ACTA ARITHMETICA
LXXIV.2 (1996)

Evaluation of divisor functions of matrices

by

GAUTAMI BHOWMIK (Nancy)

1. Introduction. The study of divisor functions of matrices arose legitimately in the context of arithmetic of matrices, and the question of the number of (possibly weighted) inequivalent factorizations of an integer matrix was asked. However, till now only partial answers were available. Nanda [6] evaluated the case of prime matrices and Narang [7] gave an evaluation for 2×2 matrices. We obtained a recursion in the size of the matrices and the weights of the divisors [1, 2] which helped us obtain a result for 3×3 matrices but no closed formula for the general case. In this paper we obtain the complete evaluation of the divisor functions by a combinatorial consideration (see Theorem 1). Because of the existence of a bijection (detailed in a forthcoming paper [3]) between the set of divisors of an $r \times r$ integer matrix and the set of subgroups of an abelian group of rank at most r, we have here a rather simple proof to obtain the number of subgroups of a finite abelian group.

Acknowledgements. I thank the Mathematisches Institut der Westfälischen Wilhelms-Universität, Münster, Germany, for providing me with the facilities for doing this work.

2. Preliminaries. We consider integer matrices, i.e. matrices with entries from $\mathbb{Z}$. We define factorization of a non-singular $r \times r$ matrix M as

$$M = M_1 M_2,$$

where taking into account the infinite cardinality of $\mathrm{GL}_r(\mathbb{Z})$, we choose M_2 to be a unique representative of a one-sided equivalence class. Our canonical choice is a matrix in non-singular Hermite Normal Form (HNF), i.e. M_2 is lower triangular with strictly positive diagonal elements and with every element in the column below the diagonal element belonging to the class modulo the diagonal element. We call M_2 a *divisor* of M.

1991 *Mathematics Subject Classification*: 05A15, 11C20, 15A36.

Associated with inequivalent factorizations of M, we define, for any complex number a, the *divisor function* as

$$\sigma_a(M) = \sum_{M=M_1M_2} (\det M_2)^a.$$

Since $\sigma_a(UMV) = \sigma_a(M)$ for U, V in $\mathrm{GL}_r(\mathbb{Z})$, we choose M to be a unique representative of a two-sided equivalence class, i.e. in Smith Normal Form (SNF). Further, the functions σ_a are *multiplicative*, i.e. $\sigma_a(AB) = \sigma_a(A)\sigma_a(B)$ whenever A and B have co-prime determinants. It is, therefore, enough to consider M to be a SNF matrix with prime power determinant. We use the notation $\langle f_1, f_2, \ldots, f_r\rangle_p$ to denote a SNF matrix whose tth diagonal entry is $p^{f_1+f_2+\ldots+f_t}$, where p is a prime number, f_1 a positive integer and $f_2, \ldots, f_r$ are non-negative integers. (We may suppress p for notational convenience.) In [1] we proved

Lemma 1.

$$\sigma_a\langle f_1, \ldots, f_r\rangle_p = p^a \sigma_a\langle f_1, \ldots, f_r - 1\rangle_p + \sigma_{a+1}\langle f_1, \ldots, f_{r-1}\rangle_p.$$

While using the above lemma recursively to evaluate σ, we encounter the problem that when f_r becomes 0, the SNF structure is destroyed and the recursion cannot be used thereafter without finding the equivalent SNF form. We could not proceed beyond the evaluation of $\sigma_0\langle f_1, f_2, f_3\rangle$ by using it.

3. Evaluations.

Now we give a recurrence which preserves the SNF structure. We use the notations 0_t for a string of t zeros and $\left[\begin{smallmatrix} k \\ t \end{smallmatrix}\right]$ for Gaussian polynomials in p, defined as

$$\begin{bmatrix} k \\ t \end{bmatrix} = \frac{\prod_{l=1}^{k}(p^l - 1)}{\prod_{m=1}^{t}(p^m - 1)\prod_{n=1}^{k-t}(p^n - 1)}.$$

We prove:

Lemma 2. *For $k \ge 1$ and $f_{r-k+1} \ge 1$, we have*

$$\sigma_a\langle f_1, \ldots, f_{r-k+1}, 0_{k-1}\rangle_p = \sum_{t=0}^{k-1} \begin{bmatrix} k \\ t \end{bmatrix} p^{a(k-t)} \sigma_{a+t}\langle f_1, \ldots, f_{r-k+1} - 1, 0_{k-t-1}\rangle_p + \sigma_{a+k}\langle f_1, \ldots, f_{r-k}\rangle_p.$$

Proof. We construct a tree to descend from $\sigma_a\langle f_1, \ldots, f_{r-k+1}, 0_{k-1}\rangle$ to $\sigma_{a+t}\langle f_1, \ldots, f_{r-k+1} - 1, 0_{k-t-1}\rangle$, $0 \le t < k$. We denote the bth element (from left) of the cth line (from top) by the symbol $\langle\!\langle c - 1, b\rangle\!\rangle$. This is done by using Lemma 1 repeatedly and realizing that, for $i + 1 \le m \le k - 2$, both

$$\sigma_{a+i}\langle f_1, \ldots, f_{r-k+1} - 1, 0_{m-i}, 1, 0_{k-m-2}\rangle$$

and

$$\sigma_{a+i+1}\langle f_1, \ldots, f_{r-k+1} - 1, 0_{m-i-1}, 1, 0_{k-m-2}\rangle,$$

i.e. elements $\langle\!\langle m+1, i+1\rangle\!\rangle$ and $\langle\!\langle m+1, i+2\rangle\!\rangle$ descend to $\langle\!\langle m+2, i+2\rangle\!\rangle$, i.e. to

$$\sigma_{a+i+1}\langle f_1, \ldots, f_{r-k+1} - 1, 0_{m-i}, 1, 0_{k-m-3}\rangle,$$

the element $\langle\!\langle m+2, i+2\rangle\!\rangle$, though in the second case, where we say that a *right turn* has taken place, there is a multiplication by p^{a+i+1} and in the first case, that of a *left turn*, such a factor does not exist. In the extreme case where $m = i$, we have $\langle\!\langle i+1, i+2\rangle\!\rangle = \sigma_{a+i+1}\langle f_1, \ldots, f_{r-k+1}, 0_{k-i-2}\rangle$. For an example where $r = 5$ and $k = 4$, see Figure 1.

We now count the number of ways in which a descent has taken place together with the powers of p that have been attached. We notice that there are exactly $k-t$ right turns to get from $\langle\!\langle 0, 1\rangle\!\rangle$ to $\langle\!\langle k, t\rangle\!\rangle$ and in each such turn the power of p attached is at most $a + t$. We consider the *partition function* $q(n, u, v)$ that counts the number of ways in which n can be written as the sum of at most u parts, each at most v. We remark that the coefficient of $p^{n+a(k-t)}$ that we obtain when we reach $\langle\!\langle k, t\rangle\!\rangle$ is precisely $q(n, k-t, t)$. We now use the fact that $\sum p^n q(n, k-t, t) = \left[{k \atop t}\right]$.

$$\begin{tikzcd}[column sep=tiny]
 & & & & \sigma_a\langle f_1, f_2, 0, 0, 0\rangle \arrow[dl, "p^a"'] \arrow[dr] & & & & \\
 & & & \sigma_a\langle f_1, f_2{-}1, 1, 0, 0\rangle \arrow[dl, "p^a"'] \arrow[dr] & & \sigma_{a+1}\langle f_1, f_2, 0, 0\rangle \arrow[dl, "p^{a+1}"'] \arrow[dr] & & & \\
 & & \sigma_a\langle f_1, f_2{-}1, 0, 1, 0\rangle \arrow[dl, "p^a"'] \arrow[dr] & & \sigma_{a+1}\langle f_1, f_2{-}1, 1, 0\rangle \arrow[dl, "p^{a+1}"'] \arrow[dr] & & \sigma_{a+2}\langle f_1, f_2, 0\rangle \arrow[dl, "p^{a+2}"'] \arrow[dr] & & \\
 & \sigma_a\langle f_1, f_2{-}1, 0, 0, 1\rangle \arrow[dl, "p^a"'] \arrow[ddr] & & \sigma_{a+1}\langle f_1, f_2{-}1, 0, 1\rangle \arrow[ddl, "p^{a+1}"'] \arrow[dr] & & \sigma_{a+2}\langle f_1, f_2{-}1, 1\rangle \arrow[dl, "p^{a+2}"'] \arrow[ddr] & & \sigma_{a+3}\langle f_1, f_2\rangle \arrow[ddl, "p^{a+3}"'] \arrow[dr] & \\
\sigma_a\langle f_1, f_2{-}1, 0, 0, 0\rangle & & & & \sigma_{a+2}\langle f_1, f_2{-}1, 0\rangle & & & & \sigma_{a+4}\langle f_1\rangle \\
 & & \sigma_{a+1}\langle f_1, f_2{-}1, 0, 0\rangle & & & & \sigma_{a+3}\langle f_1, f_2{-}1\rangle & &
\end{tikzcd}$$

Fig. 1

∎

We now evaluate the divisor functions. We use the notation $\left[{k \atop h,\ldots,t}\right]$ for Gaussian multinomials in p, defined as

$$\begin{bmatrix} k \\ h,\dots,t \end{bmatrix} = \frac{\prod_{l=1}^{k}(p^l-1)}{\prod_{i=1}^{h}(p^i-1)\dots\prod_{m=1}^{t}(p^m-1)\prod_{n=1}^{k-h-\dots-t}(p^n-1)}.$$

THEOREM 1.

$$\sigma_a\langle f_1,\dots,f_r\rangle_p = \sum \begin{bmatrix} 1 \\ t_1,\dots,t_{f_r} \end{bmatrix} \begin{bmatrix} 2-t_1-\dots-t_{f_r} \\ t_{f_r+1},\dots,t_{f_r+f_{r-1}} \end{bmatrix} \dots \dots \begin{bmatrix} r-t_1-\dots-t_{f_r+\dots+f_2} \\ t_{f_r+\dots+f_2+1},\dots,t_{f_r+\dots+f_1} \end{bmatrix} p^{\varphi(t_1,\dots,t_{f_r};a)},$$

where the summation is over all $t_1,\dots,t_{f_r}$ *such that*

$$\begin{cases} t_1+\dots+t_{f_r}\le 1, \\ t_1+\dots+t_{f_r+f_{r-1}}\le 2, \\ \vdots \\ t_1+\dots+t_{f_r+f_{r-1}+\dots+f_1}\le r, \end{cases}$$

and

$$\varphi = \sum_{k=1}^{r}\sum_{i=1}^{f_{r-k+1}}(k-t_1-t_2-\dots-t_{i+s_k})(a+t_1+t_2+\dots+t_{i-1+s_k})$$

with $s_k=\sum_{j=1}^{k-1} f_{r-j+1}$. *We let* t_0 *and all empty sums be* 0 *and if* $f_{r-k+1}=0$, *we consider the* k*-th inequality* $t_1+\dots+t_{f_r+f_{r-1}+\dots+f_{r-k+1}}\le k$ *to be empty and the* k*-th multinomial to be* 1.

The right hand side of Theorem 1, for $a=0$, counts the total number of subgroups of $\mathbb{Z}/p^{f_1}\mathbb{Z}\times\mathbb{Z}/p^{f_1+f_2}\mathbb{Z}\times\dots\times\mathbb{Z}/p^{f_1+\dots+f_r}\mathbb{Z}$ [3] and extends special cases like the number of subgroups of a given type (see, e.g. [4]) and the number of all subspaces of $(\mathbb{Z}/p\mathbb{Z})^r$ (see e.g. [5]).

Proof of Theorem 1. We use Lemma 2 repeatedly to reduce each non-zero f_i to 0, starting from the right. Let f_r be non-zero, i.e. initially $k=1$. By f_r iterations of the above lemma, we reduce f_r to 0. This involves summing over f_r variables $t_1,\dots,t_{f_r}$. We allow each t_i to vary from 0 to $1-t_1-\dots-t_{i-1}$, where in the extreme case we get 0_{-1} but locally treat it only as a symbol, since at the end of the f_r steps the number of zeros will anyway increase by 1. We then get the product of Gaussian polynomials

$$\begin{bmatrix} 1 \\ t_1 \end{bmatrix}\begin{bmatrix} 1-t_1 \\ t_2 \end{bmatrix}\dots\begin{bmatrix} 1-t_1-\dots-t_{f_r-1} \\ t_{f_r} \end{bmatrix}$$

which we combine to write the multinomial $\left[\begin{smallmatrix} 1 \\ t_1,\dots,t_{f_r} \end{smallmatrix}\right]$. Now we have to reduce $\sigma_{a+t_1+\dots+t_{f_r}}\langle f_1,\dots,f_{r-1},0_{1-t_1-\dots-t_{f_r}}\rangle$. If $f_r=0$, we have $k=2$ and now that we straight away reduce f_{r-1} to 0, the first multinomial no longer occurs. We follow the same steps recursively to complete the evaluation. ■

References

[1] G. Bhowmik, *Divisor functions of integer matrices*: *evaluations*, *average orders and applications*, Astérisque 209 (1992), 169–177.

[2] G. Bhowmik and O. Ramaré, *Average orders of multiplicative arithmetical functions of integer matrices*, Acta Arith. 66 (1994), 45–62.

[3] —, —, *Factorisation of matrices*, *partitions and Hecke algebra*, to appear.

[4] L. M. Butler, *A unimodality result in the enumeration of subgroups of a finite abelian group*, Proc. Amer. Math. Soc. 101 (1987), 771–775.

[5] A. Krieg, *Hecke Algebras*, Mem. Amer. Math. Soc. 435 (1990).

[6] V. C. Nanda, *Arithmetic functions of matrices and polynomial identities*, in: Colloq. Math. Soc. János Bolyai 34, North-Holland, 1984, 1107–1126.

[7] A. Narang, Ph. D. Thesis, Panjab University, India, 1979.

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE NANCY 1
B.P. 239
54506 VANDŒUVRE-LÈS-NANCY, FRANCE
E-mail: BHOWMIK@IECN.U-NANCY.FR

Received on 14.3.1995 (2756)