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Zeta function of subgroups of abelian groups and average orders

By *Gautami Bhowmik* at Villeneuve d'Ascq and *Jie Wu* at Vandœuvre-lès-Nancy

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§1. Introduction

The zeta function associated with the subgroups of finite abelian groups of rank at most r is difficult to investigate, except for the classical case $r = 1$. Using a correspondence between divisor classes of integer matrices and subgroups of finite abelian groups, we are able to obtain the said function precisely up to a highly convergent Dirichlet series. Then we use the zeta function to deduce the average number of subgroups of finite abelian groups.

Let $\mathscr{G} \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$ be a finite abelian group of rank r with $n_j | n_{j+1}$ for $1 \leqq j < r$ and let $\tau(\mathscr{G})$ be the number of subgroups of $\mathscr{G}$ and $r(\mathscr{G})$ the rank of $\mathscr{G}$. We introduce the *level function* $\ell_\tau^{(r)}(n) := \sum_{|\mathscr{G}|=n,\, r(\mathscr{G}) \leqq r} \tau(\mathscr{G})$, and study the associated zeta function, i.e.

$$\mathscr{D}^{(r)}(\tau, s) := \sum_{n=1}^{\infty} \ell_\tau^{(r)}(n)/n^s. \tag{1.1}$$

Since it is known that $\ell_\tau^{(r)}(n)$ is multiplicative, we can formally write the Euler product

$$\mathscr{D}^{(r)}(\tau, s) = \prod_p \sum_{\nu=0}^{\infty} \ell_\tau^{(r)}(p^\nu) p^{-\nu s},$$

where p runs through all prime numbers. In [3] the abscissa of convergence of $\mathscr{D}^{(r)}(\tau, s)$ was found to be $([r^2/4] + 1)/r$ (here $[t]$ denotes the integral part of t) while for $r = 2$ the zeta function was completely determined as

$$\mathscr{D}^{(2)}(\tau, s) = \zeta(s)^2 \zeta(2s)^2 \zeta(2s-1) \prod_p (1 + p^{-2s} - 2p^{-3s}),$$

where $\zeta(s)$ is the Riemann zeta-function. We define the summatory function of $\ell_\tau^{(r)}(n)$ as

$$T_r(x) := \sum_{n \leqq x} \ell_\tau^{(r)}(n).$$

The asymptotic behaviour of $\ell_\tau^{(2)}(n)$ was carefully investigated by Bhowmik and Wu [5], Menzer [8] and Ivić [7]. In particular we proved (1.6) of Theorem 2 below.

In the present paper we study the Dirichlet series associated with $\ell_\tau^{(r)}(n)$ and establish an asymptotic formula for $T_r(x)$ when $r \geqq 3$.

Bhowmik and Ramaré [4] have shown that there exists a bijection between the set of subgroups of a finite abelian group $\mathscr{G} \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$ and that of divisor classes of an $r \times r$ Smith Normal Form matrix $\operatorname{diag}[n_1, \ldots, n_r]$ with $n_j | n_{j+1}$. If the cardinality of the latter set is denoted by $\tau(\operatorname{diag}[n_1, \ldots, n_r])$ then for $\mathscr{G} \simeq \mathbb{Z}/p^{f_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{f_1+\cdots+f_r}\mathbb{Z}$ with $f_j \in \mathbb{Z}^+$ we have

$$(1.2) \qquad \ell_\tau^{(r)}(p^\nu) = \sum_{\substack{|\mathscr{G}|=p^\nu \\ r(\mathscr{G})\leqq r}} \tau(\mathscr{G}) = \sum_{\substack{f_1\geqq 0 \\ \nu(f_1,\ldots,f_r)=\nu}} \cdots \sum_{f_r \geqq 0} \tau(\operatorname{diag}[p^{f_1}, \ldots, p^{f_1+\cdots+f_r}]),$$

where $\nu(f_1, \ldots, f_r) = \sum_{1\leqq j\leqq r} jf_{r-j+1}$. Though the expression obtained (in [2]) for the divisor function $\tau(\operatorname{diag}[p^{f_1}, \ldots, p^{f_1+\cdots+f_r}])$, a polynomial in p with positive integral coefficients, is very complicated we have now been able to evaluate its leading term and the sum of its coefficients (see Theorem 3) and shown that we can extract enough information from these to get a zeta function precisely up to a highly convergent factor.

We let $\alpha_j := [j^2/4]$ and $\delta_j := 0$ if j is even, $\delta_j := 1$ otherwise; and obtain our Dirichlet series as:

Theorem 1. *With notations as above, we have*

$$\mathscr{D}^{(r)}(\tau, s) = \prod_{1\leqq j\leqq r} \zeta(js - \alpha_j)^{1+\delta_j}\, C_r(s),$$

where $C_r(s)$ is a Dirichlet series absolutely convergent for $\operatorname{Re} s > \alpha_r/r$. *In particular we have*

$$(1.3) \qquad \mathscr{D}^{(2)}(\tau, s) = \zeta(s)^2 \zeta(2s-1) G_2(s),$$

$$(1.4) \qquad \mathscr{D}^{(3)}(\tau, s) = \zeta(s)^2 \zeta(2s-1)\zeta(3s-2)^2 G_3(s),$$

$$(1.5a) \qquad \mathscr{D}^{(2k)}(\tau, s) = \zeta(2ks - k^2) G_{2k}(s) \quad (k \geqq 2),$$

$$(1.5b) \qquad \mathscr{D}^{(2k+1)}(\tau, s) = \zeta\big((2k+1)s - (k^2+k)\big)^2 G_{2k+1}(s) \quad (k \geqq 2),$$

where $G_r(s) = \sum_{n=1}^{\infty} g_r(n) n^{-s}$ is a Dirichlet series absolutely convergent for $\operatorname{Re} s > \alpha_r/r$.

The formula (1.3) with $G_2(s) = \zeta(2s)^3 \prod_p (1 - 2p^{-3s} - p^{-4s} + 2p^{-5s})$ was obtained by Bhowmik and Ramaré ([3], Corollary 1) whereas the other relations are new.

We notice that the cases $r = 2$ and $r = 3$ are indeed singular, for the former the largest pole, at $s = 1$, is of order 3 and the latter has its largest pole of order 5 at $s = 1$, while for $r \geqq 4$ the largest pole is of order only 1 or 2, depending on the parity of r, at $s = (\alpha_r + 1)/r > 1$.

With the help of the last theorem we can immediately see that the average value of the number of subgroups is $A_r x^{(\alpha_r+1)/r-1} \log^{\gamma_r} x$ where A_r is an effective positive constant. For finer results we need to use techniques of multiple exponential sums ([5], [12]). In the process we obtain results for weighted 5-dimensional divisor functions (see Lemmas 4.1, 4.2 below) which could have other applications as well. For what concerns the number of subgroups we prove:

Theorem 2. *With previous notations, for $k \geqq 2$, we have*

$$(1.6) \quad T_2(x) = xP_2(\log x) + O(x^{5/8} \log^4 x),$$

$$(1.7) \quad T_3(x) = xP_4(\log x) + O(x^{14/17} \log^6 x),$$

$$(1.8) \quad T_{2k}(x) = A_{2k} x^{(k^2+1)/2k} + O_{k,\varepsilon}(x^{k/2+\varepsilon}) \quad (\forall \varepsilon > 0),$$

$$(1.9) \quad T_{2k+1}(x) = x^{(k^2+k+1)/(2k+1)} P_1(\log x) + O_k(x^{(k^2+k+\mu)/(2k+1)} \log^{\beta} x),$$

where $P_j(x)$ is a jth degree polynomial, A_{2k} is an effective constant, μ and β are the exponents of x and $\log x$ in the error terms of the classical divisor problem. On the other hand for the error term $\Delta_r(x)$ of $T_r(x)$ we have

$$(1.10) \qquad \Delta_r(x) = \Omega(x^{\alpha_r/r}).$$

We notice that (1.8) is an optimal result and in (1.9) we can take $\mu = \frac{23}{73}$ (or better still $\frac{131}{416}$) and $\beta = \frac{461}{146}$ ([6] and private communication).

§2. The divisor functions

While studying the arithmetic of integer matrices we encounter a *left divisor class* of an $r \times r$ non-singular matrix M, say A, which is a canonical representative of $A \cdot \mathrm{GL}_r(\mathbb{Z})$ and for which there exists a matrix $B \in \mathrm{GL}_r(\mathbb{Z})$ such that $AB = M$. It was shown that a divisor class of M corresponds to a sub-lattice of $V(M)$, the image of the endomorphism whose matrix is M in a chosen base [4]. Since the co-kernel of the endomorphism, $\mathbb{Z}^r/V(M)$, is a finite abelian group whose invariant factors are the same as that of M and since $V(M)$ depends only on the right unimodular class of M, we consider the divisor functions (with a weight $a \in \mathbb{C}$) of M which are at the same time divisor functions of $\mathbb{Z}^r/V(M)$. Let

$$\sigma_a(M) = \sum_{A|M} (\det A)^a, \quad \tau(M) = \sigma_0(M).$$

We choose M to be a unique representative of a two-sided equivalence class, i.e. in Smith Normal Form (SNF) and A to be a unique representative of a one-sided equivalence class, i.e. in Hermite Normal Form (HNF). Further the functions $\sigma_a(M)$ are multiplicative, i.e. $\sigma_a(AB) = \sigma_a(A)\sigma_a(B)$ whenever A, B have co-prime determinants and it is enough to consider M to be an SNF matrix with a prime power determinant.

For simplicity, we write $\sigma_a\langle f_1, \dots, f_r\rangle_p$ for $\sigma_a(\mathrm{diag}[p^{f_1}, \dots, p^{f_1+\cdots+f_r}])$ and $\tau\langle f_1, \dots, f_r\rangle_p$ for $\tau(\mathrm{diag}[p^{f_1}, \dots, p^{f_1+\cdots+f_r}])$. Now $\tau\langle f_1, \dots, f_r\rangle_p$ counts the number of subgroups of the abelian p-group $\mathscr{G} \cong \mathbb{Z}/p^{f_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{f_1+\cdots+f_r}\mathbb{Z}$. In [2], Bhowmik has given a precise formula of $\sigma_a\langle f_1, \dots, f_r\rangle_p$ in terms of Gaussian multinomials. But the expression obtained is very complicated and not convenient to use. Hence we need to extract information which give us fairly precise analytic results.

For convenience we set $f_j = 0$ if $j \leqq 0$ or $j > r$, and use 0_m to denote a string of m zeros. By convention, the empty sum is 0 and the empty product is 1. We now give the leading term and the sum of coefficients of the polynomial $\sigma_a\langle f_1, \dots, f_r\rangle_p$.

Acknowledgement. The relation (2.3) in fact emerged during a discussion with V. C. Nanda.

Theorem 3. *For $a, f_1, \dots, f_r \in \mathbb{Z}^+$, the function $\sigma_a\langle f_1, \dots, f_r\rangle_p$, a polynomial in p with non-negative integral coefficients, has as leading term: $\Lambda_a^{(r)}(f_1, \dots, f_r)p^{\theta_a^{(r)}(f_1,\dots,f_r)}$, where*

$$(2.1)\qquad \theta_a^{(r)}(f_1, \dots, f_r) = \sum_{1 \leqq j \leqq r-a} \alpha_{r-j+1+a} f_j + a \sum_{r+1-a \leqq j \leqq r} (r - j + 1) f_j,$$

$$(2.2)\qquad \Lambda_a^{(r)}(f_1, \dots, f_r) = \prod_{0 \leqq j \leqq [(r-a)/2]} (f_{r-2j-a} + 1).$$

Further the sum of its coefficients, S_r, is given by

$$(2.3)\qquad S_r = \prod_{1 \leqq j \leqq r} \left(\sum_{1 \leqq k \leqq j} f_k + 1 \right).$$

In the proof of Theorem 3 we need a recursion formula proved earlier.

Lemma 2.1. *For $a, f_1, \dots, f_r \in \mathbb{Z}^+$, we have*

$$(2.4)\qquad \sigma_a\langle f_1, \dots, f_r\rangle_p = p^a \sigma_a\langle f_1, \dots, f_r - 1\rangle_p + \sigma_{a+1}\langle f_1, \dots, f_{r-1}\rangle_p.$$

If $a \geqq 0$, $f_k \geqq 1$ and $f_j = 0$ $(k < j \leqq r)$, then we have

$$(2.5)\qquad \sigma_a\langle f_1, \dots, f_k, 0_{r-k}\rangle_p = p^a \sigma_a\langle f_1, \dots, f_k - 1, 1, 0_{r-k-1}\rangle_p + \sigma_{a+1}\langle f_1, \dots, f_k, 0_{r-k-1}\rangle_p.$$

Proof. The first assertion is Theorem 2 of [1]. Applying this theorem we have

$$\sigma_a\langle f_1, \dots, f_k, 0_{r-k}\rangle_p = p^a \sigma_a\langle f_1, \dots, f_k, 0_{r-k-1}, -1\rangle_p + \sigma_{a+1}\langle f_1, \dots, f_k, 0_{r-k-1}\rangle_p$$

and the SNF of $\langle f_1, \dots, f_k, 0_{r-k-1}, -1\rangle$ is $\langle f_1, \dots, f_k - 1, 1, 0_{r-k-1}\rangle$. □

Now we are ready to prove Theorem 3.

We use induction on $v_r(f_1, \dots, f_r) = \sum_{j=1}^{r} jf_{r-j+1}$. If $v_r(f_1, \dots, f_r) = 1$, we have $f_r = 1$, $f_j = 0$ $(1 \leqq j < r)$. Thus $\sigma_a\langle 0_{r-1}, 1\rangle_p = p^a + 1$. It is easy to verify (2.1), (2.2) and (2.3).

Next we suppose $v_r(f_1, \dots, f_r) \geqq 2$ and discuss the two exhaustive possibilities.

I. *The case* $f_r \geqq 1$. By definition we have

$$\max\{v_r(f_1, \dots, f_r - 1), v_{r-1}(f_1, \dots, f_{r-1})\} < v_r(f_1, \dots, f_r).$$

From (2.4) and the recursion hypothesis we verify that $\sigma_a\langle f_1, \dots, f_r\rangle_p$ is indeed a polynomial in p with non-negative coefficients and

$$\theta_a^{(r)}(f_1, \dots, f_r) = \max\{\theta_a^{(r)}(f_1, \dots, f_r - 1) + a, \theta_{a+1}^{(r-1)}(f_1, \dots, f_{r-1})\}.$$

Applying again the recursion hypothesis, we can deduce

$$(2.6)\quad \theta_a^{(r)}(f_1, \dots, f_r - 1) + a = \sum_{1 \leqq j \leqq r-a} \alpha_{r-j+1+a} f_j + a \sum_{r+1-a \leqq j \leqq r} (r - j + 1) f_j,$$

$$\begin{aligned}(2.7)\quad \theta_{a+1}^{(r-1)}(f_1, \dots, f_{r-1}) &= \sum_{1 \leqq j \leqq r-2-a} \alpha_{r-j+1+a} f_j + (a+1) \sum_{r-1-a \leqq j \leqq r-1} (r-j) f_j \\ &= \sum_{1 \leqq j \leqq r-a} \alpha_{r-j+1+a} f_j + (a+1) \sum_{r+1-a \leqq j \leqq r-1} (r-j) f_j \\ &= \theta_a^{(r)}(f_1, \dots, f_r - 1) + a - \sum_{1 \leqq j \leqq a} j f_{r+j-a}.\end{aligned}$$

In the second equality of (2.7), we have used the relations $\alpha_{2a+2} = (a+1)^2$, $\alpha_{2a+1} = a(a+1)$. Thus the inequality $\theta_a^{(r)}(f_1, \dots, f_r - 1) + a \geqq \theta_{a+1}^{(r-1)}(f_1, \dots, f_{r-1})$ always holds. Consequently $\theta_a^{(r)}(f_1, \dots, f_r) = \theta_a^{(r)}(f_1, \dots, f_r - 1) + a$ and (2.6) show that the formula (2.1) holds.

In order to verify (2.2) we need to know when the relation

$$\theta_a^{(r)}(f_1, \dots, f_r - 1) + a = \theta_{a+1}^{(r-1)}(f_1, \dots, f_{r-1})$$

holds. From (2.7) this is only possible for $a = 0$. Hence we have

$$\Lambda_a^{(r)}(f_1, \dots, f_r) = \begin{cases} \Lambda_0^{(r)}(f_1, \dots, f_r - 1) + \Lambda_1^{(r-1)}(f_1, \dots, f_{r-1}) & \text{if } a = 0, \\ \Lambda_a^{(r)}(f_1, \dots, f_r - 1) & \text{if } a \geqq 1. \end{cases}$$

With the help of the recursion hypothesis it is a simple exercise to verify (2.2).

By (2.4) and the hypothesis, we have

$$S_r = \prod_{j=1}^{r-1}\left(\sum_{1\leqq k\leqq j} f_k + 1\right)(f_1 + \cdots + f_r) + \prod_{j=1}^{r-1}\left(\sum_{1\leqq k\leqq j} f_k + 1\right) = \prod_{j=1}^{r}\left(\sum_{1\leqq k\leqq j} f_k + 1\right).$$

This proves (2.3).

II. *The case* $f_k \geqq 1$, $f_j = 0$ $(k < j \leqq r)$. As before by using (2.5) and the recursion hypothesis we check that $\sigma_a\langle f_1, \dots, f_r\rangle_p$ is a polynomial in p with non-negative coefficients and that

$$\theta_a^{(r)}(f_1, \dots, f_k, 0_{r-k}) = \max\{\theta_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + a, \theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1})\}.$$

If $0 \leqq a \leqq r - k - 1$, again the use of the recursion hypothesis yields

$$\theta_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + a = \sum_{1\leqq j\leqq k} \alpha_{r-j+1+a} f_j - \alpha_{r-k+1+a} + \alpha_{r-k+a} + a,$$

$$\theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}) = \sum_{1\leqq j\leqq k} \alpha_{r-j+1+a} f_j.$$

Noticing that $-\alpha_{r-k+1+a} + \alpha_{r-k+a} + a = -[(r-k+a+1)/2] + a \leqq -1$, we have

$$\begin{cases} \theta_a^{(r)}(f_1, \dots, f_k, 0_{r-k}) = \theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}), \\ \Lambda_a^{(r)}(f_1, \dots, f_k, 0_{r-k}) = \Lambda_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}). \end{cases}$$

If $a = r - k$, by using the recursion hypothesis and the relations $\alpha_{2a+2} = (a+1)^2$, $\alpha_{2a+1} = a(a+1)$, we can deduce

$$\theta_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + a = \theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}) = \sum_{1\leqq j\leqq k} \alpha_{r-j+1+a} f_j.$$

Thus

$$\begin{cases} \theta_a^{(r)}(f_1, \dots, f_k, 0_{r-k}) = \theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}), \\ \Lambda_a^{(r)}(f_1, \dots, f_k, 0_{r-k}) = \Lambda_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + \Lambda_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}). \end{cases}$$

In view of the recursion hypothesis we can show that (2.1) and (2.2) are indeed true.

If $a \geqq r - k + 1$, then we have

$$\theta_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + a = \sum_{1\leqq j\leqq r-a} \alpha_{r-j+1+a} f_j + a \sum_{r+1-a\leqq j\leqq k} (r-j+1) f_j,$$

$$\begin{aligned} \theta_{a+1}^{(r-1)}(f_1, \dots, f_k, 0_{r-k-1}) &= \sum_{1\leqq j\leqq r-a} \alpha_{r-j+1+a} f_j + (a+1) \sum_{r+1-a\leqq j\leqq k} (r-j) f_j \\ &= \theta_a^{(r)}(f_1, \dots, f_k - 1, 1, 0_{r-k-1}) + a - \sum_{1\leqq j\leqq k+a-r} j f_{r+j-a}. \end{aligned}$$

Therefore $\theta_a^{(r)}(f_1,\dots,f_k-1,1,0_{r-k-1})+a \geqq \theta_{a+1}^{(r-1)}(f_1,\dots,f_k,0_{r-k-1})+1$. Thus we have

$$\begin{cases} \theta_a^{(r)}(f_1,\dots,f_k,0_{r-k}) = \theta_a^{(r)}(f_1,\dots,f_k-1,1,0_{r-k-1})+a, \\ \Lambda_a^{(r)}(f_1,\dots,f_k,0_{r-k}) = \Lambda_a^{r}(f_1,\dots,f_k-1,1,0_{r-k-1}). \end{cases}$$

From these we easily show that (2.1) and (2.2) are true.

Finally we verify (2.3). As before the sum of the coefficients of $\sigma_a\langle f_1,\dots,f_k,0_{r-k}\rangle_p$ is

$$\prod_{j=1}^{k-1}\Big(\sum_{1\leqq i\leqq j} f_i+1\Big)\Big(\sum_{1\leqq i\leqq k} f_i\Big)\Big(\sum_{1\leqq i\leqq k} f_i+1\Big)^{r-k} + \prod_{j=1}^{k-1}\Big(\sum_{1\leqq i\leqq j} f_i+1\Big)\Big(\sum_{1\leqq i\leqq k} f_i+1\Big)^{r-k}$$
$$= \prod_{j=1}^{k-1}\Big(\sum_{1\leqq i\leqq j} f_i+1\Big)\Big(\sum_{1\leqq i\leqq k} f_i+1\Big)^{r-k+1}.$$

This completes the proof. □

As a consequence we get the following information:

Corollary 2.2. *We have*

$$\limsup_{n\to\infty} \frac{\ell_\tau^{(r)}(n)}{n^{\alpha_r/r}} > 0.$$

Proof. We take $f_1=1$ and $f_2=\cdots=f_r=0$ in (2.1) to see this. The omega result (1.10) follows. For $r=2$, a more precise result was obtained in [5].

§3. Dirichlet series associated and proof of Theorem 1

From Theorem 3 we can write

$$(3.1) \qquad \tau\langle f_1,\dots,f_r\rangle_p = \Lambda(f_1,\dots,f_r)p^{\theta(f_1,\dots,f_r)} + R\langle f_1,\dots,f_r\rangle_p,$$

where

$$\Lambda(f_1,\dots,f_r) = \prod_{1\leqq j\leqq r}(\delta_j f_{r-j+1}+1), \quad \theta(f_1,\dots,f_r) = \sum_{1\leqq j\leqq r}\alpha_j f_{r-j+1}$$

and $R\langle f_1,\dots,f_r\rangle_p$ is a polynomial in p of degree at most $\theta(f_1,\dots,f_r)-1$ satisfying

$$(3.2) \qquad R\langle f_1,\dots,f_r\rangle_p \leqq (2r)^r(f_1^r+\cdots+f_r^r)p^{\theta(f_1,\dots,f_r)-1}.$$

The last relation is derivable from (2.3) knowing that the coefficients in question are positive and $f_1+\cdots+f_r \geqq 1$.

The following lemma describes the contribution of $\tau\langle f_1,\dots,f_r\rangle_p$ to the p-component of the series $\mathscr{D}^{(r)}(\tau,s)$. For simplicity we write $(f_1,\dots,f_r)_\nu$ for $f_1 \geqq 0,\dots,f_r \geqq 0$, $\nu_r(f_1,\dots,f_r) = \nu$ and put $\kappa_j(s) = js - \alpha_j$.

Lemma 3.1. *For $\sigma = \operatorname{Re} s > \alpha_r/r$, we have*

$$(3.3)\qquad \sum_{\nu=0}^{\infty} \sum_{(f_1,\dots,f_r)_\nu} \Lambda(f_1,\dots,f_r) p^{\theta(f_1,\dots,f_r)-\nu s} = \prod_{1\leqq j\leqq r} \left(1 - p^{-\kappa_j(s)}\right)^{-1-\delta_j},$$

$$(3.4)\qquad \sum_{\nu\geqq 1} \sum_{\substack{(f_1,\dots,f_r)_\nu \\ \theta(f_1,\dots,f_r)\geqq 1}} R\langle f_1,\dots,f_r\rangle_p p^{-\nu s} \leqq (2r)^r r! \sum_{1\leqq j\leqq r} p^{-\kappa_j(\sigma)-1}\Phi_j(p,\sigma),$$

where $\Phi_j(p,\sigma) = (1-p^{-\kappa_j(\sigma)})^{-r} \prod_{1\leqq k\leqq r} (1-p^{-\kappa_k(\sigma)})^{-1}$.

Proof. Obviously, for $\sigma > \alpha_r/r$, the member on the left-hand side of (3.3) is equal to

$$\prod_{1\leqq j\leqq r} \sum_{f_{r-j+1}=0}^{\infty} (\delta_j f_{r-j+1} + 1) p^{-\kappa_j(s) f_{r-j+1}}$$

and the last series equals $\delta_j(1-p^{-\kappa_j(s)})^{-2} + (1-\delta_j)(1-p^{-\kappa_j(s)})^{-1} = (1-p^{-\kappa_j(s)})^{-1-\delta_j}$. Since α_j/j is increasing, we get formula (3.3).

Using (3.2), we see that the member on the left-hand side of (3.4) is

$$(3.5)\qquad \begin{aligned} &\leqq (2r)^r \sum_{\nu\geqq 1} \sum_{(f_1,\dots,f_r)_\nu} \sum_{1\leqq j\leqq r} f_{r-j+1}^r p^{\theta(f_1,\dots,f_r)-\nu\sigma-1} \\ &\leqq (2r)^r \sum_{1\leqq j\leqq r} p^{-1} S_r^*(p^{-\kappa_j(\sigma)}) \prod_{1\leqq k\leqq r, k\neq j} (1-p^{-\kappa_k(\sigma)})^{-1}, \end{aligned}$$

where $S_r^*(y) = \sum_{n\geqq 1} n^r y^n$. We notice that

$$S_r^*(y) = y\sum_{n\geqq 0} (n+1)^r y^n \leqq y \sum_{n\geqq 0} (n+1)(n+2)\cdots(n+r) y^n = yr!/(1-y)^{r+1}.$$

Using the last inequality with $y = p^{-\kappa_j(\sigma)}$ in (3.5), we obtain (3.4) for $\sigma > \alpha_r/r$. This concludes the proof. $\square$

Now we are in a position to prove Theorem 1.

From (1.1), (1.2), (3.1) and (3.3), we can formally obtain the first assertion with

$$C_r(s) = \prod_p \left(1 + \prod_{1\leqq j\leqq r} (1-p^{-\kappa_j(s)})^{1+\delta_j} \sum_{\nu\geqq 1} \sum_{\substack{(f_1,\dots,f_r)_\nu \\ \theta(f_1,\dots,f_r)\geqq 1}} R\langle f_1,\dots,f_r\rangle_p p^{-\nu s} \right).$$

In view of (3.4) and the increase of α_j/j, the Dirichlet series $C_r(s)$ is absolutely convergent for $\operatorname{Re} s > \alpha_r/r$. This proves the first assertion. Taking $G_r(s) = C_r(s)$ for $r = 2, 3$, we obtain

(1.3) and (1.4). When $r \geqq 4$, we have (1.5) with $G_r(s) = C_r(s) \prod_{1 \leqq j < r} \zeta(js - \alpha_j)^{1+\delta_j}$. Noticing that $\alpha_r/r \geqq (\alpha_{r-1}+1)/(r-1) \geqq \cdots \geqq \alpha_1 + 1$, the series $G_r(s)$ is absolutely convergent for $\operatorname{Re} s > \alpha_r/r$.

Next we give a direct proof of (1.3) with $G_2(s) = \zeta(2s)^3 \prod_p (1 - 2p^{-3s} - p^{-4s} + 2p^{-5s})$. From Remark 1.4 of [9], we know that $\tau\langle f_1, f_2\rangle_p = \sum_{0 \leqq j \leqq f_1} (f_2 + 2j + 1) p^{f_1 - j}$. Thus we find, by interchanging the order of summation and by putting $f_1' = f_1 - j$,

$$
\begin{aligned}
(3.6) \qquad \sum_{\nu=0}^{\infty} \ell_\tau^{(2)}(p^\nu) p^{-\nu s} &= \sum_{f_1=0}^{\infty} \sum_{f_2=0}^{\infty} \sum_{j=0}^{f_1} (f_2 + 2j + 1) p^{f_1 - j - (2f_1 + f_2)s} \\
&= \sum_{j=0}^{\infty} \sum_{f_1=j}^{\infty} \sum_{f_2=0}^{\infty} (f_2 + 2j + 1) p^{f_1 - j - (2f_1 + f_2)s} \\
&= \sum_{j=0}^{\infty} \sum_{f_1'=0}^{\infty} \sum_{f_2=0}^{\infty} (f_2 + 2j + 1) p^{-2sj - (2s-1)f_1' - sf_2}.
\end{aligned}
$$

In addition we easily show that $\sum_{f_1'=0}^{\infty} p^{-(2s-1)f_1'} = (1 - p^{-(2s-1)})^{-1}$ and

$$
\begin{aligned}
\sum_{j=0}^{\infty} \sum_{f_2=0}^{\infty} (f_2 + 2j + 1) p^{-2sj - sf_2} &= \frac{2p^{-2s}}{(1 - p^{-s})(1 - p^{-2s})^2} + \frac{1}{(1 - p^{-s})^2 (1 - p^{-2s})} \\
&= \frac{1 - 2p^{-3s} - p^{-4s} + 2p^{-5s}}{(1 - p^{-s})^2 (1 - p^{-2s})^3}.
\end{aligned}
$$

Inserting these in (3.6) yields

$$
\sum_{\nu=0}^{\infty} \ell_\tau^{(2)}(p^\nu) p^{-\nu s} = \frac{1 - 2p^{-3s} - p^{-4s} + 2p^{-5s}}{(1 - p^{-s})^2 (1 - p^{-(2s-1)})(1 - p^{-2s})^3},
$$

completing the proof. □

§4. Weighted 5-dimensional divisor problems and proof of (1.7)

Before proving (1.7) we investigate weighted 5-dimensional divisor problems, which are of course of independent interest. For $\boldsymbol{a} = (a_1, \ldots, a_5) \in (\mathbb{R}^+)^5$, $\boldsymbol{b} = (b_1, \ldots, b_5) \in (\mathbb{Z}^+)^5$ with $1 \leqq b_1 \leqq \cdots \leqq b_5$, we define a weighted 5-dimensional divisor function

$$
\tau(\boldsymbol{b}, \boldsymbol{a}; n) = \sum_{n_1^{b_1} \cdots n_5^{b_5} = n} n_1^{a_1} \cdots n_5^{a_5}.
$$

Let $\pi(1, \ldots, 5)$ be the set of all permutations of $(1, \ldots, 5)$. The notation $\boldsymbol{k} \in \pi(1, \ldots, 5)$ means that $\boldsymbol{k} = (k_1, \ldots, k_5)$ runs over all permutations of $(1, \ldots, 5)$. Let $\Delta(\boldsymbol{b}, \boldsymbol{a}; x)$ be the error term in weighted 5-dimensional divisor problems

$$
D(\boldsymbol{b}, \boldsymbol{a}; x) = \sum_{n \leqq x} \tau(\boldsymbol{b}, \boldsymbol{a}; n) = \text{main term} + \Delta(\boldsymbol{b}, \boldsymbol{a}; x).
$$

We know (see (2), (3) and (4) of [11])

$$\Delta(\boldsymbol{b},\boldsymbol{a};x) = -\sum_{\boldsymbol{k}\in\pi(1,2,3,4,5)} \{\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a};x) + O(x^{\theta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a})})\},$$

where $\theta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a}) = \max_{1\leqq j\leqq 5}(a_{k_1}+\cdots+a_{k_j}+j-2)/(b_{k_1}+\cdots+b_{k_j})$ and

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a};x) = x^{a_{k_5}/b_{k_5}}\sum\nolimits_1\left(\prod_{1\leqq j\leqq 4} n_j^{a_{k_j}-a_{k_5}b_{k_j}/b_{k_5}}\right)\psi\big((x/n_1^{b_{k_1}}\cdots n_4^{b_{k_4}})^{1/b_{k_5}}\big),$$

$\psi(t) = t-[t]-\dfrac{1}{2}$ and the condition of summation of $\sum_1$ is given by

$$\mathrm{SC}(\textstyle\sum_1) \qquad n_1^{b_{k_1}}n_2^{b_{k_2}}n_3^{b_{k_3}}n_4^{b_{k_4}+b_{k_5}} \leqq x, \quad n_1(\leqq)\cdots(\leqq)n_4.$$

The notation $n_1(\leqq)n_2$ means that $n_1 = n_2$ for $b_{k_1} < b_{k_2}$, and $n_1 < n_2$ otherwise. As usual, we consider the truncated sum

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) = x^{a_{k_5}/b_{k_5}}\sum\nolimits_2\left(\prod_{1\leqq j\leqq 4} n_j^{a_{k_j}-a_{k_5}b_{k_j}/b_{k_5}}\right)\psi\big((x/n_1^{b_{k_1}}\cdots n_4^{b_{k_4}})^{1/b_{k_5}}\big)$$

where $\boldsymbol{N} = (N_1,N_2,N_3,N_4)\in\mathbb{N}^4$, and the condition of summation of $\sum_2$ is given by

$$\mathrm{SC}(\textstyle\sum_2) \qquad n_1^{b_{k_1}}n_2^{b_{k_2}}n_3^{b_{k_3}}n_4^{b_{k_4}+b_{k_5}} \leqq x, \quad n_1(\leqq)\cdots(\leqq)n_4,$$
$$N_j < n_j \leqq 2N_j \quad (1\leq j\leq 4).$$

Lemma 4.1. *With notations as above we have*

$$(4.1) \quad \Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \Xi(GN_1^2N_2^2N_3^2N_4^2)^{1/3}\mathscr{L}^2,$$

$$(4.2) \quad \Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \Xi\{(G^2N_1^8N_2^8N_3^5N_4^5)^{1/8} + (GN_1^3N_2^3N_3^2N_4)^{1/3}\}\mathscr{L}^2$$

with $G = \left(x/\prod_{1\leqq j\leqq 4} N_j^{b_{k_j}}\right)^{1/b_{k_5}}$, $\Xi = G^{a_{k_5}}\prod_{1\leqq j\leqq 4} N_j^{a_{k_j}}$ *and* $\mathscr{L} = \log x$.

Proof. By partial summation, we remove the smooth coefficients $n_j^{a_{k_j}-a_{k_5}b_{k_j}/b_{k_5}}$ to write

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \Xi\sum\nolimits_2\psi\big((x/n_1^{b_{k_1}}\cdots n_4^{b_{k_4}})^{1/b_{k_5}}\big).$$

Using (4.13) and (4.15) of Lemma 3.3 of [12], we obtain (4.1) and (4.2). □

Lemma 4.2. *There is a 4th degree polynomial Q_4 such that*

$$\sum_{n_1n_2n_3^2n_4^3n_5^3\leqq x} n_3n_4^2n_5^2 = x\,Q_4(\log x) + O(x^{14/17}\log^6 x).$$

Proof. We have $\boldsymbol{a} = (0,0,1,2,2)$, $\boldsymbol{b} = (1,1,2,3,3)$ and $\max_{\boldsymbol{k} \in \pi(1,\dots,5)} \theta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a}) \leqq \frac{4}{5}$. Thus it is sufficient to prove

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll x^{14/17}\mathscr{L}^2 \quad \text{for all } \boldsymbol{k} \in \pi(1,\dots,5).$$

We discuss all the possibilities. The first column of this table lists the possibilities for $(a_{k_1},\dots,a_{k_5})$, the second one the possibilities for $(b_{k_1},\dots,b_{k_5})$, the third and fourth ones give the values of G and Ξ that we take, while the fifth one gives the upper bound obtainable by using (4.1). In the five cases followed by a number, we have to use (4.2) as well, as is explained afterwards.

Now we consider the singularities:

1° If $(a_{k_1},a_{k_2},a_{k_3},a_{k_4},a_{k_5}) = (0,0,1,2,2)$, we have

$$(b_{k_1},b_{k_2},b_{k_3},b_{k_4},b_{k_5}) = (1,1,2,3,3),$$

$G = (x/N_1N_2N_3^2N_4^3)^{1/3}$, $\Xi = G^2N_3N_4^2$. Thus (4.1) implies

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll (x^7N_1^{-1}N_2^{-1}N_3N_4^3)^{1/9}\mathscr{L}^2 \ll (x^5/N_1N_2)^{1/6}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2$$

if $N_1N_2 \geqq x^{1/17}$. In the opposite case, (4.2) gives us

$$\begin{aligned}\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) &\ll \{(x^6N_1^2N_2^2N_3N_4^3)^{1/8} + (x^7N_1^2N_2^2N_3)^{1/9}\}\mathscr{L}^2\\ &\ll \{(x^{13}N_1^3N_2^3)^{1/16} + (x^{19}N_1^5N_2^5)^{1/24}\}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2.\end{aligned}$$

7° If $(a_{k_1},a_{k_2},a_{k_3},a_{k_4},a_{k_5}) = (1,0,0,2,2)$, we have

$$(b_{k_1},b_{k_2},b_{k_3},b_{k_4},b_{k_5}) = (2,1,1,3,3),$$

$G = (x/N_1^2N_2N_3N_4^3)^{1/3}$, $\Xi = G^2N_1N_4^2$. Thus (4.1) yields

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll (x^7N_1N_2^{-1}N_3^{-1}N_4^3)^{1/9}\mathscr{L}^2 \ll (x^5/N_1N_2)^{1/6}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2$$

if $N_1N_2 \geqq x^{1/17}$. In the other case, from (4.2) we get

$$\begin{aligned}\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) &\ll \{(x^6N_1^4N_2^2N_3^{-1}N_4^3)^{1/8} + (x^7N_1^4N_2^2N_3^{-1})^{1/9}\}\mathscr{L}^2\\ &\ll \{(x^{13}N_1^3N_2^3)^{1/16} + (x^{14}N_1^5N_2^5)^{1/18}\}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2.\end{aligned}$$

13° If $(a_{k_1},a_{k_2},a_{k_3},a_{k_4},a_{k_5}) = (0,1,0,2,2)$, we have

$$(b_{k_1},b_{k_2},b_{k_3},b_{k_4},b_{k_5}) = (1,2,1,3,3),$$

$G = (x/N_1N_2^2N_3N_4^3)^{1/3}$, $\Xi = G^2N_2N_4^2$. Thus, from (4.1), we deduce

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll (x^7N_1^{-1}N_2N_3^{-1}N_4^3)^{1/9}\mathscr{L}^2 \ll (x^5/N_1N_2)^{1/6}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2$$

$N°$	$(a_{k_i})_{1\leqq i\leqq 5}$	$(b_{k_i})_{1\leqq i\leqq 5}$	G	Ξ	upper bound
1°	$(0,0,1,2,2)$	$(1,1,2,3,3)$	$(x/N_1N_2N_3^2N_4^3)^{1/3}$	$G^2N_3N_4^2$	$x^{14/17}\mathscr{L}^2$
	$(0,2,1,0,2)$	$(1,3,2,1,3)$	$(x/N_1N_2^3N_3^2N_4)^{1/3}$	$G^2N_2^2N_3$	$x^{4/5}\mathscr{L}^2$
	$(2,0,1,0,2)$	$(3,1,2,1,3)$	$(x/N_1^3N_2N_3^2N_4)^{1/3}$	$G^2N_1^2N_3$	$x^{4/5}\mathscr{L}^2$
	$(0,2,1,2,0)$	$(1,3,2,3,1)$	$x/N_1N_2^3N_3^2N_4^3$	$N_2^2N_3N_4^2$	$x^{22/27}\mathscr{L}^2$
	$(2,0,1,2,0)$	$(3,1,2,3,1)$	$x/N_1^3N_2N_3^2N_4^3$	$N_1^2N_3N_4^2$	$x^{4/5}\mathscr{L}^2$
	$(2,2,1,0,0)$	$(3,3,2,1,1)$	$x/N_1^3N_2^3N_3^2N_4$	$N_1^2N_2^2N_3$	$x^{4/5}\mathscr{L}^2$
7°	$(1,0,0,2,2)$	$(2,1,1,3,3)$	$(x/N_1^2N_2N_3N_4^3)^{1/3}$	$G^2N_1N_4^2$	$x^{14/17}\mathscr{L}^2$
	$(1,0,2,0,2)$	$(2,1,3,1,3)$	$(x/N_1^2N_2N_3^3N_4)^{1/3}$	$G^2N_1N_3^2$	$x^{17/21}\mathscr{L}^2$
	$(1,2,0,0,2)$	$(2,3,1,1,3)$	$(x/N_1^2N_2^3N_3N_4)^{1/3}$	$G^2N_1N_2^2$	$x^{4/5}\mathscr{L}^2$
	$(1,0,2,2,0)$	$(2,1,3,3,1)$	$x/N_1^2N_2N_3^3N_4^3$	$N_1N_3^2N_4^2$	$x^{17/21}\mathscr{L}^2$
	$(1,2,0,2,0)$	$(2,3,1,3,1)$	$x/N_1^2N_2^3N_3N_4^3$	$N_1N_2^2N_4^2$	$x^{4/5}\mathscr{L}^2$
	$(1,2,2,0,0)$	$(2,3,3,1,1)$	$x/N_1^2N_2^3N_3^3N_4$	$N_1N_2^2N_3^2$	$x^{4/5}\mathscr{L}^2$
13°	$(0,1,0,2,2)$	$(1,2,1,3,3)$	$(x/N_1N_2^2N_3N_4^3)^{1/3}$	$G^2N_2N_4^2$	$x^{14/17}\mathscr{L}^2$
	$(0,1,2,0,2)$	$(1,2,3,1,3)$	$(x/N_1N_2^2N_3^3N_4)^{1/3}$	$G^2N_2N_3^2$	$x^{22/27}\mathscr{L}^2$
	$(2,1,0,0,2)$	$(3,2,1,1,3)$	$(x/N_1^3N_2^2N_3N_4)^{1/3}$	$G^2N_1^2N_2$	$x^{4/5}\mathscr{L}^2$
	$(0,1,2,2,0)$	$(1,2,3,3,1)$	$x/N_1N_2^2N_3^3N_4^3$	$N_2N_3^2N_4^2$	$x^{17/21}\mathscr{L}^2$
	$(2,1,0,2,0)$	$(3,2,1,3,1)$	$x/N_1^3N_2^2N_3N_4^3$	$N_1^2N_2N_4^2$	$x^{4/5}\mathscr{L}^2$
	$(2,1,2,0,0)$	$(3,2,3,1,1)$	$x/N_1^3N_2^2N_3^3N_4$	$N_1^2N_2N_3^2$	$x^{4/5}\mathscr{L}^2$
19°	$(0,0,2,1,2)$	$(1,1,3,2,3)$	$(x/N_1N_2N_3^3N_4^2)^{1/3}$	$G^2N_3^2N_4$	$x^{14/17}\mathscr{L}^2$
	$(0,2,0,1,2)$	$(1,3,1,2,3)$	$(x/N_1N_2^3N_3N_4^2)^{1/3}$	$G^2N_2^2N_4$	$x^{22/27}\mathscr{L}^2$
	$(2,0,0,1,2)$	$(3,1,1,2,3)$	$(x/N_1^3N_2N_3N_4^2)^{1/3}$	$G^2N_1^2N_4$	$x^{4/5}\mathscr{L}^2$
	$(0,2,2,1,0)$	$(1,3,3,2,1)$	$x/N_1N_2^3N_3^3N_4^2$	$N_2^2N_3^2N_4$	$x^{22/27}\mathscr{L}^2$
	$(2,0,2,1,0)$	$(3,1,3,2,1)$	$x/N_1^3N_2N_3^3N_4^2$	$N_1^2N_3^2N_4$	$x^{4/5}\mathscr{L}^2$
	$(2,2,0,1,0)$	$(3,3,1,2,1)$	$x/N_1^3N_2^3N_3N_4^2$	$N_1^2N_2^2N_4$	$x^{4/5}\mathscr{L}^2$
25°	$(0,0,2,2,1)$	$(1,1,3,3,2)$	$(x/N_1N_2N_3^3N_4^3)^{1/2}$	$GN_3^2N_4^2$	$x^{14/17}\mathscr{L}^2$
	$(0,2,0,2,1)$	$(1,3,1,3,2)$	$(x/N_1N_2^3N_3N_4^3)^{1/2}$	$GN_2^2N_4^2$	$x^{22/27}\mathscr{L}^2$
	$(2,0,0,2,1)$	$(3,1,1,3,2)$	$(x/N_1^3N_2N_3N_4^3)^{1/2}$	$GN_1^2N_4^2$	$x^{4/5}\mathscr{L}^2$
	$(0,2,2,0,1)$	$(1,3,3,1,2)$	$(x/N_1N_2^3N_3^3N_4)^{1/2}$	$GN_2^2N_3^2$	$x^{22/27}\mathscr{L}^2$
	$(2,0,2,0,1)$	$(3,1,3,1,2)$	$(x/N_1^3N_2N_3^3N_4)^{1/2}$	$GN_1^2N_3^2$	$x^{4/5}\mathscr{L}^2$
	$(2,2,0,0,1)$	$(3,3,1,1,2)$	$(x/N_1^3N_2^3N_3N_4)^{1/2}$	$GN_1^2N_2^2$	$x^{4/5}\mathscr{L}^2$

if $N_1N_2 \geqq x^{1/17}$. Else (4.2) gives us

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \{(x^6N_1^2N_2^4N_3^{-1}N_4^3)^{1/8} + (x^7N_1^2N_2^4N_3^{-1})^{1/9}\}\mathscr{L}^2$$
$$\ll \{(x^{13}N_1^3N_2^3)^{1/16} + (x^{44}N_1^5N_2^5)^{1/54}\}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2.$$

19° If $(a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}, a_{k_5}) = (0,0,2,1,2)$, we have

$$(b_{k_1}, b_{k_2}, b_{k_3}, b_{k_4}, b_{k_5}) = (1,1,3,2,3),$$

$G = (x/N_1N_2N_3^3N_4^2)^{1/3}$, $\Xi = G^2N_3^2N_4$. Thus (4.1) implies

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll (x^7N_1^{-1}N_2^{-1}N_3^3N_4)^{1/9}\mathscr{L}^2 \ll (x^5/N_1N_2)^{1/6}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2$$

if $N_1N_2 \geqq x^{1/17}$. In the opposite case, (4.2) gives us

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \{(x^6N_1^2N_2^2N_3^3N_4)^{1/8} + (x^7N_1^2N_2^2N_3^3N_4^{-2})^{1/9}\}\mathscr{L}^2$$
$$\ll \{(x^{13}N_1^3N_2^3)^{1/16} + (x^{19}N_1^5N_2^5)^{1/24}\}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2.$$

25° If $(a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}, a_{k_5}) = (0,0,2,2,1)$, we have

$$(b_{k_1}, b_{k_2}, b_{k_3}, b_{k_4}, b_{k_5}) = (1,1,3,3,2),$$

$G = (x/N_1N_2N_3^3N_4^3)^{1/2}$, $\Xi = GN_3^2N_4^2$. Thus from (4.1) we find

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll (x^2N_3^2N_4^2)^{1/3}\mathscr{L}^2 \ll (x^5/N_1N_2)^{1/6}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2$$

if $N_1N_2 \geqq x^{1/17}$. When not (4.2) gives us

$$\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x) \ll \{(x^5N_1^3N_2^3N_3^6N_4^6)^{1/8} + (x^2N_1N_2N_3^2N_4)^{1/3}\}\mathscr{L}^2$$
$$\ll \{(x^{13}N_1^3N_2^3)^{1/16} + (x^{19}N_1^5N_2^5)^{1/24}\}\mathscr{L}^2 \ll x^{14/17}\mathscr{L}^2.$$

This completes the proof. □

Now we are ready to prove (1.7). From (1.4), we can write

$$\ell_\tau^{(3)}(n) = \{\tau(\boldsymbol{b},\boldsymbol{a};\cdot) * g_3\}(n)$$

with $\boldsymbol{a} = (0,0,1,2,2)$ and $\boldsymbol{b} = (1,1,2,3,3)$. Noticing that $\frac{14}{17} > \frac{2}{3}$, Lemma 4.2 implies, by a simple convolution argument, the asymptotic formula (1.7). □

Remark. Here we use only two simple estimates for $\Delta(\boldsymbol{k},\boldsymbol{b},\boldsymbol{a},\boldsymbol{N};x)$, i.e. (4.1) and (4.2). Combining carefully results of [10] and [12] it is possible to obtain a better exponent (for example, $\frac{9}{11} = 0.\overline{81}$) than $\frac{14}{17} \approx 0.8235$ in Theorem 2 and Lemma 4.2. But the numerical verifications would be very complicated.

§5. The case $r \geqq 4$ and proofs of (1.8) and (1.9)

When $r \geqq 4$, the singularity of $\mathscr{D}^{(r)}(\tau, s)$ is much simpler than for $r = 2$ or 3. Therefore these cases are easier to treat. Because of the omega result (1.10) we know that even if we could move the abscissa of absolute convergence of $G_r(s)$ further (1.8) would still be the best. *There is perhaps still some room for improving* (1.9).

For $r = 2k$ $(k \geqq 2)$, the relation (1.5a) implies $\ell_\tau^{(r)}(n) = \sum_{n_1^r n_2 = n} n_1^{\alpha_r} g_r(n_2)$, where $\sum_{n=1}^{\infty} g_r(n) n^{-s}$ is absolutely convergent for $\operatorname{Re} s > \alpha_r/r$. Thus

$$(5.1)\qquad \sum_{n \leqq x} \ell_\tau^{(r)}(n) = \sum_{n_2 \leqq x} g_r(n_2)\{(x/n_2)^{(\alpha_r+1)/r}/(\alpha_r+1) + O\big((x/n_2)^{\alpha_r/r}\big)\}.$$

Obviously the contribution of $O\big((x/n_2)^{\alpha_r/r}\big)$ is $O(x^{\alpha_r/r+\varepsilon})$ $(\forall \varepsilon > 0)$. In addition we have

$$\sum_{n_2 \leqq x} g_r(n_2)/n_2^{(\alpha_r+1)/r} = \sum_{n_2=1}^{\infty} g_r(n_2)/n_2^{(\alpha_r+1)/r} + O(x^{-((\alpha_r+1)/r-\alpha_r-\varepsilon)}) \quad (\forall \varepsilon > 0).$$

Combining these with (5.1) yields (1.8).

When $r = 2k+1$ $(k \geqq 2)$, by (1.5b) of Theorem 1 we have $\ell_\tau^{(r)}(n) = \{\tau(\boldsymbol{a}, \boldsymbol{b}; \cdot) * g_r\}(n)$ with $\boldsymbol{a} = (\alpha_r, \alpha_r)$, $\boldsymbol{b} = (r, r)$, and the Dirichlet series $\sum_{n=1}^{\infty} g_r(n) n^{-s}$ is absolutely convergent for $\operatorname{Re} s > \alpha_r/r$. Since $(\alpha_r + \mu)/r > \alpha_r/r$ for $\mu = \frac{23}{73}$, the following lemma implies (1.9) by a simple convolution argument.

Lemma 5.1. *Let $r = 2k+1$ $(k \geqq 2)$, $\boldsymbol{a} = (\alpha_r, \alpha_r)$ and $\boldsymbol{b} = (r, r)$. Then we have*

$$(5.2)\qquad \sum_{n \leqq x} \tau(\boldsymbol{a}, \boldsymbol{b}; n) = \frac{x^{(\alpha_r+1)/r}\log x}{r(\alpha_r+1)} + \frac{2\gamma\alpha_r + 2\gamma - 1}{(\alpha_r+1)^2} x^{(\alpha_r+1)/r} + O(x^{(\alpha_r+\mu)/r}\log^\beta x),$$

where γ is the Euler constant and $\mu = \frac{23}{73}$, $\beta = \frac{461}{146}$.

Proof. Let $\tau(n)$ be the usual divisor function and let $D(t) = \sum_{n \leqq t} \tau(n)$. Then the member on the left-hand side of (5.2) equals

$$\sum_{n \leqq x^{1/r}} n^{\alpha_r}\tau(n) = \int_{1-}^{x^{1/r}} t^{\alpha_r}\,\mathrm{d}D(t) = x^{\alpha_r/r}\sum_{n \leqq x^{1/r}} \tau(n) - \alpha_r \int_1^{x^{1/r}} D(t) t^{\alpha_r - 1}\,\mathrm{d}t.$$

Using Huxley's estimate ([6], Corollary)

$$\sum_{n \leqq x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^{23/73}\log^{461/146} x),$$

a simple calculation gives the required result. $\square$

References

[1] *G. Bhowmik*, Divisor functions of integer matrices: evaluations, average orders and applications, Astérisque **209** (1992), 169–177.

[2] *G. Bhowmik*, Evaluation of divisor functions of matrices, Acta Arith. **74** (1996), 155–159.

[3] *G. Bhowmik* and *O. Ramaré*, Average orders of multiplicative arithmetical functions of integer matrices, Acta Arith. **66** (1994), 45–62.

[4] *G. Bhowmik* and *O. Ramaré*, Algebra of Matrix Arithmetic, J. Alg. **210** (1998), 194–215.

[5] *G. Bhowmik* and *J. Wu*, On the asymptotic behaviour of the number of subgroups of finite abelian groups, Arch. Math. **69** (1997), 95–104.

[6] *M. N. Huxley*, Exponential sums and lattice points II, Proc. London Math. Soc. **66** (1993), 279–301.

[7] *A. Ivić*, On the number of subgroups of finite abelian groups, J. Th. Nombres Bordeaux **9** (1997), no. 2, 371–381.

[8] *H. Menzer*, On the number of subgroups of finite abelian groups, in: Proc. Conf. Analytic and Elementary Number Theory (eds. W. G. Nowak and J. Schoißengeier), Universität Wien & Universität für Bodenkultur, Wien (1996), 181–188.

[9] *V. C. Nanda*, Arithmetic functions of matrices and polynomial identities, Colloq. Math. Soc. János Bolyai **34**, North-Holland (1984), 1107–1126.

[10] *P. Sargos* and *J. Wu*, Multiple exponential sums with monomials and their applications in number theory, Acta Math. Hungar. **87** (4) (2000), 333–354.

[11] *M. Vogts*, Many-dimensional generalized problems, Math. Nachr. **124** (1985), 103–121.

[12] *J. Wu*, On the distribution of square-full and cube-full integers, Mh. Math. **126** (1998), 353–367.

Départment de Mathématiques, Unité associée au CNRS, UMR 8524, Université Lille 1,
59655 Villeneuve d'Ascq Cédex, France
e-mail: bhowmik@agat.univ-lillel.fr

Laboratoire de Mathématiques, Institut Elie Cartan-UMR 7502 UHP-CNRS,
Université Henri Poincaré (Nancy 1), 54506 Vandœuvre-lès-Nancy, France
e-mail: wujie@iecn.u-nancy.fr

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