The polynomial method in Galois geometries

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OUTLINE

1. GALOIS GEOMETRIES
   - 1. Affine spaces
   - 2. Projective spaces
2. BLOCKING SETS
   - Linear blocking set
   - Multiple blocking sets in PG$(2, q)$
   - Multiple blocking sets and algebraic curves
   - Characterization result

FINITE FIELDS

- q = prime number.
  - **Prime fields** $\mathbb{F}_q = \{0, 1, \ldots, q-1\} \pmod q$.
  - Binary field $\mathbb{F}_2 = \{0, 1\}$.
  - Ternary field $\mathbb{F}_3 = \{0, 1, 2\} = \{-1, 0, 1\}$.
- **Finite fields** $\mathbb{F}_q$: q prime power.

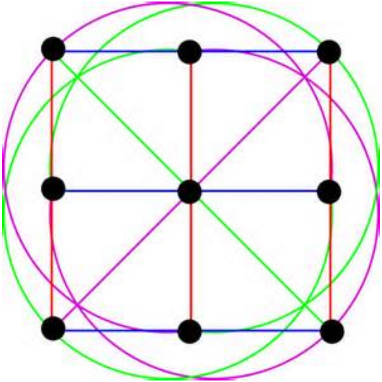
AFFINE SPACE AG(n, q)

- $V(n, q)$ = n-dimensional vector space over $\mathbb{F}_q$.
- AG(n, q) = $V(n, q)$ plus parallelism.
- k-dimensional affine subspace = (translate) of k-dimensional vector space.

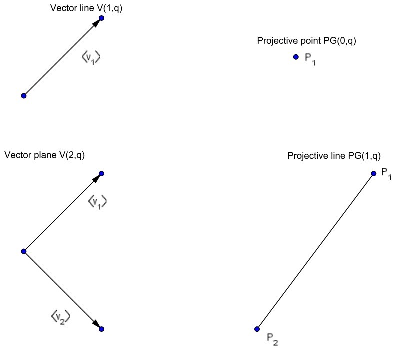
Parallelism in affine space $\mathrm{AG}(n, q)$

- Let Π_k be k-dimensional vector space of $V(n, q)$.
- $\Pi_k + b$, for $b \in V(n, q)$, are the affine k-subspaces *parallel* to Π_k.
- Two parallel affine k-subspaces are disjoint or equal.
- Parallelism leads to partitions of $\mathrm{AG}(n, q)$ into (parallel) affine k-subspaces.

AFFINE PLANE $AG(2,3)$ OF ORDER 3



FROM $V(3, q)$ TO $\mathrm{PG}(2, q)$



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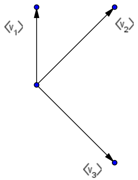
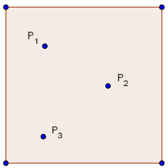
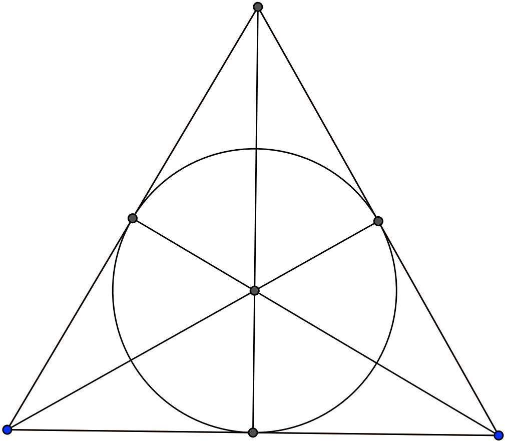
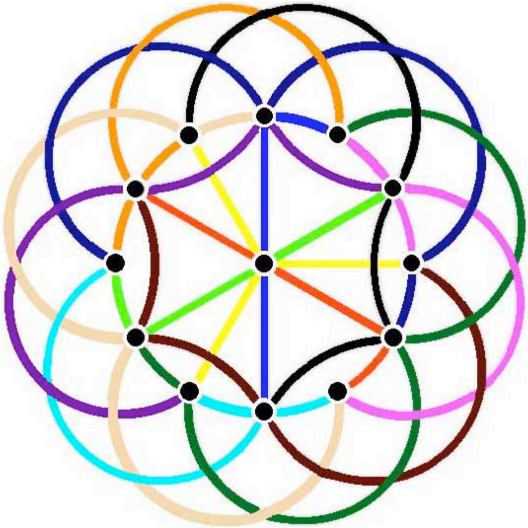




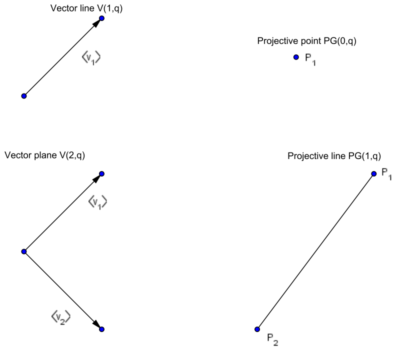
The Fano plane PG(2, 2)



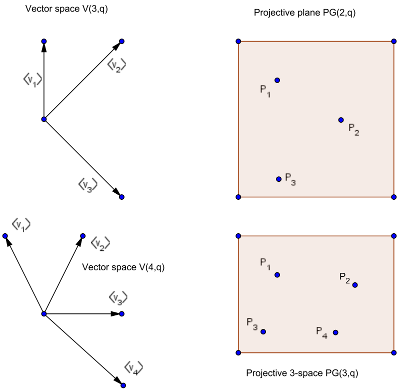
THE PLANE PG$(2,3)$



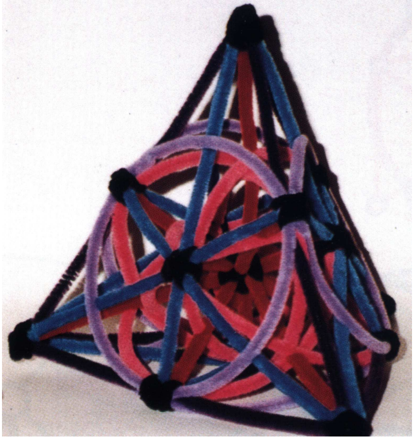
FROM $V(4, q)$ TO PG$(3, q)$



FROM $V(4, q)$ TO $\mathrm{PG}(3, q)$



$PG(3, 2)$



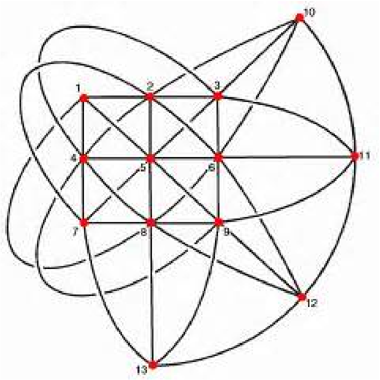
From $V(n+1,q)$ to $\mathrm{PG}(n,q)$

1. From $V(1,q)$ to $\mathrm{PG}(0,q)$ (projective point),
2. From $V(2,q)$ to $\mathrm{PG}(1,q)$ (projective line),
3. $\ldots$
4. From $V(i+1,q)$ to $\mathrm{PG}(i,q)$ (i-dimensional projective subspace),
5. $\ldots$
6. From $V(n,q)$ to $\mathrm{PG}(n-1,q)$ ($(n-1)$-dimensional subspace = hyperplane),
7. From $V(n+1,q)$ to $\mathrm{PG}(n,q)$ (n-dimensional space).

LINK BETWEEN AFFINE AND PROJECTIVE SPACES

- AG(n, q) = PG(n, q) minus one hyperplane (the hyperplane at infinity).

Link between AG(2, 3) and PG(2, 3)



Outline

Definition and example

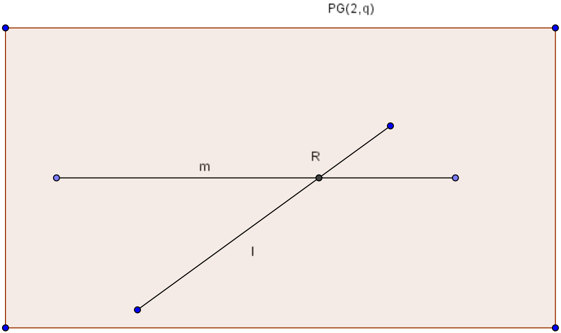
Definition

Blocking set B in PG$(2, q)$ is set of points, intersecting every line in at least one point.

Example

Line L in PG$(2, q)$.

Example



Definition

Definition

Point r of blocking set B in PG$(2, q)$ is *essential* if $B \setminus \{r\}$ is no longer blocking set.

Definition

Blocking set B is *minimal* if all of its points are essential.

Example

Line L of PG$(2, q)$ is minimal blocking set B of size $q + 1$.

Non-trivial blocking set in PG$(2, q)$

Definition

Non-trivial blocking set B in PG$(2, q)$ does not contain a line.

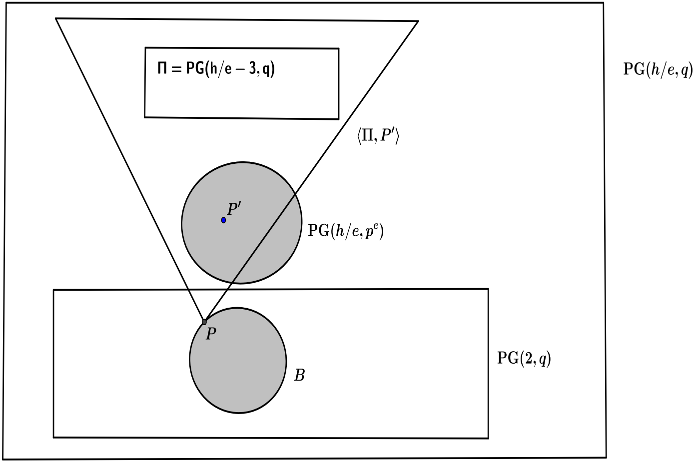
Example: Baer subplane PG$(2, \sqrt{q})$ in PG$(2, q)$, q square.

Notation: $q + r(q) + 1$ = size of smallest non-trivial blocking set in PG$(2, q)$.

- (Blokhuis) $r(q) = (q + 1)/2$ for $q > 2$ prime,
- (Bruen) $r(q) = \sqrt{q}$ for q square,
- (Blokhuis) $r(q) = q^{2/3}$ for q cube power.

Linear blocking set

- Consider PG$(2, q)$, $q = p^h$, p prime, $h \geq 1$.
- $\mathbb{F}_q$ has $\mathbb{F}_{p^e}$, $e|h$, as subfield.
- PG$(h/e, p^e)$ is naturally embedded subgeometry of PG$(h/e, q)$.
- Project PG$(h/e, p^e)$ onto plane PG$(2, q)$.
- Projection B is (linear) blocking set of PG$(2, q)$.


$\Pi = PG(h/e-3,q)$
$PG(h/e,q)$
$\langle \Pi, P' \rangle$
P'
$PG(h/e,p^e)$
P
B
$PG(2,q)$

Particular properties of linear blocking sets

- Line intersects B in $1 \pmod{p^e}$ points.
- If line L shares $1 + p^e$ points with B, then $L \cap B = \mathrm{PG}(1, p^e)$.

Theorem (Sziklai and Szőnyi)

Let B be minimal blocking set in $PG(2, q)$, $q = p^h$, p prime, $h \geq 1$, with $|B| < q + (q+3)/2$. Then

- *B intersects every line in $1 \pmod{p^e}$ points, for some $e|h$,*
- *If e is the maximal integer with this property, then $e|h$, and if line L shares $1 + p^e$ points with B, then $L \cap B = PG(1, p^e)$.*

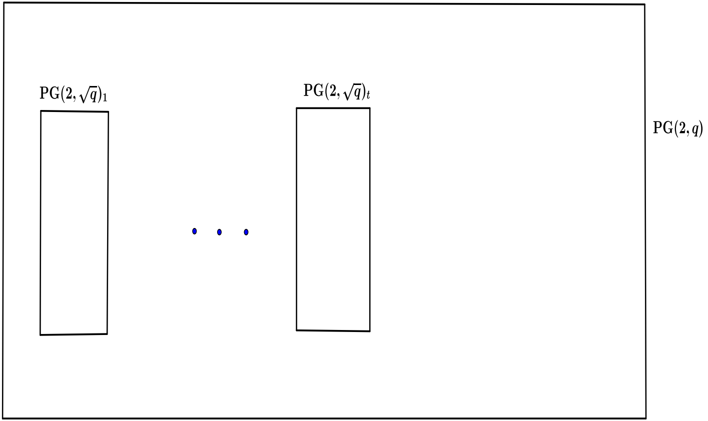
Definitions

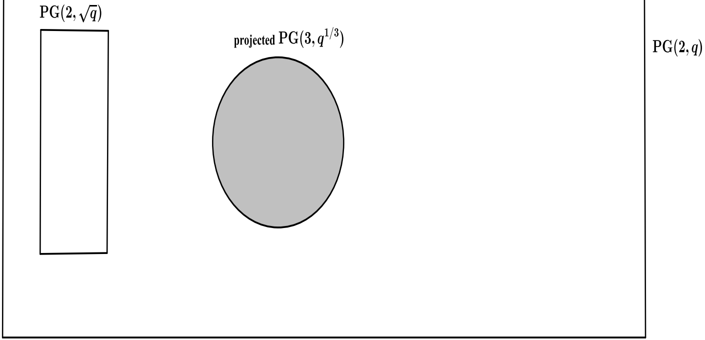
Definition

- t-Fold blocking set B in PG$(2, q)$: intersects every line in at least t points.
- Minimal t-fold blocking set: no proper subset is still t-fold blocking set.

Examples

- Union of t pairwise disjoint Baer subplanes PG$(2, \sqrt{q})$ in PG$(2, q)$, q square.
- (Polverino and Storme) Union of disjoint Baer subplane PG$(2, \sqrt{q})$ and projected subgeometry PG$(3, q^{1/3})$ in PG$(2, q)$, when q is 6-th power.
- Union of two disjoint linear non-trivial blocking sets.


$\mathrm{PG}(2,\sqrt{q})_1$
$\mathrm{PG}(2,\sqrt{q})_t$
$\mathrm{PG}(2,q)$


$\mathrm{PG}(2, \sqrt{q})$
projected $\mathrm{PG}(3, q^{1/3})$
$\mathrm{PG}(2, q)$

Setting for Rédei-polynomial

- B = t-fold blocking set in PG$(2, q)$ of size $t(q+1)+c$, with $t+c<q$.
- P point of B.
- Line ℓ = t-secant of B through P.
- Homogeneous coordinates $(X : Y : Z)$ such that
  - $P = (0:1:0) = (\infty)$,
  - $\ell : Z = 0$,
  - $B \cap \ell = \{(1 : -y_j : 0) || j = 1, \ldots, t-1\} \cup \{(0:1:0)\}$.

RÉDEI-POLYNOMIAL

- $\mathcal{A}$ = affine plane PG(2, q) $\setminus \ell$, such that $(x, y) = (x : y : 1)$,

$$B \cap \mathcal{A} = \{(a_i, b_i) || i = 1, \ldots, tq + c\}.$$

- $$F(U, V) = \prod_{j=1}^{t-1}(V + y_j) \prod_{i=1}^{tq+c}(U + a_i V + b_i).$$

  (*Rédei-polynomial*)

- $$F(U, V) = \sum_{i=0}^{t} F_i(U, V)(U^q - U)^{t-i}(V^q - V)^i,$$

  where $\deg(F_i) \leq \deg(F) - qt$.

Rédei-polynomial

- Homogeneous part of largest degree and substitute $V = 1$,

$$f(U) := \prod_{i=1}^{tq+c} (U + a_i) = \sum_{i=0}^{t} f_i(U) U^{q(t-i)},$$

  where $f_i(U) = F_{i0}(U, 1)$, and where F_{i0} is homogeneous part of $F_i(U, V)$ of highest degree.
- Since B is t-fold blocking set, f contains factor $U + y$ at least $t - 1$ times, for all $y \in \mathbb{F}_q$.
- So f is divisible by $(U^q - U)^{t-1}$. Dividing by $(U^q - U)^{t-1}$, we obtain *excess polynomial*

$$\mathrm{ex}(U) = U^q f_0(U) + f_1(U) + (t - 1) U f_0(U).$$

Rédei polynomial

Excess polynomial

$$\mathrm{ex}(U) = U^q f_0(U) + f_1(U) + (t-1)Uf_0(U)$$

contains information about lines through P having more than t points of B.

Definition

Let $\mathrm{ex}(U)$ be excess polynomial of P. Let $q = p^n$, p prime. Let $d(U) = \gcd(f_0(U), f_1(U))$. If e is largest integer for which $\mathrm{ex}(U)/d(U)$ is p^e-th power, then e is called *exponent* of P.

Rédei polynomial

Notation: $\deg(f) = f^\circ$.

Theorem (Blokhuis, Storme, Szőnyi)

Let $f \in \mathbb{F}_q[X]$, $q = p^n$, p prime, be fully reducible, $f(X) = X^q h(X) + g(X)$, where $\gcd(g, h) = 1$. Let $k = \max(g^\circ, h^\circ) < q$. Let e be maximal such that f is p^e-th power. Then:

RÉDEI POLYNOMIAL

THEOREM (BLOKHUIS, STORME, SZŐNYI)

(1) $e = n$ and $k = 0$;

(2) $e \geq 2n/3$ and $k \geq p^e$;

(3) $2n/3 > e > n/2$ and $k \geq p^{n-e/2} - (3/2)p^{n-e}$;

(4) $e = n/2$ and $k = p^e$ and $f(X) = aTr(bX + c) + d$ or $f(X) = aNorm(bX + c) + d$ for suitable constants a, b, c, d.

(5) $e = n/2$ and $k \geq p^e \left\lceil \frac{1}{4} + \sqrt{(p^e + 1)/2} \right\rceil$;

(6) $n/2 > e > n/3$ and $k \geq p^{n/2+e/2} - p^{n-e} - p^e/2$, or if $3e = n + 1$ and $p \leq 3$, then $k \geq p^e(p^e + 1)/2$;

(7) $n/3 \geq e > 0$ and $k \geq p^e\lceil (p^{n-e} + 1)/(p^e + 1)\rceil$;

(8) $e = 0$ and $k \geq (q + 1)/2$;

(9) $e = 0$, $k = 1$ and $f(X) = a(X^q - X)$.

Important lemmas

Lemma

Let B be minimal t-fold blocking set, $|B| = t(q+1) + c$ and let $P \in B$. Then at least $q - c$ lines through P intersect B in exactly t points.

Proof:

- Let $P = (0:1:0)$ and denote by e the exponent of P.
- $\mathrm{ex}(U) = U^q h(U) + g(U)$, with $h^\circ, g^\circ \leq c$.
- Let $d(U) = \gcd(h(U), g(U))$, then $\mathrm{ex}(U)/d(U) = (U^{q/p^e} h_1(U) + g_1(U))^{p^e}$.
- Number of lines that are not t-secants is at most $c + 1$.

Important Lemma

Lemma

Let B be minimal t-fold blocking set of $PG(2, q)$ of size $tq + t + c$. Let P be point of exponent e. Then

(1) *P lies on at least $2 + (q - c)/p^e$ lines meeting B in at least $p^e + t$ points;*

(2) *P lies on at least $(q - 3c)/p^e + 4$ distinct $(p^e + t)$-secants to B.*

Proof:

- Assume $d(U) = 1$.
- $\mathrm{ex}(U) = (e_1(U))^{p^e} = (U^{q/p^e} h_1(U) + g_1(U))^{p^e}$, with $g_1^\circ, h_1^\circ \leq c/p^e$.
- Then $\gcd(e_1(U), e_1'(U))$ divides $g_1(U)h_1'(U) - g_1'(U)h_1(U)$, and contains contribution of multiple roots of e_1.
- $\deg(g_1(U)h_1'(U) - g_1'(U)h_1(U)) \leq 2c/p^e - 2$.
- So, $e_1(U)$ has at least $(q - c)/p^e + 2$ distinct roots. At most $2c/p^e - 2$ of them can be multiple roots, hence $e_1(U)$ has at least $(q - 3c)/p^e + 4$ simple roots.

Setting for algebraic curves

- B = t-fold blocking set with $|B| = tq + t + c$, with $c + t < (q + 3)/2$.
- Exponent of any point in B is $e > 0$.
- (so, intuitively, every line intersects B in $t \pmod{p^e}$ points)

Definition

Let $\mathrm{ex}(U)$ be excess polynomial of P. Let $q = p^n$, p prime. Let $d(U) = \gcd(f_0(U), f_1(U))$. If e is largest integer for which $\mathrm{ex}(U)/d(U)$ is p^e-th power, then e is called *exponent* of P.

Setting for algebraic curves

- $$F(U,V)=\prod_{j=1}^{t-1}(V+y_j)\prod_{i=1}^{tq+c}(U+a_iV+b_i).$$
- $$F(U,V)=(U^q-U)^tF_0(U,V)+(U^q-U)^{t-1}(V^q-V)F_1(U,V)+$$
  $$\cdots+(V^q-V)^tF_t(U,V),$$
  where $\deg(F_i)\leq c+t-1$.

Useful lemmas

Lemma

If line $Y = -mX - b$ intersects $B \cap \mathcal{A}$ in more than t points, then $F_0(b, m) = \ldots = F_t(b, m) = 0$.

Lemma

$F_0, \ldots, F_t$ have no common divisor, dependent on U.

THEOREM

THEOREM

t-Fold blocking set B in $PG(2, q)$, $q = p^h$, p prime, $h \geq 1$, with $|B| = tq + t + c, c + t < (q + 3)/2$, intersects every line in $t \pmod p$ points.

Proof:

- Absolutely irreducible component $H(U, V)$ of $F_0(U, V)/\prod_{j=1}^{t-1}(V + y_j)$, with $\deg(H) = s$.
- $\exists i$ for which $H(U, V) \not| F_i(U, V)$.

THEOREM

(Proof, continued)

- If $H'_U \not\equiv 0$, then H has at least

$$(q+1-t)s - s(s-1)$$

  $\mathbb{F}_q$-rational points (Blokhuis, Pellikaan, Szőnyi).
- These points all belong to F_i, and Bézout's theorem gives

$$(q+1-t)s - s(s-1) \leq s(c+t-1).$$

- Gives inequality

$$c + t + (t+s) \geq q+3,$$

  and as $s \leq c$,

$$c + t \geq (q+3)/2.$$

Theorem

- If $c + t < (q + 3)/2$, then $H'_U \equiv 0$ for any component H.
- All lines intersect B in $t \pmod p$ points.

Theorem

t-Fold blocking set B in $PG(2, q)$, $q = p^h$, p prime, $h \geq 1$, with $|B| = tq + t + c, c + t < (q + 3)/2$, intersects every line in $t \pmod p$ points.

Characterization result

Let B be minimal t-fold blocking set of PG$(2, p^{6m})$ of size $t(q+1)+c$, with $2 \leq t < q^{1/4}/4$, and $c < p^{4m}\sqrt{p}/2$.

Lemma

*Point of B has exponent $4m$, $3m$ or $2m$.
Moreover, when $e = 3m$, then this point defines dual Baer subline of lines all containing at least $p^{3m} + t$ points of B.*

Rédei polynomial

Theorem (Blokhuis, Storme, Szőnyi)

(1) $e = n$ *and* $k = 0$;

(2) $e \geq 2n/3$ *and* $k \geq p^e$;

(3) $2n/3 > e > n/2$ *and* $k \geq p^{n-e/2} - (3/2)p^{n-e}$;

(4) $e = n/2$ *and* $k = p^e$ *and* $f(X) = aTr(bX + c) + d$ *or* $f(X) = aNorm(bX + c) + d$ *for suitable constants* a, b, c, d.

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(8) $e = 0$ *and* $k \geq (q + 1)/2$;

(9) $e = 0$, $k = 1$ *and* $f(X) = a(X^q - X)$.

Definition

Line containing at least $p^{4m} + t$ points of B is called *very long*, while line meeting B in at least $p^{3m} + t$ points is called *long*.

Lemma

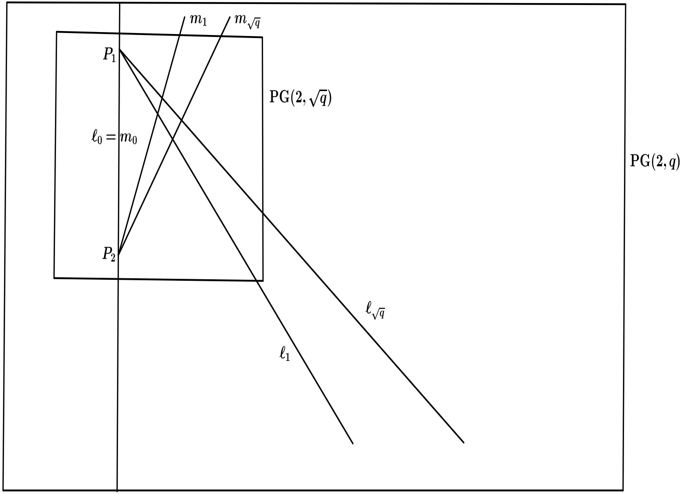
Dual Baer subline of long lines through point of exponent $3m$ is unique.

Definition

If P is point of t-fold blocking set B of exponent $3m$ defining dual Baer subline of long lines, and ℓ is one of the lines of this dual Baer subline, then we call P *special point* of ℓ.

Lemma

If line ℓ contains $2t + 1$ special points, Baer subplane contained in B.


m_1
$m_{\sqrt{q}}$
P_1
$\mathrm{PG}(2, \sqrt{q})$
$\ell_0 = m_0$
$\mathrm{PG}(2, q)$
P_2
$\ell_{\sqrt{q}}$
ℓ_1

Lemma

If there is Baer subplane S contained in B, then $B \setminus S$ is minimal $(t-1)$-fold blocking set.

From now on, line ℓ contains at most $2t$ special points.

LEMMA

B has at most c points of exponent $3m$.

LEMMA

There are at most $2t$ points of exponent $4m$.

Characterization result

Theorem (Blokhuis, Lovász, Storme, Szőnyi)

t-Fold blocking set B in $PG(2, p^{6m})$, $2 \leq t < p^{3m/2}/4$, with $|B| < tp^{6m} + p^{4m}\sqrt{p}/2 + t$, not containing Baer subplane, has size $|B| \geq tp^{6m} + tp^{4m} - O(p^{2m})$.

Thank you very much for your attention!