Biorthogonal polynomials and the coupled random matrix model

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Statistical properties of eigenvalues of random matrices taken from a probability measure

$$\frac{1}{Z_n} \exp\left(-n \operatorname{Tr} V(M)\right) dM, \qquad V \text{ is a polynomial,}$$

defined on the space of $n \times n$ Hermitian matrices M can be fully analyzed using orthogonal polynomials. In this way an almost complete picture has arisen about the possible limiting eigenvalue behaviors as $n \to \infty$, both in the macroscopic and microscopic regimes.

The coupled random matrix model is a probability measure

$$\frac{1}{Z_n} \exp\left(-n \operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)\right) dM_1 dM_2$$

defined on pairs (M_1, M_2) of $n \times n$ Hermitian matrices. Here V and W are two polynomial potentials and $\tau > 0$ is a coupling constant. The model is of interest in $2D$ quantum gravity where it is used to construct generating functions for the number of bicolored graphs on surfaces.

The role of orthogonal polynomials is now taken over by two sequences of polynomials $\left(p_j^{(n)}\right)_j$ and $\left(q_k^{(n)}\right)_k$ that satisfy the biorthogonality condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_j^{(n)}(x) q_k^{(n)}(y) e^{-n(V(x)+W(y)-\tau xy)} dx dy = \delta_{j,k}.$$

Statistical properties of the eigenvalues of M_1 are described by the polynomials $p_j^{(n)}$. Despite many contributions in the physics literature, the limiting behavior is not fully understood in the mathematical sense.

In the talk I will discuss an approach to the simplest non-trivial case $W(y) = \frac{1}{4}y^4$, which involves the following steps:

- A characterization of the polynomials $p_j^{(n)}$ as multiple orthogonal polynomials, which leads to the formulation of a 4×4 matrix valued Riemann-Hilbert problem.
- An energy minimization problem for a triple of measures (μ_1, μ_2, μ_3) where μ_1 is the asymptotic zero distribution of the polynomials $p_n^{(n)}$ as well as the limiting eigenvalue distribution of M_1.
- The steepest descent analysis of the Riemann-Hilbert problem.

This is joint work with Maurice Duits.