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On the asymptotic behaviour of the number of subgroups of finite abelian groups

By

Gautami Bhowmik and Jie Wu

Abstract. Let $\mathscr{G} \cong \mathbb{Z}/n_1\mathbb{Z} \otimes \ldots \otimes \mathbb{Z}/n_r\mathbb{Z}$ be a finite abelian group of rank r, where $n_j | n_{j+1}$ for $j = 1, \ldots, r-1$. Let $\tau(\mathscr{G})$ be the number of subgroups of $\mathscr{G}$, $|\mathscr{G}|$ the order of $\mathscr{G}$ and $r(\mathscr{G})$ the rank of $\mathscr{G}$. In this paper we investigate carefully the asymptotic behaviour of the level function $\ell_\tau^{(r)}(n) := \sum_{|\mathscr{G}|=n,\, r(\mathscr{G}) \leqq r} \tau(\mathscr{G})$ for $r = 2$. In particular we prove that

$$\sum_{n \leqq x} \ell_\tau^{(2)}(n) = A_1 x (\log x)^2 + A_2 x \log x + A_3 x + \Delta(x),$$

where A_i-s are constants, $\Delta(x) \ll x^{5/8} (\log x)^4$ and $\Delta(x) = \Omega_-(x^{1/2} (\log x)^2)$.

1. Introduction. The asymptotic behaviour of the number of subgroups of certain types of groups, like torsion free nilpotent groups, is currently being investigated (see [7], [14]). For abelian groups, the average number of their formal direct factors and their formal unitary factors are known (see [5], [11]). The asymptotic behaviour of the number of subgroups of abelian groups, however, has received little attention. Let $\mathscr{G}$ be a finite abelian group of rank r, i.e.

$$\mathscr{G} \cong \mathbb{Z}/n_1\mathbb{Z} \otimes \ldots \otimes \mathbb{Z}/n_r\mathbb{Z},$$

where $n_j | n_{j+1}$ for $j = 1, \ldots, r-1$. We write $\tau(\mathscr{G})$ for the number of subgroups of $\mathscr{G}$, $|\mathscr{G}| = n_1 \cdots n_r$ for the order of $\mathscr{G}$ and $r(\mathscr{G})$ for the rank of $\mathscr{G}$. We introduce the *level function* $\ell_\tau^{(r)}(n)$ of $\tau(\mathscr{G})$, defined by

$$\ell_\tau^{(r)}(n) := \sum_{|\mathscr{G}|=n,\, r(\mathscr{G}) \leqq r} \tau(\mathscr{G}),$$

which is the number of subgroups of finite abelian groups $\mathscr{G}$ satisfying $r(\mathscr{G}) \leqq r$ and $|\mathscr{G}| = n$. We are interested in the asymptotic behaviour of this function.

It has recently been proved that there is a natural bijection between the set of subgroups of a finite abelian group $\mathscr{G}$ of rank r and the set of divisors of an $r \times r$ Smith Normal Form (SNF) matrix $S := \mathrm{diag}[n_1, \ldots, n_r]$ (see Corollary 1 of [3]). Let $\tau(S)$ be the number of (inequivalent) factorizations of S, we have $\tau(S) = \tau(\mathscr{G})$. We define the Dirichlet series associated to $\tau(S)$ and to $\tau(\mathscr{G})$ as

$$D_S^{(r)}(\tau, s) := \sum_{S \text{ in SNF}} \frac{\tau(S)}{(n_1 \cdots n_r)^s}, \quad \mathscr{D}^{(r)}(\tau, s) := \sum_{r(\mathscr{G}) \leqq r} \frac{\tau(\mathscr{G})}{|\mathscr{G}|^s}.$$

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Obviously these two series are equal. For $r \geqq 3$, a usable Dirichlet series is still under investigation and henceforth the parameter r, when suppressed, will be understood to be 2. We know, from Corollary 1 of [2], that

$$\mathscr{D}(\tau, s) = \zeta(s)^2\zeta(2s-1)G(s), \tag{1.1}$$

where $\zeta(s)$ is the Riemann zeta-function and $G(s) := \zeta(2s)^3 \prod_p (1 - 2p^{-3s} - p^{-4s} + 2p^{-5s})$ is a Dirichlet series absolutely convergent $\Re e\, s > \frac{1}{2}$. Using (1.1), Bhowmik and Menzer [1] have proved that

$$\sum_{n \leqq x} \ell_\tau(n) = A_1 x(\log x)^2 + A_2 x \log x + A_3 x + \varDelta(x), \tag{1.2}$$

where A_i-s are computable constants and $\varDelta(x) \ll_\varepsilon x^{31/43+\varepsilon}$ $(\forall \varepsilon > 0)$.

In this paper, we first study the error term $\varDelta(x)$ more carefully. We notice that by an application of Dirichlet's hyperbola principle, the result of Huxley ([8], Corollary) concerning the divisor problem of Dirichlet implies $\varDelta(x) \ll x^{123/173}(\log x)^{607/146}$, which improves the result in [1]. Here we shall use another method to treat $\varDelta(x)$. We require a result for a weighted three dimensional divisor problem, for which effective results do not seem to be available. We use multiple exponential sum techniques to prove the following result.

Theorem 1. *We have*

$$\sum_{n_1 n_2 n_3^2 \leqq x} n_3 = B_1 x(\log x)^2 + B_2 x \log x + B_3 x + O\big(x^{5/8}(\log x)^4\big), \tag{1.3}$$

where B_j-s are some computable constants.

With the aid of (1.3), we are able to obtain, by a convolution argument, a sharper estimate.

Theorem 2. *We have $\varDelta(x) \ll x^{5/8}(\log x)^4$.*

For comparison, we have $\frac{31}{43} \approx 0.7209$, $\frac{123}{173} \approx 0.7109$, $\frac{5}{8} = 0.625$.

By considering the contribution of the second pole $s = \frac{1}{2}$ of $\mathscr{D}(\tau, s)x^s/s$, we could expect to get the following "asymptotic" formula

$$\varDelta(x) = A_4 x^{1/2}(\log x)^2 + A_5 x^{1/2} \log x + A_6 x^{1/2} + E(x),$$

where the first three terms are the residue of $\mathscr{D}(\tau, s)x^s/s$ at $s = \frac{1}{2}$ and $E(x)$ (defined by the preceding relation) is an "error term". In this direction, we can give a non trivial upper bound for the average order of $E(x)$, i.e.

Theorem 3. *For any $\varepsilon > 0$, we have*

$$\frac{1}{x}\int_0^x E(u)du \ll_\varepsilon x^{3/8+\varepsilon}.$$

As a consequence of Theorem 3, we state the following Ω-type estimate for $\varDelta(x)$.

Corollary 1. *We have $\varDelta(x) = \Omega_-\big(x^{1/2}(\log x)^2\big)$.*

For a smooth point-wise bound, we have the following result.

Theorem 4. (i) *A maximal order for the function* $\ell_\tau(n)$ *is* $C_0\sqrt{n}(\log_2 n)^6$, *i.e.*

$$\limsup_{n\to\infty} \frac{\ell_\tau(n)}{\sqrt{n}(\log_2 n)^6} = C_0,$$

where $C_0 := e^{6\gamma}\prod_p (1-1/p)^6\big(1+\sum_{j=1}^{\infty}(2j+1)(j+1)/p^j\big)$ *and* γ *is the Euler constant.*

(ii) *A minimal order for* $\ell_\tau(n)$ *is* 2.

We can establish the following Ω-type result for $E(x)$.

Corollary 2. *For infinitely many* x, *we have* $|E(x+1)-E(x)| \gg \sqrt{x}(\log_2 x)^6$. *In particular, we have* $E(x) = \Omega\big(x^{1/2}(\log_2 x)^6\big)$.

A normal order for $\log \ell_\tau^{(r)}(n)$ has been shown to be $(\log 2)\log_2 n$ in [4]. As usual, let $\omega(n)$ be the number of distinct prime factors of the integer n. The following result shows that the sum $\sum_{n\leqq x} \ell_\tau(n)$ is dominated by a small number of "abnormal" integers n, for which $\ell_\tau(n)$ is large.

Theorem 5. *For any* $\varepsilon \in (0,1)$, *we have*

$$\sum_{\substack{n\leqq x\\ |\omega(n)-3\log_2 x|>\varepsilon\log_2 x}} \ell_\tau(n) \ll x(\log x)^{2-3\eta} \tag{1.6}$$

with

$$\eta := \min\left\{\int_1^{1+\frac{1}{3}\varepsilon} \log t\,dt,\ \int_{1-\frac{1}{3}\varepsilon}^{1} \log(1/t)\,dt\right\} > 0.$$

In particular, the mean value $(1/x)\sum_{n\leqq x}\ell_\tau(n)$ *is given by the natural numbers* n *such that* $\omega(n) = 3\log_2 x + O\big(\xi(x)\sqrt{\log_2 x}\big)$, *where* $\xi(x) = o\big(\sqrt{\log_2 x}\big)$ *as* $x\to\infty$.

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2. Estimate for exponential sums. In this section, we give an estimate for exponential sums, which is essentially a general form of the first estimate in Lemma 6 of Kolesnik [10].

Lemma 2.1. *Let* $\alpha_1,\alpha_2\in\mathbb{R}$ *with* $(\alpha_2-2)(\alpha_1+2\alpha_2-2)\prod_{1\leqq j\leqq 2}\alpha_j(\alpha_j-1)(\alpha_1+\alpha_2-j) \neq 0$, $X>0$, $M_1, M_2 \geqq 1$, $\mathcal{L} := \log(XM_1M_2+2)$. *Suppose* $\mathbf{D} \subseteqq (M_1, 2M_1]\times(M_2,2M_2]$ *is a domain such that conditions* (Ω_2) *and* (Ω_3) *) *are satisfied for* $f(m_1,m_2) := (X/M_1^{\alpha_1}M_2^{\alpha_2})m_1^{\alpha_1}m_2^{\alpha_2}$ *on* $\mathbf{D}$. *Let*

$$S := \sum\sum_{(m_1,m_2)\in\mathbf{D}} e\big(f(m_1,m_2)\big).$$

Then we have

$$\begin{aligned} S \ll \big\{&(X^2M_1^3M_2^3)^{1/6} + (X^2M_1^5M_2)^{1/6} + M_1M_2^{1/2} + (X^3M_1^7)^{1/8}\\ &+ (XM_1)^{1/2} + (X^2M_1^8M_2)^{1/8} + X^{-1/2}M_2 + X^{-1}M_1M_2\big\}\mathcal{L}.\end{aligned}$$

*) For definitions of (Ω_2) and (Ω_3), see page 79–80 of [6].

For proving this lemma, we need a well known result of Kolesnik [10] in monomial case. Here we take essentially the form of Theorem 6.12 of [6].

Lemma 2.2. *Let* $\alpha_1, \alpha_2 \in \mathbb{R}$ *with* $\prod_{1 \leqq j \leqq 2} \alpha_j(\alpha_j - 1)(\alpha_j - 2)(\alpha_1 + \alpha_2 - j) \neq 0$, $M_1 \geqq M_2 \geqq 1$, $X > 0$, $\mathscr{L} := \log(XM_1M_2 + 2)$. *Let* $\mathbf{D}$, $f(m_1, m_2)$, S *be defined as in Lemma 2.1. Then we have*

$$S \ll \{(X^2M_1^3M_2^3)^{1/6} + (M_1M_2)^{5/6} + M_1M_2^{1/2} + (X^{-1}M_1^7M_2^8)^{1/8} + X^{-1/4}M_1M_2\}\mathscr{L}^{1/2}.$$

Proof. By (6.4.5) of [6], we have

$$\begin{aligned} S \ll \{&(X^2M_1^3M_2^3)^{1/6} + (M_1M_2)^{5/6} + (X^2M_1^{13}M_2^{25})^{1/26} + (M_1^5M_2^8)^{1/8} \\ &+ (X^2M_1^7M_2^{11})^{1/14} + (M_1^2M_2)^{1/2} + (X^{-1}M_1^7M_2^8)^{1/8} + (X^{-1}M_1^4M_2^4)^{1/4}\}\mathscr{L}^{1/2} \\ =:&\ (E_1 + \cdots + E_8)\mathscr{L}^{1/2}. \end{aligned}$$

Since $M_1 \geqq M_2$, it is easy to see that $E_4 \leqq E_2$ and

$$E_3 = (E_1^{63}E_7^{80})^{1/143}(M_1^{15}/M_2^{13})^{-2/143}, \quad E_5 = (E_1^{45}E_7^{32})^{1/77}(M_1^2/M_2)^{-6/77}.$$

Thus E_3, E_4, E_5 are superfluous. This proves Lemma 2.2. □

Proof of Lemma 2.1. If $M_2' := X/M_2 \leqq \frac{1}{2}$, Kusmin-Landau's inequality (Theorem 2.1 of [6]) implies the desired estimate. Next we suppose $M_2' \geqq \frac{1}{2}$. Applying Lemma 2.2 of [13] to m_2, estimating the contribution of error term by Lemma 2.3 of [13] with $n = m_1$ and removing smooth coefficients by tartial summation, we easily find that

$$S \ll X^{-1/2}M_2T + (X^{1/2} + M_1 + X^{-1/2}M_2 + X^{-1}M_1M_2)\mathscr{L}, \tag{2.1}$$

where

$$T := \sum\sum_{(m_1, m_2') \in \mathbf{D}'} e\big(f_1(m_1, m_2)\big), \quad f_1(m_1, m_2) := \tilde{\alpha}_2(X/M_1^{\beta_1}M_2'^{\beta_2})m_1^{\beta_1}m_2'^{\beta_2}$$

and $\beta_1 := \alpha_1/(1 - \alpha_2)$, $\beta_2 := -\alpha_2/(1 - \alpha_2)$, $\tilde{\alpha}_2 := |1 - \alpha_2||\alpha_2|^{-\beta_2}$, and $\mathbf{D}'$ is a suitable subregion of $(M_1, 2M_1] \times (M_2', 2M_2']$.

If $M_1 \geqq M_2'$, we can use Lemma 2.2 with $(X, M_1, M_2) = (\tilde{\alpha}_2X, M_1, M_2')$ to estimate T. This yields that

$$\begin{aligned} X^{-1/2}M_2T \ll &\{(X^2M_1^3M_2^3)^{1/6} + (X^2M_1^5M_2)^{1/6} \\ &+ M_1M_2^{1/2} + (X^3M_1^7)^{1/8} + X^{1/4}M_1\}\mathscr{L}. \end{aligned} \tag{2.2}$$

When $M_1 \leqq M_2'$, we use Lemma 2.2 with $(X, M_1, M_2) = (\tilde{\alpha}_2X, M_2', M_1)$ to write

$$\begin{aligned} X^{-1/2}M_2T \ll &\{(X^2M_1^3M_2^3)^{1/6} + (X^2M_1^5M_2)^{1/6} \\ &+ (XM_1)^{1/2} + (X^2M_1^8M_2)^{1/8} + X^{1/4}M_1\}\mathscr{L}. \end{aligned} \tag{2.3}$$

Inserting (2.2) and (2.3) in (2.1), we obtain the required result. □

3. Weighted 3-dimensional divisor problem and proof of Theorem 1. The aim of this section is to prove Theorem 1, which, as well as being of independent interest, is a key step in the proof of Theorem 2.

We first introduce some notations. Let $\mathbf{a} := (a_1, a_2, a_3) \in \mathbb{R}^3$, $\mathbf{b} := (b_1, b_2, b_3) \in (\mathbb{R}^+)^3$ with $0 < b_1 \leqq b_2 \leqq b_3$. Let $\pi(1,2,3)$ be the set of all permutations of $(1,2,3)$ and the notation $\mathbf{k} := (k_1, k_2, k_3) \in \pi(1,2,3)$ means that (k_1, k_2, k_3) runs over all permutations of $(1,2,3)$. We write $\{t\}$ for the fractional part of t and put $\psi(t) := \{t\} - \frac{1}{2}$. Defining weighted 3-dimensional divisor function

$$\tau(\mathbf{b}, \mathbf{a}; n) := \sum_{n_1^{b_1} n_2^{b_2} n_3^{b_3} = n} n_1^{a_1} n_2^{a_2} n_3^{a_3}.$$

We denote by $\Delta(\mathbf{b}, \mathbf{a}; x)$ the error term in the weighted three dimensional divisor problem

$$D(\mathbf{b}, \mathbf{a}; x) := \sum_{n \leqq x} \tau(\mathbf{b}, \mathbf{a}; n) = \text{main terms} + \Delta(\mathbf{b}, \mathbf{a}; x). \tag{3.1}$$

From (2), (3) and (4) of [16], we have

$$\Delta(\mathbf{b}, \mathbf{a}; x) = - \sum_{\mathbf{k} \in \pi(1,2,3)} \Big\{ \Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}; x) + \sum_{1 \leqq j \leqq 3} O\Big(x^{(a_{k_1} + \cdots + a_{k_j} + j - 2)/(b_{k_1} + \cdots + b_{k_j})}\Big) \Big\},$$

where

$$\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}; x) := x^{a_{k_3}/b_{k_3}} \sum\nolimits_1 n_1^{a_{k_1} - a_{k_3} b_{k_1}/b_{k_3}} n_2^{a_{k_2} - a_{k_3} b_{k_2}/b_{k_3}} \psi\Big(\big(x / n_1^{b_{k_1}} n_2^{b_{k_2}}\big)^{1/b_{k_3}} \Big)$$

and the summation condition of $\sum_1$ is given by

$$\mathrm{SC}(\textstyle\sum_1) \qquad n^{b_{k_1}} n_2^{b_{k_2} + b_{k_3}} \leqq x, \quad n_1 (\leqq) n_2.$$

The notation $n_1(\leqq)n_2$ means that $n_1 = n_2$ for $b_{k_1} < b_{k_2}$, and $n_1 < n_2$ otherwise.

As usual, for bounding $\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}; x)$, it is sufficient to consider the truncated sum

$$\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}, \mathbf{N}; x) := x^{a_{k_3}/b_{k_3}} \sum\nolimits_2 n_1^{a_{k_1} - a_{k_3} b_{k_1}/b_{k_3}} n_2^{a_{k_2} - a_{k_3} b_{k_2}/b_{k_3}} \psi\Big(\big(x / n_1^{b_{k_1}} n_2^{b_{k_2}}\big)^{1/b_{k_3}} \Big)$$

where $\mathbf{N} := (N_1, N_2) \in \mathbb{N}^2$, and the summation condition of $\sum_2$ is given by

$$\mathrm{SC}(\textstyle\sum_2) \qquad n_1^{b_{k_1}} n_2^{b_{k_2} + b_{k_3}} \leqq x, \quad n_1 (\leqq) n_2, \quad N_1 < n_1 \leqq 2N_1, \quad N_2 < n_2 \leqq 2N_2.$$

Obviously, Theorem 1 is equivalent to

$$\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}, \mathbf{N}; x) \ll x^{5/8} (\log x)^2 \tag{3.2}$$

for $\mathbf{a} = (0,0,1)$, $\mathbf{b} = (1,1,2)$ and all $\mathbf{k} \in \pi(1,2,3)$.

For this, we first establish a general estimate, which is valid for any $\mathbf{a}$ and $\mathbf{b}$, defined as in the beginning of this section. In the sequel, we write $\mathscr{L} := \log x$.

Lemma 3.1. *Under the preceding notations, we have*

$$\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}, \mathbf{N}; x) \ll \Xi \{ (G^2 N_1^5 N_2^5)^{1/8} + (G N_1^2 N_2)^{1/3} \} \mathscr{L}^2$$

with $G := \big(x / N_1^{b_{k_1}} N_2^{b_{k_2}}\big)^{1/b_{k_3}}$ *and* $\Xi := G^{a_{k_3}} N_1^{a_{k_1}} N_2^{a_{k_2}}$.

Proof. From a classic result on $\psi(t)$ (see [6], page 39), we can write, for any $H \geqq 1$, that

$$\Delta(\mathbf{k}, \mathbf{b}, \mathbf{a}, \mathbf{N}; x) \ll \Xi \Big\{ N_1 N_2 H^{-1} + \sum_{h \leqq H} |S(h)|/h \Big\}, \tag{3.3}$$

where

$$S(h) := \sum_{N_1<n_1 \leqq 2N_1} \psi_{n_1} \sum_{N_2<n_2 \leqq 2N_2} \psi_{n_2} e\Big(h\big(x/n_1^{b_{k_1}} n_2^{b_{k_2}}\big)^{b_{k_3}}\Big)$$

with $\psi_{n_j} := (n_j/N_j)^{a_{k_j}-a_{k_3}b_{k_j}/b_{k_3}}$. The Abel summation formula allows us to replace ψ_{n_j} by 1. Applying Lemma 2.1 with $(\alpha_1,\alpha_2)=(-b_{k_1}/b_{k_3},-b_{k_2}/b_{k_3})$, $(X,M_1,M_2)=(Gh,N_1,N_2)$ yields

$$\begin{aligned} S(h) \ll \big\{&(G^2h^2N_1^3N_2^3)^{1/6}+(G^2h^2N_1^5N_2)^{1/6}+N_1N_2^{1/2}+(G^3h^3N_1^7)^{1/8}+(Gh)^{1/4}N_1\\ &+(GhN_1)^{1/2}+(G^2h^2N_1^8N_2)^{1/8}+(Gh)^{-1/2}N_2+G^{-1}h^{-1}N_1N_2\big\}\mathscr{L}. \end{aligned}$$

Inserting this in (3.3), we obtain that for any $H \geqq 1$

$$\begin{aligned} \varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \Xi\big\{&(G^2N_1^3N_2^3H^2)^{1/6}+(G^2N_1^5N_2H^2)^{1/6}+(G^3N_1^7H^3)^{1/8}\\ &+(GN_1^4H)^{1/4}+(GN_1H)^{1/2}+(G^2N_1^8N_2H^2)^{1/8}+N_1N_2H^{-1}\\ &+N_1N_2^{1/2}+G^{-1/2}N_2+G^{-1}N_1N_2\big\}\mathscr{L}^2. \end{aligned}$$

In view of the term $N_1N_2H^{-1}$, this inequality also holds for $0<H<1$. By Lemma 2.4 (iii) of [13], there exists some $H>0$ such that

$$\begin{aligned} \varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \Xi\,\big\{&(G^2N_1^5N_2^5)^{1/8}+(G^2N_1^7N_2^3)^{1/8}+(G^3N_1^{10}N_2^3)^{1/11}+(GN_1^5N_2)^{1/5}\\ &+(GN_1^2N_2)^{1/3}+(G^2N_1^{10}N_2^3)^{1/10}+N_1N_2^{1/2}\\ &+G^{-1/2}N_2+G^{-1}N_1N_2\big\}\mathscr{L}^2 =: \Xi\,(E_1+E_2+\cdots+E_9)\mathscr{L}^2. \end{aligned}$$

Noticing that $\mathrm{SC}(\sum_2)$ implies $G \geqq N_2 \geqq N_1$, we easily see that E_2,E_4,E_6,E_7,E_8,E_9 can be absorbed by E_1 and $E_3=(E_1^8E_5^3)^{1/11}(N_2/N_1)^{-3/11}$. Thus E_j $(2 \leqq j \leqq 9, j \neq 5)$ are superfluous. This completes the proof of Lemma 3.1. $\square$

We are now in a position to prove (3.2). We discuss three possibilities.

1° When $(a_{k_1},a_{k_2},a_{k_3})=(0,0,1)$, we have $(b_{k_1},b_{k_2},b_{k_3})=(1,1,2)$, $G=\Xi=(x/N_1N_2)^{1/2}$. Lemma 3.1 offers immediately that

$$\varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \big\{x^{5/8}+(x^2/N_2)^{1/3}\big\}\mathscr{L}^2 \ll x^{5/8}\mathscr{L}^2$$

if $N_2 \geqq x^{1/8}$. In the contrary case, we have trivially

$$\varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \Xi N_1N_2 \ll (xN_1N_2)^{1/2} \ll (xN_2^2)^{1/2} \ll x^{5/8}.$$

2° When $(a_{k_1},a_{k_2},a_{k_3})=(0,1,0)$, we have $(b_{k_1},b_{k_2},b_{k_3})=(1,2,1)$, $G=x/N_1N_2^2$, $\Xi=N_2$. Noticing that $\mathrm{SC}(\sum_2)$ implies $N_1 \ll N_2$, $N_1N_2^3 \ll x$, Lemma 3.1 immediately gives that

$$\varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \big\{(x^2N_1^3N_2^9)^{1/8}+(xN_1N_2^2)^{1/3}\big\}\mathscr{L}^2 \ll x^{5/8}\mathscr{L}^2.$$

3° When $(a_{k_1},a_{k_2},a_{k_3})=(1,0,0)$, we have $(b_{k_1},b_{k_2},b_{k_3})=(2,1,1)$, $G=x/N_1^2N_2$, $\Xi=N_1$. Noticing that $\mathrm{SC}(\sum_2)$ implies $N_1 \ll N_2$, $N_1^2N_2^2 \ll x$, Lemma 3.1 offers immediately that

$$\varDelta(\mathbf{k},\mathbf{b},\mathbf{a},\mathbf{N};x) \ll \big\{(x^2N_1^9N_2^3)^{1/8}+(xN_1^3)^{1/3}\big\}\mathscr{L}^2 \ll x^{5/8}\mathscr{L}^2.$$

Obviously these estimates imply (3.2). This proves Theorem 1. $\square$

4. End of the proof of Theorem 2. Let $g(n)$ be the coefficient of the Dirichlet series $\zeta(2s)^3 \prod_p (1 - 2p^{-3s} - p^{-4s} + 2p^{-5s})$. In view of (1.1), we see that $\ell_\tau(n) = \{\tau(\mathbf{a}, \mathbf{b}; \cdot) * g\}(n)$ with $\mathbf{a} = (0,0,1)$ and $\mathbf{b} = (1,1,2)$. Since $\sum_{n=1}^{\infty} g(n) n^{-s}$ converges absolutely for $\Re e\, s > \frac{1}{2}$, using Theorem 1, a simple convolution argument immediately shows that

$$\sum_{n \leqq x} \ell_\tau(n) = A_1 x (\log x)^2 + A_2 x \log x + A_3 x + O\big(x^{5/8} (\log x)^4\big). \qquad \square$$

5. Proofs of Theorem 3 and Corollary 1. Let $A(x) := \sum_{n \leqq x} \ell_\tau(n) - E(x)$. By the definition of $E(x)$, we easily see that

$$A(x) = \operatorname{Res}\{\mathscr{D}(\tau, s) x^s / s, 1\} + \operatorname{Res}\Big\{\mathscr{D}(\tau, s) x^s / s, \frac{1}{2}\Big\},$$

where $\operatorname{Res}\{F(s), a\}$ is the residue of $F(s)$ at $s = a$. By Perron's formula ([15], Theorem II.2.3), we obtain

$$\int_0^x E(u) du = \frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} ds - \int_0^x A(u)\, du. \tag{5.1}$$

Let $\sigma_0 \in (\frac{1}{3}, \frac{1}{2})$. By (II.3.15) and (II.3.22) of [15], we deduce that for any $T > 10$ and any $\varepsilon > 0$

$$\int_{\sigma_0 \leqq \sigma \leqq 2,\, |t| = T} \left| \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} \right| |ds| \ll x^3 / T^{5/18 - \varepsilon},$$

and

$$\int_{\sigma = 2,\, |t| \geqq T} \left| \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} \right| |ds| \ll x^3 / T.$$

Shifting the line of integration from $\sigma = 2$ to $\sigma = \sigma_0$ and using the preceding estimates, the residue theorem implies that

$$\begin{aligned}\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} ds &= \frac{1}{2\pi i} \int_{2 - iT}^{2 + iT} \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} ds + O\Big(\frac{x^3}{T}\Big) \\ &= \int_0^x A(u) du + \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} ds + O\Big(\frac{x^3}{T^{5/18 - \varepsilon}}\Big),\end{aligned}$$

where we used the relation

$$\operatorname{Res}\{\mathscr{D}(\tau, s) x^{s+1} / s(s+1),\, 1\} + \operatorname{Res}\Big\{\mathscr{D}(\tau, s) x^{s+1} / s(s+1),\, \frac{1}{2}\Big\} = \int_0^x A(u)\, du.$$

Making $T \to \infty$ and inserting the formula obtained in (5.1), we find that

$$\int_0^x E(u)\, du = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \mathscr{D}(\tau, s) \frac{x^{s+1}}{s(s+1)} ds \ll x^{1 + \sigma_0} \int_{-\infty}^{+\infty} \frac{|\mathscr{D}(\tau, \sigma_0 + it)|}{(|t| + 1)^2} dt. \tag{5.2}$$

Observing $\frac{1}{2} < 1 - \sigma_0 < \frac{2}{3}$, the functional equation for $\zeta(s)$ ([15], Theorem II.3.3) yields

$$\mathscr{D}(\tau, \sigma_0 + it) \asymp |t|^{5/2 - 4\sigma_0} |\zeta(1 - \sigma_0 - it)|^2 |\zeta(2\sigma_0 + i2t)|^3. \tag{5.3}$$

For $\sigma \in [\frac{1}{2}, 1]$, we define $m(\sigma)$ to be the supremum of all numbers m such that

$$\int_1^T |\zeta(\sigma + it)|^m dt \ll_\varepsilon T^{1+\varepsilon}$$

holds for any $\varepsilon > 0$ and any $T \geqq 1$. Recalling $\frac{1}{2} < 1 - \sigma_0$, $2\sigma_0 < 1$ and $m(\frac{1}{2}) \geqq 6$ (Theorem 8.4 of [9]), we show, by the Hölder inequality and (5.3), that

$$\begin{aligned}\int_T^{2T} \frac{|\mathcal{D}(\tau, \sigma_0 + it)|}{t^2}\, dt &\ll T^{1/2-4\sigma_0} \Big\{\int_T^{2T} |\zeta(1 - \sigma_0 - it)|^5 dt\Big\}^{2/5} \Big\{\int_T^{2T} |\zeta(2\sigma_0 + i2t)|^5 dt\Big\}^{3/5} \\ &\ll_\varepsilon T^{3/2-4\sigma_0+\varepsilon}.\end{aligned}$$

Taking $\sigma_0 = \frac{3}{8} + \varepsilon$ and using the inequality obtained with $T = 2^k$, we find that

$$\int_1^\infty \frac{|\mathcal{D}(\tau, \sigma_0 + it)|}{t^2}\, dt = \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \frac{|\mathcal{D}(\tau, \sigma_0 + it)|}{t^2}\, dt \ll \sum_{k=0}^\infty \frac{1}{2^{3\varepsilon k}} \ll_\varepsilon 1.$$

This and (5.2) imply the required result. □

For Corollary 1, we first see that Theorem 3 yields that

$$\frac{1}{x}\int_0^x \Delta(u)\, du \sim \tfrac{1}{24} A_4 x^{1/2} (\log x)^2.$$

In addition a simple calculation shows that

$$A_4 = -\frac{1}{16}\zeta\Big(\frac{1}{2}\Big)^2 \prod_p \big(1 - 2p^{-3/2} - p^{-2} + 2p^{-5/2}\big) < 0.$$

These imply the required result. □

6. Proofs of Theorem 4 and Corollary 2. We first evaluate the coefficient $\ell_\tau(n)$. Since this function is multiplicative, it is enough to study $\ell_\tau(p^\nu)$ for prime numbers p and positive integers ν.

Lemma 6.1. *Let p be a prime number, $\nu \geqq 1$ a positive integer and $m := [\nu/2]$. Then we have*

$$\ell_\tau(p^\nu) = \sum_{j=0}^m (\nu - 2m + 2j + 1)(j + 1)p^{m-j}.$$

Proof. Obviously, we may evaluate directly $\ell_\tau(p^\nu)$ by (1.1). However, using the relation $\tau(\mathcal{G}) = \tau(S)$, we can give a more succinct proof. Let $\tau\langle p^j, p^{\nu-j}\rangle$ be the number of factorisations of the SNF matrix $\mathrm{diag}[p^j, p^{\nu-j}]$, the preceding relation implies $\ell_\tau(p^\nu) = \sum_{j=0}^m \tau\langle p^j, p^{\nu-j}\rangle$. From Remark 1.4 of [12], we find that

$$\ell_\tau(p^\nu) = \sum_{j=0}^m \sum_{k=0}^j (\nu - 2k + 1)p^k = \sum_{k=0}^m (\nu - 2k + 1)(m - k + 1)p^k,$$

which is equivalent to the required result. This proves Lemma 6.1. □

We note that $\nu - 2m + 1 \neq 0$ (it is 1 or 2 depending on whether ν is even or odd), hence $\ell_\tau(p^\nu)$ is a polynomial in p of degree m. Thus, for example, we have

$$\ell_\tau(p) = 2, \quad \ell_\tau(p^2) = p + 6, \quad \ell_\tau(p^3) = 2p + 8, \quad \ell_\tau(p^4) = p^2 + 6p + 15.$$

Now we are in a position to complete the proof of Theorem 4.

By using Lemma 6.1, we have

$$\ell_\tau(p^{2m}) = p^m\Big(1 + \sum_{1 \leqq j \leqq m-2} (2j+1)(j+1)/p^j\Big) \quad (m = 1, 2, \ldots), \tag{6.1}$$

$$\ell_\tau(p^{2m+1}) = 2p^m\Big(1 + \sum_{1 \leqq j \leqq m-2} (j+1)^2/p^j\Big) \quad (m = 0, 1, 2, \ldots). \tag{6.2}$$

From this, we easily deduce that

$$\ell_\tau(p^\nu) \leqq p^{\nu/2}\Big(1 + \sum_{j=1}^{\infty}(2j+1)(j+1)/p^j\Big).$$

Since $\ell_\tau(n)$ is multiplicative, we find that

$$\begin{aligned}\ell_\tau(n) &\leqq \sqrt{n}\prod_{p^\nu \| n}\Big(1+\sum_{j=1}^{\infty}(2j+1)(j+1)/p^j\Big) \leqq \sqrt{n}\prod_{p \leqq p_{\omega(n)}}\Big(1+\sum_{j=1}^{\infty}(2j+1)(j+1)/p^j\Big)\\ &\leqq \sqrt{n}\prod_{p \leqq p_{\omega(n)}}(1-1/p)^{-6}\prod_{p \leqq p_{\omega(n)}}(1-1/p)^6\Big(1+\sum_{j=1}^{\infty}(2j+1)(j+1)/p^j\Big),\end{aligned}$$

where $\omega(n)$ is the number of distinct prime factors of n and p_j denotes the jth prime number. By Mertens' formula and the relation $\log p_{\omega(n)} \leqq \{1 + o(1)\}\log_2 n$, it is easy to show that

$$\limsup_{n\to\infty} \frac{\ell_\tau(n)}{\sqrt{n}(\log_2 n)^6} \leqq C_0. \tag{6.3}$$

Defining $n_{k,m} := p_1^{2m} \cdots p_k^{2m}$ for $k \geqq 1$ and $m \geqq 4$, the first relation of (6.1) yields that

$$\begin{aligned}\limsup_{n\to\infty} \frac{\ell_\tau(n)}{\sqrt{n}(\log_2 n)^6} &\geqq \lim_{k\to\infty} \frac{\ell_\tau(n_{k,m})}{\sqrt{n_{k,m}}(\log_2 n_{k,m})^6}\\ &= e^{6\gamma}\prod_p (1-1/p)^6\Big(1+\sum_{1 \leqq j \leqq m-2}(2j+1)(j+1)/p^j\Big).\end{aligned}$$

We let $m \to \infty$ and together with (6.3), we obtain the first assertion of Theorem 4. The lower bound is trivial, since $\ell_\tau(n) \geqq 2$ and $\ell_\tau(p) = 2$. □

For verifying Corollary 2, we can write

$$\ell_\tau(n_{k,1}) = A(n_{k,1}) - A(n_{k,1} - 1) + E(n_{k,1}) - E(n_{k,1} - 1).$$

Since $A(n_{k,1}) - A(n_{k,1} - 1) \ll (\log n_{k,1})^2$, the first relation of (6.1) implies immediately

$$|E(n_{k,1}) - E(n_{k,1} - 1)| \gg \sqrt{n_{k,1}}(\log_2 n_{k,1})^6. \quad □$$

7. Proof of Theorem 5. For each $z > 0$, the function $\ell_\tau(n)z^{\omega(n)}$ is multiplicative and

$$\sum_{n=1}^{\infty} \ell_\tau(n)z^{\omega(n)}n^{-s} = \zeta(s)^{2z}\zeta(2s-1)^z G(s,z) \quad (\Re\mathrm{e}\, s > 1),$$

where $G(s,z)$ is a Dirichlet series absolutely convergent for $\Re e\, s > \frac{1}{2}$ and for any $z > 0$ fixed. By the Selberg–Delange method (Chapter II.5 of [15], with minor modification), we can show

$$\sum_{n \leqq x} \ell_\tau(n) z^{\omega(n)} \ll x(\log x)^{3z-1}.$$

Hence we deduce that

$$\begin{aligned}\sum_{\substack{n \leqq x \\ |\omega(n)-3\log_2 x|>\varepsilon\log_2 x}} \ell_\tau(n) &\leqq \sum_{n \leqq x} \ell_\tau(n)\Big\{\big(1+\tfrac{1}{3}\varepsilon\big)^{\omega(n)-(3+\varepsilon)\log_2 x} + \big(1-\tfrac{1}{3}\varepsilon\big)^{\omega(n)-(3-\varepsilon)\log_2 x}\Big\} \\ &\ll x(\log x)^{2-3\eta},\end{aligned}$$

where η is defined as in the formulation of Theorem 5. Combining this with (1.2) yields the second assertion. $\square$

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Anschriften der Autoren:

Gautami Bhowmik
Département de Mathématiques
B.P. 311
Université Valenciennes
59304 Valenciennes Cédex
France

Jie Wu
Laboratoire de Mathématiques
Institut Elie Cartan – UMR CNRS 9973
Université Henri Poincaré, Nancy 1
54506 Vandœuvre-lès-Nancy
France