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Davenport's Constant for Groups of the Form $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$

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Abstract. We determine Davenport's constant for all groups of the form $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$.

1. Introduction and notation

For a finite abelian group G let $D(G)$ be the Davenport's constant, that is, the least integer n such that among each sequence g_i in G there exists a non-empty subsequence g_{i_k} with sum 0. Writing G as

$$G \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r}, \quad d_1 \mid d_2 \mid \cdots \mid d_r,$$

we obtain a sequence of $\sum_i d_i - r$ elements without a zerosum subsequence, thus we have the trivial bound $D(G) \geq M(G) = \sum_i d_i - r + 1$. It has been conjectured that $D(G) = M(G)$ holds true for all finite groups, and this conjecture was proved for various special cases, including finite p-groups, and groups of rank $r \leq 2$. However, there are infinitely many counterexamples known for every rank $r \geq 4$. It is unknown whether $D(G) = M(G)$ holds true for all groups of rank 3, the authors are inclined to believe that this is always the case. The simplest undecided case up to now is $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{15}$, which was already mentioned by van Emde Boas and Kruyswijk [5]. In the present note this case is solved, more generally, we show the following.

Theorem 1. *Let d be an integer, $A \subseteq \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ be a multiset consisting of $3d + 4$ elements. Then there exists a multiset $B \subseteq A$, such that $\sum_{b \in B} b = 0$.*

Our approach is inspired by an idea of Delorme, Ordaz and Quiroz [1]. Suppose we are given a sequence A of $3d + 4$ points in $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$. Consider the image $\tilde{A}$ of this sequence under the canonical projection $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d} \to \mathbb{Z}_3^3$. If this sequence contains a family of d pairwise disjoint subsequences adding up to zero, we obtain a sequence of d elements in $\mathbb{Z}_d$, each of which is represented as a sum of certain elements in A. Among these elements we choose a subsequence adding up to 0, and find that A contains a subsequence adding up to 0. Using this method Delorme, Ordaz and Quiroz showed that for groups of the form $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ we have $D(G) \leq M(G) + 2$. Unfortunately, this inequality is the best possible,

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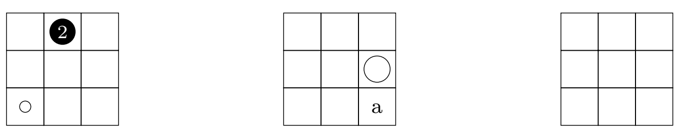
since for every $d \geq 3$ there exists a sequence $A \subseteq G$ with $3d+5$ elements, which does not contain d pairwise disjoint zerosum subsets. To remedy this, we note that for $(d,3)=1$ we have

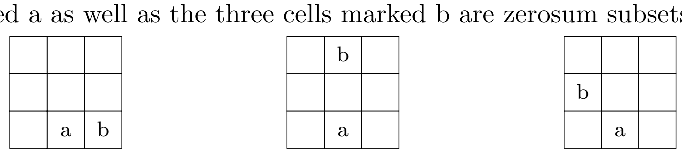
$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d} \cong \mathbb{Z}_3^3 \oplus \mathbb{Z}_d,$$

thus we can represent a sequence $A \subseteq \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$ as a pair $(\tilde{A}, f)$, where $\tilde{A} \subseteq \mathbb{Z}_3^3$ is the image of A under the canonical projection, and $f\colon \tilde{A} \to \mathbb{Z}_d$ is the function such that the element $a_i \in A$ is represented as $\big(\tilde{a}_i, f(\tilde{a}_i)\big)$ in $\mathbb{Z}_3^3 \oplus \mathbb{Z}_d$. This idea allows us to concentrate only on the small group $\mathbb{Z}_3^3$; in fact, the remainder of this article deals only with combinatorial properties of $\mathbb{Z}_3^3$.

Since the order of elements plays no rôle, we will henceforth speak of multisets instead of sequences. To visualize the combinatorial considerations, we view $\mathbb{Z}_3^3$ as the elements of a $3 \oplus 3 \oplus 3$-cube, and this cube again as three $3 \oplus 3$-squares placed side by side. The origin is placed in the lower left corner of the leftmost rectangle and coordinates are associated to cells in the order of board, row and column. Cells marked with a black circle are elements that are known to be contained in the set under consideration, cells marked with a white circle denote elements known not to be contained in the set. Cells marked with a black circle and a white number n denote elements which are known to be contained in the multiset under consideration at least n times. For example, in the following visualization of some information on a multiset $A \subseteq \mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3$, the cell marked o denotes the neutral element of the group, whereas the cell marked a is the element $(1,2,1)$, and the multiset A contains the element $(1,1,1)$, and the element $(0,1,2)$ at least twice.



One advantage of this notation is the fact that one can often read off the existence of zero-sums from the picture. For example, 3 distinct elements add up to 0 if and only if they lie on an affine line, thus, in the following picture the three cells marked a as well as the three cells marked b are zerosum subsets.



We will use this argument repeatedly without further explanation.

The technique of Delorme, Ordaz, and Quiroz requires the study of certain auxiliary functions. Denote by $D_k(G)$ [2] the least integer n, such that every multiset of n elements contains k disjoint zerosum subsets, and by $D^k(G)$ the least integer n, such that every multiset of n elements contains a zerosum subset consisting of at most k elements. Note that $D^k(G)$ is finite only if k is at least the exponent of G; the case that k equals the exponent of G has received particular interest. Adding a star always means that we ask for the least n, such that each subset of n distinct elements has the required property, for example, $D_2^*(\mathbb{Z}_4) = 4$, since 4 distinct points in $\mathbb{Z}_4$ contain the element 0 as well as the set $\{1,3\}$, and therefore 2 disjoint zerosum subsets. In particular, $D^*(G)$ is known as Olson's constant, which was determined for cyclic groups by Olson [4].

This article is organized as follows. In the next section we give several rather special results for the variations of $D(\mathbb{Z}_3^3)$ just mentioned. These results are of limited interest, but will save us a lot of work later on. In the last section we describe the splitting of the set A into the pair $(\tilde{A}, f)$, and prove Theorem 1.

Our approach is not restricted to groups of the form $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, but can be applied to all sequences of the form

$$\mathbb{Z}a_1 \oplus \mathbb{Z}_{a_2} \oplus \cdots \oplus \mathbb{Z}_{a_{r-1}} \oplus \mathbb{Z}_{a_r d}, \quad a_1 \mid a_2 \mid \cdots \mid a_r,$$

where $a_1, \dots, a_r$ are fixed, and d runs over all integers coprime to a_r. However, soon the computational effort becomes too large for a treatment as explicit as given here. In work in progress, we hope to automatize parts of the proof to deal with larger groups as well.

2. Some special values of D_k and related functions

The results of this section are summarized in the following.

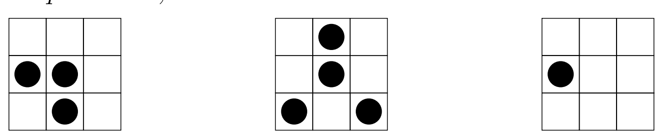
Proposition 1. *We have*

$$\begin{array}{lll} D^3(\mathbb{Z}_3^3) = 17 & D^4(\mathbb{Z}_3^3) = 10 & D^5(\mathbb{Z}_3^3) = 9 \\ D^{3*}(\mathbb{Z}_3^3) = 9 & D^{4*}(\mathbb{Z}_3^3) = 8 & D^{5*}(\mathbb{Z}_3^3) = 8 \\ D_2(\mathbb{Z}_3^3) = 11 & D_k(\mathbb{Z}_3^3) = 3k+6 \ (k \geq 3) & \\ D^*(\mathbb{Z}_3^3) = 7 & D_2^*(\mathbb{Z}_3^3) = 10 & \end{array}$$

Lemma 1. *Set $G = \mathbb{Z}_3 + \mathbb{Z}_3 + \mathbb{Z}_3$.*

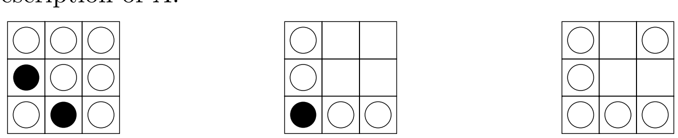
(1) *Let $A = \{a_1, \dots, a_6\}$ be a set of distinct elements of G such that there does not exist a zerosum subset Z of A with at most 3 elements. Then there are distinct indices i, j, k, such that $a_i + a_j = a_k$.*

(2) *Let $A = \{a_1, \dots, a_8\}$ be a set of distinct elements of G such that there does not exist a zerosum subset Z of A with at most 3 elements. Then, up to linear equivalence, A is the set*

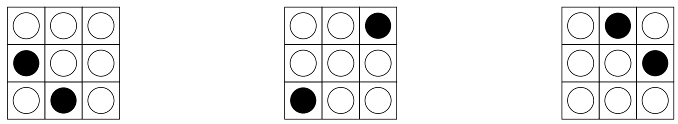


(3) [3] *Let $A = \{a_1, \dots, a_9\}$ be a set of distinct elements of G. Then there exists a zerosum subset Z of A with at most 3 elements.*

Proof. Let $A = \{a_1, \dots, a_6\}$ be a set of 6 elements, and suppose that none of the equations $x + y = z$ and $x + y + z = 0$ is solvable within A. Then A cannot be contained in a plane, thus, we may choose a basis in A. We therefore obtain the following description of A.

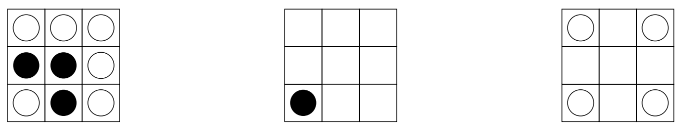


Any two of the three points $(1,1,2)$, $(1,2,1)$ and $(2,1,1)$ form together with one of the points $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$ a zerosum subset, thus, by symmetry we may assume that $(1,1,2)$ and $(1,2,1)$ are not contained in A. Moreover, if $(2,1,1)$ were in A, the only remaining position would be $(1,2,2)$, and we would have $|A| \leq 5$, that is, $(2,1,1)$ is not in A as well. From the remaining 4 positions, 3 have to be taken by elements in A, but $(1,1,1)$ and $(1,2,2)$ cannot be taken at the same time, thus, both $(2,1,2)$ and $(2,2,1)$ have to be in A. Then $(1,1,1)$ cannot be contained in A, and we obtain the following situation.

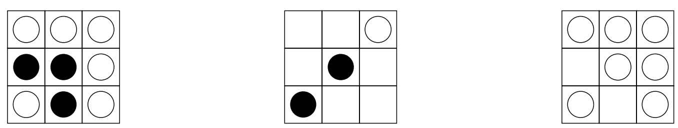


But now we have $(2,1,2)+(1,2,2)=(0,0,1)$, proving our claim.

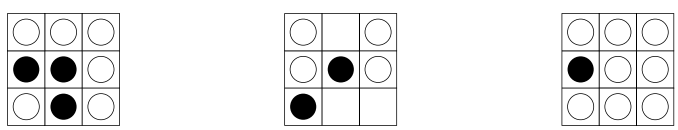
Now let A be a set of size 8 without a zerosum of length 3. By part 1 we may assume that $a_1+a_2=a_3$, moreover, not all elements of A are contained in the plane generated by a_1 and a_2, and we obtain the following situation.



Moreover, we may suppose that there are more elements in the middle layer than in the uppermost one. Suppose first that $(1,1,1)$ is in A.

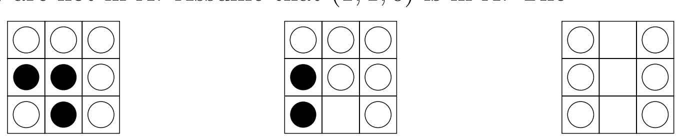


If both the remaining cells in the uppermost layer were contained in A, then no further cell in the middle layer could be contained in A; if on the other hand both cells were not contained in A, then there are three more cells in the middle layer, and it is easily seen that this implies the existence of a zerosum sequence of length 3 in the middle layer. Hence, precisely one of $(2,0,1)$ and $(2,1,0)$ is in A, and we may assume that this element is $(2,0,1)$. Then we reach the following situation.

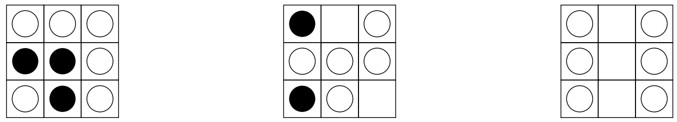


From the remaining three cells two have to be taken, but $(1,1,0)$ would yield with one of the other two cells and $(1,0,0)$ resp. $(1,1,1)$ a zerosum, thus, we obtain the constellation given in the Lemma.

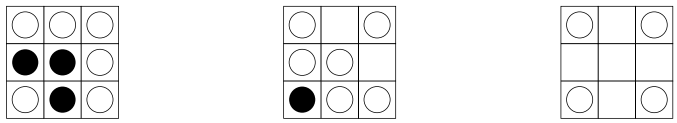
Now suppose that $(1,1,1)$ is not in A. Since any element in the two upper layers can be interchanged with $(1,0,0)$ by a linear transformation leaving the lower layer fixed, we can avoid this case unless for all elements $x,y,z\in A$ with $x+y\in A$ and $z\neq x+y$ we have $x+y+z\notin A$. In particular, in our situation this implies that we may suppose that for each $z\in A$ which is not in the lower layer both elements $z\pm(0,1,1)$ are not in A. Assume that $(1,1,0)$ is in A. The



Sine there are at least 3 elements in the middle layer, we deduce that $(1,1,0)\in A$, which contradicts the fact that there are two more elements in the uppermost layer. Next suppose that $(1,0,2)$ was in A. Then we obtain the following.



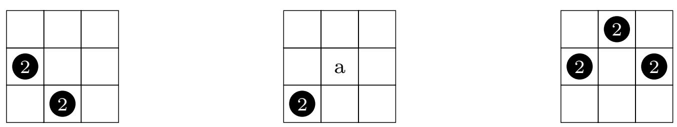
In the middle layer there has to be another element of A, however, for both possible places we see that there could be at most one other cell in the uppermost layer. Hence, $(1,0,2)$ and $(1,2,0)$ are not in A. Then we obtain the following constellation.



Here, not both cells in the middle layer can be taken, thus there have to be three cells in the uppermost layer, contrary to our assumption that there are more cells in the middle layer then in the uppermost one. Hence, the second statement of the lemma is proved. $\square$

Lemma 2. *Let A be a sequence of 14 points which does not contain a zerosum subset of length[] ≤ 3 or of length ≥ 12. Then A contains 7 distinct points, each of which is taken twice. Moreover, there exists a multiset A with these properties, and it is unique up to linear equivalence.*

Proof. The existence of A is given by the following example.

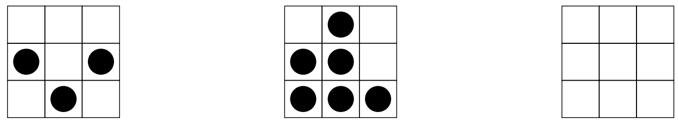


It is easy to check that A does not contain a zerosum of length ≤ 3. Next, the sum of all elements in A equals $(2,2,2)$; the inverse of this element being the element marked a in the picture above. This element is neither 0 nor contained in A, hence, there is no zerosum of length ≥ 13. Suppose that a was the sum of two elements in A. Then a is either the sum of an element in the lowest layer with an element of the middle layer, or the sum of two elements in the upper layer, but both possibilities are easily dismissed.

We now show that all sets of such 14 elements have indeed the form described above. Thus, let A be a set consisting of 14 elements, 8 of which are distinct. Denote by B the configuration described in Lemma 1, part (ii). Then up to some linear transformation, A consists of 6 points of B taken twice, and the 2 remaining points of B taken once. Note that the sum of all elements in B is 0, and the sum of any two elements is non-zero, hence, the sum of all elements in A is non-zero as well. But B is maximal among all sets of distinct elements without zerosum subsets of length ≤ 3, thus, every non-zero element, and in particular the sum of all elements in A, can be represented as the sum of 1 or 2 elements in B. Hence, by deleting 1 or 2 elements of A we obtain a zerosum subset consisting of 12 or 13 elements. $\square$

Proposition 2. *We have $D_2^*(\mathbb{Z}_3^3) = 10$.*

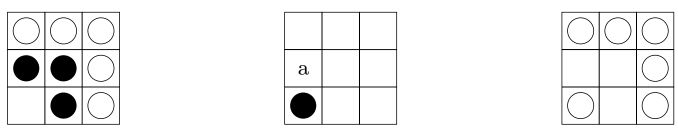
Proof. Consider the example



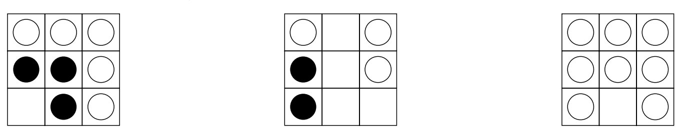
Clearly there is no zerosum subset in the lowermost layer. Hence, if there are two disjoint zero sum subsets, each of them must contain precisely 3 elements of the second layer. Now consider the sum of all elements not contained in the two zerosum subsets. This sum is equal to the sum of all elements in A, and therefore $(2, 0, 0)$, on the other hand, it equals a subset sum of the elements in the lowermost layer. However, $(2, 0, 0)$ cannot be represented by elements in the lowermost layer. Hence, $D_2^*(\mathbb{Z}_3^3) \geq 10$. On the other hand, $D^{3*}(\mathbb{Z}_3^3) = 9$, thus among 10 points there is a zerosum subset of length ≤ 3, and among the remaining 7 points, there is always another zerosum subset. □

Proposition 3. *We have $D^{*4}(\mathbb{Z}_3^3) = 8$.*

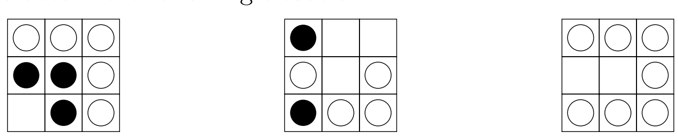
Proof. Let A be a set of 8 distinct elements which does not contain a zerosum subset of length 4. Then A cannot be contained in one plane, hence, we may choose a basis of $\mathbb{Z}_3^3$ in A, which without loss is the standard basis. Moreover, there has to be a sum in A, which we may assume,without any loss, to be $(0, 1, 1)$. Finally, we may change the third element of the basis in such a way that the middle layer contains at least as many elements as the upper layer, thus, we obtain the following picture, where at least two elements of A in the middle layer are not yet drawn.



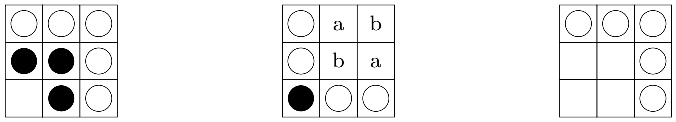
Suppose that $a = (1, 0, 1)$ is contained in A. Then several other elements of G can be excluded, since they would immediately give zerosum subsets of size ≤ 4, and we obtain the following situation.



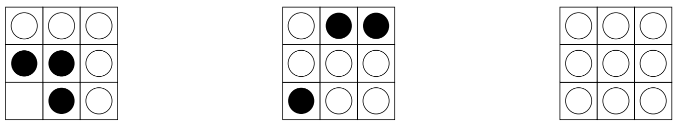
Hence, there is at most one element in the uppermost layer, that is, there are at least two more in the middle layer. But any two elements in the middle row of the middle layer would give a contradiction, hence, $(1, 2, 0)$ is in A. But then no other element of A could be in the middle layer, giving a contradiction. Hence, $(1, 0, 1)$, and by symmetry $(1, 1, 0)$ are not contained in A. Now assume that $(1, 0, 2)$ is in A. Then we obtain the following situation.



By direct inspection we see that if both possible elements in the uppermost layer are contained in A, then no additional element in the second layer could be chosen, and A would have at most 7 elements. Hence, two more elements of A in the middle layer are not yet shown. If $(1, 2, 2)$ was in A, this is impossible, thus, we find that $(1, 1, 1)$ and $(1, 1, 2)$ are both in A. Then we reach the following situation, which immediately implies $|A| = 7$, thus showing that $(1, 0, 2)$ and therefore by symmetry $(1, 2, 0)$, are not in A and hence, we obtain the following situation.



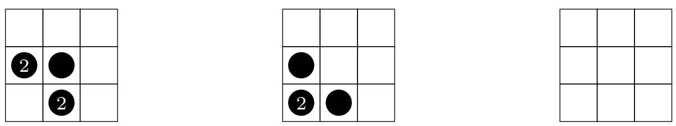
By assumption, there are two more elements in the middle layer, but on the cells marked a and b, respectively, there can be at most one element of A. Hence, without loss, we may assume that $(1, 1, 2)$ is in A, and, since $(1, 0, 0) + (1, 1, 1) + (1, 1, 2) + (0, 1, 0) = (0, 0, 0)$, that $(1, 2, 2)$ is in A as well, that is, we reach the following situation.



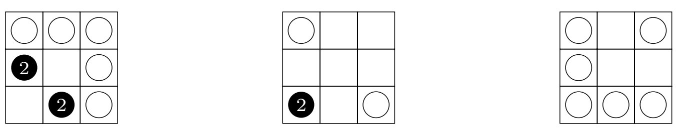
Clearly, $|A| = 6$, contrary to our assumption, and we see that the initial assumption on the existence of A is wrong. $\square$

Proposition 4. *We have $D^4(\mathbb{Z}_3^3) = 10$.*

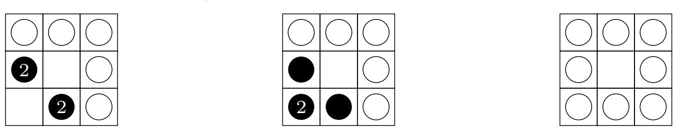
Proof. The fact that $D^4(\mathbb{Z}_3^3) \geq 10$ is proved by the following example.



Hence, it remains to show that every set of 10 elements contains a zerosum of length at most 4. By means of contraction, let A be a set consisting of 10 elements of G without a zerosum subset of size at most 4. Since $D^{*4}(\mathbb{Z}_3^3) = 8$, at least 3 elements of A are repeated, and these 3 elements form a basis of G. Hence, we have the following situation.



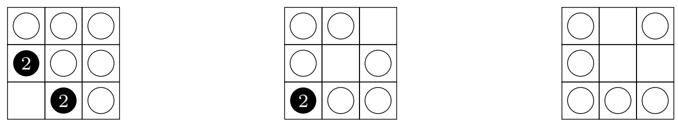
Suppose that 2 of the three elements $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$ are in A. Then we may suppose without loss that these elements are $(1, 1, 0)$ and $(1, 0, 1)$. Then we obtain the following.



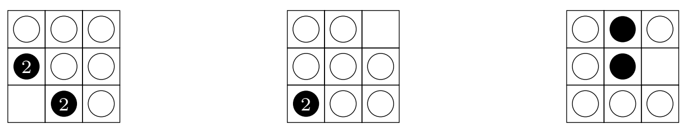
Since $2 \cdot (1, 1, 0) + (1, 0, 0) + (0, 1, 0) = (0, 0, 0)$, both $(1, 1, 0)$ and $(1, 0, 1)$ are taken at most once. Hence, there are two elements in A of the form $(a, 1, 1)$. If they are distinct, we have $(a, 1, 1)+(b, 1, 1)+(x, 1, 0)+(y, 0, 1) = (0, 0, 0)$ for appropriate values $x, y \in \{0, 1\}$, hence, there is one value which is taken twice. However, all three remaining choices lead to contradictions, and we conclude that at most one of $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$ is contained in A; without loss we may assume that $(1, 0, 1)$ and $(1, 1, 0)$ are not in A. Next we note that any two of $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ together with the elements already placed give a zerosum subset of size

3, hence, at most one of these elements can be contained in A. By symmetry we may suppose that $(1,2,1)$ is not in A; moreover, if $(0,1,1)$ was not in A, then we may also assume that $(1,1,2)$ is not in A.

We shall now assume that $(0,1,1)$ is not in A. Then we have the following situation.

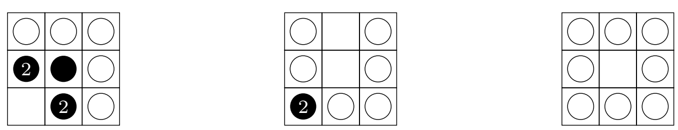


There are at most 8 elements in the lower two layers, hence, there are at least two elements in the uppermost layer; in particular, $(1,1,1)$ is not in A. Suppose that $(1,2,2)$ does not occur twice in A. Then there are at least 3 elements of A in the uppermost layer, and since $(2,1,2)$ and $(2,2,1)$ cannot both occur in A, we deduce that $(2,1,1)$ and one of $(2,1,2)$ and $(2,2,1)$ is in A; without loss we may assume the former, and obtain the following situation.



Since $(2,1,2)+(1,2,2)+2\cdot(0,0,1)=(0,0,0)$, we see that $(1,2,2)$ is not contained in A, and we find that the elements in the uppermost layer are contained twice in A. But then we obtain the contradiction $2\cdot(2,1,2)+(2,1,1)+(0,0,1)=(0,0,0)$.

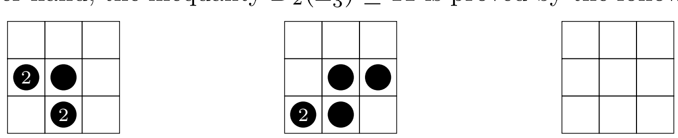
Hence, our initial assumption that $(0,1,1)$ was not in A is false, and we obtain the following situation.



Note that $(0,1,1)$ cannot be repeated in A, because $D^4(\mathbb{Z}_3^2)=6$, thus, there are 3 elements of A on the remaining three cells. But $(1,1,1)$ and $(2,1,1)$ cannot be simultaneously in A, thus, $(1,1,2)$ is in A. Because of $(1,0,0)+(0,1,0)+(1,1,1)+(1,1,2)=(0,0,0)$, $(1,1,1)$ cannot be contained in A, and therefore $(2,1,1)$ has to be contained in A. But then we obtain the zerosum $(2,1,1)+(1,1,2)+(0,1,0)$, and obtain a contradiction proving our claim. $\square$

Proposition 5. *We have $D_2(\mathbb{Z}_3^3)=11$.*

Proof. Let A be a subset of $\mathbb{Z}_3^3$ containing 11 elements. Then A contains a zerosum subset of size ≤ 4, and the complement of this zerosum is a set with ≥ 7 elements, which therefore contains another zerosum subset. Hence, $D_2(\mathbb{Z}_3^3)\leq 11$. On the other hand, the inequality $D_2(\mathbb{Z}_3^3)\geq 11$ is proved by the following example.



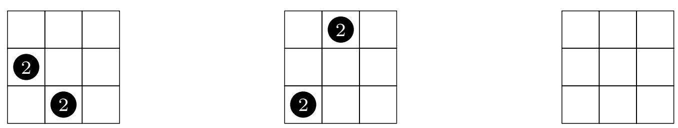
In fact, every zerosum subset contains either no elements of the middle layer, or precisely 3, thus, if there were to distinct zerosum subsets, one of them has to be contained in the lowermost layer. Obviously, it has to contain all points of this layer. But there is no zerosum subset of size three in the middle layer, and we find that this set does not contain two disjoint zerosum subsets, which proves $D_2(\mathbb{Z}_3^3) \geq 11$. $\square$

Proposition 6. *We have* $D_3(\mathbb{Z}_3^3) = 15$.

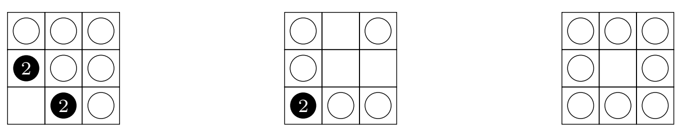
PROOF. The upper bound $D_3(\mathbb{Z}_3^3) \leq 15$ follows from Proposition 4 and 5, whereas the lower bound follows from the configuration given in Lemma 2. $\square$

Proposition 7. *We have* $D^5(\mathbb{Z}_3^3) = 9$.

PROOF. The lower bound $D^5(\mathbb{Z}_3^3) \geq 8$ is given by the following example.

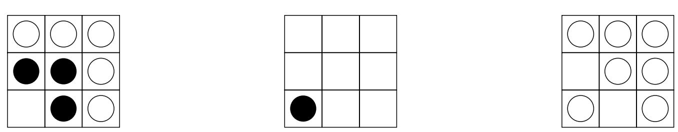


Let A be a set consisting of 9 elements without a zerosum subset of length 5. Suppose first that there are at most 5 distinct elements in A. Then there are at least 4 elements twice in A, and these elements cannot be in one plane, hence, there is a basis of elements taken twice in A. We therefore obtain the following.

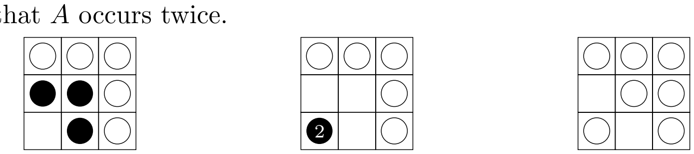


If $(2,1,1) \in A$, then there are no further elements in the middle layer, which would imply $|A| \leq 8$; thus, by symmetry, we have $(1,1,2),(1,2,1),(2,1,1) \notin A$. But then A could only have $(1,1,1)$ as a possible element left, and therefore $|A| \leq 8$.

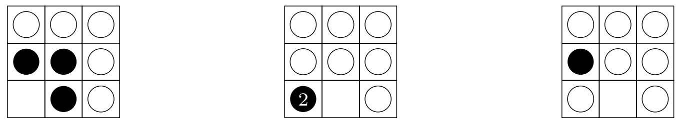
Hence, we may assume that A contains 6 distinct elements, and therefore there is some basis $\{a,b,c\} \in A$, such that $a+b \in A$, that is, we have the following situation.



Moreover, we can choose the third base element in such a way that it is either twice in A, or that no element outside the lowest plane is twice in A. Consider first the case that A occurs twice.

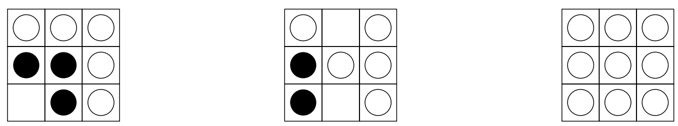


Note that $(0,1,1)$ cannot be contained twice in A, and that $(0,0,1)$ and $(0,1,0)$ cannot be both twice in A, that is, there are at most 4 elements in the lowest layer. Moreover, all elements in the middle layer not yet depicted can occur at most once, and not all three empty places can be taken, thus, there is an element in the uppermost layer; without loss we may suppose that this element is $(2,0,1)$. We then obtain the following.



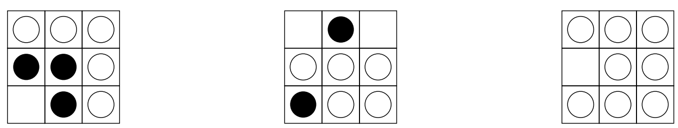
Here, $(2,0,1)$ can only be once in A, since otherwise we had the zerosum $2\cdot(2,0,1)+2\cdot(1,0,0)+(0,0,1)$, thus, there are at most 2 points in the uppermost layer. Hence, $(1,1,0)\in A$, which implies that $(2,1,0)\notin A$, and we find that $|A|\leq 8$. Thus, from now on, we shall assume that all elements outside the lowest layer occur only once in A.

Suppose that $(1,0,1)\in A$. Then we have the following.



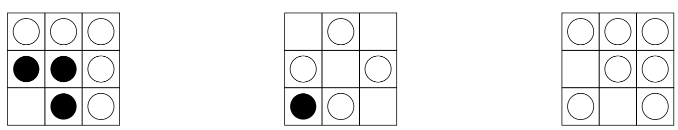
Obviously, there can only be 4 elements outside the first layer, and at most 4 inside the first layer, which gives a contradiction to $|A|=9$.

Next, suppose that $(1,1,2)\in A$. Then we obtain



Again, there are only 5 places for elements outside the first layer, and $(1,0,2)$ and $(1,2,2)$ cannot occur at the same time, thus, $(1,1,2)\notin A$ as well.

Hence, we are led to the following constellation.



There are only 7 places left for elements outside the first layer. Moreover, among these $(1,1,1)$ and $(1,2,2)$ as well as $(1,2,0)$ and $(1,0,2)$ are mutually exclusive, and we conclude that $(2,1,0),(2,0,1)\in A$, and, without loss, $(1,2,0)\in A$. But then we obtain the zerosum $(2,0,1)+(1,2,0)+(0,1,1)+(0,0,1)$, and this contradiction proves our claim. □

Proposition 8. *For $k\geq 3$ we have $D_k(\mathbb{Z}_3^3)=3k+6$. Moreover, the set of all multisets A of size $3k+5$ which do not have k disjoint zerosum subsets can be constructed as follows: Take all sets $B=\{b_1,\dots,b_7\}$ of 7 distinct points without a subsum of length ≤ 3, such that the multiset C obtained from B by taking each point twice does not contain a zerosum subset of length ≤ 12. Choose a partition $k-3=\kappa_1+\dots+\kappa_7$, and set $A=C\cup\{b_1^{3\kappa_1},\dots,b_7^{3\kappa_7}\}$.*

Proof. We first show that none of the sets described here contain k disjoint zerosum subsets. In fact, since there is no zerosum of length ≤ 3 in B, every zerosum of length 3 in A must contain the same element three times, thus, any collection of disjoint zerosum subsets can contain at most $k-2$ zerosums of length 3. next, note that the sum of all elements of A equals twice the sum of all elements of B, and that the set of elements representable by 0, 1 or 2 elements of A is equal to the set of elements representable by 0, 1, or 2 elements of $B\cup B$; thus, as in the proof of Lemma2 we see that there is no zerosum of length $\geq 3k+3$. Hence, every collection of disjoint zerosum subsets can contain $3k+2$ points at most. Thus, such a collection contains no zerosum subset of length ≤ 2, at most $k-3$ of length 3, and all together consists of at most $3k+2$ points, which implies that the total number of zerosum subsets is $\leq k-1$.

Now we show that there are no examples different from the one described here. Let A be a multiset consisting of $3k+5$ elements of $\mathbb{Z}_3^3$ which does not have k disjoint zerosum subsets. If there is a zerosum of length ≤ 2 in A, then removing this zerosum yields $k-1$ zerosums in the remaining $3(k-1)+6$ points, which gives a contradiction. Next suppose there is an element repeated 4 times in A. Then we can remove this element 3 times, and may assume by induction that the new set has the form described. Hence, it suffices to consider the case that every element in A occurs at most 3 times.

Suppose there is one element a occurring once. Then we remove as many zerosum subsets of length 3 from A as possible without removing this point. We end up with a set B of 14 or 17 points, which contains one element precisely once, and does not contain 3 resp. 4 disjoint zerosum subsets. However, we already saw that this is impossible for a set of 14elements. If $|B|=17$, then either B contains one element three times, contradicting the assumption that we removed all zerosums of length three avoiding a, or there are at least 9 distinct points in B. In the latter case let $\ell \leq 8$ be the number of points occurring twice in B. Collect each point which is twice in B and as many points that occur once in B necessary to reach 9 points in a set C. Then C contains a zerosum of length 3, we claim that removing this zerosum of B yields a set which contains one element only once. This is clear if $b<8$, for then some element occurring once in B is not contained in C, and can therefore also not be a part of the zerosum removed. If on the other hand $b=8$, then at least two points occurring twice in B are removed once, thus there are at least 2 points in the new set with multiplicity 1. In any case, we have removed $k-3$ zerosum subsets of size 3 from the beginning set A, and ended up in a set of 14 elements, one of which occurring only once. Hence, there are 3 more zerosums, contradicting the assumption that A does not contain k zerosum subsets.

Next, suppose there is a zerosum of length 3 which consists of distinct elements. Then we can remove this zerosum once or twice to reach a situation with one element occurring precisely once, a situation we just dealt with. In particular, there are at most 8 distinct elements in A. If there are only 7 distinct elements, we reached the position described in Lemma 2 and are done. Otherwise we have a set with 17 elements, 8 distinct ones among them forming the set described in Lemma 1 (ii), and one point occurs three times, whereas the other points occur exactly twice. Removing the three times repeated point once, we obtain a set of 16 elements with sum 0, thus, among the 15 remaining points there are 3 disjoint zerosums, whereas the complement of these three sums constitute a fourth one, yielding k disjoint zerosums for the original set A.

Hence, our claim follows. □

3. Proof of Theorem 1

Lemma 3. *In every set of* 5 *distinct elements of* $\mathbb{Z}_3^3$ *there is either a zerosum of length* ≤ 3*, or there are* 3 *elements* x*,* y*,* z *satisfying the equation* $x+y=z$*.*

Proof. It is easy to check that the analogous statement in two dimensions holds true for all sets of 3 elements, hence, we may assume that A does not contain

3 points in any plane passing through the origin, and we obtain the following situation.

○	○	○
●	○	○
	●	○

○	b	a
○	a	b
●	○	○

○	c	d
○	d	c
○	○	○

Without loss we may suppose that the middle plane contains at least one other element of A. If one of the cells marked a is in A, then none of the cells marked b is in A, and vice versa. On the other hand, if both cells marked a are in A, we would obtain a zerosum of length 3, and similar for b, thus, precisely one cell in the middle plane is contained in A. Moreover, if a cell marked a is in A, the cells marked d cannot be in A, and similarly with b, and we are left with the possibilities that there is precisely one element a and one c, or one b and one d.

If $(1,1,1) \in A$, both elements marked c yield zerosums of length 3, similarly, if $(2,1,1) \in A$. If $(1,1,2) \in A$, then the only remaining possibility for the last element is $(2,2,2)$, but $(1,1,2)+(2,2,2)=(0,0,1) \in A$, and the same argument applies to the case that $(2,1,1)$ is in A. Hence, no such set A can exist. □

Theorem 2. *Let n be an integer coprime to 6. Then there does not exist a multiset $A \subseteq \mathbb{Z}_3^3$ with 10 elements together with a function $f\colon A \to \mathbb{Z}_n$, such that A does not contain 2 disjoint zerosum subsets, that for every zerosum subset B of A we have $\sum_{b\in B} f(b) = 1$, and that there is some element $a \in A$ with $3f(a) = 1$.*

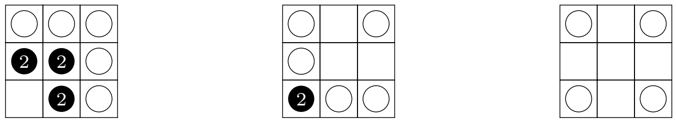
Proof of the theorem. Suppose that A and f are as in the statement. Observe that A does not have a zerosum of length ≤ 3 or ≥ 8. In particular, there are at most 8 distinct elements in A, that is, there are at least 3 elements occurring twice. We distinguish cases according to the constellation of these elements.

(i) Suppose that there exist 3 elements a, b, c occurring twice within one plane passing through the origin. Without loss we may suppose that $c = a + b$. Then we have the zerosums $2a+2b+c = a+b+2c = 0$, thus $f(a)+f(b) = f(c)$ and $3f(c) = 1$, in particular, $f(c) \neq 0$. Suppose that in A there are elements x_i outside this plane adding up to $2a$. Then we have the zerosums $a + \sum x_i = 2a + b + 2c + \sum x_i = 0$, which implies the equation

$$f(a) + \sum f(x_i) = 4f(a) + 3f(b) + \sum f(x_i),$$

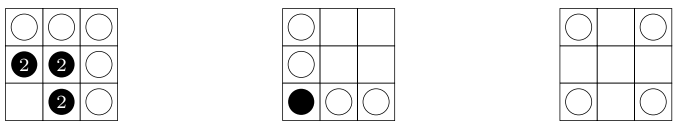
which in turn implies $f(c) = 0$, a contradiction. Next suppose that $2a + 2b$ can be represented as a sum of elements x_i of A outside the given plane. Then we have the zerosums $2a + 2b + 2c + \sum x_i$ and $c + \sum x_i$, which in the same way implies $f(c) = 0$. If $2a + b$ can be represented, we have the zerosums $a + 2b + \sum x_i$ and $2a + 2c + \sum x_i$, which implies $3f(a) = 0$, that is, $f(a) = 0$. If a can be represented, we obtain the zerosums $2a+\sum x_i$ and $b+2c+\sum x_i$, which implies $f(b) = 0$. Noting that not both $f(a)$ and $f(b)$ can vanish, since otherwise $f(c)$ would be zero, we can summarize these considerations as follows: None of $2a, 2b, 2c$ can be represented as a sum of elements of A outside the plane spanned by a and b, and if one of $a, a+2b$ is represented in such a way, none of $2a + b$, b is, and vice versa.

(ii) As an immediate consequence we obtain that A cannot consist of 5 elements, each taken twice. In fact, without loss, we have the following situation.

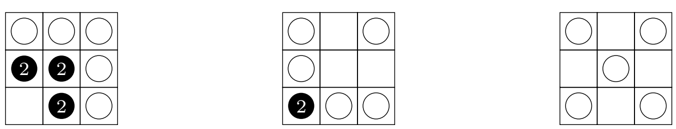


If there is an element x in the middle layer occurring twice in A, $2\cdot(1,0,0)+x$ and $(1,0,0)+2x$ are elements in the lowest layer; if there is an element x in the uppermost layer occurring twice, both $(1,0,0)+x$ and $2\cdot(1,0,0)+2x$ are elements in the lowest layer. In any case, there is some y in the lowest layer, such that both y and $2y$ can be represented as a sum of elements outside the lowermost layer. If $y=0$, we have a zerosum of length 2, if $y=(0,1,2)$, both $(0,1,2)$ and $(0,2,1)$ are representable, which gives a contradiction, and in all other cases one of $(0,0,2)$, $(0,2,0)$ and $(0,2,2)$ is representable, which gives also a contradiction.

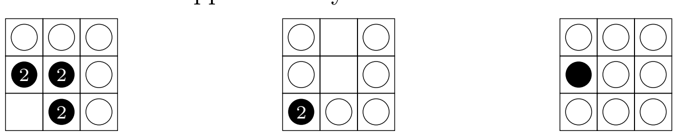
(iii) Suppose that $(0,0,1),(0,1,0)$ and $(0,1,1)$ occur twice, and that none of the equations $x+(0,1,0)=y$, $x+(0,0,1)=y$ is solvable with $x,y\in A$. Without loss we may assume that $(1,0,0)$ is in A, and that, if some other element occurs twice in A, then so does $(1,0,0)$. We obtain the following situation.



If there are two elements in the uppermost layer, we may assume without loss that $(2,1,0)\in A$. Then $(2,0,1)$ and $(2,2,1)$ cannot be in A since otherwise $(0,1,0)$ and one of $(0,0,1)$, $(0,2,1)$ is representable by elements outside the lowest layer; and $(2,1,1)$ and $(2,1,2)$ cannot be contained in A, since we would obtain a solution of the equation $x+(0,1,0=y$ in A. Hence, there is at most one element of A in the uppermost layer. On the other hand, there can be at most 3 elements in the middle layer, thus, there is precisely one element in the uppermost layer, and 3 elements in the middle layer, two of which are $(1,0,0)$. Moreover, $(2,1,1)\notin A$, since otherwise there would be no place left for the last element in the middle layer, and $(1,2,2)\notin A$, since otherwise $(0,2,2)$ was representable. We therefore get the following.

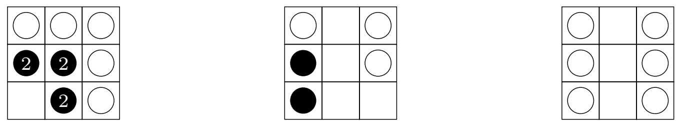


We now check that $(2,1,2)\in A$ is impossible, since there is no place for the last element in the middle layer, and we may assume without loss that $(2,1,0)$ is the unique element in the uppermost layer.

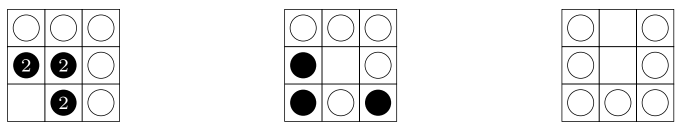


The sum of all elements in A is therefore $(2,0,1)$ or $(2,0,2)$, and both these elements can be represented as a sum of 1 or 2 elements of A; thus, A contains a zerosum of length 8 or 9, which gives a contradiction.

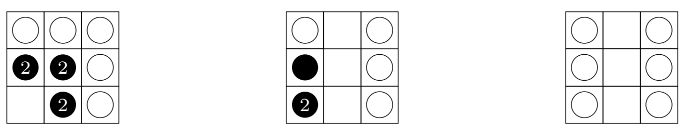
(iv) Now suppose that $(0,0,1),(0,1,0)$ and $(0,1,1)$ occur twice. By the preceding argument we know that for some elements $x,y\in A$ outside the lowermost plane we have $x+(0,1,0)=y$ or $x+(0,0,1)=y$, and we may therefore assume without loss that both $(1,0,0)$ and $(1,0,1)$ are contained in A. We obtain the following situation.



Suppose first that $(1,0,0)$ is not twice in A. Then $(1,2,0)$ is in A, since $(1,0,1)$ cannot be twice in A, and therefore all remaining elements are in the plane $(s,1,t)$, which would imply that the sum of all elements outside the lowermost plane is of the form $(0,2,t)$, which for $t\neq 1$ yields a contradiction, whereas for $t=1$ we deduce that the sum of all elements in A is $(0,0,2)$, which is a sum of two elements in A, thus, A contains a zerosum of length 8, which also yields a contradiction. Hence, we conclude that $(1,2,0)$ is in A, and obtain the following.



Now $(0,2,1)$ is a sum of elements outside the lowermost plane, hence, $(0,1,0)$ and $(0,1,2)$ are not, which implies that $(1,1,1),(2,1,1),(2,1,2),\notin A$, and that $(1,2,0)$ is only once in A, thus, A consists only of 9 elements. This contradiction shows that $(1,0,0)$ occurs twice in A, and we find the following.

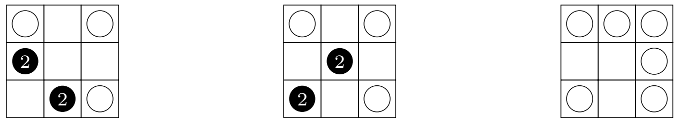


We now can represent $(0,0,1)$ as a sum of elements outside the lowermost layer, hence, $(0,1,0)$ and $(0,1,2)$ cannot be represented this way. The remaining element of A has the form $(s,1,t)$. If $t\neq 1$, we add $(1,0,0)$ once or twice, to obtain a forbidden sum in the lowermost layer, whereas if $t=1$, we add $(1,0,1)$, and, if necessary, $(1,0,0)$ to obtain a forbidden sum as well. Hence, in any case, we obtain a contradiction.

Thus, we see that the equation $x+y=z$ is not solvable among distinct elements occurring twice in A, in particular, we may assume that $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ all occur twice in A.

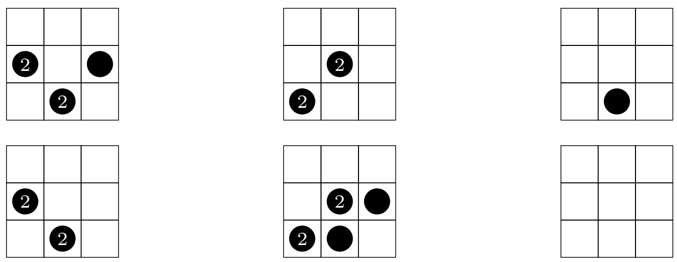
(v) Suppose that there are 4 elements occurring twice in A. We already saw that three of them may be taken to be $(0,0,1),(0,1,0)$ and $(1,0,0)$, and that the fourth cannot be contained in one of the planes generated by this three. Hence, there are 8 possibilities left. These 8 points fall into 4 equivalence classes under rotation around the spatial diagonal, and taking the fourth point to be $(2,2,2)$ would yield a zerosum of length 8. Moreover, a linear transformation fixing the plane $(0,s,t)$ and interchanging $(1,0,0)$ and $(1,1,1)$ shows that the case of the fourth point being $(1,1,1)$ is equivalent to the case $(1,2,2)$. Hence, up to symmetry we may suppose that the fourth point is $(1,1,1)$ or $(1,1,2)$.

(vi) Suppose that $(0,0,1),(0,1,0),(1,0,0)$ and $(1,1,1)$ occur twice in A. Then we have the following.

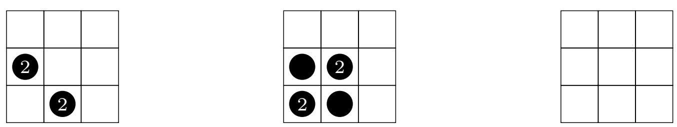


Up to symmetry all elements of $\mathbb{Z}_3^3$ except $(0,1,2)$, $(0,2,2)$ and $(1,2,2)$ can be represented as the sum of at most 2 elements already depicted, thus we deduce that the sum of all elements of A is equal to one of these, for otherwise we would obtain a zerosum of length ≥ 8. Subtracting the elements depicted we find that the sum of the two remaining elements equals $(2,0,1)$, $(2,1,1)$ or $(0,1,1)$.

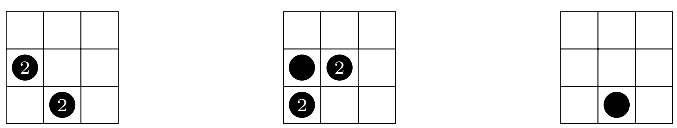
Suppose first the sum is $(2,0,1)$. Deleting all elements x such that $(2,0,1)-x$ is impossible, only the following two possibilities remain.



The same argument applied to the second case yields only one case.



Finally, the last sum gives two cases, which are symmetric to each other, thus, we only have to consider the following.



Here, the sum of all elements different from $(1,1,1)$ is 0, thus, there is a zerosum of length 8, and it suffices to consider the 3 previous cases.

We set $x=(0,0,1)$, $y=(0,1,0)$, $z=(1,0,0)$, $w=(1,1,1)$. Then we have the zerosums $2x+2y+2z+w, x+y+z+2w$, which imply that $f(w)=f(x)+f(y)+f(z)$ and $3f(w)=1$. Moreover, if $a=(r,s,t)$ is one of the other elements of A, we have the zerosum $a+[-r]z+[-s]y+[-t]x$, similarly, we have the zerosums $a+w+[-1-r]z+[-1-s]y+[-1-t]x$ and $a+2w+[-2-r]z+[-2-s]y+[-2-t]x$, and we obtain the equations

$$\begin{aligned}
&(1-[-r]+[-1-r])f(z)+(1-[-s]+[-1-s])f(y)\\
&\qquad\qquad\qquad\qquad\qquad+(1-[-t]+[-1-t])f(x)=0\\
&(2-[-r]+[-2-r])f(z)+(2-[-s]+[-2-s])f(y)\\
&\qquad\qquad\qquad\qquad\qquad+(2-[-t]+[-2-t])f(x)=0
\end{aligned}$$

Every coefficient is divisible by 3, and in the interval $[0,5]$, hence, dividing by 3 we obtain equations with all coefficients 0 and 1. We can apply this argument to both points in A occurring only once as well as to their sum, thus, in each of the four cases we obtain a $3 \oplus 6$-matrix with the property that $(f(z), f(y), f(x))$ is in the kernel of this matrix. To compute these matrices, note that

$$\frac{1-[-a]+[-1-a]}{3} = \begin{cases} 0, & a=1,2 \\ 1, & a=0 \end{cases}, \qquad \frac{2-[-a]+[-2-a]}{3} = \begin{cases} 0, & a=1 \\ 1, & a=0,2 \end{cases}$$

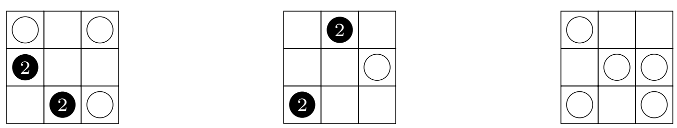
for all $a \in \mathbb{Z}_3$. We therefore obtain the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Obviously, all these matrices have rank 3, which implies that $f(x) = f(y) = f(z) = 0$, and we obtain the contradiction

$$1 = 3f(w) = 3\big(f(x) + f(y) + f(z)\big) = 0.$$

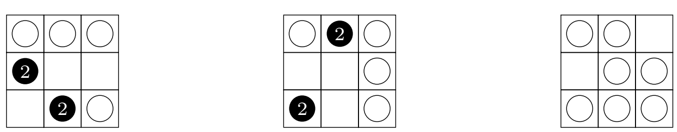
(vii) Suppose that $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ and $(1,1,2)$ occur twice in A. Then we have the following situation.



As in the previous argument we find that $(r,s,t) \in A$ implies the equations

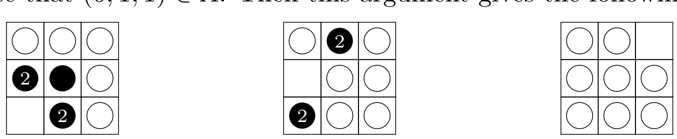
$$\begin{aligned} &(1-[-r]+[-1-r])f(z) + (1-[-s]+[-1-s])f(y) \\ &\qquad\qquad\qquad + (2[-t]+[-2-t])f(x) = 0 \\ &(2-[-r]+[-2-r])f(z) + (2-[-s]+[-2-s])f(y) \\ &\qquad\qquad\qquad + (1-[-t]+[-1-t])f(x) = 0 \end{aligned}$$

If $r = 1$ and $s, t \in \{0, 2\}$, these equations imply that $f(x) = f(y) = 0$, thus we obtain a contradiction unless all the remaining equations do not involve $f(z)$. However, this would imply that both points occurring once in A as well as their sum have z-coordinate 1, which is absurd. The same argument applies if $s = 1$ and $r, t \in \{0, 2\}$, and we deduce the following.



Moreover, applying this argument to the sum of the two remaining elements, we see that this sum cannot have x-coordinate $\neq 1$ and precisely one of y and z-coordinate equal to 1.

Suppose that $(0,1,1) \in A$. Then this argument gives the following.

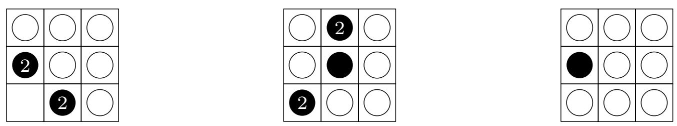


The sum of all elements equals $(1,2,1)$, thus if the remaining element is in the uppermost layer, there is a zerosum of length 9. Hence, there is only one possibility left, which leads to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which has rank 3. Hence, $(0,1,1) \notin A$.

Next suppose that $(1,1,1) \in A$. Then we have only the following possibility left.

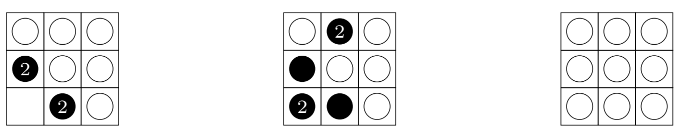


From this we obtain the matrix (deleting the zero-rows coming from $(1,1,1)$)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which has rank 3.

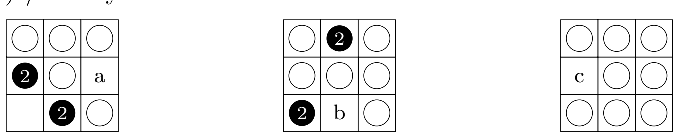
Now suppose that $(1,0,1) \in A$. Then the only possibility is



which gives the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that none of the remaining possibilities fits with $(2,2,2)$, thus $(2,2,2) \notin A$. By now we have reached



where two of the three cells marked a, b, c are in A. These elements give the following $2 \oplus 3$-matrices:

$$a \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad b \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad c \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

obviously, b cannot be in A, and the fact that the x-coordinate of $a+c$ is not 1 yields a matrix of rank 3 again. Hence, in each case we obtain a contradiction.

(viii) We have seen so far that a counterexample to our statement would have precisely 3 elements occurring twice, which generate $\mathbb{Z}_3^3$. Hence, we may suppose that $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$ appear twice in A, whereas all other elements of A occur with multiplicity 1. On one hand, having many distinct elements makes the non-existence of short zerosums a more stringent restriction, on the other hand, the number of cases increases dramatically. To deal with this number we use a simple computer program to list all possible sets of 10 elements without zerosums of length ≤ 3 or ≥ 8 having precisely 3 elements taken twice. With a rather un-sophisticated approach we obtain a list of 84 configurations, many of which are symmetric. One easily sees from this list that always one of $(0,1,1)$, $(1,0,1)$ and $(1,1,0)$ is contained in A, prescribing this element by a rotation around the spatial diagonal to be $(0,1,1)$ reduces the number of cases to 41, taking care of the remaining symmetries by hand gives the following list of 16 cases.

$(0,1,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
$(0,1,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,2)$
$(0,1,1)$	$(1,0,1)$	$(1,1,1)$	$(1,1,2)$
$(0,1,1)$	$(1,0,1)$	$(1,1,1)$	$(1,2,0)$
$(0,1,1)$	$(1,0,1)$	$(1,1,2)$	$(1,2,2)$
$(0,1,1)$	$(1,0,1)$	$(1,2,0)$	$(1,2,1)$
$(0,1,1)$	$(1,0,1)$	$(1,2,0)$	$(1,2,2)$
$(0,1,1)$	$(1,0,1)$	$(1,2,1)$	$(1,2,2)$
$(0,1,1)$	$(1,0,2)$	$(1,1,1)$	$(1,1,2)$
$(0,1,1)$	$(1,0,2)$	$(1,1,1)$	$(1,2,1)$
$(0,1,1)$	$(1,0,2)$	$(1,2,0)$	$(1,2,1)$
$(0,1,1)$	$(1,0,2)$	$(1,2,0)$	$(1,2,2)$
$(0,1,1)$	$(1,0,2)$	$(1,2,1)$	$(1,2,2)$
$(0,1,1)$	$(1,0,2)$	$(2,1,0)$	$(2,1,1)$
$(0,1,1)$	$(1,0,2)$	$(2,1,0)$	$(2,1,2)$
$(0,1,1)$	$(1,0,2)$	$(2,1,1)$	$(2,1,2)$

Let (r,s,t) be an element of A occurring once, distinct from $(0,1,1)$. Then we have the zerosums $(0,1,1)+2x+2y$, $(r,s,t)+[-t]x+[-s]y+[-t]z$ and $(0,1,1)+(r,s,t)+[-1-t]x+[-1-s]y+[-1-t]z$, which together imply the equations

$$\begin{aligned}
f\big((0,1,1)\big)+2f(y)+2f(x)&=1\\
f\big((r,s,t)\big)+[-r]f(z)+[-s]f(y)+[-t]f(x)&=1\\
f\big((0,1,1)\big)+f\big((r,s,t)\big)+[-1-r]f(z)+[-1-s]f(y)+[-1-t]f(x)&=1
\end{aligned}$$

Subtracting the third equation from the sum of the other two equations we obtain

$$(2+[-s]-[-1-s])f(y)+(2+[-t]-[-1-t])f(x)=1.$$

Note that for $a\in\mathbb{Z}_3^3$ we have

$$\frac{2+[-a]-[-1-a]}{3}=\begin{cases}0, & a=0\\ 1, & a=1,2\end{cases}.$$

Now consider the first entry in the table above. The second element is $(1,0,1)$, which gives the equation $f(x)=1$, the third element yields $f(y)=1$, whereas the third one implies $f(x)+f(y)=1$, and we obtain a contradiction. In the same way we can deal with all cases in which the third an fourth entry contains an entry 0, which is the case for all but the following.

$$\begin{array}{cccc}
(0,1,1) & (1,0,1) & (1,1,1) & (1,1,2)\\
(0,1,1) & (1,0,1) & (1,1,2) & (1,2,2)\\
(0,1,1) & (1,0,1) & (1,2,1) & (1,2,2)\\
(0,1,1) & (1,0,2) & (1,1,1) & (1,1,2)\\
(0,1,1) & (1,0,2) & (1,1,1) & (1,2,1)\\
(0,1,1) & (1,0,2) & (1,2,1) & (1,2,2)\\
(0,1,1) & (1,0,2) & (2,1,1) & (2,1,2)
\end{array}$$

Finally, there is no need to take (r,s,t) to be one of the three elements occurring once distinct from $(0,1,1)$, we could also take the sum of two distinct ones among them. Since in the 7 remaining cases we already have found equations coming from $s=0, t\neq 0$ and $s,t\neq 0$, it suffices to obtain (r,s,t) with $s\neq 0, t=0$. For the first three cases this is achieved by adding the second element to the fourth one, whereas for the other 4 cases we add the second element to the third one. Hence, in all cases we reach a contradiction, which finishes our proof. □

From Theorem 2 and Proposition 1 we can now deduce Theorem 1.

Proof of Theorem 1. We may suppose that $(6,d)=1$, since for all other cases this is already known. Let A be a set of $3d+4\geq 13$ points such that there is a function as described in Theorem 2. We claim that A contains a zerosum sequence of length ≤ 3.

We remove zero-sums of length 3 as long as possible, ending in a set with $3k+1$ points, k at most 5. We obtain $n-k+1$ zero-sums this way. If among $3k+1$ points, we can find $k-1$ disjoint zero-sums, we obtain n zero-sums, which suffices. Unfortunately, this is only the case for $k=0,1,2$. For $k=3,4,5$, we only find $k-2$ disjoint zero-sums, hence, altogether we obtain a system of $n-1$ zero-sums, each inducing an element in Z_n, such that the induced set is zero-sum free. Hence, the induced elements have to be equal.

In each case we obtain $3k+1$ points in Z_3^3 containing no system of $k-1$ disjoint zero-sums, no zero-sum of length 3 together with a function such that for every system of $k-2$ disjoint zero-sums the sum over each subset is equal to some generator for Z_n which does not depend on the choice of the system of zero-sums but is determined by any of the three-sums removed in the beginning. We may call this generator 1, thus, we always obtain a function f as described above.

Now as k decreases we have to make more and more use of f, which leads to more and more complicated arguments.

For $k=5$ things are easy, since up to linear equivalence there is only one set of 16 points containing no zero-sum of length 3 (Lemma 1(2)), and the sum over all elements in this set is 0. Since a set of size 15 contains 3 disjoint zero-sums, a set of 16 points with sum 0 contains 4 disjoint zero-sums and we are done.

For $k=4$ we have 13 points containing no zero-sum of length at most 3 and since 9 distinct points contain a zero-sum of length at most 3 (Proposition 1), 5 points have to occur with multiplicity 2 but this is ruled out at the beginning of the proof of Theorem 2 steps (i) and (ii).

For $k=3$, as above, we reach a set of 10 points admitting a function f, which does not have 2 disjoint zerosum subsets. But Theorem 2 asserts that such a function does not exist. Hence, there does not exist a zerosum free set of size $3d+4$, and we deduce $D(G) \leq 3d+4$. The other inequality is trivial. $\square$

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