Interpolation and design: from polynomials to Chebyshevian splines

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Let us consider an $(N+1)$-dimensional space $\mathcal{E}$ of sufficiently differentiable functions. Roughly speaking, **design** in the space $\mathcal{E}$ refers to the possibility of drawing a curve with a prescribed approximate shape, fixed by a polygonal line with $(N+1)$ vertices, called its *control polygon*. The curve is expected to mimic its control polygon of which it should be a smooth version. Here, **interpolation** in the space $\mathcal{E}$ refers to *Hermite interpolation*, *i.e.*, the possibility of determining a unique element in $\mathcal{E}$ with $(N+1)$ prescribed values for this element itself and its successive derivatives at given points, called *interpolation abscissæ*. The present talk surveys the strong links existing between the latter two topics which may a priori seem completely disconnected.

As is well-known, both interpolation and design are possible when $\mathcal{E}$ is the degree N polynomial space $\mathcal{P}_N$. It is the fact that any non-zero polynomial of degree at most N vanishes at most N times, counting multiplicities, which permits interpolation in $\mathcal{P}_N$. It is the presence of the Bernstein basis

$$B_i^N(x) := \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{N-i}, \quad 0 \le i \le N,$$

with all its interesting properties, which makes design possible in $\mathcal{P}_N$. Now, the actual underlying reason explaining both existence and properties of this special basis is the presence of *blossoms* in the space $\mathcal{P}_N$. Indeed, any polynomial $F \in \mathcal{P}_N$ uniquely blossoms into a function f of N variables meeting the following three requirements:

(i) f is symmetric,

(ii) f is affine in each variable,

(iii) f gives F by restriction to the diagonal.

The function f is called the blossom of F. Blossoms are wonderful tools for design, for they make the description of all design algorithms extremely simple and elegant.

Nevertheless, as degree grows, both polynomial interpolation and polynomial design quickly turn to be only theoretical possibilities. Indeed, the mimicking of the control polygon becomes not good enough, while interpolating polynomials may have nonsensical behaviour. For this reason, when N is not small, it is way more reasonable to replace the polynomial space $\mathcal{P}_N$ by an $(N+1)$-dimensional space of *polynomial splines*, *i.e.*, functions which are piecewise polynomials, two consecutive pieces joining with precribed smoothness at the corresponding *knot*. For instance, the most commonly used ones, *cubic splines*, are C^2 and have pieces of degree 3.

Polynomial spline spaces are excellent for design due to the presence of *B-spline bases* and to their properties. Among them, let us mention the fact they have small supports, which permits a local control of the spline curves. Now, again their existence as well as their interesting properties are actually due to the underlying presence of blossoms: each spline S with degree n pieces uniquely blossoms into a function s of n variables, called its blossom, meeting the same requirements as previously, except that it is defined only on a restricted set of n-tuples.

In a polynomial spline space, interpolation is possible only provided that the interpolating abscissæ and the knots of the spline space interlace according to the so-called *Schoenberg-Whitney conditions.* Again, the fact that interpolation under Schoenberg-Whitney conditions is possible is related to the existence of B-spline bases. Therefore, one can say that it is implicitly related to the existence of blossoms.

Although way much better than the polynomial case, interpolation by polynomial splines still presents some flaws: unfortunately, we may still have undesired oscillations, in particular in case there is a jump in the data. This is often referred to as *Gibbs phenomenon.* In order to make up for this inconvenience, a classical idea consists in introducing *shape parameters*, *i.e.*, some parameters on which we can play to improve the interpolating curve (or function) where necessary while keeping its general shape. In spline spaces, there are two main ways to generate such parameters. One can

1- either insert *connection matrices* at the knots, the usual smoothness being replaced by a geometrical one – this generates *geometrically continuous polynomial splines*;

2- or replace the polynomial space in which splines have their sections by a *Chebyshev space* of the same dimension. For instance one can replace the polynomial space $\mathcal{P}_3$ by the space spanned by the four functions $1, x, \cosh x, \sinh x$.

Our $(N+1)$-dimensional initial space $\mathcal{E}$ is a Chebyshev space if any non-zero element vanishes at most N times, counting multiplicities. Such spaces are thus exactly the spaces in which interpolation is possible. What about design? It is possible in an $(N+1)$-dimensional space $\mathcal{E}$ which contains constants if and only if it possesses Bernstein type bases, or if and only if it possesses blossoms, now defined in a geometrical way by means of intersections of osculating flats : any $F \in \mathcal{E}$ then blossoms into a function f of n variables satisfying the two properties (i) and (iii) above, and a modified poperty (ii). Interpolation and design are strongly linked: design is possible in $\mathcal{E}$ iff interpolation is possible in the space $D\mathcal{E}$, *i.e.*, iff the space $D\mathcal{E}$ is a Chebyshev space. In any associate spline space $\mathcal{S}$ with ordinary smoothness at the knots, blossoms automatically exist. This permits both design in $\mathcal{S}$ (existence of B-spline type bases) and interpolation under Schoenberg-Whitney conditions.

Mixing the two ideas above, one can eventually consider the general framework of splines with sections in different Chebyshev spaces, and with connection matrices at the knots. Unfortunately, blossoms do not always exist in a space $\mathcal{S}$ of such splines supposed to contain constants. Nevertheless, their existence is the necessary and sufficient condition which makes possible either design in $\mathcal{S}$ or interpolation under Schoenberg-Whitney conditions in $D\mathcal{S}$.

Unfortunately, it is quite difficult to find whether or not blossoms exist in such a general context. Sufficient conditions do exist, but they are sometimes far too restrictive. In the interesting case of splines with simple knots and sections in four-dimensional spaces, we managed to find explicit necessary and sufficient conditions for existence of blossoms. We use them to illustrate how to use the many shape parameters we have at our disposal either for design, or for interpolation, or - why not - for *interpolating design.*