ASYMPTOTICS OF GOLDBACH REPRESENTATIONS

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Dedicated to Kohji Matsumoto

Abstract. We present a historical account of the asymptotics of classical Goldbach representations with special reference to the equivalence with the Riemann Hypothesis. When the primes are chosen from an arithmetic progression comparable but weaker relationships exist with the zeros of L-functions.

1. Introduction

One of the oldest open problems today, known as the Goldbach conjecture, is to know if every even integer greater than 2 can be expressed as the sum of two prime numbers. It is known since very long that the conjecture is statistically true and it is empirically supported by calculations for all numbers the threshold of which has gone up from $10,000$ in 1855 [8] to 4×10^{18} [28] in 2014. In Section 2 we give some historical results that support the Goldbach conjecture.

Though the conjecture in its totality seems out of reach at the moment it generates a lot of mathematical activity. What follows, for the most part *expository*, concerns only a few of these aspects while many are obviously omitted.

Instead of studying directly the Goldbach function $g(n) = \sum_{p_1+p_2=n} 1$ which counts the number of representations of an integer n as the sum of two primes p_1 and p_2 and is expected to be non-zero for even $n > 2$, it is convenient to treat a smoother version using logarithms. This is easier from the point of view of analysis and the preferred function here is the *weighted* Goldbach function

$$G(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$$

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where Λ denotes von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k \text{ for some prime } p \text{ and } k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

so that $g(n)$ can be recovered from $G(n)$ by the use of partial summation and the last being sufficiently large, more precisely $G(n) > C\sqrt{n}$, would imply the Goldbach conjecture. As is common in analytic number theory we study the easier question of the average order of the Goldbach functions where the first results are at least as old as Landau's.

The dominant term in these results can be obtained easily but the oscillatory term involves infinitely many nontrivial zeros of the Riemann zeta function and any good asymptotic result involving upper bounds on the error term is conditional to the Riemann Hypothesis (RH). The lower bounds are however unconditional. We indicate some such average results in Section 2.3.

Interestingly obtaining a good average order is actually equivalent to the Riemann Hypothesis. We include the proof of the last statement in Section 4 since it has to be extracted from scattered parts in other papers.

A variation of the classical Goldbach problem is one where the summands are primes in arithmetic progressions. We present some information on the exceptional set in Section 3.1 and a proof in Section 5. In this context average orders with good error terms are necessarily conditional to the appropriate Generalised Riemann Hypothesis (GRH). Equivalences with such hypotheses till now exist only in special cases (Theorem 4) and seem difficult in general because of the possible Siegel zeros. Some of this is explained in Section 3.4.

2. Goldbach Conjecture is often true

2.1. **Hardy–Littlewood conjecture.** Hardy and Littlewood [18] pursued the ideas of Hardy–Ramanujan[19] and expected an asymptotic formula to hold for $G(n)$, which is,

Conjecture 1 (Hardy–Littlewood)**.** *The approximation $G(n) \sim J(n)$ holds for even n with*

$$J(n) := nC_2 \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}$$

and $C_2 := 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$.

The above C_2 is known as the twin prime constant and is approximately 1.32. Our interest is largely in the error term

$$F(n) := G(n) - J(n)$$

of the above conjecture, where $J(n) := 0$ for odd n. What can be said about $F(n)$? How can we estimate it to show that it is indeed small? Where are the sign changes?

To tackle these questions, it is natural to consider the average $\sum_{n\le x} F(n)$ or the second moment $\sum_{n\le x} F(n)^2$. Motivated by Conjecture 1, we should expect $o(x^3)$ for the second moment and $o(x^2)$ for the average.

2.2. **Exceptional Sets.** Using what is now called the Hardy–Littlewood–Ramanujan circle method, and assuming the GRH, Hardy and Littlewood proved [18] the estimate

$$\sum_{n\le x} F(n)^2 \ll x^{5/2+\varepsilon}$$

for the second moment.

From these nontrivial bounds for the second moment we are able to deduce estimates for the number of exceptions to the representation as the sum of two primes in the following way.

Let $E(x) = \#\{n \le x, n \in 2\mathbb{N};\ n \ne p_1 + p_2\}$ denote the size of the exceptional class depending on x, i.e. the number of even integers up to x which do not satisfy the Goldbach conjecture. For these exceptions, we have $|F(n)| \ge cn$ for some constant $c > 0$, since $J(n) \ge C_2 n$. Therefore

$$x^2 E(x) \ll \sum_{\substack{n\le x\\ 2|n\\ n\ne p_1+p_2}} n^2 \ll \sum_{n\le x} |F(n)|^2 \ll x^{5/2+\varepsilon},$$

so after a dyadic dissection

$$E(x) \ll x^{1/2+\varepsilon}$$

under the GRH.

More than 60 years later Goldston [15] improved the Hardy–Littlewood bound to

$$E(x) \ll x^{1/2}\log^3 x \qquad (1)$$

still under the assumption of GRH.

By now many authors have succeeded in obtaining nontrivial bounds for $\sum_{n\le x} F(n)^2$ unconditionally and we know that the Hardy–Littlewood conjecture is true *on average*.

Just after 1937 when Vinogradov's method [34] became available, Van der Corput, Chudakov and Estermann [6, 7, 10] independently obtained the first unconditional estimate of the type $E(x) = o(x)$. More precisely, they proved that

$$\sum_{n\le x} F(n)^2 \ll x^3 \log^{-A} x$$

so that

$$E(x) \ll x \log^{-A} x \tag{2}$$

for any $A > 0$.

Thus we know that the Goldbach conjecture is true *in a statistical sense*.

In 1975 Montgomery and Vaughan [26], improved this by using an effective form of Gallagher's work on distribution of zeros of L-functions and showed that there exists a positive effectively computable constant $\delta > 0$ such that, for all large $x > x_0(\delta)$,

$$E(x) \le x^{1-\delta}.$$

The best published proof as of now is with $\delta = 0.121$, i.e.

$$E(x) \ll x^{0.879} \tag{3}$$

by Lu [24] who obtained this by a variation of the circle method. In the very recent preprint [29] of Pintz the bound

$$E(x) \ll x^{0.72} \tag{4}$$

is achieved, i.e. $\delta = 0.28$.

Further information can be found in the survey article [33] of Vaughan.

2.3. **Average Orders.** As early as 1900 an asymptote for the average order of $g(n)$ was known due to Landau [20] who showed that

$$\sum_{n\le x} g(n) \sim \frac{1}{2}\frac{x^2}{\log^2 x}.$$

His result agreed with the conjecture that $g(n)$ should be approximated by $\frac{J(n)}{\log^2 n}$.

After almost a century Fujii [12, 13] studied the oscillating term. He first obtained, assuming the RH, that

$$\sum_{n\le x} F(n) = O(x^{3/2})$$

using the work of Gallagher of 1989.

Fujii could then extract an error term smaller than the main oscillating term [13], i.e.

Theorem 1 (Fujii's theorem)**.** *Assume RH. Then, for x sufficiently large, we have*

$$\sum_{n\le x} F(n) = -4x^{3/2}\Re\Big(\sum_{\gamma>0}\frac{x^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}\Big) + O((x\log x)^{4/3})$$

where γ denotes the imaginary parts of the zeros of the Riemann zeta function.

In 2007-8 Granville [17] also studied the average $\sum_{n\le x} F(n)$ and obtained the same error term.

It was conjectured by Egami and Matsumoto that the error term would be $O(x^{1+\epsilon})$ for ϵ positive and this was reached by Bhowmik and Schlage-Puchta [2] using the distribution of primes in short intervals to estimate exponential sums close to 0. More precisely, under the assumption of the RH, the asymptotic result therein can be stated in the form

$$\sum_{n\le x} G(n) = \frac{1}{2}x^2 + H(x) + O(x\log^5 x)$$

where $H(x) = -2\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$ is the oscillating term involving ρ, the non-trivial zeros of the Riemann zeta function.

Languasco and Zaccagnini [22] used the circle method to improve the power of the logarithm in the error term from 5 to 3.

Goldston and Yang [16] could also reach $O(x\log^3 x)$ in 2017 using the method of [2].

2.4. **Equivalence with RH.** While it is now clear that error terms for the average of $G(n)$ depends on the zeros of the Riemann zeta function $\zeta(s)$ and hence is meaningful only assuming a suitable hypothesis, it is interesting to see whether we can obtain information on the zeros of $\zeta(s)$ if we have an asymptotic expansion. This was investigated by Granville in 2007 who stated that (Theorem 1A in [17])

Theorem 2. *The RH is equivalent to the estimate*

$$\sum_{n\le x} F(n) \ll x^{3/2+o(1)}.$$

The sketch of proof in [17] is not sufficient to obtain the RH from the above asymptote. However in the recent paper by Bhowmik, Halupczok, Matsumoto and Suzuki [4] we were able to reconstruct part of the proof of this theorem and it was completed with Ruzsa in [5]. Some details are given in Section 4.

2.5. **Omega-results.** The lower bounds arise while showing that there exist n for which $G(n)$ is large. Since the two main terms in the asymptotic expansion of the average are continuous this already contributes to the error term. It is to be noted that the lower bounds obtained are uncondtional unlike the upper ones.

The first observation seems to be that of Prachar back in 1954 [30] when he proved that there are infinitely many integers n such that

$$g(n) > C\frac{n}{\log^2 n}\log\log n.$$

His study was on the lines of Erdős earlier for the number of solutions of n as the sum of 2 prime squares with combinatorial arguments for the number of primes in residue classes modulo an integer with many prime factors.

Half a century later Giordano [14] studied the irregularity of $g(n)$ depending on whether or not n is divisible by many small primes by using the Prime Number Theorem for arithmetic progression.

Having overlooked these earlier results the authors of [2] again showed that the error term in $\sum_{n\le x} G(n)$ is $\Omega(x\log\log x)$ by considering the exceptional bounded modulus for which a Siegel zero might exist and using Gallagher's density estimates.

Another way to find an omega term is to study the natural boundary of the generating function $\sum_n \frac{G(n)}{n^s}$ as was mentioned in [3].

3. Goldbach representations in arithmetic progressions

As a restricted form of the original problem one considers the possibility of representing every even number as the sum of two primes in a given residue class. Here again the conjecture is known to be almost always true. The exceptional set

$$E(x;q,a,b) = \#\{n \le x, n \in 2\mathbb{N};\ n \ne p_1 + p_2;\ p_1 \equiv a(q), p_2 \equiv b(q)\}$$

is shown, for example in [23], to satisfy, for an effectively computable positve δ

$$E(x;q,a,b) \ll \frac{x^{1-\delta}}{\phi(q)}$$

for all $q \le x^\delta$.

3.1. **Exceptional Sets.** We can apply the results of the estimation of the exceptional set of the unrestricted Goldbach representation to obtain similar results for an arithmetic progression with residue h mod q. Similar to $E(x;q,a,b)$, we define

$$E_{h,q}(x) := \#\{n \le x, n \in 2\mathbb{N};\ n \ne p_1 + p_2;\ n \equiv h(q)\}.$$

We show that elementary methods already suffice to deduce nontrivial estimates for the number of exceptions in an arithmetic progression from available nontrivial estimates for $E(x)$.

Thus a very simple deduction from (2) gives the statement that for *almost all residues* h mod q we have

$$E_{h,q}(x) \ll \frac{x}{q \log^C x} \tag{5}$$

arising from the fact that $\sum_{0 \le h < q} E_{h,q}(x) = E(x)$ so that if

$$H_q := \{0 \le h < q;\ E_{h,q}(x) > xq^{-1} \log^{-C} x\}$$

denotes the set of exceptional residues mod q, we have

$$\#H_q \frac{x}{q \log^C x} < \sum_{h \in H_q} E_{h,q}(x) \le E(x) \ll x(\log^{-2C} x),$$

and hence $\#H_q \ll q \log^{-C} x$.

We remark that the estimate (5) above, for all h mod q, is already nontrivial for modulii q with $q \gg \log^C x$ for any $C > 0$ which is equivalent to the condition

$$\frac{x}{q \log^C x} \ll \frac{x}{\log^{2C} x}.$$

Let us now introduce a handy notation. If $B \subseteq \mathbb{N}$, we write

$B(x) := \#\{n \le x;\ n \in B\}$ and $B_{h,q}(x) := \#\{n \le x;\ n \in B,\ n \equiv\ h\ (q)\}$.

Lemma 1 (Number of exceptions in progressions)**.** *Let $B \subseteq \mathbb{N}$ be such that $B(x) \ll x \log^{-A} x$ for any $A > 0$. Then for all $C > 0$ and for almost all $q \leq x^{1/2}$ we have $B_{h,q}(x) \ll xq^{-1} \log^{-C} x$, uniformly in all residues h mod q.*

A proof is given in Section 5.

From this Lemma and (2) we know that for any $C > 0$, and for almost all $q \leq x^{1/2}$ we have $E_{h,q}(x) \ll xq^{-1} \log^{-C} x$ unconditionally for all h mod q.

It follows immediately that for *almost all* modulii q with linebreak $x^{1/2} \log^{D} x \ll q \ll x^{1/2}(\log x)^{D}$, we have

$$E_{h,q}(x) \ll x^{1/2} \log^{C} x$$

for all h mod q, a bound known *for all* modulii under the assumption of the Riemann hypothesis.

Assuming RH, so that (1) is available, we conclude as in the proof of Lemma 1, that for almost all q with $x^{1/2} \ll q \leq x^{1/2}$ we have

$$E_{h,q}(x) \ll x^{7/8+\varepsilon}/q \ll x^{3/8+\varepsilon}$$

for all h mod q. (Since by (14), for the number $\#\mathcal{M}_Q$ of exceptions in question, $\#\mathcal{M}_Q \cdot x^{7/8+\varepsilon}/Q \ll x^{3/4} E(x)^{1/4} (\log x) \ll x^{7/8} \log^2 x$, so $\#\mathcal{M}_Q = o(Q)$.)

Working with the best published unconditional bound (3) we can likewise deduce that

$$E_{h,q}(x) \ll x^{0.97+\varepsilon}/q \ll x^{0.47+\varepsilon}$$

for all h mod q and almost all q with $x^{1/2} \ll q \leq x^{1/2}$.

If we were working with the latest available bound (4) instead, we would deduce

$$E_{h,q}(x) \ll x^{0.93+\varepsilon}/q \ll x^{0.43+\varepsilon}$$

for all h mod q and almost all q with $x^{1/2} \ll q \leq x^{1/2}$.

3.2. **Mean Value.** In [17], Granville studied the mean value of the Goldbach representation number $G(n)$ in an arithmetic progression, that is the sum

$$\sum_{\substack{n \leq x \\ n \equiv c\ (q)}} G(n), \tag{6}$$

in particular for the fixed residues $c = 2$ and $c = 0$ and stated the estimate $\sum_{n\le x, n\equiv 2\ (q)} F(n) \ll x^{3/2+o(1)}$. We introduced the technically useful sum [4]

$$S(x; q, a, b) = \sum_{n\le x} \sum_{\substack{\ell+m=n \\ \ell\equiv a, m\equiv b (q)}} \Lambda(\ell)\Lambda(m)$$

a special case of which, $\sum_{a\ (q)} S(x; q, a, c - a)$ is the above (6),

and proved that, for $(ab, q) = 1$,

$$S(x; q, a, b) = \frac{x^2}{2\phi(q)^2} + O(x^{1+B_q})$$

where B_q depends on the non-trivial zeros of the associated Dirichlet L-functions. The oscillating term was also extracted but we do not discuss it here.

3.3. **Equivalence with GRH-DZC.** Parallel to the equivalence of the RH and the asymptotic expansion of the classical Goldbach average, it was believed that there is an equivalence between the RH for Dirichlet L-functions $L(x, \chi)$ over all characters χ mod m which are odd squarefree divisors of q and the estimate $\sum_{n\le x, n\equiv 2\ (q)} F(n) \ll x^{3/2+o(1)}$ ([17], Thm.1B).

In [4] the same question was studied and the above claim was recovered though partially, i.e. only in the case $b = a$, and additionally under the assumption of one more conjecture on the non-trivial zeros of L-functions, called the Distinct Zero Conjecture (DZC), more precisely,

> *For any $q \ge 1$, any two distinct Dirichlet L-functions associated with characters of modulus q do not have a common non-trivial zero, except for a possible multiple zero at $s = 1/2$.*

The equivalence obtained till now does not cover all residues a and b.

Theorem 3 (from [4, Thm.1]). *Let* DZC *be true, q be odd and $(a, q) = 1$. Then, for any $\varepsilon > 0$, the asymptotic formula*

$$S(x; q, a, a) = \frac{x^2}{2\varphi(q)^2} + O_{q,\varepsilon}(x^{3/2+\varepsilon}) \tag{7}$$

is equivalent to the GRH for the functions $L(s, \chi)$ with any character χ mod q.

Continuing in the direction of possible equivalences a bit more can be said using the notation $B_\chi = \sup\{\Re\rho_\chi\}$ and $B_q = \sup\{B_\chi \mid \chi\ (q)\}$ where ρ_χ are the non-trivial zeros of $L(s,\chi)$.

Theorem 4 (from [4, Thm.3])**.** *Let q, c be integers with $(2,q) \mid c$. Assuming the GRH, we have*

$$\sum_{\substack{n\le x\\ n\equiv c\,(q)}} F(n) \ll x^{3/2}. \tag{8}$$

On the other hand if we assume that

$$\sum_{\substack{n\le x\\ n\equiv c\,(q)}} F(n) \ll_{q,\varepsilon} x^{3/2+\varepsilon} \tag{9}$$

holds for any $\varepsilon > 0$ and that there exists a zero ρ_0 of $\prod_{\chi\,(q)} L(s,\chi)$ such that

(a) $B_q = \Re\rho_0$
(b) ρ_0 belongs to a unique character $\chi_1\ (q)$
(c) the conductor q^* of $\chi_1\ (q)$ is squarefree and satisfies $(c, q^*) = 1$,

then $B_q = \Re\rho_0 \le 1/2$.

Thus the GRH can be deduced only under several additional assumptions (a),(b) and (c), from the asymptotic expansion (9).

It is worth mentioning that from (8) above, it is immediate that $\sum_{n\le x} F(n) \ll x^{3/2}$ when $q = 1$ which in turn implies that under the RH we get the bound $E(x) \ll x^{1/2}\log x$. The last is an improvement on Goldston's result (1).

3.4. **Bombieri–Vinogradov Theorem and Siegel Zeros.** Under the assumption that the GRH can be deduced for all $L(s,\chi)$ for any χ mod q from asymptotic expansions we would have a very short proof of Bombieri–Vinogadov's theorem. Let us dwell on this possibility.

Let $D_q(x) := \sum_{\substack{n\le x\\ n\equiv 2(q)}} (G(n) - J(n))$ and $\Delta_q(x) := \max^*_{a(q)} |\psi(x; q, a) - \frac{x}{\varphi(q)}|$, where trivially $\Delta_q(x) \ll xQ^{-1}\log^2 x$. Let

$$E := \{q \in \mathbb{N};\ Q < q < 2Q,\ |D_q(x)| \ge x^{3/2+\delta} \text{ for some } \delta > 0\},$$

so that $q \notin E \implies D_q(x) \ll x^{3/2+o(1)}$. From our assumption this would imply GRH for $L(s,\chi)$ with any χ mod q. Then $\Delta_q(x) \ll x^{1/2}\log^2 x$.

Hence

$$\#E\, x^{3/2+\delta} \le \sum_{q\in E} |D_q(x)|,$$

and by Lemma 2 below we would have

$$\sum_{q\in E} |D_q(x)| \le \Big(\sum_{n\le x} |G(n)-J(n)|^2\Big)^{1/2} x^{1/2}\log^{3/2} x \ll x^2 \log^{7/2} x$$

by trivially estimating the last sum as $x^3\log^4 x$. From this, we could deduce that $\#E \ll x^{1/2-\delta}\log^4 x$.

For the left hand side of the Bombieri–Vinogradov's theorem this yields

$$\sum_{Q<q\le 2Q} \Delta_q(x) = \sum_{q\in E}\Delta_q(x) + \sum_{q\notin E}\Delta_q(x)$$
$$\ll x^{1/2-\delta} x Q^{-1}\log^6 x + Q x^{1/2}\log^2 x \ll x \log^{-A} x$$

if we assume that $x^{1/2-\delta}\log^{6+A} x \ll Q \ll x^{1/2}\log^{-2-A} x$.

The elementary lemma used just above is :

Lemma 2. *For any sequence $(v_n)_{n\in\mathbb{N}}$ of complex numbers and any integer a with $a<x$ we have*

$$\sum_{Q<q\le 2Q} \Big| \sum_{\substack{n\le x\\ n\equiv a(q)}} v_n \Big| \le \Big(\sum_{n\le x} |v_n|^2\Big)^{1/2} x^{1/2}\log^{3/2} x.$$

Proof. The left hand side is equal to the sum $\sum_{Q<q\le 2Q} |\langle v,\varphi_q\rangle|$ with $\varphi_q(n) = 1$ if $n \equiv a \bmod q$ and $\varphi_q(n) = 0$ otherwise. By Halasz–Montgomery's inequality, this is

$$\le \Big(\sum_{n\le x}|v_n|^2\Big)^{1/2}\Big(\sum_{q_1,q_2}\langle\varphi_{q_1},\varphi_{q_2}\rangle\Big)^{1/2}$$

with

$$\langle\varphi_{q_1},\varphi_{q_2}\rangle = \sum_{n\le x}\sum_{q_1|n-a}\sum_{q_2|n-a} 1 = \sum_{s,0<x-a\le s\le 2x-a}\sum_{q_1|s}\sum_{q_2|s} 1 \ll x\log^3 x.$$

□

Concerning the Siegel zeros, Fei in [11] has studied a similar question. Assuming a certain version of *weak* Goldbach conjecture, namely if for all even $n>2$,

$$G(n) \ge \frac{\delta n}{\log^2 n},$$

then the possible Siegel zero β for χ mod q, where q is a prime, $q \equiv 3$ mod 4 and satisfies

$$\beta \leq 1 - \frac{c}{\log^2 q}$$

for some constant $c > 0$. Thus here we get a repulsion of the Siegel zero from the line $\Re s = 1$ while assuming (8) on average we deduce $B_q \leq 1/2$ under (a), (b) and (c), which is a repulsion of all the nontrivial zeros of L-functions mod q from the line $\Re s = 1$.

4. Proof of equivalence of RH and Goldbach average

In this section, we give an overview of the proof of Theorem 2, that the Riemann Hypothesis is equivalent to the estimate

$$\sum_{n \leq x} (G(n) - J(n)) \ll_\varepsilon x^{3/2+\varepsilon} \tag{10}$$

for any $\varepsilon > 0$.

Since $\sum_{n \leq x} J(n) - x^2/2 \ll x \log x$, we can write (10) equivalently as

$$\sum_{n \leq x} G(n) = \frac{x^2}{2} + O(x^{3/2+\varepsilon}) \tag{11}$$

for any $\varepsilon > 0$.

We concentrate on the proof of the deduction of the RH from (10) that is to obtain $B = 1/2$ for $B := \sup\{\Re\rho;\ \zeta(\rho) = 0\}$.

Step 1. Let $S(x) := \sum_{n \leq x} G(n)$ be the summatory function of $G(n)$. A key issue is the proof of the asymptotic formula

$$S(x) = x^2/2 + \sum_\rho r(\rho) \frac{x^{\rho+1}}{\rho+1} + E(x)$$

with $E(x) \ll x^{2B} \log^5 2x$ and $r(\rho) := -2/\rho$ (Theorem 2 in [4]). This involves a careful transformation of the problem to an exponential sum setting where Gallagher's lemma can be used, confer [4, Lemma 9].

Step 2. We define the Goldbach generating Dirichlet series as

$$F(s) = \sum_{n=1}^{\infty} \frac{G(n)}{n^s}.$$

This can be computed by using $S(x)$ in the integral

$$F(s) = s \int_1^\infty S(u) u^{-s-1} \, \mathrm{d}u,$$

where inserting the formula from Step 1 yields

$$F(s) = \frac{s}{2(s-2)} + \sum_\rho \frac{r(\rho)s}{(\rho+1)(s-\rho-1)} + s\int_1^\infty E(u)u^{-s-1}\,\mathrm{d}u$$
$$= \frac{1}{s-2} + \sum_\rho \frac{r(\rho)}{s-\rho-1} + s\int_1^\infty E(u)u^{-s-1}\,\mathrm{d}u + C_1, \tag{12}$$

with

$$C_1 = \frac{1}{2} + \sum_\rho \frac{r(\rho)}{\rho+1}$$

and $r(\rho) = -2/\rho$.

From the above we can read off that for $\sigma > 2$, the function $F(s)$ converges absolutely and is *analytic*.

Moreover $F(s)$ can be *continued meromorphically* to the half plane $\sigma > 2B$ since $E(u) \ll u^{2B}\log^5(2u)$.

Step 3. Assume $B < 1$. Then we have

$$1 + B = \inf\{\sigma_0 \geq \frac{3}{2} \mid F(s) - \frac{1}{s-2} \text{ is analytic on } \sigma > \sigma_0\}. \tag{13}$$

Step 2 shows that the infimum on the right is at most $2B \leq B+1$,

For the inequality in the other sense we observe that this is trivially true for $B = 1/2$, so we may assume that $1/2 < B < 1$.

Now $\max(2B, 3/2) < 1+B$ being a strict inequality, there exists a real number $\varepsilon > 0$ such that $\max(2B, 3/2) < 1+B-\varepsilon$ holds true. Then by the definition of B, there exists a zero ρ such that $1/2 < B-\varepsilon < \Re\rho$.

From the formula for $F(s)$ from Step 2, the function has a pole at $\rho+1$ with residue $r(\rho) = -2/\rho \neq 0$ in the half plane $\sigma > 1+B-\varepsilon > 3/2$, and we conclude that

$$1 + B - \varepsilon \leq \inf\{\sigma_0 \geq \frac{3}{2} \mid F(s) - \frac{1}{s-2} \text{ is analytic on } \sigma > \sigma_0\}.$$

Letting $\varepsilon \to 0$, we obtain the desired goal.

Step 4. Now let $D(x) = \sum_{n\leq x} G(n) - \frac{x^2}{2}$, and we have $D(x) \ll x^{3/2+\varepsilon}$ from (11). Hence, as in the proof of (12),

$$F(s) - \frac{1}{s-2} = s\int_1^\infty D(u)u^{-s-1}\,\mathrm{d}u + \frac{1}{2}$$

for $\sigma > 2$, where the right-hand side gives an analytic function on $\sigma > 3/2$ since $D(u) \ll u^{3/2+\varepsilon}$ from (11).

Therefore, by (13) from Step 3, we conclude that $B \le 1/2$ provided that $B < 1$, hence RH.

Step 5. We now need to exclude the possibility that (11) could imply $B = 1$ (see [5]). For this, let $|z| < 1$ and consider the power series

$$f(z) = \sum_{n \ge 1} \Lambda(n) z^n \text{ and } f^2(z) = \sum_{n \ge 1} G(n) z^n,$$

so that

$$\frac{1}{1-z} f^2(z) = \sum_{n \ge 1} S(n) z^n,$$

again with the summatory function $S(x) = \sum_{m \le x} G(m)$ of the Goldbach function $G(m)$. Then

$$\frac{1}{1-z} f^2(z) = \frac{1}{(1-z)^3} + O(N^{5/2+\varepsilon})$$

on the circle $|z| = e^{-1/N}$, which can be reformulated as

$$f^2(z) = \frac{1}{(1-z)^2} + O(|1-z| N^{5/2+\varepsilon}).$$

This yields an asymptotic formula on the major arc $|1-z| \le cN^{-C/3}$. Taking the complex $s$quare root yields

$$f(z) = \pm \frac{1}{1-z} + O(|1-z|^2 N^{5/2+\varepsilon}).$$

Due to continuity and non-negativity of the coefficients of $f(z)$, we have the plus sign throughout the whole major arc.

Now by Cauchy's integral formula, we obtain

$$\psi(N) = \frac{1}{2\pi i} \int_{|z|=R} f(z) K(z) \, \mathrm{d}z, \quad N = \int_{|z|=R} \frac{1}{1-z} K(z) \, \mathrm{d}z$$

for the kernel

$$K(z) = z^{-N-1} + z^{-N} + \cdots + z^{-2} = z^{-N-1} \frac{1-z^N}{1-z}.$$

The contribution of $f(z)$ to this integral is $O(N^{5/6})$ on the major arc, and only $O(N^{11/12+\varepsilon})$ for the minor arc (which needs a little care to prove).

Comparing this with the explicit formula for $\psi(N)$, we conclude that $B < 11/12 < 1$.

5. Proof of Lemma on the number of exceptions in progressions

For a positive integer N, for $a_1, \dots, a_N \in \mathbb{C}$ and a residue h mod q denote

$$Z := \sum_{n \le N} a_n \text{ and } Z(q,h) := \sum_{\substack{n \le N \\ n \equiv h \ (q)}} a_n.$$

We start with the proof of the following

Theorem 5. *For any real $H > 0$ and $Q > 1$ we have*

$$\sum_{Q<m\le 2Q} m \max_{h \bmod m} |Z(m,h)|^2$$
$$\le (N^2+Q^2)\frac{\log Q}{H} \max_{n\le N} |a_n|^2 + (N+Q^2)H \log Q \sum_{n \le N} |a_n|^2.$$

Proof. We write $m \sim Q$ for $Q < m \le 2Q$. Split the left hand side of the theorem into $E_1 + E_2$ with

$$E_1 := \sum_{\substack{m\sim Q \\ \tau(m)>H}} m \max_{h \bmod m} |Z(m,h)|^2$$

and

$$E_2 := \sum_{\substack{m\sim Q \\ \tau(m)\le H}} m \max_{h \bmod m} |Z(m,h)|^2,$$

where $\tau(m)$ denotes the number of divisors of m.

Consider first E_1. Let

$$A := \#\{m \sim Q;\ \tau(m) > H\},$$

then

$$AH < \sum_{\substack{m\sim Q \\ \tau(m)>H}} \tau(m) \le \sum_{m \le 2Q} \tau(m) \ll Q \log Q,$$

so

$$A \ll \frac{Q \log Q}{H}.$$

Since $Z(m,h) \ll (\frac{N}{m}+1) \max_{n\le N} |a_n|$ we get

$$E_1 \ll \sum_{\substack{m\sim Q \\ \tau(m)>H}} m \max_h |Z(m,h)|^2 \ll \sum_{\substack{m\sim Q \\ \tau(m)>H}} m \left(\frac{N^2}{m^2}+1\right) \max_{n\le N} |a_n|^2$$

$$\ll A\Big(\frac{N^2}{Q}+Q\Big)\max_{n\le N}|a_n|^2 \ll \Big(\frac{N^2}{H}+\frac{Q^2}{H}\Big)\log Q\max_{n\le N}|a_n|^2.$$

This is the first summand on the right hand side of Theorem 5.

Now we look at E_2. From Theorem 6 below we have

$$E_2=\sum_{\substack{m\sim Q\\ \tau(m)\le H}} m\max_{0<h\le m}|Z(m,h)|^2\le\sum_{d\le 2Q}M'_d\sum_{\substack{0<b\le d\\ (b,d)=1}}\Big|T\Big(\frac{b}{d}\Big)\Big|^2.$$

Now we estimate

$$M'_d=\sum_{\substack{m\sim Q,d|m\\ \tau(m)\le H}}\frac{\tau(m)d}{m}\ll H\log Q$$

and an application of the *large sieve inequality* yields

$$E_2\ll H\log Q\ (N+Q^2)\sum_{n\le N}|a_n|^2,$$

which is the second term on the right hand side of Theorem 5. □

We use the following result.

Theorem 6. *We have the estimates*

$$\sum_{m\in\mathcal{M}} m\max_{h \bmod m}\Big|Z(m,h)-\frac{Z}{m}\Big|^2\le\sum_{d=2}^{\infty}M'_d\sum_{\substack{0<b<d\\ (b,d)=1}}\Big|T\Big(\frac{b}{d}\Big)\Big|^2$$

and

$$\sum_{m\in\mathcal{M}} m\max_{h \bmod m}|Z(m,h)|^2\le\sum_{d=1}^{\infty}M'_d\sum_{\substack{0<b\le d\\ (b,d)=1}}\Big|T\Big(\frac{b}{d}\Big)\Big|^2$$

with

$$M'_d:=\sum_{t,td\in\mathcal{M}}\frac{\tau(dt)}{t}.$$

Here τ denotes the divisor function. Note that $M'_d\ll\tau(d)\log^2 Q$ if $\mathcal{M}\subseteq\{1,\dots,\lfloor Q\rfloor\}$ for a real number $Q>1$.

Proof. Let

$$f_h(m):=\sum_{d|m}\mu(d)\frac{m}{d}Z\Big(\frac{m}{d},h\Big),$$

then we have by Möbius inversion

$$\sum_m m \max_h |Z(m,h) - Z/m|^2 = \sum_m \frac{1}{m} \max_h |mZ(m,h) - Z|^2$$
$$= \sum_m \frac{1}{m} \max_h \Big| \sum_{d|m} f_h(d) - Z \Big|^2 = \sum_m \frac{1}{m} \max_h \Big| \sum_{\substack{d|m \\ d\neq 1}} f_h(d) \Big|^2$$
$$\leq \sum_m \frac{\tau(m)}{m} \sum_{\substack{d|m \\ d\neq 1}} \max_h |f_h(d)|^2.$$

Now we note that $f_h(d)$ is d-periodic in h for $d|m$, since $Z(t, h+d) = Z(t,h)$ for $t|d$, so

$$f_{h+dl}(d) = \sum_{t|d} \mu(t) \frac{d}{t} Z\Big(\frac{d}{t}, h + dl\Big) = \sum_{t|d} \mu(t) \frac{d}{t} Z\Big(\frac{d}{t}, h\Big) = f_h(d)$$
$$\text{for all } l \in \mathbb{Z},$$

therefore the maximum remains the same if taken only over h with $0 < h \leq d$. We estimate this maximum by $\sum_{0<h\leq d}$, therefore an upper estimate for $\max_{0<h\leq q} |f_h(d)|^2$ is

$$\sum_{h=1}^{d} |f_h(d)|^2 = d \sum_{\substack{0\leq b\leq d \\ (b,d)=1}} \Big| T\Big(\frac{b}{d}\Big)\Big|^2$$

by Montgomery's formula [25], namely

$$q \sum_{h=1}^{q} \Big| \sum_{d|q} \frac{\mu(d)}{d} Z\Big(\frac{q}{d}, h\Big)\Big|^2 = \sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \Big| T\Big(\frac{a}{q}\Big)\Big|^2$$

for the exponential sum

$$T(\alpha) := \sum_{n\leq N} a_n e(\alpha n).$$

So we get

$$\sum_m m \max_h |Z(m,h) - Z/m|^2 \leq \sum_{d=2}^{\infty} M'_d \sum_{\substack{0<b<d \\ (b,d)=1}} \Big| T\Big(\frac{b}{d}\Big)\Big|^2$$

with

$$M'_d := \sum_{m\in\mathcal{M}, d|m} \frac{\tau(m)d}{m} = \sum_{t, dt\in\mathcal{M}} \frac{\tau(td)}{t}.$$

This shows the first inequality of Theorem 6.

The only change for the proof of the second inequality when replacing $|Z(m,h) - Z/m|^2$ by $|Z(m,h)|^2$ is to include the summand for $d = 1$ in the sums over d. □

Now we give the proof of Lemma 1.

Proof. Let $Q \leq x^{1/2}$ and $N = \lfloor x \rfloor$. Then Theorem 5, used with the indicator function for B as sequence $(a_n)_{n\leq N}$, shows

$$\sum_{q\sim Q} \max_{h \bmod q} B_{h,q}(x) \ll (x^2H^{-1} + HxB(x))^{1/2} \log^{1/2} Q.$$

The optimal choice for H is $H = (x/B(x))^{1/2}$, and therefore we have

$$\sum_{q\sim Q} \max_{h \bmod q} B_{h,q}(x) \ll x^{3/4}B(x)^{1/4} \log^{1/2} Q. \tag{14}$$

So if $B(x)$ is small, we expect $B_{h,q}(x)$ to be small too.

Consider therefore for $C > 0$ the number of exceptional modulii

$$\mathcal{M}_Q := \{q \sim Q;\ B_{h,q}(x) > \frac{x}{q \log^C x} \text{ for any } h \bmod q\}.$$

It follows that

$$\begin{aligned} \#\mathcal{M}_Q \cdot \frac{x}{Q\log^C x} &\ll \sum_{q\in\mathcal{M}_Q} \max_h B_{h,q}(x) \\ &\ll x^{3/4}B(x)^{1/4}\log^{1/2} Q \ll \frac{x}{\log^{2C+1} x}, \end{aligned}$$

so $\#\mathcal{M}_Q \ll \frac{Q}{\log^{C+1} x}$.

If we split $[1, x^{1/2}]$ into $\ll \log x$ many dyadic intervals, we get

$$\#\{q \leq x^{1/2};\ B_{h,q}(x) > \frac{x}{q\log^C x} \text{ for any } h \bmod q\} \ll \frac{x^{1/2}}{\log^C x}.$$

This proves the Lemma. □

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