Geodesic regression and cubic splines on shape spaces

Francois-Xavier Vialard

joint work with Marc Niethammer (geodesic regression) and Alain Trouvé (cubic splines)

Ceremade, Université Paris Dauphine

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Outline

Motivation

- Developing geometrical and statistical tools to analyse biomedical shapes,
- Developing the associated numerical algorithms.

Example of problems of interest

Given two shapes, find a diffeomorphism of $\mathbb{R}^3$ *that maps one shape onto the other*

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Different data types and different way of representing them.

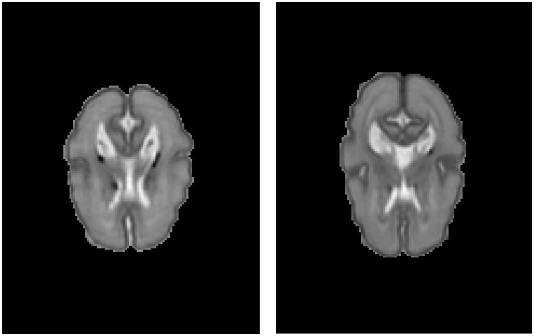

Figure: Two slices of 3D brain images of the same subject at different ages

Example of problems of interest

Given two shapes, find a diffeomorphism of $\mathbb{R}^3$ that maps one shape onto the other

Deformation by a diffeomorphism

Figure: Diffeomorphic deformation of the image

About Computational Anatomy

Old problems:

1. to find a framework for registration of biological shapes,
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Idea pioneered by Grenander and al. (80's), then developed by M.Miller, A.Trouvé, L.Younes.

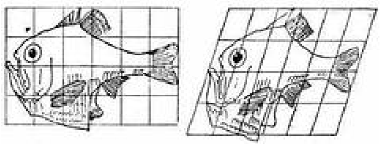

Figure: deforming the shape of a fish by D'Arcy Thompson, author of *On Growth and Forms* (1917)

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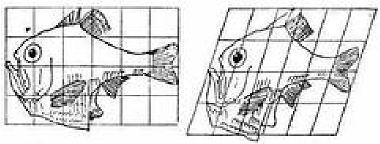

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New problems like study of Spatiotemporal evolution of shapes within a diffeomorphic approach

A Riemannian approach to diffeomorphic registration

Several diffeomorphic registration methods are available:

- Free-form deformations B-spline-based diffeomorphisms by D. Rueckert
- Log-demons (X.Pennec et al.)
- Large Deformations by Diffeomorphisms (M. Miller,A. Trouvé, L. Younes)

Only the last one provides a Riemannian framework.

A Riemannian approach to diffeomorphic registration

- $v_t \in V$ a time dependent vector field on $\mathbb{R}^n$.
- $\phi_t \in Diff$, the flow defined by $\partial_t \phi_t = v_t(\phi_t)$.

Action of the group of diffeomorphism G_0 (flow at time 1):

$$\Pi : G_0 \times \mathcal{C} \to \mathcal{C}\,,$$
$$\Pi(\phi, X) \doteq \phi.X$$

Right-invariant metric on G_0: $d(\phi_{0,1}, \mathrm{Id})^2 = \frac{1}{2}\int_0^1 |v_t|_V^2 dt$.

Inexact matching: taking noise into account

Minimizing

$$\mathcal{J}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2} d(\phi_{0,1}.A, B)^2 .$$

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In the case of landmarks:

$$\mathcal{J}(\phi) = \frac{1}{2}\int_0^1 |v_t|_V^2 dt + \frac{1}{2\sigma^2} \sum_{i=1}^{k} \|\phi(x_i) - y_i\|^2 \,,$$

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In the case of images:

$$d(\phi_{0,1}.I_0, I_{target})^2 = \int_U |I_0 \circ \phi_{1,0} - I_{target}|^2 dx \,.$$

Main issues for practical applications:

- choice of the metric (prior),
- choice of the similarity measure.

A Riemannian framework

Proposition

The inexact matching functional

$$\mathcal{J}(v) = \int_0^1 |v_t|_V^2 dt + \frac{1}{\sigma^2} d(\phi_{0,1}.A, B)^2$$

leads to geodesics on the orbit of A for the induced Riemannian metric.

Proposition

Left-action $G \times Q \mapsto Q$ of a group G endowed with a right-invariant metric induces a Riemannian metric on the orbits of the action and the map $\Pi_{q_0} : G \ni g \mapsto g \cdot q_0 \in Q$ is a Riemannian submersion.
Consequence: Geodesics downstairs horizontally lift to geodesics upstairs.

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- Statistics on the initial momentum.

Bayesian interpretation

The prior in the functional

$$\mathcal{J}(v) = \int_0^1 |v_t|_V^2 dt + \frac{1}{\sigma^2} d(\phi_{0,1}.A, B)^2$$

suggests a white noise in time for generic evolutions.

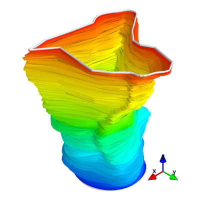

Figure: Kunita flows

→ Not realistic for evolutions of biological shapes.

Interpolating sparse longitudinal shape data

What we aim to do:

Within a diffeomorphic framework:

Let $(S^i_{t^i_0}, \ldots, S^i_{t^i_k})_{i\in[1,n]}$ be a $n-$sample of shape sequences indexed by the time $(t^i_0, \ldots, t^i_n) \subset [0,1]$.

Having in mind biological shapes, at least two problems

$\diamond$ To find a deterministic framework to treat each sample.
(in which space to study these data?)

$\diamond$ To develop a probabilistic framework to do statistics.
(classification into normal and abnormal growth)

A natural attempt

How to interpolate a sequence of data $(S_0, \ldots, S_{t_k})$ (images, surfaces, landmarks ...)

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When $k = 1 \longrightarrow$ standard registration problem of two images: Geodesic on a diffeomorphism group - LDDMM framework (M.Miller, A.Trouvé, L.Younes, F.Beg,...)

$$\mathcal{F}(v) = \frac{1}{2} \int_0^1 |v_t|_V^2 \, dt + |\phi_1 . S_0 - S_{t_1}|^2 ,$$

$$\begin{cases} \phi_0 = Id \\ \dot{\phi}_t = v_t(\phi_t) . \end{cases} \tag{1}$$

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Extending it to $k > 1$,

$$\mathcal{F}(v) = \frac{1}{2} \int_0^{t_k} |v_t|_V^2 \, dt + \sum_{j=1}^{k} |\phi_{t_j} . S_0 - S_{t_j}|^2 \,,$$

$\Longrightarrow$ piecewise geodesics in the group of diffeomorphisms

Illustration on 3D images

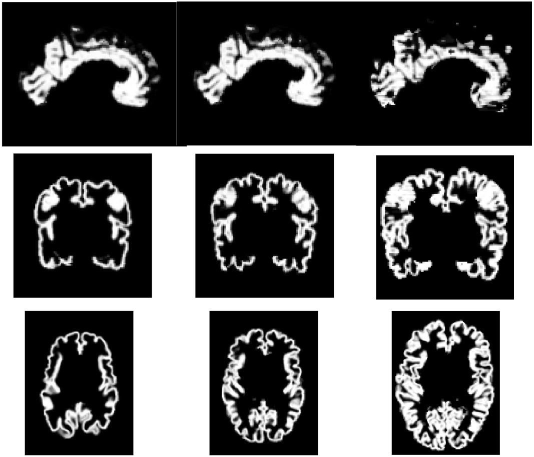

Figure: Slices of 3D volumic images: 33 / 36 / 43 weeks of gestational age of the same subject.

Illustration on 3D images

Video courtesy of Laurent Risser

Figure: Video courtesy of Laurent Risser

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Figure: Representation of the surface - Back of the brain

How to smoothly interpolate longitudinal data

In the Euclidean space:

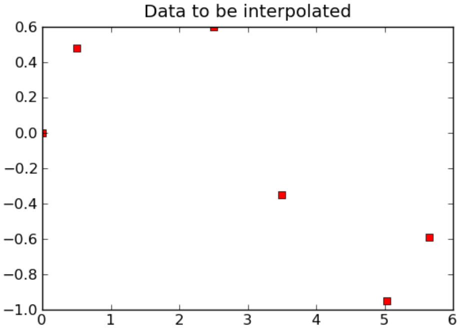

Figure: Sparse data from a sinus curve

How to smoothly interpolate longitudinal data

In the Euclidean space:

Minimizing the L^2 norm of the **speed** $\rightarrow$ piecewise linear interpolation

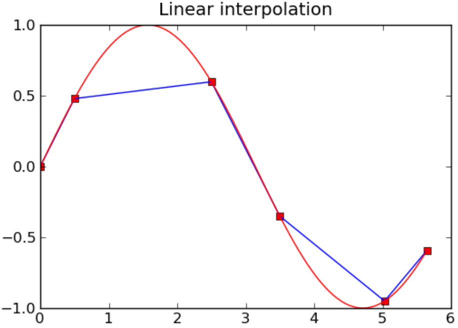

Figure: Linear interpolation of the data.

How to smoothly interpolate longitudinal data

In the Euclidean space:

Enforcing the geodesicity constraint

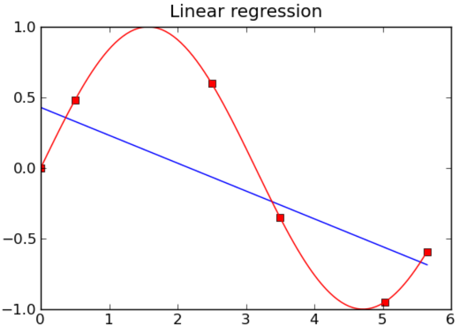

Figure: Cubic spline interpolation of the data.

How to smoothly interpolate longitudinal data

In the Euclidean space:

Minimizing the L^2 norm of the **acceleration** $\rightarrow$ cubic spline interpolation

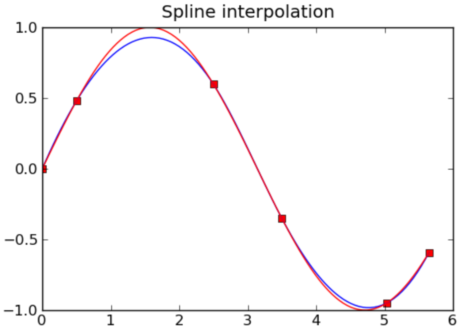

Figure: Cubic spline interpolation of the data.

What is acceleration in our context?

First attempt, on the group in the matching functional:

$$\mathcal{F}(v) = \frac{1}{2}\int_0^1 |v_t|_V^2 \, dt + |\phi_1.S_0 - S_{t_1}|^2 \,, \tag{2}$$

Replace the L^2 norm of the speed:

$$\frac{1}{2}\int_0^1 |v_t|_V^2 \, dt \tag{3}$$

by the L^2 norm of the acceleration of the vector field:

$$\frac{1}{2}\int_0^1 |\frac{d}{dt} v_t|_V^2 \, dt + |\phi_1.S_0 - S_{t_1}|^2 \,, \tag{4}$$

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Null cost for this norm $\longrightarrow v_t \equiv v_0$: Incoherent

Correct notion of acceleration

Acceleration on a Riemannian manifold M: let $c : I \to M$ be a C^2 curve. The notion of acceleration is:

$$\frac{D}{dt}\dot{c}(t) = \nabla_{\dot{c}}\dot{c}(= \ddot{c}_k + \sum_{i,j} \dot{c}_i \Gamma^k_{i,j} \dot{c}_j) \tag{5}$$

with ∇ the Levi-Civita connection.

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Riemannian splines: Crouch, Silva-Leite (90's)

$$\text{On } SO(3) \ \inf_c \int_0^1 \frac{1}{2} |\nabla_{\dot{c}_t} \dot{c}_t|^2_M dt\,. \tag{6}$$

subject to $c(i) = c_i$ and $\dot{c}(i) = v_i$ for $i = 0, 1$.

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Elastic Riemannian splines:

$$\inf_c \int_0^1 \frac{1}{2} |\nabla_{\dot{c}_t} \dot{c}_t|^2_M + \frac{\alpha}{2} |\dot{c}_t|^2_M dt\,. \tag{6}$$

subject to $c(i) = c_i$ and $\dot{c}(i) = v_i$ for $i = 0, 1$.

A modeling question

The Euler-Lagrange equation for Riemannian cubics is

$$\nabla^3_{\dot{c}} \dot{c} + R(\nabla_{\dot{c}} \dot{c}, \dot{c}) \dot{c} = 0 \,, \tag{7}$$

where R is the curvature tensor of the metric.

Remarks

If $\pi : M \mapsto B$ is a Riemannian submersion then:
geodesics lift to geodesics.
Probably not true for Riemannian cubics ...

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If $\pi : M \mapsto B$ is a Riemannian submersion then:
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Probably not true for Riemannian cubics . . .

In our context of a group action, $G \times M \mapsto M$:
$\Pi_{q_0} : G \ni g \mapsto g \cdot q_0 \in Q$ is a Riemannian submersion

Question

Higher-order on the group (upstairs) or higher-order on the orbit (downstairs)?

The convenient Hamiltonian setting

Hamiltonian equations of geodesics for landmarks:

$$\textbf{Geodesics} \quad \begin{cases} \dot{p} = -\partial_q H(p,q) \\ \dot{q} = \partial_p H(p,q) \end{cases} \tag{8}$$

with $H(p,q) = H(p_1,\dots,p_n,q_1,\dots,q_n) \doteq \frac{1}{2}\sum_{i,j=1}^{n} p_i k(q_i,q_j)p_j$ and k is the kernel for spatial correlation.

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Lemma

On a general Riemannian manifold,

$$\nabla_{\dot{q}}\dot{q} = K(q)(\dot{p} + \partial_q H(p,q)) \tag{9}$$

where $\dot{q} = K(q)p$ with $K(q)$ being the identification given by the metric between T_q^*Q and T_qQ.

Splines on shape spaces

We introduce a forcing term u as:

$$\textbf{Perturbed geodesics} \quad \begin{cases} \dot{p}_t = -\partial_q H(p_t, q_t) + u_t \\ \dot{q}_t = \partial_p H(p_t, q_t) \end{cases} \tag{10}$$

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Definition (Shape Splines)

Shape splines are defined as minimizer of the following functional:

$$\inf_u J(u) \doteq \frac{1}{2} \int_0^{t_k} \|u_t\|_X^2 \, dt + \sum_{j=1}^{k} |q_{t_j} - x_{t_j}|^2 \, . \quad (11)$$

subject to (q, p) perturbed geodesic through u_t for a freely chosen norm $\| \cdot \|_X$ on T_q^*.

Simulations

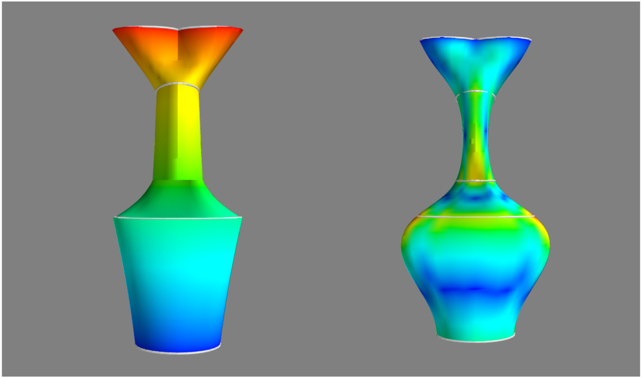

Figure: Comparison between piecewise geodesic interpolation and spline interpolation

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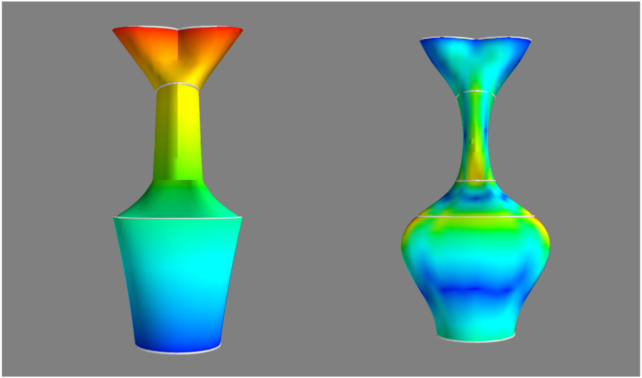

Figure: Comparison between piecewise geodesic interpolation and spline interpolation

- Matching of 4 timepoints from an initial template.
- $|\cdot|_X$ is the Euclidean metric.
- Smooth interpolation in time.

Information contained in the acceleration and extrapolation

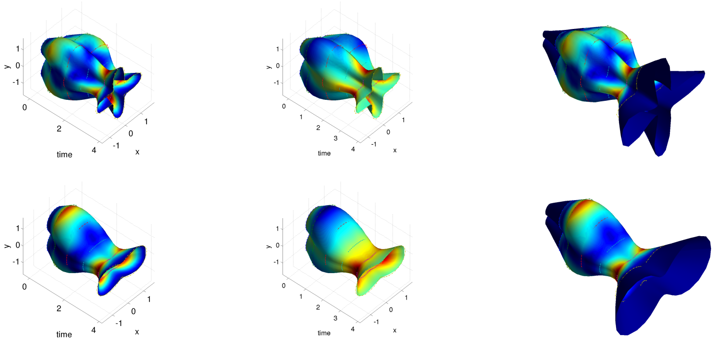

Figure: On each row: two different examples of the spline interpolation. In the first column, the norm of the control is represented whereas the signed normal component of the control is represented in the second one. The last column represents the extrapolation.

Robustness to noise

Due to the spatial regularisation of the kernel:

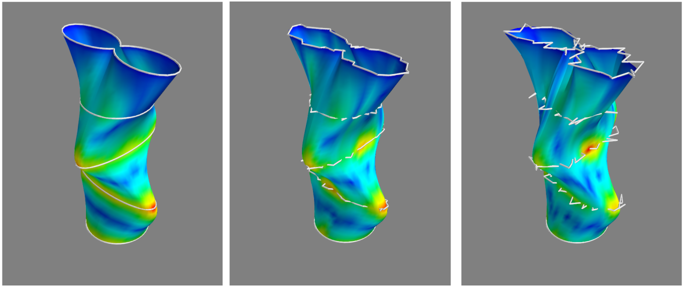

Figure: Gaussian noise added to the position of 50 landmarks

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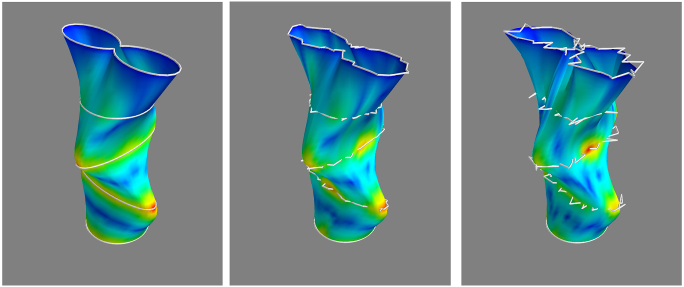

Figure: Gaussian noise added to the position of 50 landmarks

- Left: no noise.
- Center: standard deviation of 0.02.
- Right: standard deviation of 0.09.

Stochastics

A stochastic model:

Theorem

If k is C^1, the solutions of the stochastic differential equation defined by

$$\begin{cases} dp_t = -\partial_x H_0(p_t, x_t)dt + u_t(x_t)dt + \varepsilon(p_t, x_t)dB_t \\ dx_t = \partial_p H_0(p_t, x_t)dt. \end{cases} \quad (12)$$

are non exploding with few assumptions on u_t and ε.

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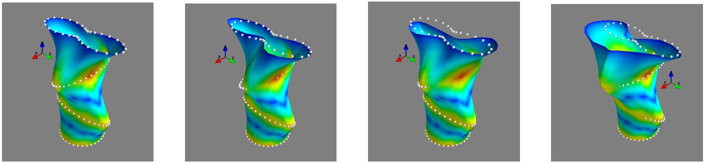

Figure: The first figure represents a calibrated spline interpolation and the three others are white noise perturbations ot the spline interpolation with respectively $\sqrt{n}\epsilon$ set to $0.25, 0.5$ and 0.75.

Simple PCA on the forcing term

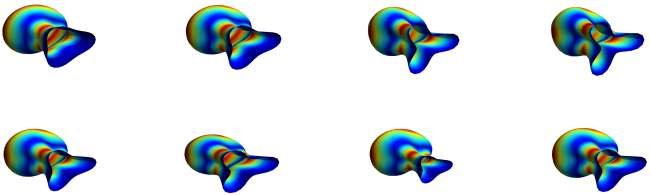

Figure: Top row: Four examples of time evolution reconstructions from the observations at 6 time points (not represented here) in the learning set. Bottom row: The simulated evolution generated from a PCA model learn from the pairs (p_0^k, u^k). The comparison between the two rows shows that the synthetised evolutions from the PCA analysis are visually good.

Ongoing work

- Extension to infinite dimensions (diffeomorphism group and images)
- Spline regression on the space of real images and statistical studies.

Geodesic regression

Enforcing the geodesicity constraint, shooting methods on the space of images:

$$\mathcal{S}(P(0)) = \frac{\lambda}{2}\langle \nabla I(0)P(0), K \star \nabla I(0)P(0)\rangle_{L^2} + \frac{1}{2}\|I(1) - J\|^2_{L^2} . \tag{13}$$

with:

$$\begin{cases} \partial_t I + v \cdot \nabla I = 0, \\ \partial_t P + \nabla \cdot (vP) = 0, \\ v + K \star (P\nabla I) = 0. \end{cases} \tag{14}$$

Adjoint equations

Proposition

The gradient of $\mathcal{S}$ *is given by:*
$\nabla_{P(0)}\mathcal{S} = -\hat{P}(0) + \nabla I(0) \cdot K \star (P(0)\nabla I(0))$ *where* $\hat{P}(0)$ *is given by the solution the backward PDE in time:*

$$\begin{cases} \partial_t \hat{I} + \nabla \cdot (v\hat{I}) + \nabla \cdot (P\hat{v}) = 0\,, \\ \partial_t \hat{P} + v \cdot \nabla \hat{P} - \nabla I \cdot \hat{v} = 0\,, \\ \hat{v} + K \star (\hat{I}\nabla I - P\nabla \hat{P}) = 0\,, \end{cases} \tag{15}$$

subject to the initial conditions:

$$\begin{cases} \hat{I}(1) = J - I(1)\,, \\ \hat{P}(1) = 0\,, \end{cases} \tag{16}$$

A key point: Integral formulation

Gradient descent based on an integral formulation:

Theorem

Let $I(0), J \in H^2(\Omega, \mathbb{R})$ be two images and K be a C^2 kernel on Ω. For any $P(0) \in L^2(\Omega)$, let (I, P) be the solution of the shooting equations with initial conditions $I(0), P(0)$. Then, the corresponding adjoint equations have a unique solution $(\hat{I}, \hat{P})$ in $C^0([0,1], H^1(\Omega) \times H^1(\Omega))$ such that

$$\begin{cases} \hat{P}(t) = \hat{P}(1) \circ \phi_{t,1} - \int_t^1 [\nabla I(s) \cdot \hat{v}(s)] \circ \phi_{t,s} \, ds \,, \\ \hat{I}(t) = \mathrm{Jac}(\phi_{t,1}) \hat{I}(1) \circ \phi_{t,1} \\ \qquad + \int_t^1 \mathrm{Jac}(\phi_{t,s}) [\nabla \cdot (P(s) \hat{v}(s))] \circ \phi_{t,s} \, ds \,. \end{cases} \tag{17}$$

with:

$$\begin{cases} \hat{v}(t) = K \star [P(t) \nabla \hat{P}(t) - \hat{I}(t) \nabla I(t)] \,, \\ P(t) = \mathrm{Jac}(\phi_{t,0}) P(0) \circ \phi_{t,0} \,, \\ I(t) = I(0) \circ \phi_{t,0} \,, \end{cases} \tag{18}$$

where $\phi_{s,t}$ is the flow of $v(t) = -K \star P(t) \nabla I(t)$.

Numerical examples on points

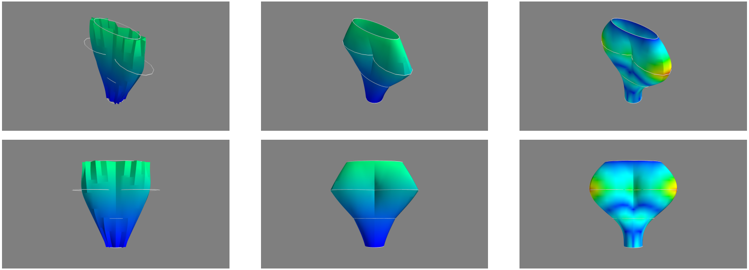

Figure:

- First Column: Geodesic Regression
- Second column: Linear Interpolation
- Third Column: Spline Interpolation

Papers

I warmly thank Colin Cotter, Darryl Holm, David Meier, Marc Niethammer, Laurent Risser and Alain Trouvé for this work.

- Shape Splines and Stochastic Shape Evolutions: A Second-Order Point of View. QAM (Trouvé A. and Vialard F.X.)
- Diffeomorphic 3D Image Registration via Geodesic Shooting using an Efficient Adjoint Calculation. (IJCV, 2011) Vialard F.X., Risser L., Rueckert D., Cotter C.J.
- Geodesic Regression for Image Time-Series. Niethammer M., Huang Y., Vialard F.-X., MICCAI 2011