Three-fold symmetric polynomials with an algebraic classical behaviour

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Definition

A monic Orthogonal Polynomial Sequence (OPS) $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_0, P_nP_k\rangle = N_n\delta_{n,k}\ ,\ \text{with}\ \ N_n \neq 0.$$

where u_0 is the first element of the corresponding dual sequence.

- Equivalently, $\{P_n\}_{n\geqslant 0}$ is an OPS for u_0 iff

$$\langle u_0, x^m P_n\rangle = \begin{cases} 0 & \text{if}\ \ n > m, \\ N_n & \text{if}\ \ n = m,\ \ \text{for}\ \ n \geq 0. \end{cases}$$

- It always satisfies the second order recurrence relation

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$ and

$$\beta_n = \frac{\langle u_0, xP_n^2\rangle}{\langle u_0, P_n^2\rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle} \neq 0,\ n\in\mathbb{N}$$

Consider a sequence $\{\widetilde{P}_{\vec{n}}\}$ with $\vec{n} = (n_1, n_2)$ and $\deg \widetilde{P}_{\vec{n}}(x) = n_1 + n_2$ such that

$$\langle u_0, x^k \widetilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_1} x^k \widetilde{P}_{\vec{n}}(x) W_0(x) \mathrm{d}x = 0 \ , \ \ k = 0, 1, \ldots, n_1 - 1$$
$$\langle u_1, x^k \widetilde{P}_{\vec{n}}(x) \rangle = \int_{\Delta_2} x^k \widetilde{P}_{\vec{n}}(x) W_1(x) \mathrm{d}x = 0 \ , \ \ k = 0, 1, \ldots, n_2 - 1$$

Multiple orthogonal polynomials - type II

Consider a sequence $\{\widetilde{P}_{\vec{n}}\}$ with $\vec{n} = (n_1, n_2)$ and $\deg \widetilde{P}_{\vec{n}}(x) = n_1 + n_2$ such that

$$\langle u_0, x^k \widetilde{P}_{\vec{n}}(x)\rangle = \int_{\Delta_1} x^k \widetilde{P}_{\vec{n}}(x) W_0(x)\mathrm{d}x = 0 \ , \ \ k = 0, 1, \ldots, n_1 - 1$$
$$\langle u_1, x^k \widetilde{P}_{\vec{n}}(x)\rangle = \int_{\Delta_2} x^k \widetilde{P}_{\vec{n}}(x) W_1(x)\mathrm{d}x = 0 \ , \ \ k = 0, 1, \ldots, n_2 - 1$$

Now, if we construct a sequence $\{\widetilde{P}_n\}_{n\geq 0}$ such that

$$P_{2n}(x) = \widetilde{P}_{n,n}(x)$$
$$P_{2n+1}(x) = \widetilde{P}_{n,n+1}(x)$$

then $\{P_n\}_{n\geq 0}$ is a 2-OPS.

Definition

Consider a vector linear functional $\mathbf{u} = (u_0, u_1)$ defined on $\mathcal{P}$ in $\mathbb{C}$. The sequence of polynomials $\{P_n\}_{n\geq 0}$, where $\deg P_n = n$, is said to be 2-orthogonal to $\mathbf{u} = (u_0, u_1)$ if

$$< u_0, x^m P_n >= \begin{cases} 0 & \text{for} \quad n \geq 2m+1 \\ N_{2m} \neq 0 & \text{for} \quad n = 2m \end{cases} \tag{1}$$

$$< u_1, x^m P_n >= \begin{cases} 0 & \text{for} \quad n \geq 2m+2 \\ N_{2m+1} \neq 0 & \text{for} \quad n = 2m+1 \end{cases} \tag{2}$$

2-orthogonal polynomials

The monic 2-OPS $\{P_n\}_{n\geq 0}$ for $\mathbf{u} = (u_0, u_1)$ satisfies a third order recurrence relation (see Van Iseghem'88, Maroni'89)

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x) \tag{3}$$

with $P_0(x) = 1,\ \ P_1(x) = x - \beta_0$ and $P_2(x) = (x - \beta_1)P_1(x) - \alpha_1$.

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with $P_0(x) = 1$, $P_1(x) = x - \beta_0$ and $P_2(x) = (x - \beta_1)P_1(x) - \alpha_1$.

Expressions for the recurrence coefficients follow immediately from the definition. For instance,

$$\gamma_{2n+1} = \frac{< u_0, x^{n+1}P_{2n+2} >}{< u_0, x^n P_{2n} >}, \quad \gamma_{2n+2} = \frac{< u_1, x^{n+1}P_{2n+3} >}{< u_1, x^n P_{2n+1} >}, \quad n \geq 0.$$

Conversely, we also have

$$N_{2n} :=< u_0, x^{n+1}P_{2n+2} >= \prod_{k=0}^{n} \gamma_{2k+1}$$

and

$$N_{2n+1} :=< u_1, x^{n+1}P_{2n+3} >= \prod_{k=0}^{n} \gamma_{2k+2}, \quad \text{for} \quad n \geq 0.$$

Example 1: 2-orthogonal polynomials for Bessel weights

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x)$$

with

$$\begin{aligned}
\beta_n &= 3n^2 + (2\alpha + 2\beta + 3)n + (1+\alpha)(1+\beta) \\
\alpha_n &= n(3n + \alpha + \beta)(n+\alpha)(n+\beta), \ \ n \geq 1, \\
\gamma_n &= n(n+1)(n+\alpha+1)(n+\alpha)(n+\beta+1)(n+\beta), \ \ n \geq 2,
\end{aligned}$$

They satisfy the 3rd order recurrence relation

$$x^2 P_n''' + (3 + \alpha + \beta) x P_n'' + ((\alpha+1)(\beta+1) - x) P_n' = -n P_n$$

and are 2-OPS for $U = (u_0, u_1)$ satisfying

$$x^2 u_0'' - (\alpha + \beta - 1) x u_0' - (x - \alpha\beta) u_0 = 0 \quad , \quad (\alpha+1)(\beta+1) u_1 = -(x u_0)'$$

Such vector functional $U = (u_0, u_1)$ admits the following integral representation

$$\begin{aligned}
< u_0, f(x) > &= \tfrac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x) x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x}) \mathrm{d}x, \\
< u_1, f(x) > &= \tfrac{2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{+\infty} f(x) \left(x^{(\alpha+\beta)/2} K_{\alpha-\beta}(2\sqrt{x}) \right)' \mathrm{d}x,
\end{aligned}$$

(See Ben Cheikh&Douak'00 and Van Assche&Yakubovich'00.)

The sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ satisfying the recurrence relation

$$P_{n+1}(x) = xP_n(x) - \frac{4}{27}P_{n-2}(x)$$

is 2-orthogonal with respect to $U = (u_0, u_1)$ such that

$$\begin{cases} (x^3-1)u_0'' + \frac{3}{2}x^2u_0' - \frac{1}{2}xu_0 = 0 \\ u_1 = 3(x^3-1)u_0' - \frac{3}{2}x^2u_0 \end{cases}$$

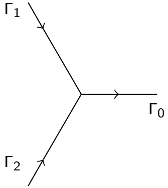
Such vector functional admits an integral representation on the real line as follows

$$\begin{aligned} < u_0, f(x) > &= \int_0^1 f(x)\frac{9\sqrt{3}}{4\pi}\left[(1+\sqrt{1-x^3})^{1/3} - (1-\sqrt{1-x^3})^{1/3}\right]\mathrm{d}x \\ &\quad + \int_0^{+\infty} f(x)3\mathrm{e}^{-x}\left[\lambda_1\sqrt{x}\cos(\sqrt{3}x) + \lambda_2x^2\sin(\sqrt{3}x)\right]\mathrm{d}x, \\ < u_1, f(x) > &= \int f(x)\mathcal{U}_1(x)\mathrm{d}x, \end{aligned}$$

(See Douak&Maroni'97 for further details.)

Consider the monic polynomials $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0\cup\Gamma_1} x^j P_{n,m}(x)\exp\left(-x^3+tx\right)dx = 0,\ j=0,\dots,n-1,$$
$$\int_{\Gamma_0\cup\Gamma_2} x^j P_{n,m}(x)\exp\left(-x^3+tx\right)dx = 0,\ j=0,\dots,m-1,$$



with $\Gamma_k=\{z\in\mathbb{C} : \arg z = e^{2k\pi i/3}\},\ k=0,1,2.$
(see Van Assche & Filipuk & Zhang (2015))

Rodrigues' formula:

$$\mathrm{e}^{-x^3+tx}P_{n,n+m}(x) = \frac{(-1)^n}{3^n}\frac{d^n}{dx^n}\left(\mathrm{e}^{-x^3+tx}P_{0,m}(x)\right)$$
$$\mathrm{e}^{-x^3+tx}P_{n+m,n}(x) = \frac{(-1)^n}{3^n}\frac{d^n}{dx^n}\left(\mathrm{e}^{-x^3+tx}P_{m,0}(x)\right)$$

where $P_{m,0}$ and $P_{0,m}$ are orthogonal polynomials...

and $\{P_{k,k}\}_k$ is 2-OPS.

(Case $t=0$ already in Pólya and Szegő (1925).
Special case of Gould-Hopper polynomials (1962).)

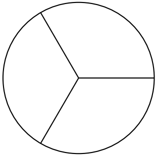
Definition

A monic polynomial sequence $\{B_n\}_{n\geq 0}$ is 3-fold symmetric if and only if

$$B_n(\mathrm{e}^{\frac{2i\pi}{3}}x) = \mathrm{e}^{\frac{2in\pi}{3}} B_n(x)$$

and

$$B_n(\mathrm{e}^{\frac{4i\pi}{3}}x) = \mathrm{e}^{\frac{4in\pi}{3}} B_n(x), \;\; n \geq 0.$$



In other words, this is to say that there exist three sequences $\{B_n^{[j]}\}_{n\geq 0}$ with $j \in \{0, 1, 2\}$ such that

$$\begin{aligned} B_{3n}(x) &= B_n^{[0]}(x^3), \\ B_{3n+1}(x) &= xB_n^{[1]}(x^3), \\ B_{3n+2}(x) &= x^2 B_n^{[2]}(x^3), \end{aligned}$$

(The sequences $\{B_n^{[j]}\}_{n\geq 0}$ are the components of the cubic decomposition of the 3-fold symmetric sequence $\{B_n\}_{n\geq 0}$.)

(see Barrucand&Dickinson'66)

Whilst we are dealing with 3-fold symmetric and 2-orthogonal sequences, we recall the following result.

Theorem (Douak & Maroni'92)

Let $\{P_n\}_{n\geq 0}$ be a 2-orthogonal polynomial sequence for $U=(u_0,u_1)$. Then, $\{P_n\}_{n\geq 0}$ is 3-fold symmetric iff if satisfies the third order recurrence relation

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x), \;\; n \geq 2,$$

with $P_0(x)=1, \;\; P_1(x)=x$ and $P_2(x)=x^2$.

Moreover, we have

Lemma (Douak & Maroni'92)

If the a 3-fold symmetric sequence $\{P_n\}_{n\geq 0}$ is 2-orthogonal, then the three components in the cubic decomposition of $\{P_n\}_{n\geq 0}$ are also 2-orthogonal fulfilling the recurrence relations:

$$P_{n+1}^{[k]}(x) = (x - \beta_n^{[k]})P_n^{[k]}(x) - \alpha_n^{[k]} P_{n-1}^{[k]}(x) - \gamma_{n-1}^{[k]} P_{n-2}^{[k]}(x),$$

where

$$\begin{aligned}
\beta_n^{[k]} &= \gamma_{3n-1+k} + \gamma_{3n+k} + \gamma_{3n+1+k}, \;\; n \geq 0, \\
\alpha_n^{[k]} &= \gamma_{3n-2+k}\gamma_{3n+k} + \gamma_{3n-1+k}\gamma_{3n-3+k} + \gamma_{3n-2+k}\gamma_{3n-1+k}, \;\; n \geq 1, \\
\gamma_n^{[k]} &= \gamma_{3n-2+k}\gamma_{3n+k}\gamma_{3n+2+k} \neq 0, \;\; n \geq 2,
\end{aligned}$$

for each $k = 0, 1, 2$.

Theorem. (Aptekarev *et al.*'00)
If $\gamma_n > 0$ for $n \geq 1$ in

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x),$$

then $\{P_n\}_{n\geq 0}$ is a 2-OPS w.r.t. the vector of linear functionals (u_0, u_1) and

$$< u_0, f(x) >= \int_S f(x)\mathrm{d}\mu_0(x) \tag{4}$$

$$< u_1, f(x) >= \int_S f(x)\mathrm{d}\mu_1(x) \tag{5}$$

where S represents the starlike set

$$S := \bigcup_{k=0}^{2} \Gamma_k \qquad \text{with} \qquad \Gamma_k = [0, \mathrm{e}^{2\pi ik/3}\infty),$$

and the measures have a common support which is a subset of S and are invariant under rotations of $2\pi/3$.

Theorem. (Ben Romdhane'08)
Let $\{P_n\}_{n\geq 0}$ be a 2-OPS satisfying

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x).$$

If $\gamma_n > 0$, then the following statements hold

(a) If x is a zero of P_{3n+j}, then $\omega^k x$ are also zeros of P_{3n+j} with $\omega = \mathrm{e}^{2\pi i/3}$

(b) 0 is a zero of P_{3n+j} of multiplicity j when $j = 1, 2$

(c) P_{3n+j} has n distinct positive real zeros

$$0 < x_{n,1}^{(j)} < \ldots < x_{n,n}^{(j)}$$

(d) Between two real zeros of P_{3n+j+3} there exist only one zero of P_{3n+j+2} and only one zero of P_{3n+j+1}, ie,

$$x_{n,k}^{(j+2)} < x_{n,k+1}^{(j)} < x_{n,k+1}^{(j+1)} < x_{n,k+1}^{(j+2)}$$

Theorem. (AL & Van Assche'18)
Let $\{P_n\}_{n\geq 0}$ be a 2-OPS satisfying

$$P_{n+1}(x) = xP_n(x) - \gamma_{n-1}P_{n-2}(x).$$

If $\gamma_n > 0$ and, additionally,

$$\gamma_{2n} = c_0 n^{\alpha} + o(n^{\alpha}) \quad \text{and} \quad \gamma_{2n+1} = c_1 n^{\alpha} + o(n^{\alpha})$$

for large n, with $c_0, c_1 > 0$ and $\alpha \geq 0$, then the largest zero in absolute value $|x_{n,n}|$ behaves as

$$|x_{n,n}| \leq \frac{3}{2^{2/3}} c^{1/3} n^{\alpha/3} + o(n^{\alpha/3}), \ \ n \geq 1, \tag{6}$$

where $c = \max\{c_0, c_1\}$.

Proof. Consider the Hessenberg matrix

$$\mathbf{H}_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & \ddots & & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & \gamma_{n-2} & 0 & 0 \end{pmatrix}$$

Hence,

$$\mathbf{H}_n \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} = x \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-1}(x) \end{pmatrix} - P_n(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and each zero of $P_n(x)$ is an eigenvalue of the matrix $\mathbf{H}_n$.

The spectral radius of the matrix $\mathbf{H}_n$,

$$\rho(\mathbf{H}_n) = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } \mathbf{H}_n\},$$

is bounded from above by $||\mathbf{H}_n||$ where $||\cdot||$ denotes a matrix norm. In particular

$$||\mathbf{H}_n||_S = ||S^{-1}\mathbf{H}_n S||_\infty = \max_{1\le i\le n}\left\{\sum_{j=1}^{n}\left|(S^{-1}\mathbf{H}_n S)_{i,j}\right|\right\},$$

where $S = \mathrm{diag}(d_1,\ldots,d_k,\ldots,d_n)$ is non-singular matrix and $(S^{-1}\mathbf{H}_n S)_{i,j}$ if the ith row and jth column entry of the product matrix $S^{-1}\mathbf{H}_n S$ we obtain

$$||\mathbf{H}_n||_S = \max\left\{\frac{d_2}{d_1},\frac{d_3}{d_2},\frac{d_4+d_1\gamma_1}{d_3},\ldots,\frac{d_k+d_{k-3}\gamma_{k-3}}{d_{k-1}},\ldots,\frac{d_{n-2}\gamma_{n-2}}{d_n}\right\}.$$

Setting $d_k = d^k (k!)^{\alpha/3} \neq 0$, for some $d>0$, brings

$$||\mathbf{H}_n||_S \le 2^{\alpha/3}\left(d+\frac{c}{d^2}\right)n^{\alpha/3} + o(n^{\alpha/3}) \quad \text{as} \quad n\to+\infty.$$

The choice of $d=(2c)^{1/3}$ provides a minimum to $\left(d+\frac{c}{d^2}\right)$ and this gives

$$||\mathbf{H}_n||_S \le \frac{3}{4^{1/3}}\left(c\ n^{\alpha}\right)^{1/3} + o(n^{\alpha/3}) \quad \text{as} \quad n\to+\infty.$$

$\square$

Definition

A monic 2-OPS $\{P_n\}_{n\geq 0}$ is "**classical**" in Hahn's sense when the sequence of its derivatives $\{Q_n\}_{n\geq 0}$, with

$$Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)$$

is also a 2-OPS.

Hence, as a monic 2-OPS, the sequence $\{Q_n\}_{n\geq 0}$ satisfies a third order recurrence relation:

$$Q_{n+1}(x) = (x - \widetilde{\beta}_n) Q_n(x) - \widetilde{\alpha}_n Q_{n-1}(x) - \widetilde{\gamma}_{n-1} Q_{n-2}(x), \; n \geq 2, \qquad (7)$$

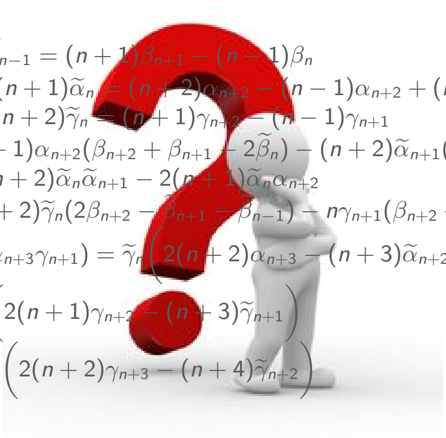
with $Q_0 = 1$, $Q_1(x) = x - \widetilde{\beta}_0$ and $Q_2(x) = (x - \widetilde{\beta}_1) Q_1(x) - \widetilde{\alpha}_1$.

"Classical" 2-orthogonal polynomials

Between the two recurrence relations

$$
\begin{aligned}
P_{n+1}(x) &= (x-\beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x)\\
Q_{n+1}(x) &= (x-\widetilde{\beta}_n)Q_n(x) - \widetilde{\alpha}_n Q_{n-1}(x) - \widetilde{\gamma}_{n-1}Q_{n-2}(x), \;\; n \geq 2,
\end{aligned}
$$

it follows a nonlinear system of equations



$$
\begin{aligned}
&(n+2)\widetilde{\beta}_n - n\widetilde{\beta}_{n-1} = (n+1)\beta_{n+1} - (n-1)\beta_n\\
&(n+3)\widetilde{\alpha}_{n+1} - (n+1)\widetilde{\alpha}_n = (n+2)\alpha_{n+2} - (n-1)\alpha_{n+2} + (n+1)(\beta_{n+1} - \widetilde{\beta}_n)^2\\
&(n+4)\widetilde{\gamma}_{n+1} - (n+2)\widetilde{\gamma}_n = (n+1)\gamma_{n+2} - (n-1)\gamma_{n+1}\\
&\qquad\qquad + (n+1)\alpha_{n+2}(\beta_{n+2} + \beta_{n+1} - 2\widetilde{\beta}_n) - (n+2)\widetilde{\alpha}_{n+1}(2\beta_{n+2} - \widetilde{\beta}_{n+1} - \widetilde{\beta}_n)\\
&n\alpha_{n+1}\alpha_{n+2} + (n+2)\widetilde{\alpha}_n\widetilde{\alpha}_{n+1} - 2(n+1)\widetilde{\alpha}_n\alpha_{n+2}\\
&\qquad\qquad = (n+2)\widetilde{\gamma}_n(2\beta_{n+2} - \widetilde{\beta}_{n+1} - \widetilde{\beta}_{n-1}) - n\gamma_{n+1}(\beta_{n+2} + \beta_n - 2\widetilde{\beta}_{n-1})\\
&n(\alpha_{n+1}\gamma_{n+2} + \alpha_{n+3}\gamma_{n+1}) = \widetilde{\gamma}_n\Big(2(n+2)\alpha_{n+3} - (n+3)\widetilde{\alpha}_{n+2}\Big)\\
&\qquad\qquad + \widetilde{\alpha}_n\Big(2(n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\Big)\\
&n\gamma_{n+1}\gamma_{n+3} = \widetilde{\gamma}_n\Big(2(n+2)\gamma_{n+3} - (n+4)\widetilde{\gamma}_{n+2}\Big)
\end{aligned}
$$

On the other hand, the 2-orthogonality of $\{P_n\}_{n\geq 0}$ for $U=(u_0,u_1)$ and the 2-orthogonality of $\{Q_n\}_{n\geq 0}$ for $V=(v_0,v_1)$ implies

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \tag{8}$$

and also that

$$\begin{bmatrix} v_0' \\ v_1' \end{bmatrix} = -\mathbf{\Psi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}. \tag{9}$$

with

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix} \quad \text{and} \quad \mathbf{\Psi} = \begin{bmatrix} 0 & 1 \\ \psi(x) & \zeta \end{bmatrix}$$

where $\psi(x) = \frac{2}{\gamma_1} P_1(x)$ and $\zeta = -\frac{2\alpha_1}{\gamma_1}$,

whilst $\deg\{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1$ and $\deg \phi_{1,0} \leq 2$.

Theorem. (Maroni& Douak'92, Maroni'99)
The monic 2-OPS $\{P_n\}_{n\geq 0}$ for $U = (u_0, u_1)$ is "classical" iff there are polynomials ψ and $\phi_{i,j}$, with $i, j \in \{0, 1\}$, and a constant ζ such that

$$\left(\begin{bmatrix} \phi_{0,0} & \phi_{0,1} \\ \phi_{1,0} & \phi_{1,1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \begin{bmatrix} 0 & 1 \\ \psi(x) & \zeta \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (10)$$

where $\deg\{\phi_{0,0}, \phi_{0,1}, \phi_{1,1}\} \leq 1$, $\deg \phi_{1,0} \leq 2$ *and* $\deg \psi = 1$.

Relation (11a) reads as follows

$$\left(\mathbf{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \mathbf{\Psi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\{P_n\}_{n\geqslant 0}$ is three-fold symmetric, then so is $\{Q_n\}_{n\geqslant 0}$ where

$$Q_n(x) := \frac{1}{n+1} P'_{n+1}(x), \;\; n \geqslant 0.$$

This means that for a *three-fold symmetric Hahn-classical polynomial sequence* $\{P_n\}_{n\geqslant 0}$ then $\{Q_n\}_{n\geqslant 0}$ is three-fold and satisfies

$$Q_{n+1}(x) = xQ_n(x) - \widetilde{\gamma}_{n-1} Q_{n-2}, \quad \text{for} \quad n \geqslant 2,$$

with initial conditions $Q_k(x) = x^k$ for $k = 0, 1, 2$.

in this case we have

Theorem. (AL&Van Assche'18) Let $\{P_n(x)\}_{n\geq 0}$ be a three-fold symmetric 2-OPS for (u_0, u_1). The following are equivalent:

(a) $\{P_n(x)\}_{n\geq 0}$ is a three-fold symmetric "classical" 2-orthogonal polynomial sequence.

(b) The vector functional (u_0, u_1) satisfies the matrix differential equation

$$\left(\mathbf{\Phi}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right)' + \mathbf{\Psi}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11a}$$

where

$$\Phi = \begin{bmatrix} \vartheta_1 & (1-\vartheta_1)x \\ \dfrac{2}{\gamma_1}(1-\vartheta_2)\,x^2 & 2\vartheta_2 - 1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Psi} = \begin{bmatrix} 0 & 1 \\ \frac{2}{\gamma_1}x & 0 \end{bmatrix} \tag{11b}$$

for some constants $\vartheta_1 = \frac{3\widetilde{\gamma}_1}{\gamma_2}$ and $\vartheta_2 = \frac{2\widetilde{\gamma}_2}{\gamma_3}$ such that $\vartheta_1, \vartheta_2 \neq \frac{n-1}{n}$.

(c) There exists a sequence of numbers $\{\widetilde{\gamma}_{n+1}\}_{n\geq 0}$ such that

$$P_{n+3}(x) = Q_{n+3}(x) + \Big((n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\Big)Q_n(x) \tag{12}$$

with initial conditions $P_k(x) = Q_k(x) = x^k$ for $k = 0, 1, 2$.

Proof. (a) $\Rightarrow$ (c): consequence of the rec. rel. of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$.
(c) $\Rightarrow$ (b): If $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are the dual sequences of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, resp., then

$$v_n' = -(n+1)u_{n+1} \tag{13}$$

$$v_n = u_n + \Big((n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\Big)u_{n+3}. \tag{14}$$

The 2-orthogonality of $\{P_n\}_{n\geq 0}$ implies

$$u_2 = \frac{x}{\gamma_1}u_0, \quad u_3 = -\frac{1}{\gamma_2}u_0 + \frac{x}{\gamma_2}u_1, \quad u_4 = \frac{x^2}{\gamma_1\gamma_3}u_0 - \frac{1}{\gamma_3}u_1$$

If we take $n=0$ and $n=1$ in (13) we obtain

$$\begin{bmatrix} v_0' \\ v_1' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{\gamma_1}x & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

With $n=0$ and $n=1$ in (14) leads to

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} \vartheta_1 & (1-\vartheta_1)x \\ \frac{2}{\gamma_1}(1-\vartheta_2)\,x^2 & 2\vartheta_2 - 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Proof. (cont.)
The proof of (b) $\Rightarrow$ (a) is essentially about showing that $\{Q_n\}_{n\geq 0}$ is 2-orthogonal with respect to

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} \vartheta_1 & (1-\vartheta_1)x \\ \frac{2}{\gamma_1}(1-\vartheta_2)\,x^2 & 2\vartheta_2 - 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$\square$

Proof. (cont.)
The proof of (b) $\Rightarrow$ (a) is essentially about showing that $\{Q_n\}_{n\geq 0}$ is 2-orthogonal with respect to

$$\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} \vartheta_1 & (1-\vartheta_1)x \\ \frac{2}{\gamma_1}(1-\vartheta_2)\, x^2 & 2\vartheta_2 - 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$\square$

The Pearson equation

$$\left(\mathbf{\Phi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)' + \mathbf{\Psi} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gives

$$\widetilde{\gamma}_n = \frac{n}{n+2} \vartheta_n \gamma_{n+1}$$

with

$$\vartheta_{2n+1} = \left(\frac{1-(n+1)(1-\vartheta_1)}{1-n(1-\vartheta_1)} \right) \quad \text{and} \quad \vartheta_{2n+2} = \left(\frac{1-(n+1)(1-\vartheta_2)}{1-n(1-\vartheta_2)} \right). \tag{15}$$

If we replace each P in

$$xP_n = P_{n+1} + \gamma_{n-1} P_{n-2}$$

by the corresponding expression given in

$$P_{n+3}(x) = \; Q_{n+3}(x) + \Big((n+1)\gamma_{n+2} - (n+3)\widetilde{\gamma}_{n+1}\Big) Q_n(x)$$

to then use the recurrence relation

$$xQ_n = Q_{n+1} + \widetilde{\gamma}_{n-1} Q_{n-2} \quad \text{where} \quad \widetilde{\gamma}_{n-1} = \frac{n-1}{n+1} \vartheta_{n-1} \gamma_n$$

we obtain

$$\vartheta_{n+2} + \frac{1}{\vartheta_n} = 2, \;\; n \geqslant 1,$$

and

$$\gamma_{n+2} = \frac{n+3}{n+1} \frac{\Big(n(\vartheta_n - 1) + 1\Big)}{\Big((n+4)(\vartheta_{n+1} - 1) + 1\Big)} \gamma_{n+1} \neq 0$$

Lemma (Douak&Maroni'97)

If a 2-symmetric 2-OPS $\{P_n\}_{n\geq 0}$ is "classical", then each polynomial is a solution of the third order differential equation

$$(a_n x^3 - b_n) P'''_{n+1} + c_n x^2 P''_{n+1} + d_n x P'_{n+1} = e_n P_{n+1}$$

where

$$\begin{aligned}
a_n &= (\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
b_n &= \tfrac{\gamma_{n+3}((n+3)\vartheta_{n+2}-(n+2))((n+4)\vartheta_{n+1}-(n+3))((n+5)\vartheta_{n+2}-(n+4))}{(n+3)(n+4)} \\
c_n &= \vartheta_n\vartheta_{n+1} - 1 - (n-3)(\vartheta_n - 1)(\vartheta_{n+1} - 1) \\
d_n &= n\vartheta_{n+1} - (n-1)\vartheta_n(2\vartheta_{n+1} - 1) \\
e_n &= n\vartheta_{n+1}, \quad \textit{for any} \quad n \geq 1,
\end{aligned}$$

with $a_0 = b_0 = c_0 = d_0 = e_0 = 0$.

Here

$$\vartheta_{2n+1} = \left(\frac{1-(n+1)(1-\vartheta_1)}{1-n(1-\vartheta_1)}\right) \quad \text{and} \quad \vartheta_{2n+2} = \left(\frac{1-(n+1)(1-\vartheta_2)}{1-n(1-\vartheta_2)}\right).$$

Proposition. (AL & Van Assche'18) The 2-OPS $\{P_n(x)\}_{n\geq 0}$ with respect to the vector linear functional $\mathbf{U} = (u_0, u_1)$ satisfy the Hahn's property if and only if there are coefficients $\vartheta_1, \vartheta_2 \neq \frac{n-1}{n}$, such that $\mathbf{U} = (u_0, u_1)$ satisfies

$$\Big(\phi(x)u_0\Big)'' + \left(\frac{2}{\gamma_1}(\vartheta_2 + \vartheta_1 - 2)x^2 u_0\right)' + \frac{2}{\gamma_1}(\vartheta_1 - 2)\, x u_0 = 0 \tag{16}$$

and

$$\begin{cases} (\vartheta_1 - 2)(2\vartheta_2 - 1)\, u_1 = \phi(x)u_0' - \frac{2}{\gamma_1}(\vartheta_1 - 1)(2\vartheta_2 - 3)\, x^2 u_0, & \textit{if} \quad \vartheta_1 \neq 2, \\ x\, u_1' = 2u_0', & \textit{if} \quad \vartheta_1 = 2, \end{cases}$$

where

$$\phi(x) = \left(\vartheta_1(2\vartheta_2 - 1) - \frac{2}{\gamma_1}(\vartheta_1 - 1)(\vartheta_2 - 1)\, x^3\right). \tag{17}$$

and from this we have

Theorem. (AL & Van Assche'18) For a "classical" threefold symmetric $\{P_n\}_{n\geq 0}$ 2-orthogonal with respect to (u_0, u_1) and satisfying the rec. rel. with $\gamma_{n+1} > 0$:

$$\langle u_k, f(x)\rangle$$
$$= \frac{1}{3}\left(\int_0^b f(x)\mathcal{U}_k(x)\mathrm{d}x + \omega^{2k-1}\int_0^{b\omega} f(x)\mathcal{U}_k(\omega^2 x)\mathrm{d}x + \omega^{1-2k}\int_0^{b\omega^2} f(x)\mathcal{U}_k(\omega x)\mathrm{d}x\right),$$

with $\omega = \mathrm{e}^{2\pi i/3}$ and $b = \lim_{n\to\infty}\left(\frac{27}{4}\gamma_n\right)$, provided that $\mathcal{U}_0(x)$ and $\mathcal{U}_1(x)$

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with $\omega = \mathrm{e}^{2\pi i/3}$ and $b = \lim_{n\to\infty}\left(\frac{27}{4}\gamma_n\right)$, provided that $\mathcal{U}_0(x)$ and $\mathcal{U}_1(x)$

$$\boxed{\begin{cases} \left(\phi(x)\mathcal{U}_0(x)\right)'' + \left(\frac{2(\vartheta_2+\vartheta_1-2)}{\gamma_1}x^2\mathcal{U}_0(x)\right)' + \frac{2(\vartheta_1-2)}{\gamma_1}x\mathcal{U}_0(x) = \lambda_0 g_0(x) \\ (\vartheta_1-2)(2\vartheta_2-1)\mathcal{U}_1(x) = \phi(x)\mathcal{U}_0'(x) - \frac{2(\vartheta_1-1)(2\vartheta_2-3)}{\gamma_1}x^2\mathcal{U}_0(x) + \lambda_1 g_1(x) \\ x\mathcal{U}_1'(x) = 2\mathcal{U}_0'(x) \qquad \text{if} \qquad \vartheta_1 = 2 \end{cases}}$$

with $\phi(x) = \left(\vartheta_1(2\vartheta_2-1) - \frac{2(\vartheta_1-1)(\vartheta_2-1)}{\gamma_1}x^3\right)$, satisfying

$$\boxed{\lim_{x\to b} f(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathcal{U}_0(x) = 0, \quad \text{and} \quad \int_0^b \mathcal{U}_0(x)\mathrm{d}x = 1}$$

$\lambda_k \in \mathbb{C}$ and $\int_\Gamma x^n g_k(x)\mathrm{d}x = 0$.

There are four cases to single out:

Case A: $\vartheta_1 = \vartheta_2 = 1$. This implies that $\vartheta_n = 1$ for all $n \geq 0$.

Case B$_1$: $\vartheta_1 \neq 1$ but $\vartheta_2 = 1$ so that by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ it follows

$$\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu} \quad \text{and} \quad \vartheta_{2n} = 1\ , \quad n \geq 1.$$

Case B$_2$: $\vartheta_1 = 1$ but $\vartheta_2 \neq 1$ so that by setting $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows

$$\vartheta_{2n-1} = 1 \quad \text{and} \quad \vartheta_{2n} = \frac{n+\rho+1}{n+\rho}\ , \quad n \geq 1.$$

Case C: $\vartheta_1 \neq 1$ and $\vartheta_2 \neq 1$ and hence by setting $\vartheta_1 = \frac{\mu+2}{\mu+1}$ and $\vartheta_2 = \frac{\rho+2}{\rho+1}$ it follows

$$\vartheta_{2n-1} = \frac{n+\mu+1}{n+\mu} \quad \text{and} \quad \vartheta_{2n} = \frac{n+\rho+1}{n+\rho}\ , \quad n \geq 1.$$

Case A: Appell polynomials

In this case we have $Q_n(x) := \frac{1}{n+1} P'_{n+1}(x) = P_n(x)$. Additionally

$$\gamma_{n+1} = (n+1)(n+2)\frac{\gamma_1}{2}, \qquad \text{and} \qquad \begin{cases} u_0'' - \dfrac{2}{\gamma_1} x \; u_0 = 0 \\ \\ u_1 = -u_0' \end{cases}$$

With the choice $\gamma_1 = 2$, it follows that

$$\gamma_{n+1} = (n+1)(n+2), \qquad \text{and} \qquad \begin{cases} u_0'' - x \; u_0 = 0 \\ \\ u_1 = -u_0' \end{cases}$$

and

$$-P'''_{n+1}(x) + xP'_{n+1}(x) = nP_{n+1}(x), \;\; n \geq 0.$$

Remark. The polynomials appear in the Vorob'ev-Yablonski polynomials associated with rational solutions of Painlevé II equations (Clarkson & Mansfield'03)

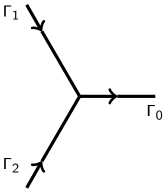
Integral representation (AL&Van Assche)

$$\begin{aligned} < u_0, f > &= \int_\Gamma f(x) W_0(x) dx, \text{ for all } f \in \mathcal{P}, \\ < u_1, f > &= \int_\Gamma f(x) W_1(x) dx, \text{ for all } f \in \mathcal{P}, \end{aligned}$$

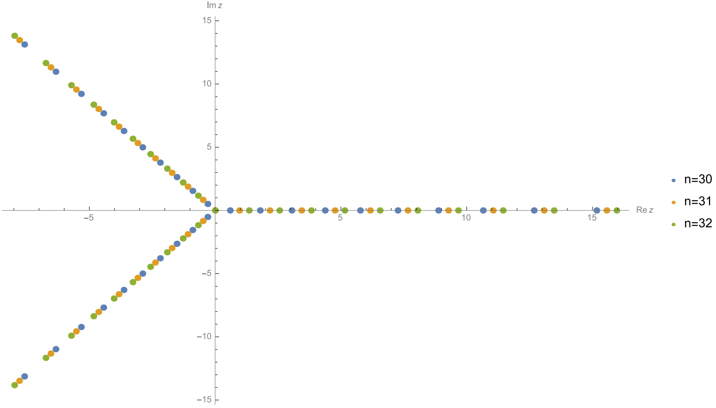
where $W_0 : \Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \longrightarrow \mathbb{R}$ defined by

$$W_0(x) = \mathrm{Ai}(x) \mathbb{I}_{\Gamma_0} - \mathrm{e}^{-2\pi i/3} \mathrm{Ai}(\mathrm{e}^{-2\pi i/3} x) \mathbb{I}_{\Gamma_1} - \mathrm{e}^{2\pi i/3} \mathrm{Ai}(\mathrm{e}^{2\pi i/3} x) \mathbb{I}_{\Gamma_2}$$

with $\Gamma_k = \left\{ w : \arg(w) = \frac{2k\pi}{3} \right\}$, with $k = 0, 1, 2,$



where the orientations of Γ_k are all taken from left to right



Remarks.

- All the zeros of $P_n(x)$ are located on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$
- In each Γ_k, between two zeros of P_{n+2} there is one zero of P_n and P_{n+1}.

Case B_1

- Here we have

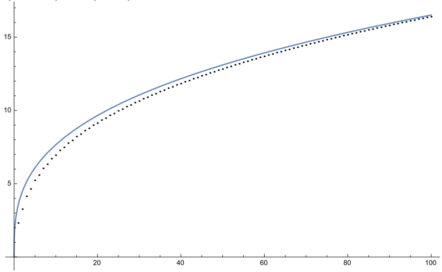
$$\gamma_{2n} = \frac{n(2n+1)(n+\mu)(\mu+2)}{(3n+\mu-1)(3n+\mu+2)}\gamma_1 = \frac{2\gamma_1(\mu+2)}{9} n + o(n), \quad n \geq 1,$$

$$\gamma_{2n+1} = \frac{(n+1)(2n+1)(\mu+2)}{(3n+\mu+2)}\gamma_1 = \frac{2\gamma_1(\mu+2)}{3} n + o(n), \quad n \geq 0,$$

- For $\mu > 0$, then $\gamma_n > 0$ for all $n \geq 1$.
- The largest real zero $x_{n,n}^{(j)}$ of P_{3n+j} is bounded from above by

$$x_{n,n}^{(j)} \leqslant \frac{3^{2/3}}{2^{1/3}}(\gamma_1(\mu+2))^{1/3} n^{1/3} + o(n^{1/3})$$

Fig. largest zero of the first 300 polynomials $(\gamma_1, \mu) = (2, 3)$



Case B_1: the zeros of P_n

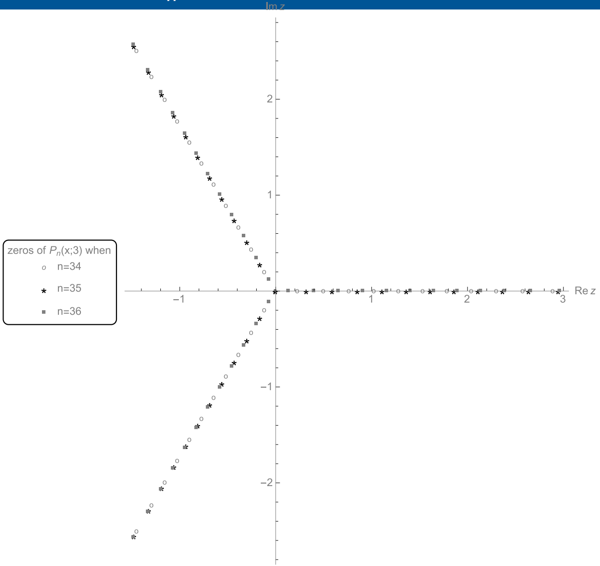


Figure: Zeros of $P_{34}(x;\mu)$ (circle), $P_{35}(x;\mu)$ (star) and $P_{36}(x;\mu)$ (square) with $\mu = 3$, where $P_n(x;\mu)$ is the 2-OPS studied in case B1.

With the choice of $\gamma_1 = 2$, when $\mu > 0$

$$\begin{cases} \frac{1}{3} u_0'' + x^2 u_0' - (\mu - 2) x u_0 = 0 \\ \\ u_1 = -\frac{(\mu+2)}{\mu} \left(u_0' + 3x^2 u_0 \right) . \end{cases}$$

and for $\mu = 0$:

$$\begin{cases} u_0' + 3x^2 u_0 = 0 \\ \\ x u_1' = 2 u_0' \end{cases}$$

Theorem (AL & Van Assche)

The linear 3*-fold symmetric 2-orthogonal vector functional* (u_0, u_1) *admit the following integral representation:*

$$\langle u_k, f(x)\rangle = \frac{1}{3}\left(\int_0^{\infty} f(x)\mathcal{U}_k(x)\mathrm{d}x + \omega^{2k-1}\int_0^{\infty\omega} f(x)\mathcal{U}_k(\omega^2 x)\mathrm{d}x + \omega^{1-2k}\int_0^{\infty\omega^2} f(x)\mathcal{U}_k(\omega x)\mathrm{d}x\right),$$

with $k = 0, 1$ *and*

$$\mathcal{U}_0(x) := \mathcal{U}_0(x;\mu) = \frac{3\Gamma(\frac{\mu+2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}\mathrm{e}^{-x^3}\mathbf{U}(\tfrac{\mu}{3}, \tfrac{2}{3}; x^3),$$

$$\mathcal{U}_1(x) := \mathcal{U}_1(x;\mu) = \frac{9\Gamma\left(\frac{\mu+5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}x^2\ e^{-x^3}\ \mathbf{U}\left(\tfrac{\mu}{3}+1, \tfrac{5}{3}, x^3\right), \quad \textit{for} \quad \mu \neq 0$$

$$\mathcal{U}_1(x;0) = 3\sqrt{3}\Gamma\left(\tfrac{2}{3}\right)\ \Gamma\left(\tfrac{2}{3}, x^3\right)$$

Here

$$\mathbf{U}(a,b;x) = \frac{1}{\Gamma(a)}\int_0^{\infty} t^{a-1}(t+1)^{-a+b-1}\mathrm{e}^{-tx}\mathrm{d}t \quad \textit{and} \quad \mathbf{U}(0,b;x) = 1$$

case B_1: proof of the integral representation

Proof (idea). We seek an integral representation for u_0, that is, we seek a weight function $\mathcal{U}_0(x)$ and a path $\mathcal{C}$ so that

$$< u_0, f(x) >= \int_{\mathcal{C}} f(x)\mathcal{U}_0(x)\mathrm{d}x,$$

is valid for any polynomial f. In particular, we must have

$$< u_0, x^n >= \int_{\mathcal{C}} x^n\mathcal{U}_0(x)\mathrm{d}x, \;\; n \geq 0.$$

The functional equation $(\mu+2)u_0'' + x^2 u_0' - (\mu-2)xu_0 = 0$ implies that $\mathcal{U}_0$ must be a solution of the differential equation

$$(\mu+2)\mathcal{U}_0'' + x^2\mathcal{U}_0' - (\mu-2)x\mathcal{U}_0 = \lambda g(x)$$

where λ is a complex constant and $g(x)$ is a function such that

$$\int_{\mathcal{C}} x^n g(x)(x)\mathrm{d}x = 0, \;\; n \geq 0.$$

With $\lambda = 0$, it follows that

$$\mathcal{U}_0(x) = c_1 \; {}_1F_1\left(\frac{2-\mu}{3}, \frac{2}{3}; t\right) + c_2 t^{1/3} \; {}_1F_1\left(1-\frac{\mu}{3}, \frac{4}{3}; t\right)$$

The choice of the constants c_1 and c_2 as well as the path of integration is dictated by the conditions

$$< u_0, x^n >= \int_{\mathcal{C}} x^n \mathcal{U}_0(x) \mathrm{d}x, \;\; n \geq 0,$$

and $$\left[(\mu+2) \left(f'(x) - f(x) \right) \mathcal{U}_0'(x) - x^2 f(x) \mathcal{U}_0(x) \right] \Big|_{\mathcal{C}} = 0, \text{ for any } f \in \mathcal{P}.$$

From DLMF (relations (13.2.39) and (13.2.41)) we deduce

$$\mathrm{e}^{-z} \mathbf{U}(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \, {}_1F_1\left(b-a, b; -z\right) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \, {}_1F_1\left(1-a, 2-b; -z\right)$$

which are valid when b is not an integer.

Thus, with $c_1 = \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1+\mu}{3})} K$ and $c_2 = \frac{-\mu \; \Gamma(\frac{2}{3})}{\Gamma(\frac{\mu}{3}+1)} K$ and $\mathcal{C} = \Gamma$, the result follows.

$\square$

The particular choice of $\mu = 1$ produces

$$\gamma_{2n} = \frac{2}{9}(2n+1)(\mu+2), \quad n \geq 1,$$

$$\gamma_{2n+1} = \frac{2}{3}(2n+1)(\mu+2), \quad n \geq 0,$$

whilst the weight functions become

$$\mathcal{U}_0(x;1) = \frac{\sqrt{x}}{2\sqrt{3}\pi^{3/2}}\, e^{-\frac{x^3}{18}}\, K_{\frac{1}{6}}\left(\frac{x^3}{18}\right)$$

$$\mathcal{U}_1(x;1) = \frac{x^2}{4\sqrt{3}\pi^{3/2}\left(x^3\right)^{5/6}}\, e^{-\frac{x^3}{18}}\left(\left(x^3+6\right)K_{\frac{1}{6}}\left(\frac{x^3}{18}\right) - x^3 K_{\frac{7}{6}}\left(\frac{x^3}{18}\right)\right)$$

where $K_\nu(z)$ represents the modified Bessel function of second kind.

With $\mu = 2$, we have

$$\gamma_{2n} = \frac{4n(2n+1)(n+2)}{(3n+1)(3n+4)}\gamma_1, \quad n \geq 1,$$

$$\gamma_{2n+1} = \frac{4(n+1)(2n+1)}{(3n+4)}\gamma_1, \quad n \geq 0,$$

whilst the integral representation becomes

$$\mathcal{U}_0(x;2) = \frac{\sqrt{3}\ \Gamma(\frac{4}{3})}{2\ \pi\ 3^{\frac{1}{3}}4^{\frac{1}{3}}}\ \Gamma\left(\tfrac{1}{3}, \tfrac{1}{12}x^3\right)$$

$$\mathcal{U}_1(x;2) = \frac{\sqrt[6]{3}\Gamma\left(\frac{4}{3}\right)}{\sqrt[3]{4}\pi}\left(\frac{1}{2}x^2\ \Gamma\left(\tfrac{1}{3}, \tfrac{1}{12}x^3\right) - \sqrt[3]{18}\ \mathrm{e}^{-\frac{x^3}{12}}\right)$$

where $\Gamma(\alpha, z)$ represents the incomplete Gamma function:
$\Gamma(\alpha, z) = \int_z^{+\infty} t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t$ provided that $\alpha > 0$.

3rd order differential equation:

$$-\gamma_1(\mu+2)P_n'''(x)+2x^2P_n''(x)+2x\left(\mu+\frac{3}{4}\left((-1)^n+3\right)-\frac{n}{2}\right)P_n'(x)$$
$$=2n\left(\mu+\frac{n}{2}+\frac{3(-1)^n}{4}+\frac{5}{4}\right)P_n(x)$$

from which we deduce

$$P_n^{[0]}(x;\mu) \quad = \quad \frac{(-1)^n(3\mu+6)^n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{n}{2}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{5}{12}\right)_n}\,{}_2F_2\left(\begin{array}{c}-n,\ \frac{2\mu+3n}{6}+\frac{(-1)^n}{4}+\frac{5}{12}\\ \frac{1}{3},\ \frac{2}{3}\end{array};\frac{x}{3(\mu+2)}\right)$$

$$P_n^{[1]}(x;\mu) \quad = \quad \frac{(-1)^n(3\mu+6)^n\left(\frac{2}{3}\right)_n\left(\frac{4}{3}\right)_n}{\left(\frac{n}{2}+\frac{(-1)^{n+1}}{4}+\frac{\mu}{3}+\frac{11}{12}\right)_n}\,{}_2F_2\left(\begin{array}{c}-n,\ \frac{2\mu+3n}{6}+\frac{(-1)^{n+1}}{4}+\frac{11}{12}\\ \frac{2}{3},\ \frac{4}{3}\end{array};\frac{x}{3(\mu+2)}\right)$$

$$P_n^{[2]}(x;\mu) \quad = \quad \frac{(-1)^n(3\mu+6)^n\left(\frac{4}{3}\right)_n\left(\frac{5}{3}\right)_n}{\left(\frac{n}{2}+\frac{(-1)^n}{4}+\frac{\mu}{3}+\frac{17}{12}\right)_n}\,{}_2F_2\left(\begin{array}{c}-n,\ \frac{2\mu+3n}{6}+\frac{(-1)^n}{4}+\frac{17}{12}\\ \frac{4}{3},\ \frac{5}{3}\end{array};\frac{x}{3(\mu+2)}\right)$$

In this case we have

$$\gamma_{2n} = \frac{n(2n+1)(\rho+3)}{(3n+\rho)}\gamma_1, \quad n \geq 1,$$

$$\gamma_{2n+1} = \frac{(n+1)(2n+1)(n+\rho)(\rho+3)}{(3n+\rho+3)(3n+\rho)}\gamma_1, \quad n \geq 0.$$

With the choice of $\gamma_1 = \frac{2}{3(\rho+3)}$, we obtain

In this case we have

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With the choice of $\gamma_1 = \frac{2}{3(\rho+3)}$, we obtain

$$Q_n^{\text{case B}_2}(x;\mu) = P_n^{\text{case B}_1}(x;\mu+1), \quad \text{for all} \quad n \geq 0,$$

while

$$Q_n^{\text{case B}_1}(x;\mu) = P_n^{\text{case B}_2}(x;\mu+2), \quad \text{for all} \quad n \geq 0,$$

which brings

$$\frac{1}{(n+2)(n+1)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_{n+2}(x;\mu) = P_n(x;\mu+3)$$

We set

$$\gamma_1 = \frac{2}{(\mu+2)(\rho+3)}$$

so that

$$\gamma_{2n} := \gamma_{2n}(\mu,\rho) = \frac{2n(2n+1)(n+\mu)}{(3n+\mu-1)(3n+\mu+2)(3n+\rho)}, \quad n \geq 1,$$

$$\gamma_{2n+1} := \gamma_{2n}(\mu,\rho) = \frac{2(n+1)(2n+1)(n+\rho)}{(3n+\mu+2)(3n+\rho)(3n+\rho+3)}, \quad n \geq 0.$$

Besides, we have

$$\begin{cases} \left(1-x^3\right) u_0'' + x^2(\mu+\rho-4)u_0' - (\mu-2)(\rho-1)xu_0 = 0, & \\ \frac{\mu}{(\mu+2)} u_1 = \left(x^3-1\right) u_0' - (\rho-1)x^2 \; u_0, & \text{for} \quad \mu > -1, \\ xu_1' = 2u_0', & \text{for} \quad \mu = 0. \end{cases}$$

Some of these are related to polynomials introduced by Pincherle (1890) and later extended by Humbert (1920), which were also related to ${}_3F_2$ functions by Baker (1920).

Here we have

$$\langle u_k, f(x)\rangle = \frac{1}{3}\left(\int_0^1 f(x)\mathcal{U}_k(x)\mathrm{d}x + \omega^{2k-1}\int_0^{\omega} f(x)\mathcal{U}_k(\omega^2 x)\mathrm{d}x + \omega^{1-2k}\int_0^{\omega^2} f(x)\mathcal{U}_k(\omega x)\mathrm{d}x\right)$$

with

$$\begin{aligned}\mathcal{U}_0(x) &:= \mathcal{U}_0(x;\mu,\rho)\\ &= \frac{3\Gamma\left(\frac{\mu+2}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}(1-x^3)^{\frac{\mu+\rho-1}{3}}\ {}_2F_1\left(\begin{matrix}\frac{\mu}{3}, \frac{\rho+1}{3}\\ \frac{\mu+\rho+2}{3}\end{matrix}; 1-x^3\right),\end{aligned}$$

$$\begin{aligned}\mathcal{U}_1(x) &:= \mathcal{U}_1(x;\mu,\rho)\\ &= \frac{3\Gamma\left(\frac{\mu+5}{3}\right)\Gamma\left(\frac{\rho}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{\mu+\rho+2}{3}\right)}x^2(1-x^3)^{\frac{\mu+\rho-1}{3}}\,{}_2F_1\left(\begin{matrix}\frac{\mu}{3}+1, \frac{\rho+1}{3}\\ \frac{\mu+\rho+2}{3}\end{matrix}; 1-x^3\right).\end{aligned}$$

Humbert polynomials: when $\mu = \frac{3\nu-1}{2}$ and $\rho = \frac{3\nu}{2}$, this 2-OPS satisfies

$$P_{n+2}(x; \tfrac{3\nu-1}{2}, \tfrac{3\nu}{2}) \\ = xP_{n+1}(x; \tfrac{3\nu-1}{2}, \tfrac{3\nu}{2}) - \frac{4}{27}\frac{n(n+1)(3\nu+n-1)}{(\nu+n-1)(\nu+n)(\nu+n+1)} P_{n-1}(x; \tfrac{3\nu-1}{2}, \tfrac{3\nu}{2})$$

"Chebyshev"-type polynomials: when $\nu = 1 \ \Rightarrow (\mu, \rho) = (1, 3/2)$:

$$P_{n+2}(x; 1, \tfrac{3}{2}) = xP_{n+1}(x; 1, \tfrac{3}{2}) - \tfrac{4}{27} P_{n-1}(x; 1, \tfrac{3}{2})$$

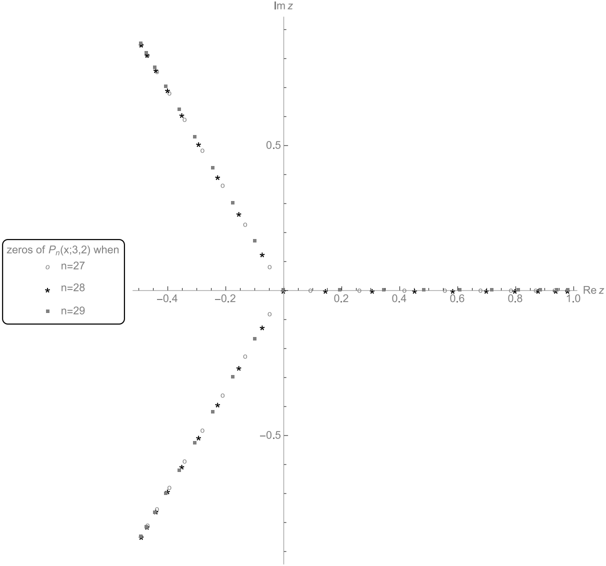
and here

$$\mathcal{U}_0(x) = \frac{9\sqrt{3}}{4\pi}\left(\left(1+\sqrt{1-x^3}\right)^{1/3} - \left(1-\sqrt{1-x^3}\right)^{1/3} \right)$$

$$\mathcal{U}_1(x) = \frac{27\sqrt{3}}{8\pi}\left(\sqrt{1-x^3}\left[\left(1+\sqrt{1-x^3}\right)^{2/3} - \left(1-\sqrt{1-x^3}\right)^{2/3} \right] \right.$$

$$
\begin{aligned}
P_{3n}(x;\mu,\rho) &= \frac{(-1)^n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{5}{12}\right)_n \left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{1}{4}\right)_n} \\
&\quad {}_3F_2\left(\begin{matrix} -n,\ \frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{5}{12},\ \frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{1}{4} \\ \frac{1}{3},\ \frac{2}{3} \end{matrix};x^3\right) \\
P_{3n+1}(x;\mu,\rho) &= x\,\frac{(-1)^n \left(\frac{2}{3}\right)_n \left(\frac{4}{3}\right)_n}{\left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{11}{12}\right)_n \left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{3}{4}\right)_n} \\
&\quad {}_3F_2\left(\begin{matrix} -n,\ \frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{11}{12},\ \frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{3}{4} \\ \frac{2}{3},\ \frac{4}{3} \end{matrix};x^3\right) \\
P_{3n+2}(x;\mu,\rho) &= x^2\,\frac{(-1)^n \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n}{\left(\frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{17}{12}\right)_n \left(\frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{5}{4}\right)_n} \\
&\quad {}_3F_2\left(\begin{matrix} -n,\ \frac{n}{2}+\frac{1}{4}(-1)^{3n}+\frac{\mu}{3}+\frac{17}{12},\ \frac{n}{2}-\frac{1}{4}(-1)^{3n}+\frac{\rho}{3}+\frac{5}{4} \\ \frac{4}{3},\ \frac{5}{3} \end{matrix};x^3\right)
\end{aligned}
$$

Case C: zeros of $P_n(x; 3, 2)$



THANK YOU!