

# AVERAGE GOLDBACH AND THE QUASI-RIEMANN HYPOTHESIS

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ABSTRACT. We prove that a good average order on the Goldbach generating function implies that the real parts of the non-trivial zeros of the Riemann zeta function are strictly less than 1. This together with existing results establishes an equivalence between such asymptotics and the Riemann Hypothesis.

## 1. INTRODUCTION

The relation between the average order of the Goldbach generating function and the Riemann Hypothesis was introduced by Granville [4]. In an attempt to generalise such relations to sums of primes in arithmetic progressions it was observed that asymptotic results on the average number of restricted representations of integers assuming a conjecture on distinct zeros of Dirichlet L-functions did not obviously yield an equivalence with the generalised Riemann Hypothesis since the case of the real part of non-trivial zeros being close to 1 could not be eliminated. In this note we succeed in doing so for the Riemann zeta function in a simple way and establish an equivalence of average orders of the original Goldbach function with what we call a quasi-Riemann hypothesis, that the real parts of the non-trivial zeros of the Riemann zeta function are strictly less than  $c_0, c_0 < 1$ . This completes the proof of Theorem 1A in [4] i.e. a good asymptotic order of the Goldbach function is indeed equivalent to the Riemann Hypothesis. The same idea has since been applied to the restricted sums of two primes both in the same congruence class ([1], Section 8).

The study of the average order of the function  $g(n) = \sum_{n=p_1+p_2} n$  goes back a long way, at least to Landau [6] who obtained, for even  $n$  and primes  $p_i$ ,  $\sum_{n \leq x} g(n) \sim \frac{x^2}{2(\log x)^2}$  which agrees with Hardy and Littlewood's conjecture [5] that  $g(n)$  should be approximated by  $\mathfrak{S}(n) \frac{n}{(\log n)^2}$  for the singular series  $\mathfrak{S}$ . Under the Riemann Hypothesis, the slightly more convenient summatory function

$$G(n) = \sum_{k_1+k_2=n} \Lambda(k_1)\Lambda(k_2),$$

where  $\Lambda$  is the von Mangoldt function was studied by Fujii [3] and Granville (op.cit) who showed that its summatory function can be expressed as

$$(1.1) \quad S(x) = \sum_{n \leq x} G(n) = \frac{x^2}{2} + O(x^{3/2}).$$

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The error term was eventually improved to its conjectured value  $O(x^{1+\varepsilon})$  [2] but here it is enough for us to assume any error term less than  $x^2$  and from the square root of the associated power series estimate well the Chebyshev function,  $\psi(x) = \sum_{n \leq x} \Lambda(n)$ , using a Dirichlet kernel.

## 2. QUASI-RIEMANN HYPOTHESIS

**Theorem 2.1.** *If the asymptotic relation*

$$(2.1) \quad S(x) = \frac{x^2}{2} + O(x^{2-\delta}), \quad \delta > 0$$

*holds, then there exists  $0 < \delta' < 1$  such that for the non-trivial zeros  $\rho$  of the Riemann zeta function  $\operatorname{Re} \rho < 1 - \delta'$ .*

*Proof.* We consider the power series

$$F(z) = \sum_{n=1}^{\infty} \Lambda(n) z^n, \quad |z| < 1$$

whose square is written in terms of the Goldbach function as

$$F(z)^2 = \sum_{n=1}^{\infty} G(n) z^n.$$

Writing the above as a telescoping series with  $G(n) = S(n) - S(n-1)$  we get

$$F(z)^2 = (1-z) \sum_{n=1}^{\infty} S(n) z^n.$$

From the assumption (2.1) on the Goldbach asymptotic order we infer

$$(2.2) \quad \sum_{n=1}^{\infty} S(n) z^n = \sum_n \left( \frac{n^2}{2} + O(n^{2-\delta}) \right) z^n = \sum_n \frac{n^2}{2} z^n + O \left( \sum_n n^{2-\delta} |z|^n \right).$$

We have

$$\sum_n \frac{n^2}{2} z^n = \frac{1}{(1-z)^3} - \frac{3}{2(1-z)^2} + \frac{1}{2(1-z)},$$

while the error term of (2.2) is estimated as

$$\sum_{n=1}^{\infty} n^{2-\delta} |z|^n \ll (1-|z|)^{\delta-3}$$

by comparison with the power series expansion of the right hand side and thus

$$\sum_n S(n) z^n = \frac{1}{(1-z)^3} + O \left( \frac{1}{(1-|z|)^{3-\delta}} \right)$$

which implies

$$F(z)^2 = (1-z) \sum_{n=1}^{\infty} S(n) z^n = \frac{1}{(1-z)^2} + O \left( \frac{|1-z|}{(1-|z|)^{3-\delta}} \right).$$

To estimate  $\psi(N)$  we consider the circle  $|z| = R = 1 - 1/N$  for a large positive integer  $N$  and rewrite the last formula as

$$(2.3) \quad F(z)^2 = \frac{1}{(1-z)^2} + O(|1-z|N^{3-\delta}).$$

The error term is less than the absolute value of the main term for

$$(2.4) \quad |1-z| < N^{\frac{\delta}{3}-1},$$

and in this range we take the complex square root of (2.3) to obtain

$$(2.5) \quad F(z) = \frac{1}{1-z} + O(|1-z|^2 N^{3-\delta}).$$

Kernel. We introduce the function

$$K(z) = z^{-N-1} \frac{1-z^N}{1-z}.$$

We have  $K(z) \ll |1-z|^{-1}$  and we use it to estimate the partial sums

$$(2.6) \quad \psi(N) = \frac{1}{2\pi i} \int_{|z|=R} F(z)K(z)dz = N + \frac{1}{2\pi i} \int_{|z|=R} \left( F(z) - \frac{1}{1-z} \right) K(z)dz.$$

Major Arc. On the part of the circle (2.4) the integrand is

$$O(|1-z|N^{3-\delta}) = O(N^{2-2\delta/3})$$

so the contribution to the integral above is  $O(N^{1-\delta/3})$  from the power series estimate obtained in (2.5).

Minor Arc. On the rest of the circle, i.e. the arc  $|1-z| > N^{\delta/3-1}$  we use the Cauchy-Schwarz inequality. For  $F$  we apply the estimate for the full circle

$$\begin{aligned} \int_{|z|=R} \left| F(z) - \frac{1}{1-z} \right|^2 |dz| &= \int_0^{2\pi} \sum_n (\Lambda(n) - 1) R^n e^{int} \sum_m (\Lambda(m) - 1) R^m e^{-imt} \\ &= 2\pi \sum_{n=1}^{\infty} (\Lambda(n) - 1)^2 \left( 1 - \frac{1}{N} \right)^{2n} = O(N \log N). \end{aligned}$$

We estimate the *square integral of the kernel* as follows.

$$I = \int_{\substack{|z|=R \\ |1-z| > N^{\delta/3-1}}} |K(z)|^2 |dz| \ll \int_{\substack{|z|=R \\ |1-z| > N^{\delta/3-1}}} \frac{|dz|}{|1-z|^2}.$$

With the parametrization  $z = Re^{it}$  the last integral becomes

$$2 \int_{t_0}^{\pi} \frac{1}{|1 - Re^{it}|^2} dt,$$

where  $t_0$  is defined by  $|1 - Re^{it_0}| = N^{\delta/3-1}$ .

For the minor arc, we have

$$|1 - Re^{it}|^2 = 1 + R^2 - 2R \cos t = (1 - R)^2 + 4R \sin^2 \frac{t}{2} \begin{cases} < t^2 + N^{-2} \\ > t^2/3, \end{cases}$$

whence  $t_0 \gg N^{\delta/3-1}$  and the integral satisfies

$$I \ll \int_{t_0}^{\pi} t^{-2} dt < t_0^{-1} \ll N^{1-\delta/3}.$$

Now the Cauchy Schwarz inequality gives the bound for the minor arc as

$$\left( \int_{|z|=R} \left| F(z) - \frac{1}{(1-z)} \right|^2 dz \right)^{1/2} I^{1/2} \ll N^{1-\delta/6} (\log N)^{1/2}.$$

The major and minor arcs together give

$$\psi(N) - N \ll N^{1-\delta/6} (\log N)^{1/2},$$

which proves the theorem with  $\delta' = \delta/6$ . □

### 3. REMARKS

1. Using a better kernel and a more careful calculation we can improve the bound to  $\delta' = \delta/3$ . It is of little importance in this context as the methods of [1,2,4] give  $\delta' = \delta$  as soon as we have  $\delta' < 1$ , through an explicit expression of  $S(x)$  using roots of the zeta-function. We are unable to get  $\delta' = \delta$  directly by the simple method of this paper and we think it cannot be done. We used very little of the properties of the Liouville function and our method works equally for any function  $f$  satisfying  $|f(n)| = O(n^\varepsilon)$  for all  $\varepsilon > 0$  and

$$\sum_{m+n \leq x} f(m)f(n) = cx^2 + O(x^{2-\delta}).$$

We doubt that

$$\sum_{n \leq x} f(n) = c'x + O(x^{1-\delta+o(1)})$$

follows in this generality.

2. In general we are yet unable to establish an equivalence between average orders of restricted Goldbach functions and appropriate quasi Riemann Hypotheses for  $L$ -functions of [4] or [1] though a particular case is handled in [1] using the method of this note.

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