Taylor remainder $I_m\varphi$ defined by

$$I_m\varphi(x,h) = \frac{1}{m!}\sum_{|\alpha|=m+1} h^\alpha \int_0^1 (1-t)^m \left(\partial_h^\alpha\varphi\right)(x,th)dt. \tag{24}$$

Then we will study the existence of the limit:

$$\left\langle \overline{t},\varphi\right\rangle = \lim_{\varepsilon\rightarrow 0}\left\langle t(\chi-\chi_{\varepsilon^{-1}}), I_m\varphi\right\rangle + \left\langle t(1-\chi),\varphi\right\rangle. \tag{25}$$

First, observe that if the support of φ does not meet I, then φ equals its Taylor remainder $I_m\varphi$ since φ vanishes at infinite order on the subspace I and formula 25 is well defined and coincides with $\langle t,\varphi\rangle$. Therefore if $\overline{t}$ were defined, it would be an extension of t.

Theorem 3.3. *Let $s \in \mathbb{R}$ s.t. $-m-1 < s+d \leqslant -m, m \in \mathbb{N}$, $U \subset \mathbb{R}^{n+d}$ a convex set, $\Gamma \subset T^\bullet U$ a closed conic set stable by scaling. If $t \in \mathcal{D}'(U\setminus I)$ is weakly homogeneous of degree s in $\mathcal{D}'_\Gamma(U\setminus I)$, then formula (25) defines an extension $\overline{t}$ of t and $WF(\overline{t}) \subset WF(t)\cup N^*(I)\cup\Xi$ where*

$$\Xi = \{(x,0;\xi,\eta) | (x,h) \in \textit{supp }\psi,\ \ (x,h;\xi,0)\in\Gamma\}.$$

Proof. Before we state our theorem, let us describe the central new ingredient of our proof. In appendix, we will study the Schwartz kernel of the operator $I_m \in \mathcal{D}'(\mathbb{R}^{n+d}\times\mathbb{R}^{n+d})$ realizing the projection on the Taylor remainder. We work in $\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$ with coordinates (x,h,x',h') and we note $I_m((x,h),(x',h'))$ this Schwartz kernel.

Step 1 The distribution $I_m(.,.)$ plays the same role in the proof of Theorem 3.3 as $\delta_\Delta(.,.)$ in the proof of Theorem 3.1 and we prove in appendix (Lemma 9.1) that:

$$I_m(.,.) = \sum_{|\alpha|=m+1} h^\alpha R_\alpha(.,.) \text{ where } \forall\alpha, R_\alpha(.,.)\in\mathcal{D}'(\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}) \tag{26}$$

$$WF\left(R_\alpha(.,.)\right) \subset \{(x,h,x,th;\xi,t\eta,-\xi,-\eta) | t\in[0,1],(\xi,\eta)\neq(0,0)\}. \tag{27}$$

Step 2 Let $\varphi \in \mathcal{D}(U)$ be a test function, we have to establish the convergence of the formula

$$\langle t(\chi - \chi_{\varepsilon^{-1}}), I_m(\varphi)\rangle + \langle t(1-\chi), \varphi\rangle$$

when $\varepsilon \to 0$. As in the proof of Theorem 3.1, we use the partition of unity to derive an equivalent formula for $\lim_{\varepsilon\to 0} \langle t(\chi - \chi_{\varepsilon^{-1}}), I_m(\varphi)\rangle$ in terms of the family $\lambda^{-s}t_\lambda$:

$$\lim_{\varepsilon\to 0} \langle t(\chi - \chi_{\varepsilon^{-1}}), I_m(\varphi)\rangle \quad = \quad \int_0^1 \frac{d\lambda}{\lambda} \lambda^{s+d+m+1} \left\langle (\lambda^{-s}t_\lambda)\psi, \sum_{|\alpha|=m+1} h^\alpha R_\alpha(\varphi)_\lambda \right\rangle \tag{28}$$

where $R_\alpha(\varphi)_\lambda(x,h) = \frac{1}{m!}\int_0^1 (1-t)^m \partial_h^\alpha \varphi(x, t\lambda h)dt$ and the r.h.s. of (28) is absolutely convergent since $s+d+m+1 > 0$. It remains to prove that the limit when $\varepsilon \to 0$ is a well defined distribution. The proof is similar to the proof in Theorem 3.1 except that we should use the fact that the seminorms of $\psi \sum_{|\alpha|=m+1} h^\alpha R_\alpha(\varphi)_\lambda$ are controlled by the seminorms of φ by Taylor's formula for the integral remainder.

Step 3 We are reduced to prove the boundedness of the family distributions parametrized by $\varepsilon \in (0,1]$

$$t(\chi - \chi_{\varepsilon^{-1}}) I_m(x', h') \quad = \quad \int_{\mathbb{R}\times\mathbb{R}^{n+d}} d\lambda dx dh 1_{[\varepsilon,1]}(\lambda) \lambda^{s+d+m} \lambda^{-s} t(x, \lambda h)\psi(x,h)\lambda^{-m-1} I_m((x,\lambda h),(x',h'))$$

in $\mathcal{D}'_\Lambda$ where $\Lambda = WF(t) \cup \Xi \cup N^*(I)$.

We can repeat exactly the same proof as for Theorem 3.1 using parallel notations. Set $B_{1,\varepsilon}(\lambda, x, h) = 1_{[\varepsilon,1]}(\lambda)\lambda^{s+d+m}\lambda^{-s}t(x,\lambda h)\psi(x,h)$ then by Lemma 9.3, the family $(B_{1,\varepsilon}(\lambda,x,h)\otimes 1(x',h'))_{\varepsilon\in(0,1]}$ is bounded in $\mathcal{D}'_{\Lambda_1}(U\times U\times\mathbb{R})$ where:

$$\Lambda_1 = \{ \begin{pmatrix} \lambda & ; & \widehat{\lambda} \\ x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \\ x' & ; & 0 \\ h' & ; & 0 \end{pmatrix} | \begin{pmatrix} x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \end{pmatrix} \in \Gamma \cup \underline{0}, (x,h) \in \text{supp } \psi \}$$

Equations 26 together with the pull–back theorem of Hörmander imply:

$$WF\left(\lambda^{-m-1} I_m(\Phi, .)\right) \quad \subset \quad \Lambda_2 = \{ \begin{pmatrix} \lambda & ; & -\langle h, \eta\rangle \\ x & ; & -\xi \\ h & ; & -t\eta \\ x' & ; & \xi \\ h' & ; & \eta \end{pmatrix} |(x, th) = (x', h'), t \in [0,\lambda], (\xi,\eta) \neq (0,0)\}.$$

Note that $\Lambda_1 \cap -\Lambda_2 = \emptyset$ implies that the family of products $\left(B_{1,\varepsilon}(\lambda,x,h)\lambda^{-m-1}I_m((x,\lambda h),(x',h'))\right)_\varepsilon$ is bounded in $\mathcal{D}'_{\Lambda_1+\Lambda_2\cup\Lambda_1\cup\Lambda_2}$ by hypocontinuity of the Hörmander product [4, Thm 7.1]. As in the proof of proposition 3.1, we have $t(\chi - \chi_{\varepsilon^{-1}})I_m = \pi_3(B_{1,\varepsilon}\lambda^{-m-1}I_m(\Phi,(x',h')))$ therefore, in order to conclude, it suffices to control the family $\left(\pi_{3*}(B_{1,\varepsilon}\lambda^{-m-1}I_m(\Phi,(x',h')))\right)_\varepsilon$ in $\mathcal{D}'_\Lambda$ where $\Lambda = WF(t)\cup\Xi\cup N^*(I)$, using continuity of the push–forward [4, Theorem 7.3], we have the following estimate:

$$(\Lambda \cap T^*_I U) \subset \pi_{3*}(\Lambda_1 + \Lambda_2 \cup \Lambda_1 \cup \Lambda_2) \cap T^*_I U. \tag{29}$$

We study the closed conic set $\pi_{3*}(\Lambda_1+\Lambda_2)\cap T^*_I U$:

$$(x', 0; \xi, \eta) \in \pi_{3*}(\Lambda_1 + \Lambda_2)$$

$$\Leftrightarrow \quad \left\{ \begin{array}{c} \widehat{\lambda} - \langle h, \eta\rangle = 0 \\ \widehat{\xi} - \xi = 0 \\ \widehat{\eta} - t\eta = 0 \end{array} |\exists t \in [0,\lambda] \text{ s.t. } (x, th) = (x', 0), (x,h) \in \text{supp } \psi, \begin{pmatrix} x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \end{pmatrix} \in \Gamma \cup \underline{0} \right\}$$

has a solution. Note that $\left\{\begin{array}{c}(x,h)\in \text{supp }\psi\\ (x,th)=(x',0)\\ \widehat{\eta}-t\eta=0\end{array}\right\} \implies |h|\neq 0, t=0, \widehat{\eta}=0$. Therefore

$$\begin{array}{ll} & (x',0;\xi,\eta)\in\pi_{3*}\left(\Lambda_1+\Lambda_2\right)\\ \Leftrightarrow & \left\{\left(\begin{array}{ccc} x & ; & \xi\\ h & ; & 0\end{array}\right)\in\Gamma\cup\underline{0}, (x,h)\in\text{supp }\psi\right\}\\ \Leftrightarrow & \pi_{3*}\left(\Lambda_1+\Lambda_2\right)\cap T^*_IU\subset\Xi.\end{array}$$

It is immediate that $\pi_{3*}\Lambda_1=\emptyset$, finally

$$\left(\begin{array}{ccc} x' & ; & \xi\\ 0 & ; & \eta\end{array}\right)\in\pi_{3*}\Lambda_2\cap T^*_IU \quad\Leftrightarrow\quad \left\{\begin{array}{c}\langle h,\eta\rangle=0\\ \xi=0\\ t\eta=0\end{array}\right\} \text{ for } (x,th)=(x',0), t\in[0,\lambda], (x,h)\in\text{supp }\psi$$
$$\implies \xi=0\implies \pi_{3*}\Lambda_2\cap T^*_IU\subset N^*(I).$$

Finally, we can summarize the bounds that we obtained:

$$\Lambda\cap T^*_I\mathbb{R}^{n+d} \quad\subset\quad \Xi\cup N^*(I) \tag{30}$$

which establishes the claim of our theorem. □

We want to show that our extension is weakly homogeneous in $\mathcal{D}'_\Gamma$.

Proposition 3.3. *Under the assumptions of proposition (3.3), if s is not an integer then the extension map $t\in E_s\left(\mathcal{D}'_\Gamma\left(U\setminus I\right)\right)\longmapsto\overline{t}\in E_s(\mathcal{D}'_{\Gamma\cup N^*(I)\cup\Xi}(U))$ is bounded.*

Proposition 3.4. *Under the assumptions of proposition (3.3), if $s+d$ is a* ***non positive integer*** *then*

- *the extension map $t\in E_s\left(\mathcal{D}'_\Gamma\left(U\setminus I\right)\right)\longmapsto\overline{t}\in E_{s'}(\mathcal{D}'_{\Gamma\cup N^*(I)\cup\Xi}(U)),\forall s'<s$ is bounded,*
- *the family of distributions $\lambda^{-s}(\log\lambda)^{-1}\overline{t}_\lambda$ is bounded in $\mathcal{D}'_{\Gamma\cup N^*(I)\cup\Xi}(U)$.*

Proof. Choose a test function φ. To check the homogeneity of the renormalized integral is a little tricky since we have to take the scaling of counterterms into account. When we scale the test function φ then we should scale simultaneously the Taylor polynomial $(P_m\varphi)_\lambda$ and the remainder $(I_m\varphi)_\lambda$:

$$\varphi_\lambda=(P_m\varphi)_\lambda+(I_m\varphi)_\lambda=P_m\varphi_\lambda+I_m\varphi_\lambda.$$

We want to know to which scale space $E_{s'}(\mathcal{D}'_{\Gamma\cup N^*(I)})$ the distribution $\overline{t}$ belongs:

$$\begin{aligned}\mu^{-s'}\left\langle\overline{t}_\mu,\varphi\right\rangle &= \mu^{-s'}\int_0^1\frac{d\lambda}{\lambda}\mu^{-d}\left\langle t\psi_{\lambda^{-1}},(I_m\varphi)_{\mu^{-1}}\right\rangle\\ &= \mu^{-s'}\int_0^1\frac{d\lambda}{\lambda}\lambda^d\mu^{-d}\left\langle t_\lambda\psi,(I_m\varphi)_{\lambda\mu^{-1}}\right\rangle.\end{aligned}$$

For the moment, we find that:

$$\mu^{-s'}\left\langle\overline{t}_\mu,\varphi\right\rangle=\mu^{s-s'}\int_0^1\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle\left(\lambda^{-s}t_\lambda\right)\psi,(I_m\varphi)_{\frac{\lambda}{\mu}}\right\rangle.$$

The test function φ is supported in $\{|h|\leqslant R\}$ therefore $\varphi_{\frac{\lambda}{\mu}}$ is supported on $|h|\leqslant\frac{\mu R}{\lambda}$ thus when $\frac{R\mu}{\lambda}\leqslant a\Leftrightarrow\frac{R\mu}{a}\leqslant\lambda$, the support of $\varphi_{\frac{\lambda}{\mu}}$ does not meet the support of $\lambda^{-s}t_\lambda\psi$ because ψ is supported on $a\leqslant|h|$, whereas the polynomial part $P_m\varphi$ is supported everywhere since it is a Taylor polynomial. Consequently, we must split the scaled distribution $\mu^{-s}\overline{t}_\mu=I_1^\mu+I_2^\mu$ in two parts, where:

$$\langle I_1^\mu,\varphi\rangle=\int_0^{\frac{R\mu}{a}}\frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d}\left\langle\left(\lambda^{-s}t_\lambda\right)\psi,(I_m\varphi)_{\frac{\lambda}{\mu}}\right\rangle$$

$$= \int_0^{\frac{R\mu}{a}} \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{(d+s+m+1)} \left\langle \left(\lambda^{-s}t_\lambda\right)\psi, \sum_{|\alpha|=m+1} h^\alpha R_{\alpha,\frac{\lambda}{\mu}} \right\rangle.$$

$$\langle I_2^\mu, \varphi\rangle = \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d} \underbrace{\left\langle \lambda^{-s}t_\lambda, \varphi_{\frac{\lambda}{\mu}} - (P_m\varphi)_{\frac{\lambda}{\mu}} \right\rangle}_{\text{no contribution of } \varphi_{\frac{\lambda}{\mu}} \text{ since } \frac{R\mu}{a}\leqslant\lambda}.$$

We make a simple variable change for I_1^μ:

$$\langle I_1^\mu, \varphi\rangle = \int_0^{\frac{R}{a}} \frac{d\lambda}{\lambda}\lambda^{(d+s+m+1)} \left\langle (\lambda\mu)^{-s}t_{\lambda\mu}\psi, \sum_{|\alpha|=m+1} h^\alpha R_{\alpha,\lambda} \right\rangle$$

then following the proof of proposition 3.1, we note that

$$I_1^\mu = \int_{\mathbb{R}\times\mathbb{R}^{n+d}} d\lambda dx dh \lambda^{s+d+m}\lambda^{-s}\Phi^*(\mu^{-s}t_\mu)(\lambda,x,h)1_{[0,\frac{R}{a}]}\psi(x,h)\lambda^{-m-1}I_m(\Phi(.),.). \tag{31}$$

Therefore, we can repeat the proof of proposition 3.3 for the bounded family $(\mu^{-s}t_\mu)_\mu$ in $\mathcal{D}'_\Gamma(U)$ and we deduce that $(I_1^\mu)_\mu$ is bounded in $\mathcal{D}'_\Lambda$ where $\Lambda = \Gamma \cup N^*(I) \cup \Xi$.

Notice that in the second term I_2^μ only the counterterm $P_m\varphi$ contributes

$$I_2^\mu = \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{s+d} \left\langle \lambda^{-s}t_\lambda\psi, -(P_m\varphi)_{\frac{\lambda}{\mu}} \right\rangle$$

$$= \int_{\frac{R\mu}{a}}^1 \frac{d\lambda}{\lambda} \left\langle \lambda^{-s}t_\lambda\psi, -\sum_{|\alpha|\leqslant m} \left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|} \frac{h^\alpha}{\alpha!}\pi^*\left(i^*\partial_h^\alpha\varphi\right) \right\rangle.$$

We reformulate I_2^μ as

$$I_2^\mu = -\int_{\mathbb{R}\times\mathbb{R}^{n+d}} \frac{d\lambda}{\lambda} dx dh \lambda^{-s}\Phi^* t(\lambda,x,h)1_{[\frac{R\mu}{a},1]}\psi(x,h) \sum_{|\alpha|\leqslant m} \left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|} \frac{h^\alpha}{\alpha!}\pi^*\left(i^*\partial_h^\alpha\delta_\Delta(.,.)\right) \tag{32}$$

Then notice that by assumption $s+d\leqslant -m$ and $|\alpha|$ ranges from 0 to m which implies that we always have $s+d+|\alpha|\leqslant 0$.

If $s+d<m$ then for all α such that $0\leqslant|\alpha|\leqslant m$ we have the inequality $s+d+|\alpha|<0$, hence the family of functions $1_{[\frac{R\mu}{a},1]}\left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|}\lambda^{-1}$ is integrable w.r.t the variable λ uniformly in the parameter μ since:

$$\|1_{[\frac{R\mu}{a},1]}\left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|}\lambda^{-1}\|_{L^1(\mathbb{R})} = \frac{1}{|s+d+|\alpha||}\underbrace{\left|\left(\frac{1}{\mu}\right)^{s+d+|\alpha|} - \left(\frac{R}{a}\right)^{s+d+|\alpha|}\right|}_{\text{no blow up when } \mu\rightarrow 0}.$$

Therefore the family $(\lambda^{-s-1}\Phi^* t(\lambda,x,h)1_{[\frac{R\mu}{a},1]}\psi(x,h)\left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|}\frac{h^\alpha}{\alpha!})_{\mu\in(0,1)}$ is bounded in $\mathcal{D}'_V$ where $V = \{(\lambda,x,h;\tau,\xi,\eta)\in \dot{T}^*(\mathbb{R}\times\mathbb{R}^{n+d}); \lambda\in[0,1], (x,h;\xi,\eta)\in\Gamma\cup\underline{0}\}$ by Proposition 9.3 and we can repeat the proof of proposition 3.3 where the Schwartz kernel $I_m(\Phi(.),.)$ should be replaced with the distribution $\pi^*\left(i^*\partial_h^\alpha\delta_\Delta(.,.)\right)$ whose wave front set is calculated in Lemma 9.2 in appendix, the proof of Proposition 3.3 still applies in our case since $WF\left(\pi^*\left(i^*\partial_h^\alpha\delta_\Delta(.,.)\right)\right)\subset WF\left(I_m(\Phi(.),.)\right)$. However if $s+d+m=0$ then for $|\alpha|=m$, we find that the family of functions

$$\left(1_{[\frac{R\mu}{a},1]}\left(\frac{\lambda}{\mu}\right)^{s+d+|\alpha|}\lambda^{-1} = \lambda^{-1}1_{[\frac{R\mu}{a},1]}\right)_{\mu\in(0,1]}$$

is no longer bounded in the $L^1_\lambda([0,1])$ for $\mu\in(0,1]$ but exhibits a **logarithmic divergence**:

$$\forall\mu\in(0,1], \|1_{[\frac{R\mu}{a},1]}\lambda^{-1}\|_{L^1(\mathbb{R})} = \log(\frac{R\mu}{a}) \leqslant \log\mu + \log(\frac{R}{a}).$$

Then it is easy to conclude that $(\log\lambda)^{-1}\lambda^{-s}\bar{t}_\lambda$ is bounded in $\mathcal{D}'_{\Xi\cup N^*(I)\cup\Gamma}(U)$. □

3.3 The general extension in the flat case.

For the sequel, we recall that $\chi \in C^\infty(\mathbb{R}^{n+d})$ is our partition of unity used to construct the extension and $\psi = -h\frac{d\chi}{dh}$.

Theorem 3.4. *Let $s \in \mathbb{R}$, $\Gamma \subset T^\bullet U$ a closed conic set stable by scaling. If $t \in \mathcal{D}'(U \setminus I)$ is weakly homogeneous of degree s in $\mathcal{D}'_\Gamma(U \setminus I)$, then*

1. *there is an extension $\overline{t} \in \mathcal{D}'(U)$ of t where:*

$$WF(\overline{t}) \subset WF(t) \cup N^*(I) \cup \Xi, \ \ \Xi = \{(x, 0; \xi, \eta) | (x, h) \in supp\ \psi, (x, h; \xi, 0) \in \Gamma\}.$$

2. *$\overline{t}$ is in $E_{s,\Gamma\cup\Xi\cup N^*(I)}(U)$ if $-s - d \notin \mathbb{N}$ and $\overline{t} \in E_{s',\Gamma\cup\Xi\cup N^*(I)}(U), s' < s$ otherwise.*

We give here the proof of an important particular case of the above theorem:

Theorem 3.5. *Under the assumptions of the above theorem if $\left(\overline{\Gamma} \cap T^\bullet_I U\right) \subset N^*(I)$ then*

1. *there is an extension $\overline{t} \in \mathcal{D}'(U)$ of t where:*

$$WF(\overline{t}) \subset WF(t) \cup N^*(I).$$

2. *$\overline{t}$ is in $E_{s,\Gamma\cup N^*(I)}(U)$ if $-s - d \notin \mathbb{N}$ and $\overline{t} \in E_{s',\Gamma\cup N^*(I)}(U), s' < s$ otherwise.*

Proof. The proof proceeds in two steps. First, we show that there exists a neighborhood V of $I = \{h = 0\}$ such that $\forall (x, h; \xi, \eta) \in T^\bullet V \cap \Gamma$, $\eta \neq 0$. In the second part, we explain that by carefully choosing χ in such a way that supp $\chi \subset V$, the subset Ξ will be empty.

Step 1, we prove that for all compact set K there is some neighborhood V of I such that $\Gamma \cap T^\bullet_{K\cap V} U$ does not meet the set $\{(x, h; \xi, 0) | \xi \neq 0\}$. Then it follows immediately by a covering argument that there exists a neighborhood V of $I = \{h = 0\}$ such that $\forall (x, h; \xi, \eta) \in T^\bullet V \cap \Gamma$, $\eta \neq 0$. By contradiction assume there is some compact set K such that for all $V_n = \{|h| \leqslant n^{-1}\}$, there is some $(x_n, h_n; \xi_n, 0) \in T^\bullet_{K\cap V_n} U \cap \Gamma$. By extracting a convergent subsequence one easily concludes that there would be a sequence $(x_n, h_n; \xi_n, 0) \to (x, 0; \xi, 0) \in \Gamma$, contradiction !

Step 2 We choose a function χ which equals 1 in some neighborhood of I and χ is supported in V. Therefore the function $\psi = -\rho\chi$ is supported in V. But the set $\Gamma \cap T^*_V U$ does not meet the set $\{(x, h; \xi, 0) | \xi \neq 0\}$ therefore the set $\Xi = \{(x, 0; \xi, \eta) | (x, h) \in \text{supp } \psi(x, h; \xi, 0) \in \Gamma\}$ is empty and the conclusion follows. ☐

4 The extension theorem for $E_{s,I}$.

We are now ready to prove Theorem 0.5 and some part of the claim of Theorem 0.4:

Theorem 4.1. *Let U be an open neighborhood of $I \subset M$, if $t \in E_{s,I}(U \setminus I)$ then there exists an extension $\overline{t}$ in $E_{s',I}(U)$ where $s' = s$ if $-s - d \notin \mathbb{N}$ and $s' < s$ otherwise.*

Proof. $t \in E_{s,I}(U \setminus I)$ implies that for all $p \in I$, there is some open chart $\psi : V_p \subset U \mapsto \mathbb{R}^{n+d}$, $\psi(I) \subset \mathbb{R}^n \times \{0\}$ where $\lambda^{-s}(\psi_* t)(x, \lambda h)$ is bounded in $\mathcal{D}'(\psi(V_p \setminus I))$. Moreover, we must choose V_p in such a way that its image $U = \psi(V_p) \subset \mathbb{R}^{n+d}$ is of the form $U_1 \times U_2$ where $U_1 \subset \mathbb{R}^n, U_2 \subset \mathbb{R}^d$ and $\lambda U_2 \subset U_2, \forall \lambda \in [0, 1]$. $\cup_{p\in I} V_p$ forms an open cover of I, consider a locally finite subcover $\cup_{a\in A} V_a$ and denote by $(\psi_a)_{a\in A}$ the corresponding charts. For every $a \in A$, Theorem 3.4 yields an extension $\overline{\psi_{a*} t}$ of $\psi_{a*} t$ in $E_{s',I}(\psi_a(V_a))$ and by diffeomorphism invariance of $E_{s',I}$ (Theorem 0.3), the element $\psi_a^* \overline{\psi_{a*} t}$ belongs to $E_{s',I}(V_a)$. Choose a partition of unity $(\varphi_a)_a$ subordinated to the open cover $\cup_{a\in A} V_a$, then an extension of t reads $\sum_{a\in A} \varphi_a \psi_a^* \overline{\psi_{a*} t} + \left(1 - \sum_{a\in A} \varphi_a\right) t$ and belongs to $E_{s',I}(U)$ by the gluing property for $E_{s',I}$. ☐

4.1 A converse result.

Before we move on, let us prove a converse theorem, namely that given any distribution $t \in \mathcal{D}'\left(\mathbb{R}^{n+d}\right)$, for all relatively compact subset U, we can find $s_0 \in \mathbb{R}$ such that for all $s \leqslant s_0$, $t \in E_{s,I}(U)$, this means morally that any distribution has "finite scaling degree" along an arbitrary vector subspace. We also have the property that $\forall s_1 \leqslant s_2, t \in E_{s_2,I} \implies t \in E_{s_1,I}$. This means that the spaces $E_{s,I}$ are **filtered**. We work in $\mathbb{R}^{n+d}$ where $I = \mathbb{R}^n \times \{0\}$ and $\rho = h^j \frac{\partial}{\partial h^j}$:

Proposition 4.1. *Let U be a relatively compact convex open set and $t \in \mathcal{D}'(\mathbb{R}^{n+d})$. If t is of order k on U, then $t \in E_{s,I}(U)$ for all $s \leqslant d+k$, where d is the* ***codimension*** *of $I \subset \mathbb{R}^{n+d}$. In particular any compactly supported distribution is in $E_{s,I}(\mathbb{R}^{n+d})$ for some s.*

Proof. First notice if a function $\varphi \in \mathcal{D}(U)$, then the family of scaled functions $(\varphi_{\lambda^{-1}})_{\lambda \in (0,1]}$ has support contained in a compact set $K = \{(x, \lambda h) | (x,h) \in \text{supp } \varphi, \lambda \in (0,1]\}$. We recall that for any distribution t, there exists k, C_K such that

$$\forall \varphi \in \mathcal{D}_K(U), |\langle t, \varphi \rangle| \leqslant C_K \pi_{K,k}(\varphi).$$

$$|\langle t_\lambda, \varphi \rangle| = |\lambda^{-d} \langle t, \varphi_{\lambda^{-1}} \rangle| \leqslant C_K \lambda^{-d} \pi_{K,k}(\varphi_{\lambda^{-1}}) \leqslant C_K \lambda^{-d-k} \pi_{K,k}(\varphi).$$

So we find that $\lambda^{d+k} \langle t_\lambda, \varphi \rangle$ is bounded which yields the conclusion. □

Then Theorem 0.4 follows from Proposition 4.1 and the diffeomorphism invariance of $E_{s,I}$.

5 The subspace $E_{s,N^*(I)}(U)$.

It is a central assumption of our extension theorems that the family $(\lambda^{-s} t_\lambda)_\lambda$ is bounded in $\mathcal{D}'_\Gamma$ and we found that in the particular case where $\overline{\Gamma}|_I \subset N^*(I)$ then the wave front set of the extension is minimal i.e.

$$WF(\bar{t}) \subset WF(t) \cup N^*(I). \tag{33}$$

In this section, we generalize the previous situation to manifolds. We define a subspace $E_{s,N^*(I)}$ of E_s which contains distributions t such that their extension $\bar{t}$ satisfies $WF(\bar{t}) \subset WF(t) \cup N^*(I)$.

5.1 The conormal landing condition.

Definition 5.1. *Let U be an open neighborhood of I. A closed conic set $\Gamma \subset T^\bullet(U \setminus I)$ (resp $\Gamma \subset T^\bullet U$) is said to satisfy the* conormal landing condition *if $(\overline{\Gamma} \cap T^\bullet_I U) \subset N^* I$ (resp $(\Gamma \cap T^\bullet_I U) \subset N^* I$) where $\overline{\Gamma}$ is the closure of Γ in $T^\bullet U$.*

The *conormal landing condition* which concerns the closure of Γ over $T^*_I U$ is clearly intrinsic and does not depend on chosen coordinates. The following is a stability result for sets which satisfy the conormal landing condition.

Lemma 5.1. *Let U be some open neighborhood of I, $\Gamma \subset T^\bullet(U \setminus I)$, and $\Phi \in C^\infty([0,1] \times U, U)$ be such that $\Phi(\lambda, .)$ is a germ of diffeomorphism along I, $\Phi(\lambda, .)|_I$ is the identity map for all $\lambda \in (0,1]$ and*

$$\forall (x,h;\xi,\eta) \in N^*(I), (\Phi_\lambda^{-1}(x,h); (\xi,\eta) \circ d\Phi_\lambda) = (x,h;\xi,\eta).$$

If Γ satisfies the conormal landing condition then the cone Γ' defined as

$$\Gamma' = \bigcup_{\lambda \in (0,1]} \Phi(\lambda)^* \Gamma \tag{34}$$

also does.

In the terminology of Lemma 9.4 in appendix, the condition of the above Lemma means that the cotangent lift $T^*\Phi(\lambda, .)$ restricted to $N^*(I)$ acts as the identity map.

Proof. Let $(x, 0; \xi, \eta)$ be in the closure of Γ', then there exists a sequence $(\lambda_n, x_n, h_n; \xi_n, \eta_n)_n$ such that $(\Phi^{-1}_{\lambda_n}(x_n, h_n); (\xi_n, \eta_n) \circ d\Phi_{\lambda_n}) \rightarrow (x, 0; \xi, \eta)$. By compactness of $[0, 1]$, we can always extract a subsequence so that $\lambda_n \rightarrow \lambda_0 \in [0, 1]$. Then necessarily $(\Phi^{-1}_{\lambda_0}(x_n, h_n); (\xi_n, \eta_n) \circ d\Phi_{\lambda_0}) \rightarrow (x, 0; \xi, \eta)$ which implies that $(x_n, h_n; \xi_n, \eta_n) \rightarrow (\Phi_{\lambda_0}(x, 0); (\xi, \eta) \circ d\Phi^{-1}_{\lambda_0}) = (x, 0; \xi, \eta)$ since the cotangent lift $T^*\Phi(\lambda_0, .)|_{N^*(I)}$ is the identity map and $T^*\Phi(\lambda_0, .)|$ is a diffeomorphism. □

5.2 Construction of $E_{s,N^*(I)}$.

We keep the notations of the above subsection. We give a preliminary definition of the space $E^\rho_s(\mathcal{D}'_\Gamma(U))$ for ρ-convex open sets U and a given closed cone $\Gamma \subset T^\bullet U$ which depends on the choice of ρ.

Definition 5.2. *Let U be ρ convex set, $\Gamma \subset T^\bullet U$ a closed cone, then $E^\rho_s(\mathcal{D}'_\Gamma(U))$ is defined as the space of distribution t such that the family $\left(\lambda^{-s} e^{\log \lambda \rho *} t\right)_{\lambda \in (0,1]}$ is bounded in $\mathcal{D}'_\Gamma(U)$.*

We next define a localized version of the above space around an element $p \in I$.

Definition 5.3. *t belongs to $E^\rho_{s,N^*(I),p}$ if there exists a ρ-convex open set U s.t. $\overline{U}$ is a neighborhood of p and $t \in E^\rho_s(\mathcal{D}'_\Gamma(U))$ for some $\Gamma \subset T^\bullet U$ which satisfies the conormal landing condition.*

Theorem 5.1. *Let $t \in \mathcal{D}'(M \setminus I)$ and $p \in I$. If t belongs to $E^\rho_{s,N^*(I),p}$ for some Euler vector field ρ then it is so for any Euler vector field.*

Proof. Let ρ_1, ρ_2 be two Euler vector fields and t belongs to $E^{\rho_1}_{s,N^*(I),p}$. It suffices to establish that the family $\left(\lambda^{-s} e^{\log \lambda \rho_2 *} t\right)_\lambda$ is bounded in $\mathcal{D}'_{\Gamma_2}(V'_p \setminus I)$ for some neighborhood V'_p of p and Γ_2 satisfying the conormal landing condition. We use Proposition 1.1 which states that locally there exists a smooth family of germs of diffeomorphisms $\Phi(\lambda) : V_p \mapsto M$ such that $\forall \lambda \in [0, 1], \Phi(\lambda)(p) = p$ and $\Phi(\lambda)$ relates the two scalings:

$$e^{\log \lambda \rho_2 *} = \Phi(\lambda)^* e^{\log \lambda \rho_1 *}.$$

Assume that V_p is chosen small enough so that $\lambda^{-s} e^{\log \lambda \rho_1 *} t$ is bounded in $\mathcal{D}'_{\Gamma_1}(V_p \setminus I)$, then by [4, Theorem 6.9], we deduce that the family

$$\left(\Phi(\lambda)^* \left(\lambda^{-s} e^{\log \lambda \rho_1 *} t\right)\right)_\lambda = \left(\lambda^{-s} e^{\log \lambda \rho_2 *} t\right)_\lambda$$

is in fact bounded in $\mathcal{D}'_{\Gamma_2}(V'_p \setminus I)$ for some smaller neighborhood V'_p of p and with Γ_2 given by the equation

$$\Gamma_2 = \bigcup_{\lambda \in [0,1]} \Phi(\lambda)^* \Gamma_1.$$

By Lemma 9.5 proved in appendix, the family $\Phi(\lambda)$ satisfies:

$$\forall (x, h; \xi, \eta) \in N^*(I), (\Phi^{-1}_\lambda(x, y); (\xi, \eta) \circ d\Phi_\lambda) = (x, h; \xi, \eta).$$

which implies by Lemma 5.1 that Γ_2 satisfies the conormal landing condition concluding our proof. □

The previous theorem allows us to define spaces $E_{s,N^*(I),p}, E_{s,N^*(I)}$ which makes no mention of the choice of Euler vector field ρ:

Definition 5.4. *A distribution $t \in \mathcal{D}'(U)$ belongs to $E_{s,N^*(I),p}(U)$ if $t \in E^\rho_{s,N^*(I),p}$ for some Euler vector field ρ. We define $E_{s,N^*(I)}(U)$ as the space of all distributions $t \in \mathcal{D}'(U)$ such that $t \in E_{s,N^*(I),p}(U)$ for all $p \in I \cap int(\overline{U})$.*

It is immediate to deduce from Theorem 5.1 and definition 5.4 that $E_{s,N^*(I)}$ satisfies the same restriction and gluing properties as $E_{s,I}$.

We prove that $E_{s,N^*(I)}(U)$ satisfies a property of diffeomorphism invariance:

Theorem 5.2. *Let I (resp I') be a closed embedded submanifold of M (resp M'), $U \subset M$ (resp $U' \subset M'$) open and $\Phi : U' \mapsto U$ a diffeomorphism s.t. $\Phi(U' \cap I') = I \cap U$. Then $\Phi^* E_{s,N^*(I)}(U) = E_{s,N^*(I')}(U')$.*

Proof. By Theorem 5.1, we can localize the proof at all points $p \in I \cap int(\overline{U})$. Let $p \in I \cap int(\overline{U})$, then $t \in E_{s,N^*(I)}(U)$ implies by definition that $t \in E_s^\rho(\mathcal{D}'_\Gamma(V))$, where $int(\overline{V})$ is a neighborhood of p, some Euler ρ and Γ satisfying the conormal landing condition, which means that:

$$\lambda^{-s} e^{\log \lambda \rho *} t \text{ bounded } \mathcal{D}'_\Gamma(V)$$

$$\Leftrightarrow \lambda^{-s} \Phi^* e^{\log \lambda \rho *} \Phi^{-1*} (\Phi^* t) \text{ bounded in } \mathcal{D}'_{\Phi^*\Gamma}(\Phi^{-1}(V))$$

because the pull–back by a diffeomorphism is a bounded map from $\mathcal{D}'_\Gamma(V) \mapsto \mathcal{D}'_{\Phi^*\Gamma}(\Phi^{-1}(V))$,

$$\Leftrightarrow \lambda^{-s} e^{\log \lambda (\Phi_*^{-1} \rho) *} (\Phi^* t) \text{ bounded in } \mathcal{D}'_{\Phi^*\Gamma}(\Phi^{-1}(V))$$

where the vector field $\Phi_*^{-1}\rho$ is Euler by 0.1. Therefore $\Phi^* t$ is in $E_s^{\Phi_*^{-1}\rho}(\mathcal{D}'_{\Phi^*\Gamma}(\Phi^{-1}(V)))$ at $p' = \Phi^{-1}(p)$ where $\Phi^*\Gamma$ also satisfies the conormal landing condition hence $\Phi^* t$ is locally in $E_{s,N^*(I),p}$ at p and repeating the proof for all $p \in I \cap int(\overline{U})$ yields the claim. □

6 The extension theorem for $E_{s,N^*(I)}$.

We are now ready to prove Theorem 0.8:

Theorem 6.1. *Let $U \subset M$ be some open neighborhood of I. If $t \in E_{s,N^*(I)}(U \setminus I)$ then there exists an extension $\bar{t}$ with $WF(\bar{t}) \subset WF(t) \cup N^*(I)$ and $\bar{t} \in E_{s',N^*(I)}(U)$, where $s' = s$ if $s + d \notin -\mathbb{N}$ and $s' < s$ otherwise.*

Proof. $t \in E_{s,N^*(I)}(U \setminus I)$ implies that for all $p \in I$, there is some open chart $\psi : V_p \subset U \mapsto \mathbb{R}^{n+d}$, $\psi(I) \subset \mathbb{R}^n \times \{0\}$ where $\lambda^{-s}(\psi_* t)(x, \lambda h)$ is bounded in $\mathcal{D}'_\Gamma(\psi(V_p \setminus I))$ for Γ satisfying the conormal landing condition. Moreover, we must choose V_p in such a way that its image $U = \psi(V_p) \subset \mathbb{R}^{n+d}$ is of the form $U_1 \times U_2$ where $U_1 \subset \mathbb{R}^n, U_2 \subset \mathbb{R}^d$ and $\lambda U_2 \subset U_2, \forall \lambda \in [0,1]$. $\cup_{p \in I} V_p$ forms an open cover of I, consider a locally finite subcover $\cup_{a \in A} V_a$ and denote by $(\psi_a)_{a \in A}$ the corresponding charts. For every $a \in A$, Theorem 3.4 yields an extension $\overline{\psi_{a*} t}$ of $\psi_{a*} t$ in $E_{s',N^*(I)}(\psi_a(V_a))$ and by diffeomorphism invariance of $E_{s',N^*(I)}$ (Theorem 5.2), the element $\psi_a^* \overline{\psi_{a*} t}$ belongs to $E_{s',N^*(I)}(V_a)$. Choose a partition of unity $(\varphi_a)_a$ subordinated to the open cover $\cup_{a \in A} V_a$, then an extension of t reads $\sum_{a \in A} \varphi_a \psi_a^* \overline{\psi_{a*} t} + \left(1 - \sum_{a \in A} \varphi_a\right) t$ and belongs to $E_{s',N^*(I)}(U)$ by the gluing property for $E_{s',N^*(I)}$. □

7 Renormalized products.

In this section, we have a fixed Euler vector field ρ and we scale only w.r.t. the flow generated by ρ. We can now prove our Theorem 0.9 of renormalization of the product, we denote by $E_s^\rho(\mathcal{D}'_\Gamma(U))$ the space of distributions t s.t. the family $\left(\lambda^{-s} e^{\log \lambda \rho *} t\right)_{\lambda \in (0,1]}$ is bounded in $\mathcal{D}'_\Gamma(U)$ for some ρ-convex set U and some cone Γ stable by scaling:

Theorem 7.1. *Let ρ be some Euler vector field, U some neighborhood of I, (Γ_1, Γ_2) two cones in $T^\bullet(U \setminus I)$ which satisfy the conormal landing condition and $\Gamma_1 \cap -\Gamma_2 = \emptyset$. Set $\Gamma = (\Gamma_1 + \Gamma_2) \cup \Gamma_1 \cup \Gamma_2$. If $\Gamma_1 + \Gamma_2$ satisfies the conormal landing condition then there exists a bilinear map $\mathcal{R}$ satisfying the following properties:*

- $\mathcal{R} : (u_1, u_2) \in E_{s_1}^\rho\left(\mathcal{D}'_{\Gamma_1}(U \setminus I)\right) \times E_{s_2}^\rho\left(\mathcal{D}'_{\Gamma_2}(U \setminus I)\right) \mapsto \mathcal{R}(u_1 u_2) \in E_{s,N^*(I)}(U), \forall s < s_1 + s_2$
- $\mathcal{R}(u_1 u_2) = u_1 u_2$ *on* $U \setminus I$
- $\mathcal{R}(u_1 u_2) \in \mathcal{D}'_{\Gamma \cup N^*(I)}(U)$.

Proof. The families $\left(\lambda^{-s_i} e^{\log \lambda \rho *} u_i\right)_{\lambda \in (0,1]}$ are bounded in $\mathcal{D}'_{\Gamma_i}(U \setminus I)$. By hypocontinuity of the Hörmander product [4, Theorem 7.1], the family $\left(\lambda^{-s_1 - s_2} e^{\log \lambda \rho *} (u_1 u_2)\right)_{\lambda \in (0,1]}$ is bounded in $\mathcal{D}'_\Gamma(U \setminus I)$, Γ still satisfies the conormal landing condition by assumption then it follows by Theorem 0.8 that $u_1 u_2$ admits an extension $\mathcal{R}(u_1 u_2)$ in $E_{s,N^*(I)}(U)$ and $\mathcal{R}(u_1 u_2) \in \mathcal{D}'_{\Gamma \cup N^*(I)}(U)$. □

8 Renormalization ambiguities.

8.1 Removable singularity theorems.

First, we would like to start this section by a simple removable singularity theorem in the spirit of [25, Theorems 5.2 and 6.1]. In a renormalization procedure, there is always an ambiguity which is the ambiguity of the extension of the distribution. Indeed, two extensions always differ by a distribution supported on I. The removable singularity theorem states that if $s + d > 0$ and if we demand that $t \in E_{s,I}(U \setminus I)$ should extend to $\bar{t} \in E_{s,I}(U)$, then the extension is **unique**. Otherwise, if $-m - 1 < s + d \leqslant -m$, then we bound the transversal order of the ambiguity. We fix the coordinate system (x^i, h^j) in $\mathbb{R}^{n+d}$ and $I = \{h = 0\}$. The collection of coordinate functions $(h^j)_{1 \leqslant j \leqslant d}$ defines a canonical collection of transverse vector fields $(\partial_{h^j})_j$. We denote by δ_I the unique distribution such that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+d}), \langle \delta_I, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x, 0) d^n x. \tag{35}$$

If $t \in \mathcal{D}'(\mathbb{R}^{n+d})$ with supp $t \subset I$, then by [46, Theorems 36,37 p. 101–102] or [32, Theorem 2.3.5] there exist unique distributions (once the system of transverse vector fields ∂_{h^j} is fixed) $t_\alpha \in \mathcal{D}'(\mathbb{R}^n)$, where each compact intersects supp t_α for a finite number of multi–indices α, such that $t(x, h) = \sum_\alpha t_\alpha(x) \partial_h^\alpha \delta_I(h)$ or) where the ∂_h^α are derivatives in the **transverse** directions.

Theorem 8.1. *Let $t \in E_{s,I}(U \setminus I)$ and $\bar{t} \in E_{s',I}(U \setminus I)$ its extension given by Theorem (3.1) and Theorem (3.2) $s' = s$ when $-s - d \notin \mathbb{N}$ or $\forall s' < s$ otherwise. Then $\tilde{t}$ is an extension in $E_{s',I}(U)$ if and only if*

$$\tilde{t}(x, h) = \bar{t}(x, h) + \sum_{\alpha \leqslant m} t_\alpha(x) \partial_h^\alpha \delta_I(h), \tag{36}$$

where m is the integer part of $-s - d$. In particular when $s + d > 0$ the extension is unique.

Remark: when $-s - d$ is a nonnegative integer, the counterterm is in $E_{s,I}$ whereas the extension is in $E_{s',I}, \forall s' < s$.

Proof. We scale an elementary distribution $\partial_h^\alpha \delta_I$:

$$\langle (\partial_h^\alpha \delta_I)_\lambda, \varphi \rangle = \lambda^{-d} \langle \partial_h^\alpha \delta_I, \varphi_{\lambda^{-1}} \rangle = (-1)^{|\alpha|} \lambda^{-d-|\alpha|} \langle \partial_h^\alpha \delta_I, \varphi \rangle$$

hence $\lambda^{-s}(\partial^\alpha \delta_I)_\lambda = \lambda^{-d-|\alpha|-s} \partial_h^\alpha \delta_I$ is bounded iff $d + s + |\alpha| \leqslant 0 \implies d + s \leqslant -|\alpha|$. When $s + d > 0$, $\forall \alpha, \partial_h^\alpha \delta_I \notin E_{s,I}$ hence any two extensions in $E_{s,I}(U)$ cannot differ by a local counterterm of the form $\sum_\alpha t_\alpha \partial_h^\alpha \delta_I$. When $-m-1 < d+s \leqslant -m$ then $\lambda^{-s}(\partial_h^\alpha \delta_I)_\lambda$ is bounded iff $s+d+|\alpha| \leqslant 0 \Leftrightarrow -m \leqslant -|\alpha| \Leftrightarrow |\alpha| \leqslant m$. We deduce that $\partial_h^\alpha \delta_I \in E_{s,I}$ for all $\alpha \leqslant m$ which means that the scaling degree **bounds** the order $|\alpha|$ of the derivatives in the transverse directions. Assume there are two extensions in $E_{s,I}$, their difference is of the form $u = \sum_\alpha u_\alpha \partial_h^\alpha \delta_I$ by the structure theorem (36) p. 101 in [46] and is also in $E_{s,I}$ which means their difference equals $u = \sum_{|\alpha| \leqslant m} u_\alpha \partial_h^\alpha \delta_I$. ☐

8.2 Counterterms on manifolds and conormal distributions.

What happens in the case of manifolds ? From the point of view of L. Schwartz, the only thing to keep in mind is that a distribution supported on a submanifold I is always well defined locally and the representation of this distribution is unique once we fix a system of coordinate functions $(h^j)_j$ which are transverse to I [46, Theorem 37 p. 102]. For any distribution $t_\alpha \in \mathcal{D}'(I)$, if we denote by $i : I \hookrightarrow M$ the canonical embedding of I in M then $i_* t_\alpha$ is the push-forward of t_α in M:

$$\forall \varphi \in \mathcal{D}(M), \langle i_* t_\alpha, \varphi \rangle = \langle t_\alpha, \varphi \circ i \rangle.$$

The next lemma completes Theorem 8.1. Here the idea is that we add a constraint on the **local counterterm** t, namely that $WF(t)$ is contained in the conormal of I. Then we prove that the coefficients t_α appearing in the Schwartz representation (36) are in fact **smooth** functions.

Lemma 8.1. *Let $t \in \mathcal{D}'(M)$ such that t is supported on I, then*
1) t has a unique decomposition as locally finite linear combinations of transversal derivatives of push-forward to M of distributions t_α in $\mathcal{D}'(I)$: $t = \sum_\alpha \partial_h^\alpha (i_* t_\alpha)$,
and 2) $WF(t)$ is contained in the conormal of I if and only if $\forall\alpha$, t_α is smooth.

Proof. If $(t_\alpha)_\alpha$ are smooth then the wave front set of the push–forward $i_* t_\alpha$ is contained in the normal of the embedding i denoted by N_i [4, 2.3.1] which is nothing but the conormal bundle $N^*(I)$ [4, Example 2.9]. To prove the converse, in local coordinates, let

$$t(x,h) = \sum_\alpha \partial_h^\alpha (t_\alpha(x)\delta_I(h)) = \sum_\alpha t_\alpha(x)\partial_h^\alpha \delta_I(h).$$

Assume t_α is not smooth then $WF(t_\alpha)$ would be **non empty**. Then $WF(t_\alpha)$ contains an element $(x_0;\xi_0)$. Pick $\chi \in \mathcal{D}(R^n)$ such that $\chi(x_0) \neq 0$ then

$$\mathcal{F}(t_\alpha \chi \partial_h^\alpha \delta_I)(\xi,\eta) = \widehat{t_\alpha \chi}(\xi)(-i\eta)^\alpha,$$

hence we find a codirection $(\lambda\xi_0, \lambda\eta), \xi_0 \neq 0$ in which the product $\widehat{t_\alpha\chi}\widehat{\partial_h^\alpha \delta_I}$ is not rapidly decreasing, hence there is a point $(x,0)$ such that $(x,0;\xi_0,\eta_0) \in WF(t)$ [32, Lemma 8.2.1] which is in contradiction with the fact that $WF(t) \subset N^*(I) = \{(x,0,0,\eta) | \eta \neq 0\}$. □

Combining with Theorem 8.1, we obtain:

Corollary 8.1. *Let $t \in \mathcal{D}'(\mathbb{R}^{n+d})$ and supp $t \subset I$. If $WF(t) \subset N^*(I)$ and $t \in E_{s,N^*(I)}(\mathbb{R}^{n+d})$, $-m-1 < s+d \leqslant -m$, then $t(x,h) = \sum_\alpha t_\alpha(x)\partial_h^\alpha \delta_I(h)$, where $\forall\alpha$, $t_\alpha \in C^\infty(\mathbb{R}^n)$ and $|\alpha| \leqslant m$.*

Corollary 8.2. *Let M be a smooth manifold and I a closed embedded submanifold. For $-m-1 < s+d \leqslant -m$, the space of distributions $t \in E_{s,N^*(I)}(M)$ such that supp $t \in I$ and $WF(t)$ is contained in the conormal of I is a finitely generated module of* ***rank*** *$\frac{m+d!}{m!d!}$ over the ring $C^\infty(I)$.*

Proof. In each local chart (x,h) where $I = \{h=0\}$, $t = \sum_\alpha t_\alpha(x)\partial_h^\alpha \delta_I(h)$ where the lenght $|\alpha|$ is bounded by m by the above corollary and $\forall\alpha$, $t_\alpha \in C^\infty(I)$. This improves on the result given by the structure theorem of Laurent Schwartz since we now know that the t_α are smooth. □

9 Appendix.

Wave front set of the kernels of the operators I_m, R_α. In this part, we calculate the wave front set of the kernels of the operators I_m, R_α introduced in the proof of Theorem 3.3. Recall $I = \mathbb{R}^n \times \{0\}$ is the vector subspace $\{h=0\}$, we define the projection $\pi : (x,h) \in \mathbb{R}^{n+d} \longmapsto (x,0) \in \mathbb{R}^n \times \{0\}$, the inclusion $i : \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^{n+d}$, the operator I_m of projection on the Taylor remainder of degree m:

$$I_m := \varphi \in C^\infty(\mathbb{R}^{n+d}) \longmapsto I_m\varphi = \varphi - P_m\varphi \in C^\infty(\mathbb{R}^{n+d})$$

$$P_m\varphi = \sum_{|\alpha| \leqslant m} \frac{h^\alpha}{\alpha!}\pi^*\left(i^*\partial_h^\alpha \varphi\right)$$

$$I_m\varphi = \frac{1}{m!}\sum_{|\alpha|=m+1} h^\alpha \int_0^1 (1-t)^m \left(\partial_h^\alpha \varphi\right)_t dt.$$

We also introduce the operators $(R_\alpha)_{\{|\alpha|=m+1\}}$:

$$I_m = \sum_{|\alpha|=m+1} h^\alpha R_\alpha.$$

We next explain how to calculate the Schwartz kernels of I_m, R_α which are distributions in $\mathcal{D}'(\mathbb{R}^{n+d} \times \mathbb{R}^{n+d})$ and their wave front set. We double the space $\mathbb{R}^{n+d}$ and we work in $\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}$ with coordinates (x,h,x',h'). We denote by $\delta \in \mathcal{D}'(\mathbb{R}^{n+d})$ the delta distribution supported at $(0,0) \in \mathbb{R}^{n+d}$ and $\delta_\Delta(.,.) \in$

$\mathcal{D}'(\mathbb{R}^{n+d}\times\mathbb{R}^{n+d})$ the delta distribution supported on the diagonal Δ in $\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$ where we have the relation $\delta_\Delta((x,h),(x',h'))=\delta(x-x',h-h')$. The Schwartz kernel of I_m is the distribution defined as:

$$\begin{aligned} I_m(.,.) &= \delta_\Delta(.,.)-\sum_{|\alpha|\leqslant m}\frac{h^\alpha}{\alpha!}\pi^*\left(i^*\partial_h^\alpha\delta_\Delta(.,.)\right) && (37)\\ &= \frac{1}{m!}\sum_{|\alpha|=m+1}h^\alpha\int_0^1(1-t)^m\partial_h^\alpha\delta_\Delta(\Phi(t,.),.)dt, && (38)\end{aligned}$$

where $\Phi(t,x,h)=(x,th)$. We also need to define Schwartz kernels R_α:

$$\begin{aligned} R_\alpha(.,.) &= \frac{1}{m!}\int_0^1(1-t)^m\partial_h^\alpha\delta(\Phi(t,.),.)dt\\ \text{where } I_m(.,.) &= \sum_{|\alpha|=m+1}h^\alpha R_\alpha(.,.).\end{aligned}$$

Lemma 9.1. *Let $I_m(.,.)$ and $R_\alpha(.,.)$ be defined as above then*

$$WF\left(R_\alpha(.,.)\right)\subset\{(x,h,x,th;\xi,t\eta,-\xi,-\eta)|t\in[0,1],(\xi,\eta)\neq(0,0)\}. \quad (39)$$

and

$$WF(I_m(.,.))\subset\{(x,h,x,th;\xi,t\eta,-\xi,-\eta)|t\in[0,1],(\xi,\eta)\neq(0,0)\}. \quad (40)$$

Proof. Let us calculate $WFI_m(.,.)$, the idea is to work in "extended phase space" $[0,1]\times\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$ with coordinates (t,x,h,x',h'). Consider the map

$$\Phi:=(t,x,h,x',h')\in[0,1]\times\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}\longmapsto(x,th,x',h')\in\mathbb{R}^{n+d}\times\mathbb{R}^{n+d},$$

then $(\Phi^*\delta)(t,x,h,x',h')=\delta((x,th),(x',h'))$ and application of the pull–back theorem [4, Proposition 6.1] implies that

$$WF(\Phi^*\partial_h^\alpha\delta(.,.)))\subset\{(t,x,h,x',h';\tau,\xi,t\eta,-\xi,-\eta)|(x,th)=(x',h')\text{ and }\tau=\langle h.\eta\rangle,(\xi,\eta)\neq(0,0)\}. \quad (41)$$

We also note that $m!R_\alpha$ is just the integral of $f=1_{[0,1]}(1-t)^m\Phi^*\partial_h^\alpha\delta(.,.))dt$ over $[0,1]$, in other words, it is the push–forward of f by the projection $\mathbf{p}:\mathbb{R}\times\mathbb{R}^{2(n+d)}\mapsto\mathbb{R}^{2(n+d)}$. From the bound (41) on $WF(\Phi^*\partial_h^\alpha\delta(.,.)))$ and the behaviour of wave front sets under product, we find the rough upper bound:

$$WF(f)\subset\Xi=\{(t,x,h,x',h';\tau,\xi,t\eta,-\xi,-\eta)|(x,th)=(x',h'),(\xi,\eta)\neq(0,0)\}.$$

Finally, from the relation $R_\alpha=\frac{\mathbf{p}_*f}{m!}$, $f=1_{[0,1]}(1-t)^m\Phi^*\partial_h^\alpha\delta(.,.))dt$ we find that

$$\begin{aligned} WFR_\alpha(.,.) &\subset \mathbf{p}_*WF(f)\subset\mathbf{p}_*\Xi\\ \Longrightarrow WFR_\alpha(.,.) &\subset \{(x,h,x,th;\xi,t\eta,-\xi,-\eta)|t\in[0,1],(\xi,\eta)\neq(0,0)\}.\end{aligned}$$

□

We also need the wave front set of the Schwartz kernel of the operator $\varphi\longmapsto P_m\varphi$ which projects φ on its "Taylor polynomial":

$$\forall|\alpha|\leqslant m, WF\left(\pi^*(i^*\partial_h^\alpha\delta_\Delta(.,.))\right)\subset\{(x,h,x,0;\xi,0,-\xi,-\eta)|(\xi,\eta)\neq(0,0)\}. \quad (42)$$

Note the important fact that $WF\left(\pi^*(i^*\partial_h^\alpha\delta_\Delta(.,.))\right)\subset WF(I_m(.,.))$.

Lemma 9.2. *Let $\delta_\Delta((x,h),(x',h'))$ be the delta function of the diagonal $\Delta\subset\mathbb{R}^{n+d}\times\mathbb{R}^{n+d}$, $i:x\longmapsto(x,0)$ the inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+d}$ and π the projection $(x,h)\in\mathbb{R}^{n+d}\longmapsto x\in\mathbb{R}^n$. The Schwartz kernel of the linear map $\varphi\longmapsto\pi^*(i^*\partial_h^\alpha\varphi)$ is $\pi^*(i^*\partial_h^\alpha\delta_\Delta)$,*

$$WF(\pi^*(i^*\partial_h^\alpha\delta_\Delta))\subset\{(x,h,x,0;\xi,0,-\xi,-\eta)|(\xi,\eta)\neq(0,0)\}. \quad (43)$$

Proof. First, we have: $WF(i^*\partial_h^\alpha\delta_\Delta)\subset\{(x,x,0;\xi,-\xi,-\eta),\ (\xi,\eta)\neq(0,0)\}$, then

$$\begin{aligned} WF(\pi^*(i^*\partial_h^\alpha\delta_\Delta)) &\subset\{(x,h,x',h';\xi,0,\xi',\eta')|(x,x',h';\xi,\xi',\eta')\in WF(i^*\partial_h^\alpha\delta_\Delta)\}\\ &=\{(x,h,x,0;\xi,0,-\xi,-\eta)|(\xi,\eta)\neq(0,0)\}.\end{aligned}$$

□

Technical Lemma. In this part, we prove the main technical Lemma which is essential in the proof of the main Theorems of section 3 and we follow its terminology and notations.

Lemma 9.3. *Let $U \subset \mathbb{R}^{n+d}$ be a convex set, for $\varepsilon \geqslant 0$, $1_{[\varepsilon,1]}$ is the indicator function of $[\varepsilon, 1]$. Set*

$$V = \{ \begin{pmatrix} \lambda & ; & \widehat{\lambda} \\ x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \end{pmatrix} | \begin{pmatrix} x & ; & \widehat{\xi} \\ h & ; & \widehat{\eta} \end{pmatrix} \in \Gamma \cup \underline{0}, (x,h) \in supp\ \psi \}. \quad (44)$$

Let B be some bounded subset in $E_s(\mathcal{D}'_\Gamma(U))$. For all function $f \in L^1([0,1]) \cap C^\infty(0,1)$, for all $t \in B$, the family $\left(f 1_{[\varepsilon,1]}\lambda^{-s}\Phi^* t\right)_{\varepsilon\in[0,1], t\in B}$ is bounded in $\mathcal{D}'_V(\mathbb{R}\times U)$.

Proof. We first prove that $\left(f 1_{[\varepsilon,1]}\lambda^{-s}\Phi^* t\right)_{\varepsilon\in[0,1]}$ is weakly bounded in $\mathcal{D}'(\mathbb{R}\times U)$. $\lambda^{-s}t_\lambda$ is bounded in $\mathcal{D}'(U)$ therefore by the uniform boundedness principle in Fréchet space [40],

$$\forall K \subset U \text{ compact }, \exists m \in \mathbb{N}, \exists C > 0, \forall \varphi \in \mathcal{D}_K(U), \sup_{\lambda\in[0,1]} |\left\langle \lambda^{-s}t_\lambda, \varphi\right\rangle| \leqslant C\pi_{m,K}(\varphi).$$

If t is in a bounded subset B of $E_s(\mathcal{D}'_\Gamma(U))$, then one can choose the constant C independent of $t \in B$. It follows easily that for all subset of the form $(\mathbb{R}\times K) \subset (\mathbb{R}\times U)$:

$$\exists m \in \mathbb{N}, C \geqslant 0, \text{ such that } \forall \varphi \in \mathcal{D}_{\mathbb{R}\times K}(\mathbb{R}\times U), \forall\varepsilon \geqslant 0,$$

$$\begin{aligned} |\int_{[\varepsilon,1]\times\mathbb{R}^{n+d}} f(\lambda)\lambda^{-s}t(x,\lambda h)\varphi(\lambda,x,h)d\lambda dx dh| &\leqslant \|f\|_{L^1([0,1])} \sup_{\lambda\in[0,1]} |\left\langle \lambda^{-s}t_\lambda, \varphi(\lambda,.)\right\rangle| \\ &\leqslant C\|f\|_{L^1([0,1])} \sup_{\lambda\in[0,1]} \pi_{m,K}(\varphi(\lambda,.)) \\ &\leqslant C\|f\|_{L^1([0,1])}\pi_{m,[0,1]\times K}(\varphi). \end{aligned}$$

For all $(\lambda, x, h; \tau, \xi, \eta) \notin V$, there is a conic set $W \subset \mathbb{R}^{n+d}\setminus\{0\}$, a test function $\varphi_2 \in \mathcal{D}(U)$ such that $(x, h; \xi, \eta) \in \text{supp } \varphi_2 \times W$ and $(\text{supp } \varphi_2 \times W) \cap \Gamma = \emptyset$. Let $\varphi(\lambda, x, h) = \varphi_1(\lambda)\varphi_2(x, h)$ for some $\varphi_1, \varphi_1(\lambda) \neq 0$ in $\mathcal{D}(\mathbb{R})$ and we define a conic neighborhood W' of (τ_0, ξ_0, η_0) as follows $W' = \{(\tau, \xi, \eta)||\tau| \leqslant 2\frac{|\tau_0|}{|\xi_0|+|\eta_0|}(|\xi| + |\eta|), (\xi, \eta) \in W\}$. We find that $\forall(\tau, \xi, \eta) \in W'$:

$$\begin{aligned} |\int_\varepsilon^1 d\lambda f(\lambda)\left\langle \lambda^{-s}t_\lambda, \varphi_2 e^{i(x.\xi+h.\eta)}\right\rangle \varphi_1(\lambda)e^{i\lambda.\tau}| &= |\int_\varepsilon^1 d\lambda \widehat{\lambda^{-s}t_\lambda\varphi_2}(\xi,\eta)f(\lambda)\varphi_1(\lambda)e^{i\lambda.\tau}| \\ &\leqslant \|\varphi_1\|_{L^\infty(\mathbb{R})}\|f\|_{L^1[0,1]}\|\lambda^{-s}t_\lambda\|_{N,W,\varphi_2}(1+|\xi|+|\eta|)^{-N} \\ &\leqslant C\|\varphi_1\|_{L^\infty(\mathbb{R})}\|f\|_{L^1[0,1]}\|\lambda^{-s}t_\lambda\|_{N,W,\varphi_2}(1+|\tau|+|\xi|+|\eta|)^{-N} \end{aligned}$$

where $C = (1 + 2\frac{|\tau_0|}{|\xi_0|+|\eta_0|})^N$. Therefore, $\forall(\lambda, x, h; \tau, \xi, \eta) \notin \Lambda, \exists\chi \in \mathcal{D}(\mathbb{R}\times U)$ and a closed conic set W' such that $\chi(\lambda, x, h) \neq 0$, $(\text{supp } \chi \times W') \cap \Lambda = \emptyset$ and the following estimate is satisfied:

$$\forall N, \exists C, \|f\lambda^{-s}\Phi^* t\|_{N,W',\chi} \leqslant C \sup_{\lambda\in[0,1]} \|\lambda^{-s}t_\lambda\|_{N,W,\varphi} \quad (45)$$

for some continuous seminorm $\sup_{\lambda\in[0,1]} \|\lambda^{-s}t_\lambda\|_{N,W,\varphi}$ of $E_s(\mathcal{D}'_\Gamma(U))$ and where the constant C does not depend on t.

It follows easily from the above that the family $\left(f 1_{[\varepsilon,1]}\lambda^{-s}\Phi^* t\right)_{\varepsilon\in(0,1]}$ is bounded in $\mathcal{D}'_V(\mathbb{R}\times U)$. □

9.0.1 The symplectic geometry of the vector fields tangent to I and of the diffeomorphisms leaving I invariant.

We will work at the infinitesimal level within the class $\mathfrak{g}$ of vector fields tangent to I defined by Hörmander [33, Lemma (18.2.5)]. First recall their definition in coordinates (x, h) where $I = \{h = 0\}$: the vector fields X tangent to I are of the form

$$h^j a^i_j(x,h)\partial_{h^i} + b^i(x,h)\partial_{x^i}$$

and they form an infinite dimensional Lie algebra denoted by $\mathfrak{g}$ which is a Lie subalgebra of $Vect(M)$. Actually, these vector fields form a module over the ring $C^\infty(M)$ finitely generated by the vector fields $h^i\partial_{h^j}, \partial_{x^i}$. This module is naturally filtered by the vanishing order of the vector field on I.

Definition 9.1. *Let $\mathcal{I}$ be the ideal of functions vanishing on I. For $k \in \mathbb{N}$, let F_k be the submodule of vector fields tangent to I defined as follows, $X \in F_k$ if $X\mathcal{I} \subset \mathcal{I}^{k+1}$.*

This definition of the filtration is completely coordinate invariant. We also immediately have $F_{k+1} \subset F_k$. Note that $F_0 = \mathfrak{g}$.

Cotangent lift of vector fields. We recall the following fact, any vector field $X \in Vect(M)$ lifts functorially to a *Hamiltonian vector field* $X^* \in Vect(T^*M)$ by the following procedure which is beautifully described in [8, p. 34]:

$$X = a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial h^j} \in Vect(M) \overset{\sigma}{\mapsto} \sigma(X) = a^i\xi_i + b^j\eta_j \in C^\infty(T^*M)$$

$$\mapsto X^* = \{\sigma(X), .\} = a^i\frac{\partial}{\partial x^i} + b^j\frac{\partial}{\partial h^j} - \frac{\partial(a^i\xi_i + b^j\eta_j)}{\partial x^i}\frac{\partial}{\partial\xi_i} - \frac{\partial(a^i\xi_i + b^j\eta_j)}{\partial h^j}\frac{\partial}{\partial\eta_j},$$

where $\{.,.\}$ is the Poisson bracket of T^*M.

Lemma 9.4. *Let X be a vector field in $\mathfrak{g}$. If $X \in F_1$, then X^* vanishes on the conormal $N^*(I)$ of I and $N^*(I)$ is contained in the set of fixed points of the symplectomorphism e^{X^*}.*

Proof. If $X \in F_1$, then $\sigma(X) = h^jh^ia^l_{ji}(x,h)\eta_l + h^ib^l_i(x,h)\xi_l$ where a^l_{ji}, b^l_i are smooth functions on T^*M by the Hadamard lemma. The symplectic gradient X^* is given by the formula

$$X^* = \frac{\partial\sigma(X)}{\partial\xi_i}\partial_{x^i} - \frac{\partial\sigma(X)}{\partial x^i}\partial_{\xi_i} + \frac{\partial\sigma(X)}{\partial\eta_i}\partial_{h^i} - \frac{\partial\sigma(X)}{\partial h^i}\partial_{\eta_i},$$

thus $X^* = 0$ when $\xi = 0, h = 0$ which means $X^* = 0$ on the conormal $N^*(I)$. ☐

Lemma 9.5. *Let ρ_1, ρ_2 be two Euler vector fields and $\Phi(\lambda) = e^{-\log\lambda\rho_1} \circ e^{\log\lambda\rho_2}$. Then the cotangent lift $T^*\Phi(\lambda)$ restricted to $N^*(I)$ is the identity map:*

$$T^*\Phi(\lambda)|_{N^*(I)} = Id|_{N^*(I)}.$$

Proof. Let us set

$$\Phi(\lambda) = e^{-\log\lambda\rho_1} \circ e^{\log\lambda\rho_2} \tag{46}$$

which is a family of diffeomorphisms which depends smoothly in $\lambda \in [0,1]$ according to 1.1. The proof is similar to the proof of proposition 1.1, $\Phi(\lambda)$ satisfies the differential equation:

$$\lambda\frac{d\Phi(\lambda)}{d\lambda} = e^{-\log\lambda\rho_1}(\rho_2 - \rho_1)e^{\log\lambda\rho_1}\Phi(\lambda) \text{ where } \Phi(1) = Id \tag{47}$$

we reformulated this differential equation as

$$\frac{d\Phi(\lambda)}{d\lambda} = X(\lambda)\Phi(\lambda), \Phi(1) = Id \tag{48}$$

where the vector field $X(\lambda) = \frac{1}{\lambda}e^{-\log\lambda\rho_1}(\rho_2 - \rho_1)e^{\log\lambda\rho_1}$ depends smoothly in $\lambda \in [0,1]$. The cotangent lift $T^*\Phi_\lambda$ satisfies the differential equation

$$\frac{dT^*\Phi(\lambda)}{d\lambda} = X^*(\lambda)T^*\Phi(\lambda), T^*\Phi(1) = Id \tag{49}$$

Notice that $\forall\lambda \in [0,1], X(\lambda) \in F_1$ which implies that for all λ the lifted Hamiltonian vector field $X^*(\lambda)$ will vanish on $N^*(I)$ by the lemma (9.4). Since $T^*\Phi(1) = Id$ obviously fixes the conormal, this immediately implies that $\forall\lambda, T^*\Phi(\lambda)|_{N^*(I)} = Id|_{N^*(I)}$. ☐

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