ANALYTIC CONTINUATION OF SOME ZETA FUNCTIONS

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1. Introduction

The contents of this paper were presented as lectures at the Miura Winter School on Zeta and L-functions held in 2008. Though the analytic continuation of zeta functions beyond its region of absolute convergence is a fundamental question, in general not much is known about the conditions that guarantee a meromorphic continuation. It is also interesting to know how far such a function can be continued, that is where the natural boundary of analytic continuation lies.

The choice of functions that are considered here are 'arbitrary', that is a matter of personal taste and expertise. Most of the work reported is on what I have studied or actually contributed to together with my co-authors. The word 'some' in the title is to indicate that though the paper is expository, it is not exhaustive. Only outlines of proofs have sometimes been provided.

In the first part we consider Euler products. One of the most important applications of zeta functions is the asymptotic estimation of the sum of its coefficients via Perron's formula, that is, the use of the equation

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\sum_{n\ge 1}\frac{a_n}{n^s}\Big)\frac{x^s}{s}\,ds.$$

To use this relation, one usually shifts the path of integration to the left, thereby reducing the contribution of the term x^s. This becomes possible only if the function $D(s) = \sum \frac{a_n}{n^s}$ is holomorphic on the new path. In Section 3 details of certain examples from height zeta functions and zeta functions of groups have been given.

Clearly all zeta functions do not have Euler product expansions, one important class of examples being multiple zeta functions which have been studied often in recent years. Not many general methods exist and here I treat the case of the Goldbach generating function associated to $G_r(n)$, the number of representations of n as the sum of r primes

$$\sum_{k_1=1}^{\infty}\cdots\sum_{k_r=1}^{\infty}\frac{\Lambda(k_1)\dots\Lambda(k_r)}{(k_1+k_2+\cdots+k_r)^s} = \sum_{n=1}^{\infty}\frac{G_r(n)}{n^s}.$$

where Λ is the classical von-Mangoldt function.

In almost all examples the natural boundary, if it can be obtained, corresponds to the intuitively expected boundary and this can in fact be proved in a probabilistic sense. However one of the difficulties in actually obtaining the boundary is that our analyses often depend on the distribution of zeros of the Riemann zeta function, and thus on yet unproved hypotheses (see, for example, Theorem 3 or Theorem 7 below).

I would like to thank Jean-Pierre Kahane for his comments on Theorem 4 and to Kohji Matsumoto for honouring me with a kanji name. Qu'ils soient ici remerciés !

2. Euler products

Many Dirichlet-series occurring in practice satisfy an Euler product and if this product is simple we often get some information on the domain of convergence of the Dirichlet series. Among such cases is the product over all primes p of a polynomial in p^{-s}. One of the oldest ideas is due to Estermann [20] who obtained a precise criterion for the continuation to the whole complex plane of the Euler product of an integer polynomial in p^{-s}. He proved the existence of the following dichotomy :

Theorem 1. *Let*

$$h(X) = 1 + a_1 X + \cdots + a_d X^d = \prod_{j=1}^{d}(1-\alpha_j X) \in \mathbb{Z}[X]$$

then $Z(h;s) = \prod_p h(p^{-s})$ *is absolutely convergent for* $\Re(s) > 1$ *and can be meromorphically continued to the half plane* $\Re(s) > 0$*. If* $h(X)$ *is a product of cyclotomic polynomials, i.e. if* $|\alpha_j| = 1$ *for every* j*, then and only then can* $Z(h;s)$ *be continued to the whole complex plane. In all other cases the imaginary axis is the natural boundary.*

The strategy of his proof was to show that every point on the line $\Re s = 0$ is an accumulation point of poles or zeros of $\mathbb{Z}$. Estermann's method was subsequently generalised by many authors.

Dahlquist [14], for example, extended the above case to h being any analytic function with isolated singularities within the unit circle. He used the concept of vertex numbers and showed that except for the case where $h(p^{-s})$ has a finite number of factors of the form $(1-p^{-\nu s})^{-\beta_\nu}$, there is a natural boundary of the zeta function at $\Re s = 0$.

Later, Kurokawa [26] continued on the idea of Estermann to cases where h depends on the traces of representations of a topological group and solved Linnik's problem for the analytic continuation of scalar products of the Hecke-L series $L(s;\chi_i)$ where χ_i are Grösssencharakters (not necessarily of finite order) of finite extensions of an algebraic number field. His result can be stated more precisely as :

Let $F/\mathbb{Q}$ be a finite extension and K_i/F be r finite extensions of degree n_i each. The scalar product $L(s;\chi_1,\ldots,\chi_r)$ has the imaginary axis as the natural boundary except when

$$\begin{aligned}(n_1,\ldots,n_r) &= (1,\ldots\ldots,1,\star) \qquad \text{or}\\ &= (1,\ldots,1,2,2),\end{aligned}$$

in which case $L(s;\chi_1,\ldots,\chi_r)$ can be continued to the whole of $\mathbb{C}$ (ibid. Part II, Theorem 4).

There is of course, no reason to believe that the natural boundary would always be a line. In an example involving the Euler-phi function [32]

$$Z(s) = \sum_{n=1}^{\infty}\frac{1}{\phi(n)^s} = \prod_p\left(1+(p-1)^{-s}(1-p^{-s})^{-1}\right),$$

the boundary of continuation is an open, simply connected, dense set of the half-plane $\Re s > -1$.

The question of analytic continuation of Euler products in several variables occur naturally in very many contexts. To cite just one example, in the study of strings over $p-$adic fields [10], products of 5-point amplitudes for the open strings are considered, where the amplitudes are defined as $p-$adic integrals

$$A_5^p(k_i) = \int_{\mathbb{Q}_p^2} \mid x \mid^{k_1 k_2} \mid y \mid^{k_1 k_3} \mid 1-x \mid^{k_2 k_4} \mid 1-y \mid^{k_3 k_4} \mid x-y \mid^{k_2 k_3} dxdy.$$

The product $\prod_p A_5^p$ can be analytically continued to the whole of $\mathbb{C}$, which gives interesting relations of such amplitudes with real ones.

We would thus like a multivariable Estermann type of theorem. For this we need some notation [4]. Let us consider n-variable integer polynomials h_k and let

$$h(X_1, \ldots, X_n, X_{n+1}) = 1 + \sum_{k=0}^{d} h_k(X_1, \ldots, X_n) X_{n+1}^k.$$

The exponents of the monomial occurring in this expression determine a polyhedron in $\mathbb{R}^n$ and enable us to give a description of the domain of convergence for the Euler product in n complex variables $Z(h; s_1, \ldots, s_n) = \prod_{p \text{ prime}} h(p^{-s_1}, \ldots, p^{-s_n})$. Thus we define, for $\delta \in \mathbb{R}$,

$$V(h;\delta) := \bigcap_{k=0}^{d} \{s \in \mathbb{C}^n \mid \Re(\langle \alpha, s \rangle) > k + \delta \quad \forall\, \alpha \in Ext(h_k)\}$$

where $Ext(h_k)$ is the set of those points which do not belong to the interior of any closed segment of the Newton polyhedron of h_k. We show that the geometry of the natural boundary is that of a tube over a convex set with piecewise linear boundary and give a criterion for its existence which is analogous to Theorem 1.

A polynomial h in several variables is called cyclotomic, if there exists a finite set of non-negative integers $m_{i,j}$ and a finite set of integers $(\gamma_j)_{j=1,\ldots,q}$ such that:

$$h(X) = \prod_{j=1}^{q} (1 - X_1^{m_{1,j}} \ldots X_n^{m_{n,j}})^{\gamma_j}.$$

In [4] we prove that either h is cyclotomic, or it determines a natural boundary of meromorphy, i.e.

Theorem 2. *The Euler product*

$$Z(h;s) = Z(h; s_1, \ldots, s_n) = \prod_p h(p^{-s_1}, \ldots, p^{-s_n})$$

converges absolutely in the domain $V(h;1)$ and can be meromorphically continued to the domain $V(h;0)$.

Moreover, $Z(h;s)$ can be continued to the whole complex space $\mathbb{C}^n$ if and only if h is cyclotomic. In all other cases $V(h;0)$ is a natural boundary.

Using Newton polyhedra we can write the above as a product of Riemann zeta functions and a holomorphic function in $V(h;1/r)$, for every natural number r, i.e.

$$Z(h;s) = \Big(\prod_{1 \le |m| \le N_r} \zeta(\langle m, s \rangle)^{\gamma(m)} \Big) G_{1/r}(s)$$

where m is a n-tuple of positive integers, $\{N_r\}$ an increasing sequence of positive integers and $G(s)$ an absolutely convergent Euler product. We then treat separately the cases where the set $\{m : \gamma(m) \neq 0\}$ is finite or infinite to show that a meromorphic continuation to $V(h;\delta)$ is not possible for any $\delta < 0$.

A result similar to the above theorem can also be obtained for Euler products of analytic functions on the unit poly-disc $P(1)$ in $\mathbb{C}^n$ rather than polynomials (op.cit. Theorem 4). However Theorem 2 is in general not enough to treat Euler products of the form $\prod_p h(p, p^{-s})$ which occur, for example, in zeta functions of groups and height zeta functions. In certain cases authors have been able to find natural boundaries of such Euler products while even for an apparently simple case like $f(s) = \prod_p \left(1 + p^{-s} + p^{1-2s}\right)$ [15] we might be unable to provide a complete answer (see the next section).

In fact it does not suffice to prove that each point is a limit point of poles or zeros of the single factors, since poles and zeros could cancel. In certain situations it is possible to find conditions which ensure that too much cancellation among potential singularities is impossible and thereby get information on series like the one just cited. For instance, in [6] we obtain :

Theorem 3. *Assume the Riemann ζ-function has infinitely many zeros off the line $\frac{1}{2}+it$. Suppose that f is a function of the form $f(s) = \prod_{\nu \geq 1} \zeta(\nu(s-\frac{1}{2})+\frac{1}{2})^{n_\nu}$ where the exponents n_ν are rational integers and the series $\sum \frac{n_\nu}{2^{\epsilon\nu}}$ converges absolutely for every $\epsilon > 0$. Then f is holomorphic in the half plane $\Re s > 1$ and has meromorphic continuation in the half plane $\Re s > \frac{1}{2}$. Denote by $\mathcal{P}$ the set of prime numbers p, such that $n_p > 0$, and suppose that for all $\epsilon > 0$ we have $\mathcal{P}((1+\epsilon)x) - \mathcal{P}(x) \gg x^{\frac{\sqrt{5}-1}{2}} \log^2 x$. Then the line $\Im s = \frac{1}{2}$ is the natural boundary of f; more precisely, every point of this line is accumulation point of zeros of f.*

To get this result we need some combinatorial geometry on the lines of Dahlquist [14]. The following is a sketch of the argument to get the above natural boundary. By assumption of the falsity of the Riemann hypothesis, for every $\epsilon > 0$ and every t there is a zero $\rho = \sigma + iT$ of ζ, such that $\mathcal{P}(T/t) - \mathcal{P}(T/((1+\epsilon)t)) \gg (T/t)^\theta \log^2(T/t)$, where $\theta = \frac{\sqrt{5}-1}{2}$. Instead of showing that this particular ρ cannot be cancelled out by poles or zeros of other factors,, we show that not all zeros can be cancelled out. If $\frac{\rho-1/2}{p} + \frac{1}{2}$ is not a zero of f for any $p \in \mathcal{P}$ and any $\frac{T}{p} \in [t, (1+\epsilon)t]$, using combinatorial arguments we reach the contradictory conclusion that $\theta < \frac{\sqrt{5}-1}{2}$. So in every square of the form $\{s : \Re\, s \in [\frac{1}{2}, \frac{1}{2}+\epsilon], \Im\, s \in [t, t+\epsilon]\}$, there is a zero of f.

Concerning general Euler products of polynomials in p and p^{-s}, there exists a conjecture [16].

Conjecture 1. *Let $W(x,y) = \sum_{n,m} a_{n,m} x^n y^m$ be an integral polynomial with $W(x,0) = 1$. Then $D(s) = \prod_p W(p, p^{-s})$ is meromorphically continuable to the whole complex plane if and only if it is a finite product of Riemann ζ-functions. Moreover, in the latter case if $\beta = \max\{\frac{n}{m} : a_{n,m} \neq 0\}$, then $\Re\, s = \beta$ is the natural boundary of D.*

Though all known examples confirm this it is still far from being resolved. In fact we believe that any refinement of Estermann's method is not enough to prove this conjecture [8].

We define an obstructing point z to be a complex number with $\Re z = \beta$, such that there exists a sequence of complex numbers z_i, $\Re z_i > \beta$, $z_i \to z$, such that D has a pole or a zero in z_i for all i. Obviously, each obstructing point is an essential singularity for D, the converse not being true in general.

Since D may not be convergent on the half-plane $\Re s > \beta$, to continue it meromorphically it is written as a product of Riemann ζ-functions and a function $R(s)$ holomorphic, zero-free, and bounded on every half-plane $\Re s > \beta + \epsilon$. Thus there exist integers $c_{n,m}$ such that

$$D(s) = \prod_{n,m} \zeta(ns+m)^{c_{n,m}} \times R(s).$$

When approximating $D(s)$ by a product of Riemann ζ-functions, the main contribution comes from monomials $a_{n,m}x^n y^m$ with $\frac{n}{m} = \beta$. We collect these monomials together in $\tilde{W}$ that is, we have

$$W(x,y) = \tilde{W}(x,y) + \sum\nolimits^*_{n,m} a_{n,m}x^n y^m,$$

where $\sum^*$ means summation over all pairs n, m with $\frac{n}{m} < \beta$ (in [17] the terminology 'ghost polynomial' is used).

We can classify such polynomials into exclusive, non-empty cases as follows :

1. $W = \tilde{W}$ and W is cyclotomic; in this case, D is a finite product of Riemann ζ-functions;
2. $\tilde{W}$ is not cyclotomic; in this case, every point of the line $\Re s = \beta$ is an obstruction point;
3. $W \neq \tilde{W}$, $\tilde{W}$ is cyclotomic, and there are infinitely many pairs n, m with $a_{n,m} \neq 0$ and $\frac{n}{m} < \beta < \frac{n+1}{m}$; in this case, β is an obstruction point;
4. $W \neq \tilde{W}$, $\tilde{W}$ is cyclotomic, there are only finitely many pairs n, m with $a_{n,m} \neq 0$ and $\frac{n}{m} < \beta < \frac{n+1}{m}$, but there are infinitely many primes p such that the equation $W(p, p^{-s}) = 0$ has a solution s_0 with $\Re s_0 > \beta$; in this case every point of the line $\Re s = \beta$ is an obstruction point;
5. None of the above; in this case, no point on the line $\Re s = \beta$ is an obstruction point.

In the third case we need an understanding of the zeros of the Riemann-zeta function to have information about the meromorphic continuation and as we will see in the next section that this may only give conditional answers. However in the last case we might be able to say nothing about the analytic continuation as we will see in the example of $D(s) = \prod_p (1 - p^{2-s} + p^{-s})$.

We would need some really new ideas to understand Euler products of polynomials in p and p^{-s}.

2.1. **A random series.** From a probabilistic point of view, it is usual to study random Dirichlet series and show that almost surely they have natural boundaries. Such generic conditions comfort us in the belief that for a Dirichlet series there should be meromorphic continuation up to an expected domain.

Often in the definition of a random series the coefficients are random (for example in Kahane [25] or Quéffelec [29]). In the following [6] we use random variables in the exponent to resemble the Euler products $W(p, p^{-s})$ discussed before.

We call a function regular in a domain if it is meromorphic up to a discrete set of branch points in the domain, that is, it is holomorphic with the exception of poles and branch points. We can now state the following probabilistic result :

Theorem 4. *Let $(a_\nu),(b_\nu),(c_\nu)$ be real sequences, such that $a_\nu, b_\nu \to \infty$, and set $\sigma_h = \limsup_{\nu\to\infty} -\frac{b_\nu}{a_\nu}$. Let ϵ_ν be a sequence of independent real random variables, such that*

$$\liminf_{\nu\to\infty} \max_{x\in\mathbb{R}} P(\epsilon_\nu = x) = 0,$$

and suppose that for $\sigma > \sigma_h$ the series

$$\sum_{\nu=1}^{\infty} \frac{|c_\nu + \epsilon_\nu|}{2^{a_\nu\sigma+b_\nu}}$$

converges almost surely. Then with probability 1 the function

$$Z(s) = \prod_{\nu=1}^{\infty} \zeta(a_\nu s + b_\nu)^{c_\nu+\epsilon_\nu}$$

is regular in the half-plane $\Re\ s > \sigma_h$ and has the line $\Re\ s = \sigma_h$ as its natural boundary.

To give an idea of the arguments used in the proof we let $s_0 = \sigma_h + it$ be a point on the supposed boundary with $t \neq 0$ rational, and consider the square S with side length $\frac{2}{n}$ centred in s_0. For $\epsilon > 0$ given, we show that with probability $> 1-\epsilon$ the function Z is either not meromorphic on S, or has a zero or a pole in S. Then for a suitably chosen index μ we consider

$$Z_\mu(s) = \prod_{\nu\neq\mu}^{\infty} \zeta(a_\nu s + b_\nu)^{c_\nu+\epsilon_\nu}.$$

such that if Z is meromorphic on S, so is Z_μ. Let D_1 be the divisor of the restriction of Z_μ to S, and let D_2 be the divisor of $\zeta(a_\mu s + b_\mu)$ restricted to S. We show that $D_1 + (c_\mu + \epsilon_\mu)D_2$ is non-trivial with probability $> 1-\epsilon$. The number of zeros of $\zeta(a_\mu s + b_\mu)$ in S equals $N(T+h) - N(T)$, where N denotes the number of zeros of ζ with imaginary part $\leq T$, and T and h are certain real numbers satisfying $T \geq 1000$ and $h \geq 6$. Using a classical estimate [2], we can show that D_2 is non-trivial.

We note that in the initial statement of the above theorem, the term 'holomorphic' appeared instead of 'regular' (Theorem 3, ibid.). In fact, as pointed out by J-P. Kahane, the finite product of ζ-functions dominating the behaviour of Z in a half-plane $\Re\ s > \sigma_h + \epsilon$ can yield branch points at all poles and zeros of the involved ζ-functions.

3. Examples

3.1. **Zeta function of a symplectic group.** The local zeta function associated to the algebraic group $\mathcal{G}$ is defined as

$$Z_p(\mathcal{G}, s) = \int_{\mathcal{G}_p^+} |\det(g)|_p^{-s}\, d\mu$$

where $\mathcal{G}_p^+ = \mathcal{G}(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$, $|\ .\ |_p$ denotes the p-adic valuation and μ is the normalised Haar measure on $\mathcal{G}(\mathbb{Z}_p)$. In [17], du Sautoy and Grunewald prove that the natural boundary of the zeta function $Z(\mathcal{G}, s)$ of the symplectic group GSp_6 given by [24]

$$Z(s/3) \quad = \quad \zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6)\prod_p \left(1 + p^{1-s} + p^{2-s} + p^{3-s} + p^{4-s} + p^{5-2s}\right)$$

has a natural boundary at $\Re\, s = \frac{4}{3}$. To show that every point on the boundary is an accumulation point of zeros, we could consider the equation $1+(1+V+V^2+V^3)U+V^3U^2 = 0$ where $V = p^{-1},\ U = p^{4-s}$. For V small, we can neglect the powers of V and then there exists a solution for the above equation in $U_0 = -1 + V + \mathcal{O}(p^{-2})$ for p large enough. Thus for every integer n there is a solution

$$s = 4 - \frac{\log(1 - p^{-1} + \mathcal{O}(p^{-2}))}{\log p} + \frac{(2n-1)\pi i}{\log p}.$$

Now for a large prime p and a fixed point A with $\Re\, s = 4$ on the boundary we can find a sequence of integers n_p such that $\frac{(2n_p-1)\pi}{\log p} \to \Im(A)$. Further the fact that

$$-\frac{\log(1 - p^{-1} + \mathcal{O}(p^{-2}))}{\log p} > 0$$

for large enough p means that $Z(s/3)$ cannot be continued beyond its assumed boundary $\Re\, s = 4$.

Notice that this is an example of the 'lucky' situation we encountered in the fourth case of the classification of $\prod W(p, p^{-s})$.

3.2. **Height zeta functions.** Several people in the recent past have studied the analytic properties of height zeta functions associated to counting rational points on algebraic varieties. Of particular interest is the case of a variety with ample anti-canonical bundle (called a Fano variety) V over a number field k whose k-rational points are Zariski dense in V, for a height function H defined naturally over the anti-canonical sheaf. Here an important motivation is Manin's conjecture that, for U a suitably defined open subset of V,

$$|\{x \in U(k) : H(x) \le t\}| \sim Ct(\log t)^{r-1}$$

as $t \to \infty$. In the above, C is a non zero constant and r the rank of the Picard group of V. There is a further conjecture, due to Peyre [28], on the constant C relating it to the Tamagawa measure.

We will concentrate on $\mathbb{P}^n$, the projective n-space over the field $\mathbb{Q}$ with the classical normalised height function $H_n : \mathbb{P}^{n-1} \to \mathbb{R}_{>0}$ defined by $H(x) = \max_i\{|x_i|\}$, where $\gcd(x_1, \ldots, x_{n+1}) = 1$. (Other definitions of the height exist but we shall not treat them here. The interested reader could see, for example, [19]).

We now give details of analytic continuation and boundaries of a few zeta functions in the above context which have Euler products in several variables.

3.2.1. *A cubic surface.* For studying the above case, it is possible to first choose the anti-canonical line bundle and assume that it be ample. This then determines a projective embedding of the desingularised model of the variety using a fan decomposition into finitely many simplical integral cones (for details see, for example, [12] or [30]). The zeta function is then defined, for $\Re s$ large enough, as

$$Z_U(s) = \sum_{x \in U} \frac{1}{H(x)^s}.$$

De la Breteche and Swinnerton-Dyer [13] proved that the zeta function associated to singular cubic surfaces has a natural boundary at $\Re s = 3/4$. We follow the treatment of [11] to give a summary of their original proof where they study the multi-variable function

$$Z(s_1, s_2, s_3) = \sum_{\substack{x_1x_2x_3=x_4^3 \\ gcd(x_1,x_2,x_3)=1}} \frac{1}{x_1^{s_1}x_2^{s_2}x_3^{s_3}}$$

outside the union of three lines in the hyper-surface $x_4 = 0$. The Euler product of this function is given by

$$\prod_p \frac{1 + \sum(1-p^{3s_i})p^{-(2s_j+s_k)} - p^{-(3s_1+3s_2+3s_3)}}{(1-p^{3s_1})(1-p^{3s_2})(1-p^{3s_3})}$$

where in the sum each of i, j, k take the values $1, 2, 3$. The above is then written with the help of functions 'convenient' for $\Re s > \frac{3}{4}$ and $F(s)$ which involves the Euler product of a rational function in two variables

$$W(x, y) = 1 + (1-x^3y)(x^6y^{-2} + x^5y^{-1} + x^4 + x^2y^2 + xy^3 + y^4) - x^9y^3$$

with $x = p^{-1/4},\ y = p^{3/4-s}$ which is convergent in a certain strip. Here again the authors succeed in establishing that every point on the assumed boundary is the limit point of a subset of zeros of the function

$$F(s) = \prod_p W(p^{-1/4}, p^{3/4-s}) \prod_{j\in J}(1 - p^{-(1+j(s-1))})$$

with $J = \{-2, -1, 0, 2, 3, 4\}$. For a fixed prime p, the number of zeros with $\Re s > \frac{3}{4}$ of W, i.e.

$$\frac{3}{4} + \frac{1}{4\sqrt{2}p^{1/4}\log p} + \mathcal{O}(\frac{1}{p^{3/4}\log p})$$

is large. Now for $\Re s > \frac{3}{4} + \frac{1}{N}$, there exist suitably chosen finite number of integers $b(k, k')$ such that

$$F(s) = \prod_{\substack{k,k' \\ k-k'/4+k'/N>1}} \zeta(k + k'(s-1))^{b(k,k')} \prod_p W_N(p^{-1/4}, p^{3/4-s})$$

where

$$W_N(p^{-1/4}, p^{3/4-s}) = W(p^{-1/4}, p^{3/4-s}) \prod_{\substack{k,k' \\ k-k'/4+k'/N>1}} (1 - p^{(k+k'(s-1))})^{b(k,k')}.$$

The zeros of $W_N(p^{-1/4}, p^{3/4-s})$ and $W(p^{-1/4}, p^{3/4-s})$ are the same. Further for every real τ one can construct a sub-sequence of its zeros which converge to $\frac{3}{4} + i\tau$ and which are not poles of

$$\prod_{\substack{k,k' \\ k-k'/4+k'/N>1}} \zeta(k + k'(s-1))^{b(k,k')}.$$

These zeros are again the zeros of $F(s)$ and therefore no continuation is possible beyond the assumed boundary.

For what concerns the asymptotics, it is known that

$$|\{x \in U : H(x) \le t\}| = tQ(\log t) + \mathcal{O}(t^{7/8+\epsilon})$$

where the degree of Q is 6 and the leading coefficient is $\frac{1}{6}\prod_p\{(1-1/p)^7(1+7/p+1/p^2)\}$.

3.2.2. *An n-fold product.* In [4], we consider instead an implicit projective embedding determined by a finite set of equations and do not need a fan decomposition.

Let X be a toric variety and $A_{d,n}=A$ a $d\times n$ integer matrix all of whose row sums are zero. The rational points of the toric variety are defined by

$$X(A):=\{(x_1,\cdots,x_n)\in\mathbb{P}^{n-1}(\mathbb{Q}):\prod_{i:a_{j,i}\geq 0}x_i^{a_{j,i}}=\prod_{i:a_{j,i}<0}x_i^{-a_{j,i}}\ \forall j\}$$

and the maximal torus $U(A)$ comprises of those elements of $X(A)$ the product of whose coordinates is non-zero. Each point in the maximal torus corresponds to a unique n-tuple of co-prime positive integers which we denote by $(m_1,\cdots,m_n)$.

We define a multivariable zeta function, for $\Re\, s_i>1$, comparable to the one used for toric varieties in [12] as

$$Z_A(s)=\sum_{m_i\in\mathbb{N}}\frac{F_A(m_1,\cdots,m_n)}{m_1^{s_1}\dots m_n^{s_n}},$$

where

$$\begin{aligned} F_A(m_1,\cdots,m_n) &= 1 \qquad \text{if gcd } (m_1,\cdots,m_n)=1,\ \prod_i m_i^{a_{j,i}}=1\ \forall j,\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

The defining equations are multiplicative and we thus get an Euler product expansion of an analytic function in n complex variables

$$Z_A(s)=\prod_p h_A(p^{s_1},\dots,p^{s_n}).$$

Now the function $h_A(X)$ is expressed as a rational function

$$\prod_{\nu\in K}(1-X^\nu)^{-c(\nu)}W(X)$$

for positive integers $c(\nu)$, a finite index set K and an integer n-variable polynomial W. We can prove, using Theorem 2, that $Z_A(s)$ has a natural boundary. In fact, it is possible to explicitly describe the whole boundary of analytic continuation (see [4], Theorem 6). The description of the analytic continuation of this zeta function can now be used to deduce the asymptotic properties of the height density function on $U(A)$ because of the equation

$$|\{x\in U(A):H(x)=\max_i|m_i(x)|\leq t\}|=C(A)\sum_{m_i\leq t}F_A(m_1,\cdots,m_n)$$

where $C(A)$ is a computable constant.

As a special case, we get asymptotic results for the number of n-fold products of relatively prime positive integers that equal the nth power of an integer. Batyrev and Tschinkel [3] showed that this problem is equivalent to the asymptotic description of the height density function on the maximal torus of the hyper-surface $x_1\cdots x_n=x_{n+1}^n$. Now that there is only one equation involved, i.e. $d=1$, we use the matrix $A_n=(1,\cdots,1,-n)$ and the rational points are

$$U(A_n):=\{(x_1,\cdots,x_{n+1})\in\mathbb{P}^n(\mathbb{Q}):x_1\cdots x_n=x_{n+1}^n,,\ x_1\cdots x_n\neq 0\}.$$

To express $h_{A_n}(X)$ precisely as a rational function on the unit poly-disc $P(1)$ we notice that if

$$h_{A_n}(X) = \sum_{\alpha} X^{\alpha}$$

for $\alpha \in \mathbb{N}_0^{n+1}$, to satisfy the definition of F_{A_n} we require that $A_n(\alpha) = 0$. Thus for a n-tuple r, we use the notation $|r|$ for its weight, i.e. the sum of its n coordinates and for all r such that $|r|/n$ is a non-negative integer, we let $l(r) = (r_1, \ldots, r_n, |r|/n)$. Further we ensure that the condition of coprimality of the p^{α_i} is met and obtain

$$h_{A_n}(X) = (\prod_{i=1}^{n}(1 - X_i^n X_{n+1})^{-1}) \sum_{|r|/n \in \mathbb{N}_0} X_1^{r_1} \ldots X_n^{r_n} X_{n+1}^{|r|/n}.$$

The sum in the expression above is not cyclotomic and this gives the natural boundary of

$$Z_{A_n}(s) = R \times \prod_p (\sum_{r \in D_n} \frac{1}{p^{\langle l(r), s\rangle}})$$

for R a finite product of Riemann zeta functions, to be

$$V(0) = \{s \in \mathbb{C}^{n+1} \mid \Re(\langle l(r), s\rangle) > 0 \quad \forall\, r \in D_n\}.$$

Using the above analytic properties and a multivariable Tauberian theorem [12] we prove that

Theorem 5. *There exists $\theta > 0$ such that*

$$|\{x \in U(A_n) : H(x) \le t\}| = tQ_n(\log t) + \mathcal{O}(t^{1-\theta})$$

where $Q_n(\log t)$ is a non-vanishing polynomial of degree $d_n = \binom{2n-1}{n} - n - 1$.

Actually we can describe the last polynomial rather precisely for all $n \ge 3$ ([4], Theorem 7).

3.3. **Unlucky cases.** In the last two subsections we could give satisfactory descriptions of the analytic behaviour of the Euler products. This need not always be possible. In the following we can only show the existence of a conditional natural boundary [6].

Proposition 1. *Suppose that there are infinitely many zeros of ζ off the line $\frac{1}{2}+it$. Then the function*

$$f(s) = \prod_p \left(1 + p^{-s} + p^{1-2s}\right)$$

has meromorphic continuation to the half plane $\Re s > \frac{1}{2}$, and the line $\Re s = \frac{1}{2}$ is the natural boundary of f.

This is an example of case (3) of our classification of the previous section. We notice that the real parts of the zeros of $f(s)$ are exactly $\frac{1}{2}$ and thus we can not construct a sub-sequence of zeros or poles which would converge on each point of the presumed natural boundary $\Re s = \frac{1}{2}$. The conditional result above is attained by expressing $f(s)$ as a product of functions 'convenient' for $\Re s > \frac{1}{2}$ and

$$\prod_{m \ge 1} \frac{\zeta((4m+1)s - 2m)}{\zeta((4m+3)s - 2m - 1)},$$

which is of the type considered in Theorem 3.

We next consider the Euler product $D(s) = \prod_p (1 - p^{2-s} + p^{-s})$ which can be written as

$$D(s) = \prod_p (1 - p^{2-s}) \prod_p (1 + \frac{p^{-s}}{1 - p^{2-s}}) = \zeta(s-2) D^*(s).$$

We expect a natural boundary at $\Re\, s = 2$, or at least an essential singularity at $s = 2$, and our only method to prove this is to approach this point from the right. But for $\Re\, s = \sigma > 2$ we estimate the second product as

$$\sum_p \sum_{n \geq 1} |p^{2n-(2n+1)s}| \leq \sum_p p^{-2} \sum_{n \geq 1} p^{-n(\sigma-2)} \leq \sum_p p^{-2} \frac{1}{1 - 2^{\sigma-2}}.$$

So the product for D^* converges absolutely in the half-plane $\Re\, s > 2$, in particular, D^* does not have any zeros or poles in this half-plane. This example falls under case (5) of the classification mentioned. It is worse than Proposition 1 where we could not unconditionally prove the existence of zeros or poles clustering on the assumed boundary whereas here such zeros or poles do not even exist.

4. No Euler products

There are numerous contexts in which we come across zeta functions that do not have an Euler product. We cite just two examples. The first, mentioned because it comes from a context quite different from the other examples we treated, is that of Dirichlet series generated by finite automata.

Roughly speaking, a sequence (u_n) with values in a finite set is d-automatic if we can compute the n-th term of the sequence by feeding the base d representation of n to a finite state machine. One of the best known among 2-automatic cases is the Thue-Morse sequence,

$$01\ 10\ 1001\ 10010110\ \cdots$$

generated by the substitution maps $0 \to 01,\ 1 \to 10$. The Dirichlet series $\sum_{n=0}^{\infty} \frac{u_n}{n^s}$ corresponding to a d-automatic sequence can be meromorphically continued to the whole complex plane. Among consequences it is proved [1] that automatic sequences have logarithmic densities. It would be interesting to know how Dirichlet series associated to non automatic sequences (like the infinite Fibonacci word generated by the substitutions $0 \to 01,\ 1 \to 0$) behave.

The second example is in several variables. The Euler-Zagier sum defined as

$$\zeta_r(s_1, \cdots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} m_1^{-s_1} (m_1 + m_2)^{-s_2} \cdots (m_1 + \cdots + m_r)^{-s_r}$$

has been studied with much enthusiasm. This function can be analytically continued to the whole $\mathbb{C}^r$ space. Matsumoto introduced the generalised multiple zeta function

$$\zeta_r((s_1, \cdots, s_r); (\alpha_1, \cdots, \alpha_r), (w_1, \cdots, w_r)) = \sum_{j=1}^{r} \sum_{m_j=0}^{\infty} \prod_{i=1}^{r} (\alpha_i + m_1 w_1 \cdots + m_i w_i)^{-s_i}$$

where w_i, m_i are complex parameters with branches of logarithms suitably defined. This too can be continued as a meromorphic function to the whole $\mathbb{C}^r$ plane. We do not wish to elaborate on this subject but the interested reader can find details elsewhere (see, for example, the expository paper [27] for references).

4.1. **Goldbach zeta function.** Here we consider the number $G_r(n)$, $r \geq 2$, of representations of n as the sum of r primes. Egami and Matsumoto[18] introduced the generating function

$$\Phi_r(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_r=1}^{\infty} \frac{\Lambda(k_1)\ldots\Lambda(k_r)}{(k_1+k_2+\cdots+k_r)^s} = \sum_{n=1}^{\infty} \frac{G_r(n)}{n^s}$$

using the von Mangoldt function Λ. This series is absolutely convergent for $\Re\, s > r$, and has a simple pole at $s = r$. It is clear that to study the analytic properties in this context it is necessary to have information on the zero-free region of ζ, the Riemann zeta function, and the presence of even one zero of ζ may prevent us from having useful information. All results that we will talk about will therefore be under the assumption of the Riemann Hypothesis (RH).

We can show that from the analytic point of view, under RH, Φ_r is determined by the case $r = 2$ [7].

Theorem 6. *Suppose that the Riemann Hypothesis is true. Then for any $r \geq 3$ there exist polynomials $f_r(s), g_r(s), h_r(s)$, such that*

$$\Phi_r(s) = f_r(s)\zeta(s-r+1) + g_r(s)\frac{\zeta'}{\zeta}(s-r+1) + h_r(s)\Phi_2(s-r+2) + R(s),$$

where $R(s)$ is holomorphic in the half-plane $\Re s > r-1$ and uniformly bounded in each half-strip of the form $\Re s > r+1$, $T < \Im s < T+1$, with $T > 0$.

This is done by computing the function using the circle method which give the three main terms. A bound (under RH) for

$$\sum_{n\leq x} \Lambda(n) e^{2\pi i\alpha n} - \sum_{n\leq x} e^{2\pi i\alpha n}$$

gives an error term of order $\mathcal{O}(x^{r-1-\delta})$ for some δ positive for all but the above three terms.

It is thus important to consider the situation of $r = 2$. We recall that this case occurs in the consideration of the Goldbach conjecture that every even integer larger than 2 is the sum of two primes. To study this problem often it is natural to consider the corresponding problem for Λ and try to show that $G_2(n) > C\sqrt{n}$.

Now, assuming the RH, the authors in [18] described the analytic continuation of Φ_2 and for obtaining a natural boundary they used unproved assumptions on the distribution of the imaginary parts of zeros of ζ. In this context we denote the set of imaginary parts of non-trivial zeros of ζ by Γ. The belief that the positive elements in Γ are rationally independent is folkloric and Fujii[21] used the following special case :

Conjecture 2. *Suppose that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4 \neq 0$ with $\gamma_i \in \Gamma$. Then $\{\gamma_1, \gamma_2\} = \{\gamma_3, \gamma_4\}$.*

In [18] an effective version of the above conjecture is formulated, i.e.

Conjecture 3. *There is some $\alpha < \frac{\pi}{2}$, such that for $\gamma_1, \ldots, \gamma_4 \in \Gamma$ we have either $\{\gamma_1, \gamma_2\} = \{\gamma_3, \gamma_4\}$, or*

$$|(\gamma_1+\gamma_2) - (\gamma_3+\gamma_4)| \geq \exp\big(-\alpha(|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|)\big),$$

and it is proven that:

Theorem 7. *Suppose the Riemann hypothesis holds true. Then $\Phi_2(s)$ can be meromorphically continued into the half-plane $\Re s > 1$ with an infinitude of poles on the line $\frac{3}{2}+it$. If in addition Conjecture 3 holds true, then the line $\Re\, s = 1$ is the natural boundary of Φ_2. More precisely, the set of points $1+i\kappa$ with $\lim_{\sigma\searrow 1} |\Phi(\sigma+\kappa)| = \infty$ is dense on $\mathbb{R}$.*

In [ibid.] the authors conjectured that under the same assumptions the domain of meromorphic continuation of Φ_r should be the half-plane $\Re s > r-1$. Notice that a direct consequence of Theorem 6 confirms the following :

Theorem 8. *If the RH holds true, then $\Phi_r(s)$ has a natural boundary at $\Re s = r-1$ for all $r \geq 2$ if and only if $\Phi_2(s)$ has a natural boundary at $\Re s = 1$.*

In [7] it is also shown that if the RH and Conjecture 2 hold true, then $\Phi_2(s)$ does have a natural boundary at $\Re s = 1$ and a singularity can be described precisely as

Theorem 9. *If the RH holds true, then Φ_2 has a singularity at $2\rho_1$, where $\rho_1 = \frac{1}{2} + 14.1347\ldots i$ is the first root of ζ. Moreover,*

$$\lim_{\sigma\searrow 0}(\sigma-1)|\phi_2(2\rho_1+\sigma)| > 0.$$

This last result helps us obtain an Ω- result for $G_r(n)$. We consider the oscillating term

$$H_r(x) = -r\sum_\rho \frac{x^{r-1+\rho}}{\rho(1+\rho)\ldots(r-1+\rho)}$$

where the summation runs over all non-trivial zeros of ζ. The generating Dirichlet series for

$$\sum_{n\leq x} G_r(n) - \frac{1}{r!}x^r - H_r(x)$$

has a singularity at $2\rho_1 + r - 1$, which gives the following :

Corollary 1. *Suppose that RH holds true. Then we have*

$$\sum_{n\leq x} G_r(n) = \frac{1}{r!}x^r + H_r(x) + \Omega(x^{r-1}).$$

In fact the quality of the error term does not improve with increasing r. We mention a few historical facts about the error term. Fujii[21] obtained under the RH

$$\sum_{n\leq x} G_2(n) = x^2/2 + \mathcal{O}(x^{3/2})$$

which he later improved [22], by explicitly writing the oscillating term, to

$$\sum_{n\leq x} G_2(n) = x^2/2 + H_2(x) + \mathcal{O}((x\log x)^{4/3}).$$

Further, in [9] we used the distribution of primes in short intervals to estimate exponential sums close to the point 0 and proved that

Theorem 10. *Suppose that the RH is true. Then we have*

$$\sum_{n\leq x} G_2(n) = \frac{1}{2}x^2 + H_2(x) + \mathcal{O}(x\log^5 x),$$

and

$$\sum_{n\leq x} G_2(n) = \frac{1}{2}x^2 + H_2(x) + \Omega(x\log\log x).$$

which confirms the conjectural value of the error term [18, Conj. 2.2].

Recently Granville[23] used the error term $\mathcal{O}((x\log x)^{4/3})$ to obtain a new characterisation of the RH, i.e. for the 'twin prime constant' C_2, the Riemann Hypothesis is equivalent to the estimate

$$\sum_{n\leq x}(G_2(n) - nC_2\prod_{p|n}\frac{p-1}{p-2}) \ll x^{3/2+o(1)}.$$

Using the GRH one could similarly find bounds for the exponential sums in question which would give

Theorem 11. *The Generalised Riemann Hypothesis for Dirichlet L-functions $L(s,\chi)$, χ mod q, is equivalent to the estimate*

$$\sum_{n\leq x,\ q|n} G_2(n) = \frac{1}{\phi(q)}\sum_{n\leq x} G_2(n) + \mathcal{O}(x^{1+o(1)}),$$

as announced in [ibid.Theorem 1C].

5. Consequences

One of the amusing consequences of the existence of a natural boundary is to suggest that there is a 'natural' limit to what can be achieved for asymptotic results associated to Dirichlet series by using complex analysis. A natural boundary could show the non-existence of certain asymptotic results involving error terms and thus imply the existence of an inverse result, i.e. an Ω-term. Usually when proving an Ω-result we first derive an explicit formula with oscillating terms and then show that these terms cannot cancel each other out for all choices of the parameters. In [6] we show that even if we allow for infinite oscillatory sums to be part of the main terms, we still get lower bounds for the error terms. Thus a natural boundary at $\Re s = \sigma$ precludes the existence of an explicit formula with main terms over the zeros of the Riemann zeta function and an error term $\mathcal{O}(x^\sigma)$. We state this more precisely as :

Proposition 2. *Let a_n be a sequence of complex numbers such that the generating Dirichlet-series has a natural boundary at $\Re s = \sigma_h$. Then there does not exist an explicit formula of the form*

$$A(x) := \sum_{n\leq x} a_n = \sum_{\rho\in\mathcal{R}} c_\rho x^\rho + \mathcal{O}(x^\theta)$$

for any sequence c with $|c_\rho| \ll (1+|\rho|)^c$ and $|\mathcal{R}\cap\{s : \Re s > \theta, |\Im s| < T\}| \ll T^c$ and for any $\theta < \sigma_h$. In particular, for any sequence α_i, β_i, $1\leq i\leq k$ and any $\epsilon > 0$ we have

$$A(x) = \sum \alpha_i x^{\beta_i} + \Omega(x^{\sigma_h-\epsilon}).$$

In practice, the integral taken over the shifted path need not always converge and we may not be able to obtain an explicit formula. This can happen even when the series is meromorphic in the entire plane, for example, the age-old divisor problem where we have an Ω-estimate of size $x^{1/4}$ though the corresponding Dirichlet-series $\zeta^2(s)$ is meromorphic on $\mathbb{C}$.

However, in certain cases we can actually obtain explicit formulae if we find bounds on the growth of the Dirichlet-series. We consider a case of an Euler product $\prod_p W(p, p^{-s})$ which we have already encountered as the p-adic zeta function of GSp_6. This can be interpreted as a counting function [5] by establishing a bijection between right cosets of $2t \times 2t$ symplectic matrices and sub modules of finite index of $\mathbb{Z}^{2t}$ which are equal to their duals and called polarised.

Theorem 12. *Denote by a_n the number of polarised sub modules of $\mathbb{Z}^6$ of order n. Then we have for every $\epsilon > 0$*

$$A(x) := \sum_{n\geq 1} a_n e^{-n/x} = c_1 x^{7/3} + c_2 x^2 + c_3 x^{5/3} + \sum_\rho \alpha_\rho x^{\frac{\rho+8}{6}} + \mathcal{O}(x^{4/3+\epsilon}), \tag{1}$$

where ρ runs over all zeros of ζ, and the coefficients c_1, c_2, c_3, and α_ρ are numerically computable constants. Moreover, the error term cannot be improved to $\mathcal{O}(x^{4/3-\epsilon})$ for any fixed $\epsilon > 0$.

The interpretation above allows us to use the zeta function $Z(GSp_6, s)$ as the generating function for a_n. Applying the Mellin transform we obtain

$$A(x) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} Z(s)\Gamma(s)x^s \, ds.$$

For σ and $\epsilon > 0$ fixed, we have $\Gamma(\sigma + it) \ll e^{-(\frac{\pi}{2}-\epsilon)t}$. We now choose a path (following Turán [31, Appendix G]) to shift the integration. The integral on this new path is bounded above by $x^{4/3+\epsilon}$. Hence, we obtain the formula

$$A(x) = \sum_{\Re\rho > 4/3+\epsilon} \Gamma(\rho)x^\rho \mathrm{res}_{s=\rho} Z(s) + \mathcal{O}(x^{4/3+\epsilon}),$$

where ρ runs over the poles of $Z(s)$, and all complex numbers $4/3+\rho/6$. We already saw that $\Re\, s = \frac{4}{3}$ is the natural boundary for $Z(s)$ and as in Proposition 2, we now get an Ω-result.

The moral of the story is not necessarily to get the best possible Ω-result but to show that a non-trivial result is obtainable by this method.

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